Equation (7) can readily be solved by iteration. Most importantly, it provides a convenient means of performing a sensitivity analysis that is always an essential part of the decision analysis. One can also easily discern from (7) the structural dependence of the optimal target a^* upon the outcome function (through k, k^o, k^u), the utility function (through c), and the distribution of the uncertain state (through y, M, S^2). In particular, when the utility function ν is linear so that $c = 0$, then (7) reduces to

$$a^* = y + M + Q^{-1}\left(\frac{k^u}{k^u + k^o}\right)S, \tag{9}$$

which is a special case of (3), under our assumptions 1 and 2.

III. Derivation of Optimal Solution

The derivation of (7)–(8) is accomplished in four steps; details of lengthy but straightforward transformations are omitted.

1) Utility of State-Target Pairs: While expression (5) is convenient for the estimation of parameter b, an alternative form is advantageous for decision analysis:

$$\nu(x) = pe^{-cx} - q, \tag{10}$$

where c is given by (8c), $q = 1/[\exp(-b) - 1]$, and $p = q\exp(cx^o)$. With $u = \nu(B)$ denoting the composition of (10) and (2), after dropping the inessential scaling constant q and introducing the relation $r = y + \omega$, we have

$$u(y+\omega, a) = \begin{cases} \alpha^o e^{ck^o a} e^{-c(k+k^o)\omega} & \text{if } \omega \leqslant a - y, \\ \alpha^u e^{-ck^u a} e^{-c(k-k^u)\omega} & \text{if } \omega \geqslant a - y, \end{cases} \tag{11}$$

where

$$\begin{aligned} \alpha^o &= pe^{-c[(k+k^o)y - kr^o]}, \\ \alpha^u &= pe^{-c[(k-k^u)y - kr^o]}. \end{aligned} \tag{12}$$

2) Utility of Decision: Given u and density g corresponding to distribution G, the utility of any decision a is specified by

$$\begin{aligned} U(a) &= \int_\Omega u(y+\omega, a) g(\omega)\, d\omega \\ &= \alpha^o e^{ck^o a} \int_{-\infty}^{a-y} e^{-c(k+k^o)\omega} g(\omega)\, d\omega \\ &\quad + \alpha^u e^{-ck^u a} \int_{a-y}^{\infty} e^{-c(k-k^u)\omega} g(\omega)\, d\omega. \end{aligned} \tag{13}$$

The first derivative of U is

$$\begin{aligned} \frac{dU(a)}{da} &= \alpha^o ck^o e^{ck^o a} \int_{-\infty}^{a-y} e^{-c(k+k^o)\omega} g(\omega)\, d\omega \\ &\quad - \alpha^u ck^u e^{-ck^u a} \int_{a-y}^{\infty} e^{-c(k-k^u)\omega} g(\omega)\, d\omega, \end{aligned} \tag{14}$$

as the other two terms cancel out.

3) Normality Assumption: We shall now make use of the assumption $G = N(M, S^2)$, so that for any constant β,

$$e^{-\beta\omega} g(\omega) = e^{-\beta(M - \beta S^2/2)} h(\omega), \tag{15}$$

where h is the density corresponding to distribution $H = N(M - \beta S^2, S^2)$. By applying (15) to (14), and solving both integrals, we obtain

$$\frac{dU(a)}{da} = A^o e^{ck^o a} H^o(a) - A^u e^{-ck^u a}[1 - H^u(a)], \tag{16}$$

where

$$\begin{aligned} A^o &= \alpha^o ck^o \exp\{-c(k+k^o)[M - c(k+k^o)S^2/2]\}, \\ A^u &= \alpha^u ck^u \exp\{-c(k-k^u)[M - c(k-k^u)S^2/2]\}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} H^o &= N(m^o, S^2), \\ H^u &= N(m^u, S^2), \end{aligned} \tag{18}$$

with means m^o and m^u specified by (8a) and (8b).

4) Optimality Condition: The optimality condition $dU(a^*)/da = 0$ derived from (16) takes the form

$$e^{c(k^o + k^u)a^*} = \frac{A^u}{A^o} \frac{1 - H^u(a^*)}{H^o(a^*)}. \tag{19}$$

It remains now to demonstrate that

$$\begin{aligned} \frac{A^u}{A^o} &= \frac{k^u}{k^o} \exp\{c(k^o + k^u) \\ &\quad \cdot [y + M - c(2k + k^o - k^u)S^2/2]\} \\ &= \frac{k^u}{k^o} \exp[c(k^o + k^u)(m^o + m^u)/2], \end{aligned} \tag{20}$$

and that

$$c(k^o + k^u) = (m^u - m^o)/S^2, \tag{21}$$

where m^o and m^u are defined by (8a) and (8b). By inserting (20) and (21) into (19), we obtain

$$\exp\left[\frac{m^u - m^o}{2S}\left(\frac{a^* - m^o}{S} + \frac{a^* - m^u}{S}\right)\right] = \frac{k^u}{k^o} \frac{1 - H^u(a^*)}{H^o(a^*)}. \tag{22}$$

Finally, substituting H^u and H^o by the standard normal distribution Q and rearranging the terms results in (7).

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The Interactive Step Trade-Off Method (ISTM) for Multiobjective Optimization

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***Abstract*—A new method to solve multiobjective optimization problems (MOP's), called the interactive step trade-off method (ISTM), is proposed here. With the help of an auxiliary problem $AP(\epsilon^{l-1})$, the preferred solutions of MOP's will be found on their efficient solution faces in ISTM. Using local trade-off information presented by the analyst, the decisionmaker (DM) makes decisions step by step, which heuristically directs the analyst to look for efficient solutions of MOP's, along preference directions, in proper step sizes, until the preferred solutions are found. The interaction in the method is practical, clear, and easy to understand. The method may be used in the design of decision support systems in such fields as the production management of industrial enterprises.**

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I. Introduction

In the past two decades, many kinds of methods to solve MOP's have been proposed from different points of view [11], [15], [16]. Recently, however, more attention has been paid to interactive methods because of their practicality [12], [32]. Several well-known methods have been presented by, for instance, Y. Y. Haimes *et al.* [13], Benayoun *et al.* [2], Geoffrion *et al.* [10], Belenson and Kapur [1], and Zionts and Wallenius [33]. Some methods have been successfully applied to many fields, such as the planning of water resources, forest management, production planning, etc.

Generally, a multiobjective optimization problem (MOP) can be expressed as follows:

$$\begin{cases} \max F(X) = \left[F_1(X), \quad \cdots, \quad F_p(X)\right]^T \\ \text{s.t. } X \in \Omega, \quad \Omega \triangleq \{X | g_i(X) \leqslant 0, \quad i = 1, \cdots, m\} \end{cases} \tag{1}$$

where X is an n-dimensional decision vector, Ω a nonempty constraint domain (also called decision space), and $F_l(X)$ ($l = 1, \cdots, p$) the objective functions (OF), which are usually conflicting with each other and noncommensurable. Therefore it is generally impossible in MOP's to obtain an optimal solution at which all objective functions are optimized. We can only obtain preferred efficient solutions.

MOP can be used as one of the basic quantitative models of a decision support system (DSS). As we known, a DSS is designed to help, rather than replace, the decisionmaker (DM) to make decisions in complex situations. Hence any method for solving MOP's in a DSS must possess the following two functions: 1) generate efficient solutions, and 2) provide the DM with helpful trade-off information. Although some proposed interactive methods may have both functions, their interactive procedures between analysts and the DM seem indirect and not practical enough for DM to provide direct preference information.

Haimes and Chankong [3], [13], [14] used the trade-off rates between objective functions as local trade-off information in their SWT method, so that the DM is able to effectively make decision analysis to provide useful preference information. STEM, proposed by Benayoun *et al.*, possessed the advantage of simple and clear interactive procedure [8], [15]. By synthesizing the authors' experience in designing a decision support system for production planning of an oil refinery, this paper attempts to combine some advantages of the two methods mentioned and to propose an interactive method (ISTM) which enables the DM to directly express his preference information in interactive procedures.

Three basic steps are included in ISTM, as in other interactive methods. First, an efficient solution and the corresponding local trade-off information (there may be different contents in local trade-off information for different methods) are provided by the analyst. Then the DM determines the preference direction and step size. Again the analyst looks for a new efficient solution according to the preference information. The new solution should dominate the previous one. In ISTM, the efficient solution and the local trade-off information, the current values of objective functions and the trade-off rates between them, are obtained by solving an auxiliary problem $AP(\epsilon^{t-1})$, which is fundamental to ISTM. $AP\epsilon^{t-1})$ is first defined in the paper, and then relationships between the optimal solutions of $AP(\epsilon^{t-1})$ and the efficient solutions of the original problem are explored. The basic idea of designing $AP(\epsilon^{t-1})$ comes from a reasonable classification of the objective space given by W. Michalowski and A. Piotrowski [21], and by K. Musselman and J. Talavage [22]. Then the relationships between the Kuhn–Tucker multipliers (or simplex multipliers) of $AP(\epsilon^{t-1})$ and the trade-off rates are analyzed. Subsequently, steps of ISTM algorithm are given. Finally, an example is discussed to illustrate the use of the algorithm.

II. The Auxiliary Problem and its Properties

A. Basic Concepts and Assumptions

Regarding efficient solutions of MOP's, we define the following basic concepts.

Definition 1: X^0 is called an efficient (noninferior or Pareto-optimal) solution of MOP if there does not exist any $X \in \Omega$, $X \neq X^0$, so that $F(X) \geqslant F(X^0)$ and $F(X) \neq F(X^0)$.

Definition 2: X^0 is called a weakly efficient solution of MOP if there does not exist any $X \in \Omega$, $X \neq X^0$, so that $F(X) > F(X^0)$.

Definition 3: The set of all efficient solutions in Ω is called the efficient solution face of MOP.

The investigation here is based on the following assumptions.

1) Objective functions $F_l(X)$ ($l = 1, \cdots, p$) and constraint functions $g_i(X)$ ($i = 1, \cdots, m$) are all twice continuously differentiable.
2) Decision space Ω is a compact set which is closed and bound.
3) The preferred solution of an MOP must be on its efficient solution face.
4) The DM can judge the relative importance among objective functions and is able to decide whether it is worth substituting one unit of $F_k(X)$ with several units of $F_l(X)$ ($l, k = 1, \cdots, p$; $l \neq k$).

B. The Definition of Auxiliary Problem $AP(\epsilon^{t-1})$

Suppose t represents the number of interaction, and given an efficient solution of the MOP X^{t-1} and its corresponding objective values $F_l(X^{t-1})$ ($l = 1, \cdots, p$), which are not preferred by the DM; then the set of objective indices can be classified into the following three subsets.

- W the index subset of objective functions which should be improved from $F_i(X^{t-1})$.
- R the index subset of objective functions which should be maintained at least at the current level, $F_j(X^{t-1})$.
- Z the index subset of objective functions which should be decreased from $F_k(X^{t-1})$.

Let

$$\begin{cases} W = \{i | i = i_1, i_2, \cdots, i_w\} \\ R = \{j | j = j_1, j_2, \cdots, j_r\} \\ Z = \{k | k = k_1, k_2, \cdots, k_z\}. \end{cases} \tag{2}$$

Then $WURUZ = \{1, 2, \cdots, p\}$ and $W \cap R \cap Z = \phi$. Suppose $Xa = [X^T u_{i_1}, \cdots, u_{i_w}]^T$ and $\epsilon_k^{t-1} = F_k(X^{t-1}) - dF_k(X^{t-1})$, where $dF_k(X^{t-1})$ is the reduced value of $F_k(X)$, $dF_k(X^{t-1}) \geqslant 0$, and $u_i (i \in W)$ is an auxiliary variable, then the auxiliary problem $AP(\epsilon^{t-1})$ can be defined as

$$\begin{cases} \max u = \sum_{i \in W} \sigma_i u_i \\ \text{s.t. } Xa \in \Omega a \end{cases} \tag{3}$$

$$\Omega a \triangleq \left\{ Xa \;\middle|\; \begin{array}{ll} F_i(X) - h_i \cdot u_i \geqslant F_i(X^{t-1}), \quad u_i \geqslant 0, & i \in W \\ F_j(X) \geqslant F_j(X^{t-1}), & j \in R \\ F_k(X) \geqslant \epsilon_k^{t-1}, & k \in Z \\ X \in \Omega & \end{array} \right\}$$

where u_i is maximized to improve $F_i(X)$ as greatly as possible, u is the auxiliary objective function, and σ_i is a positive weighting factor, which is determined according to the relative importance of the objective functions in subset W. Normally, we let $\sigma_i = 1$ ($i \in W$). In (3), $\epsilon^{t-1} = [\epsilon_{k_1}^{t-1}, \cdots, \epsilon_{k_z}^{t-1}]^T$ and h_i is a weighting factor defined as

$$h_i = d_i - c_i \qquad i \in W. \tag{4}$$

In (4), d_i expresses the ideal value of the ith objective function and c_i the lower bound of the ith objective function. We assume that D strictly dominates C, or $d_i > c_i$ $(i \in W)$, where

$$D = [d_{i_1}, \cdots, d_{i_w}]^T, \qquad C = [c_{i_1}, \cdots, c_{i_w}]^T. \tag{5}$$

Hence $h_i > 0$. If it is impossible for the DM to determine the values of d_i and c_i, a simple rule introduced in the section IV can be adopted to find them. In the subsequent discussion, let $t = 1$ for convenience. All the conclusions for $t = 1$, however, are also true for $t > 1$.

There are three advantages of using $AP(\epsilon^0)$ to look for the preferred solutions of the MOP.

1) $AP(\epsilon^0)$ clearly reflects the interactive direction, that is, the DM can assign the three index subsets of objective functions and can select the values of $dF_k(X^0)$. For real complex problems, we have proposed an expert system approach to perform the assignment of the three indices and the selection of the values of $dF_k(X^0)$ [29], [30].
2) The search domain of $AP(\epsilon^0)$ is greatly reduced by making full use of assumption 3).
3) It becomes possible to provide trade-off information to the DM in trade-off methods, to help the DM select proper $dF_k(X^0)$.

C. The Properties of $AP(\epsilon^0)$'s Optimal Solutions

It can be proved that $AP(\epsilon^0)$'s optimal solutions possess the properties described by following lemmas 1 and 2 and theorem 1.

Lemma 1:

1) Ωa is nonempty if X^0 is a feasible solution of MOP.
2) If Xa^1 is an optimal solution of $AP(\epsilon^0)$, where $Xa^1 = [(X^1)^T, u_{i_1}^1, \cdots, u_{i_w}^1]^T$, then $F_i(X^1) - h_i u_i^1 = F_i(X^0)$, $i \in W$.

Proof:

1) Let $Xa^0 = [(X^0)^T 0 \cdots 0]^T$. Since X^0 is supposed to be a feasible solution of MOP, naturally $X^0 \in \Omega$, and hence $Xa^0 \in \Omega a$. Therefore there exists at least one point Xa^0 in Ωa, that is, Ωa is nonempty.
2) If the conclusion is not true for any $i \in W$, such as i_λ, i_τ $(1 \leqslant \lambda \leqslant \tau \leqslant w)$, considering $Xa^1 \in \Omega a$, we then have

$$F_i(X^1) - h_i u_i^1 > F_i(X^0) \qquad \text{when } i = i_\lambda, i_\tau.$$

Then a vector Xa',

$$Xa' = \left[(X^1)^T u_{i_1}^1 \cdots u_{i_{\lambda-1}}^1 u_{i_\lambda}' u_{i_{\lambda+1}}^1 \cdots u_{i_{\tau-1}}^1 u_{i_\tau}' u_{i_{\tau+1}}^1 \cdots u_{i_w}^1\right]^T$$

can be found, so that

$$F_i(X^1) - h_i u_i' = F_i(X^0) \qquad \text{when } i = i_\lambda, i_\tau.$$

Hence

$$F_i(X^0) = F_i(X^1) - h_i u_i' < F_i(X^1) - h_i u_i^1 \qquad \text{when } i = i_\lambda, i_\tau$$

or

$$h_i u_i' > h_i u_i^1 \qquad \text{when } i = i_\lambda, i_\tau.$$

Notice that $h_i > 0$ $(i \in W)$; we obtain $u_i' > u_i^1$ $(i = i_\lambda, i_\tau)$, which means

$$u_{i_1}^1 + \cdots + u_{i_{\lambda-1}}^1 + u_{i_\lambda}' + u_{i_{\lambda+1}}^1 + \cdots + u_{i_{\tau-1}}^1 + u_{i_\tau}' + u_{i_{\tau+1}}^1 + \cdots + u_{i_w}^1 > u_{i_1}^1 + u_{i_2}^1 + \cdots + u_{i_w}^1 = \max_{Xa \in \Omega a} \sum_{i \in W} u_i.$$

It is easy to prove $Xa' \in \Omega a$. As a result, we conclude that Xa^1 is not the optimal solution of $AP(\epsilon^0)$, which contradicts our assumption.

Lemma 2: If $\hat{X}a \in \Omega a$, where $\hat{X}a = [\hat{X}^T, \hat{u}_{i_1}, \cdots, \hat{u}_{i_w}]^T$, and $F_i(X) - h_i \cdot u_i = F_i(X^0)$ for all points in a neighborhood of $\hat{X}$, then there exists a feasible direction $\hat{d}_i$ emanating from $\hat{X}a$, such that

$$\frac{\partial F_i(\hat{X}, \hat{d}_i)}{\partial \hat{u}_i} = \lim_{\Delta \hat{u} \to 0} \frac{\Delta F_i(\hat{X}, \hat{d}_i)}{\Delta \hat{u}_i} = h_i.$$

Proof: According to lemma 1, there must exist $a\hat{X}a \in \Omega a$ (such as an optimal solution of $AP(\epsilon^0)$, Xa^1), such that

$$F_i(\hat{X}) - h_i \hat{u}_i = F_i(X^0).$$

Since we assume that decision space Ω is a compact set (assumption 2)) and objective functions $F_l(X)$ $(l = 1, \cdots, p)$ are all twice continuously differentiable (assumption 1)), the set Ωa is also compact. Then there exists a nonempty compact subset $\Omega a^i \subset \Omega a$ in a neighborhood of $\hat{X}$, such that for any $Xa \in \Omega a^i$, we have

$$F_i(X) - h_i u_i = F_i(X^0). \tag{6}$$

Suppose $\hat{d}_i$ is a feasible direction emanating from the point $\hat{X}$, we then define a vector

$$\hat{X}a = \left[(\hat{X} + \alpha_i \hat{d}_i)^T, \quad \hat{u}_{i_1}, \cdots, (\hat{u}_i + \Delta \hat{u}_i), \cdots, \hat{u}_{i_w}\right]^T$$

where $0 \leqslant \alpha_i \leqslant \bar{\alpha}_i (\bar{\alpha}_i > 0)$ and $\alpha_i \to 0$ when $\Delta \hat{u}_i \to 0$. Then, if we select α_i and a feasible direction $\hat{d}_i$ for a small increment $\Delta \hat{u}_i$ so that $\hat{X}a \in \Omega a^i$, we will have

$$F_i(\hat{X} + \alpha_i \hat{d}_i) - h_i(\hat{u}_i + \Delta \hat{u}_i) = F_i(X^0). \tag{7}$$

Because an optimal solution of $AP(\epsilon^0)$ is also an efficient solution of the MOP, and all efficient solutions of the MOP can be found by solving $AP(\epsilon^0)$ (see the following theorem 1), it is obvious from lemma 1 that such a feasible direction $\hat{d}_i$ always exists on the efficient solution face of MOP in a neighborhood of $\hat{X}$ (here, suppose $\hat{X}$ is an efficient solution), which is sufficient for our interactive algorithm, designed in the following sections. In fact, if we add some very small perturbations into the right-side values ϵ_k^0 of $AP(\epsilon^0)$ $(k \in Z)$, we can obtain other optimal solutions of $AP(\epsilon^0)$, being located on the efficient solution face of the MOP in a neighborhood of $\hat{X}$ and satisfying (7). In other words, the compact subset Ωa^i can be constructed on the efficient solution face of the MOP in the neighborhood of $\hat{X}$.

Therefore, along the direction $\hat{d}_i$, we obtain

$$\begin{aligned}\Delta F_i(\hat{X}, \hat{d}_i) &= F_i(\hat{X} + \alpha_i \hat{d}_i) - F_i(\hat{X}) \\ &= h_i(\hat{u}_i + \Delta \hat{u}_i) + F_i(X^0) - [h_i \hat{u}_i + F_i(X^0)] \\ &= h_i \Delta \hat{u}_i.\end{aligned}$$

Notice that $\Delta F_i(\hat{X}, \hat{d}_i) \to 0$ when $\alpha_i \to 0$; we therefore conclude

$$\lim_{\Delta \hat{u}_i \to 0} \frac{\Delta F_i(\hat{X}, \hat{d}_i)}{\Delta \hat{u}_i} = h_i.$$

Lemma 2 tells us that in interaction, trade-off should be made along directions on the efficient solution face of the MOP if a preferred solution is searched for using $AP(\epsilon^0)$. However, such a conclusion depends on the following theorem.

Theorem 1:

1) If Xa^1 is an optimal solution of $AP(\epsilon^0)$, $Xa^1 = [(X^1)^T u_{i_1}^1 \cdots u_{i_w}^1]^T$, X^1 is also a weakly efficient solution of the MOP.
2) If Xa^1 is the unique optimal solution, X^1 is then the efficient solution.
3) Any efficient solution of the MOP can be generated by solving $AP(\epsilon^0)$ through proper choices of W, R, and Z, and the right-side values of objective constraints in Ωa.

Proof: Define two subspaces Ω_0 and Ω_b

$$\Omega_0 = \left\{ Xa \;\middle|\; \begin{pmatrix} F_i(X) - h_i u_i \\ F_j(X) \\ F_k(X) \end{pmatrix} \geqslant \begin{pmatrix} F_i(X^0) \\ F_j(X^0) \\ \epsilon_k^0 \end{pmatrix} \quad \begin{matrix} u_i \geqslant 0, & i \in W \\ & j \in R \\ & k \in Z \end{matrix} \right\}$$

$$\Omega_b = \left\{ Xa \;\middle|\; \begin{pmatrix} F_i(X) - h_i u_i \\ F_j(X) \\ F_k(X) \end{pmatrix} \ngeqslant \begin{pmatrix} F_i(X^0) \\ F_j(X^0) \\ \epsilon_k^0 \end{pmatrix} \quad \begin{matrix} u_i \geqslant 0, & i \in W \\ & j \in R \\ & k \in Z \end{matrix} \right\}$$

where the symbol "$\ngeqslant$" means that there exists at least one "$<$" relationship between the elements in a vector inequality. Evidently, the following equation is true:

$$\Omega_0 \cup \Omega_b = R^{n+w}.$$

For convenience, let's define Ω of (1) again. Let $Xe = [X^T, 0, \cdots, 0]$ so that Xe and Xa have the same dimension. Then Ω is defined as the following equivalent form:

$$\Omega = \{ Xe | g_i(Xe) \leqslant 0, \qquad i = 1, \cdots, m \}.$$

Obviously, $\Omega \subseteq R^{n+w}$. Hence

$$\begin{aligned} \Omega &= \Omega \cap (\Omega_0 \cup \Omega_b) \\ &= (\Omega \cap \Omega_0) \cup (\Omega \cap \Omega_b) \\ &= \Omega_a \cup \Omega_\perp \end{aligned}$$

where

$$\Omega_a = \Omega \cap \Omega_0, \qquad \Omega_\perp = \Omega \cap \Omega_b.$$

1) Let $Xa^1 = [(X^1)^T u_{i_1}^1 \cdots u_{i_w}^1]^T$ be an optimal solution of $AP(\epsilon^0)$. If $X \in \Omega_\perp$, there must be at least one 1 ($1 \in W$, or $1 \in R$, or $1 \in Z$), such that

$$F_\iota(X) < F_\iota(X^0) \qquad \text{when } 1 \in R$$

or

$$F_\iota(X) < \epsilon_\iota^0 \qquad \text{when } 1 \in Z$$

or

$$F_\iota(X) - h_\iota u_\iota < F_\iota(X^0) \text{ for any } u_\iota \geqslant 0 \quad \text{when } 1 \in W$$

so

$$F_\iota(X) < F_\iota(X^0) \qquad \text{when } 1 \in W.$$

Hence, for any $X \in \Omega_\perp$, we have

$$F_\iota(X^1) \geqslant F_\iota(X^0) > F_\iota(X) \qquad \text{when } 1 \in R$$

or

$$F_\iota(X^1) \geqslant \epsilon_\iota^0 > F_\iota(X) \qquad \text{when } 1 \in Z$$

or

$$F_\iota(X^1) - h_\iota u_\iota^1 = F_\iota(X^0) > F_\iota(X) \quad \text{when } 1 \in W$$

so

$$F_\iota(X^1) > F_\iota(X) + h_\iota u_\iota^1 \geqslant F_\iota(X) \quad \text{when } 1 \in W$$

where $h_\iota > 0$ and $u_\iota^1 \geqslant 0$ are considered. The preceding relations mean that it is impossible that $F(X) > F(X^1)$ for any $X \in \Omega_\perp$.

If $X \in \Omega_a$, let $Xa' = [X^T u_{i_1}' \cdots u_{w}']^T \in \Omega_a$; then

$$F_i(X) - h_i u_i' \geqslant F_i(X^0) \qquad i \in W$$

or

$$F_i(X) \geqslant F_i(X^0) + h_i u_i' \qquad i \in W.$$

If the symbol "$\ngtr$" means that there exists at least one pair of elements with the "$\leqslant$" relationship, the following vector inequality will be true:

$$\begin{pmatrix} F_{i_1}(X) \\ \vdots \\ F_{i_w}(X) \end{pmatrix} \ngtr \begin{pmatrix} F_{i_1}(X^0) + h_{i_1} u_{i_1}^1 \\ \vdots \\ F_{i_w}(X^0) + h_{i_w} u_{i_w}^1 \end{pmatrix}.$$

Otherwise, a group of u_i'' could be found where $u_i'' > u_i^1$ ($i \in W$), such that

$$F_i(X) = F_i(X^0) + h_i u_i'' \qquad i \in W.$$

Obviously, $Xa'' = [X^T u_{i_1}'' \cdots u_{i_w}'']^T \in \Omega_a$. On the other hand, since

$$\sum_{i \in W} u_i'' > \sum_{i \in W} u_i^1 = \max_{Xa \in \Omega_a} \sum_{i \in W} u_i$$

we conclude that Xa^1 is not the optimal solution of $AP(\epsilon^0)$, which contradicts the assumption. According to lemma 1, we obtain

$$\begin{pmatrix} F_{i_1}(X) \\ \vdots \\ F_{i_w}(X) \end{pmatrix} \ngtr \begin{pmatrix} F_{i_1}(X^0) + h_{i_1} u_{i_1}^1 \\ \vdots \\ F_{i_w}(X^0) + h_{i_w} u_{i_w}^1 \end{pmatrix} = \begin{pmatrix} F_{i_1}(X^1) \\ \vdots \\ F_{i_w}(X^1) \end{pmatrix}$$

which shows that $F(X) \ngtr F(X^1)$ when $X \in \Omega_a$.

As a whole, there exists no $X \in \Omega_a \cup \Omega_\perp = \Omega (X \neq X^1)$, such that $F(X) > F(X^1)$.

2) If $X \in \Omega_\perp$, it is easy to prove $F(X) \ngeqslant F(X^1)$, according to the definition of $\Omega_\perp$, and similar reasoning for 1).

If $X \in \Omega_a$, but $X \neq X^1$, we must have

$$\begin{pmatrix} F_{i_1}(X) \\ \vdots \\ F_{i_w}(X) \end{pmatrix} \ngeqslant \begin{pmatrix} F_{i_1}(X^0) + h_{i_1} u_{i_1}^1 \\ \vdots \\ F_{i_w}(X^0) + h_{i_w} u_{i_w}^1 \end{pmatrix} = \begin{pmatrix} F_{i_1}(X^1) \\ \vdots \\ F_{i_w}(X^1) \end{pmatrix}$$

Otherwise, only two cases might occur. First, the relationships $F_i(X) = F_i(X^1)$ ($i \in W$) might be true, which, however, would contradict the assumption of Xa^1 being the unique optimal solution of $AP(\epsilon^0)$.

Secondly, at least one "$>$" relationship might exist between some pair of elements, and "$=$" relationships would be true between the other elements. In this case, we can obtain the conclusion that Xa^1 is not the optimal solution of $AP(\epsilon^0)$ according to the similar reasoning for 1). Hence, if Xa^1 is the unique optimal solution of $AP(\epsilon^0)$, there exists no $X \in \Omega$ ($X \neq X^1$), such that $F(X) \geqslant F(X^1)$.

3) If W is chosen to include only one objective function index, and the right-side vector of objective constraints in Ω_a is considered to be arbitrarily given values, then $AP(\epsilon^0)$ is the same as ϵ-constraint problems, as defined in [3], [4], and [14]. Therefore the proof of the conclusion is also the same as the proof of theorem 1 in [14].

Conclusions 1) and 2) in the preceding theorem 1 show that the preferred solution will be an efficient (weakly efficient) solution of the MOP if $AP(\epsilon^0)$ is used as the interactive model.

Then the algorithm of this paper will look only for the preferred solution on the efficient solution face of the MOP. The DM determines search directions and selects step sizes according to local trade-off information, which is determined, step by step, by the optimal solutions of $AP(\epsilon^0)$ and the trade-off rates analyzed in the following.

III. Trade-Off Rates and Decision Analysis

A. Trade-Off Rates

In the MOP, the following trade-off problems must be answered. Is it worth decreasing some objective functions so as to increase some other objective functions? If so, how much is reasonable? Following [14], these concepts can be quantitatively described as the trade-off rates.

Definition 4: Given an efficient solution of the MOP, X^1, and a feasible direction, d^1, emanating from X^1 (i.e., there exists $\alpha_0 > 0$ so that $X^1 + \alpha d^1 \in \Omega$ for $0 \leqslant \alpha \leqslant \alpha_0$). Define $T_{i,k}(X^1, d^1)$ as

$$T_{i,k}(X^1,d^1) = \lim_{\alpha \to 0} \frac{F_i(X^1+\alpha d^1) - F_i(X^1)}{F_k(X^1+\alpha d^1) - F_k(X^1)} = \frac{\partial F_i(X^1,d^1)}{\partial F_k(X^1,d^1)}. \tag{8}$$

If there exist a d_0^l and a $\bar{\alpha} > 0$ so that $F_l(X^1 + \alpha d_0^l) = F_l(X^1)$ $(l \neq i,k,\ 0 \leqslant \alpha \leqslant \bar{\alpha})$, then we call the corresponding $T_{i,k}(X^1, d^1)$ the partial trade-off rate of objective functions $F_k(X)$ and $F_i(X)$ in X^1 (or, the trade-off rate).

B. Trade-Off Rates and Kuhn–Tucker Multipliers of $AP(\epsilon^0)$

The relationships between trade-off rates and Kuhn–Tucker multipliers can be obtained from the sensitivity theorem [3], [20]. With the help of the definition of $AP(\epsilon^0)$ and theorem 4.30 in [3], we can prove the following theorem.

Theorem 2: Given the optimal solution of $AP(\epsilon^0)$, Xa^1, and the corresponding Kuhn–Tucker multipliers, λ_k^1 $(k \in Z)$, for the constraints $F_k(X) \geqslant \epsilon_k^0$ $(k \in Z)$, if

1) Xa^1 is a regular point for the constraints of $AP(\epsilon^0)$,
2) Xa^1 satisfies the second-order sufficient conditions of optimality for $AP(\epsilon^0)$, and
3) the constraints $F_k(X) \geqslant \epsilon_k^0$ $(k \in Z)$ are not degenerate in Xa^1; then

$$\lambda_k^1 = -\left.\frac{\partial u(X)}{\partial \epsilon_k^0}\right|_{Xa = Xa^1} \qquad k \in Z. \tag{9}$$

The definitions of regular points, nondegenerate constraints, and the second-order sufficient conditions in theorem 2 can be found in [3] and [20]. In $AP(\epsilon^0)$, condition 3) is always considered to be satisfied.

Considering the formulation of u, we can get

$$\partial u = \sum_{i \in W} \partial u_i. \tag{10}$$

According to lemma 1, there is

$$F_i(X^1) - h_i \cdot u_i^1 = F_i(X^0) \qquad i \in W \tag{11}$$

or

$$u_i^1 = \frac{1}{h_i}\left(F_i(X^1) - F_i(X^0)\right). \tag{12}$$

Suppose (11) is true for all X in a neighborhood of X^1, along a feasible direction d^1, emanating from X^1. Then, combining (9), (10), and (11), and noticing lemma 2, we can obtain

$$\sum_{i \in W} \frac{1}{h_i} \frac{\partial F_i(X^1,d^1)}{\partial F_k(X^1,d^1)} = -\lambda_k^1 \qquad k \in Z$$

or

$$\sum_{i \in W} \frac{1}{h_i} T_{i,k}(X^1,d^1) = -\lambda_k^1 \qquad k \in Z. \tag{13}$$

If we independently consider the trade-off relationship between $F_i(X)$ and $F_k(X)$, then

$$T_{i,k}(X^1,d^1) = -h_i \cdot \lambda_k^1. \tag{14}$$

Equation (14) means that $h_i \cdot \lambda_k^1 \cdot dF_k(X^1)$ units of $F_i(X)$ will be increased for the decrease of $dF_k(X^1)$ units of $F_k(X)$ in the neighborhood of Xa^1 along the direction d^1 if only $F_k(X)$ is decreased from $F_k(X^1)$ to $F_k(X^1 + \alpha d^1)$. Here we consider only the influence on $F_i(X)$. Thus, when the optimal solution of $AP(\epsilon^0)$ and the corresponding Kuhn–Tucker multipliers λ_k^1 $(k \in Z)$ are obtained, the DM is asked the following questions.

Questions Q_k^i: Suppose that all $F_l(X)$ $(l = 1,\cdots,p,\ l \neq i,k)$ are kept at the levels of $F_l(X^1)$. If you decrease $F_k(X)$ from $F_k(X^1)$ to $F_k(X^1) - dF_k(X^1)$, you will obtain another efficient solution X^2 by using the ISTM, so that $F_i(X^2)$ will be $F_i(X^1) + h_i \cdot \lambda_k^1 dF_k(X^1)$. Do you think the trade-offs worthwhile?

For every $k \in Z$, the DM is required to answer w questions (i.e., $Q_k^{i_1} \cdots Q_k^{i_w}$).

C. Trade-Off Rates and Simplex Multipliers of $AP(\epsilon^0)$

If the MOP is linear, the simplex method can be used to solve $AP(\epsilon^0)$. Given the optimal basic vector Xb^1, define the corresponding simplex multiplier vector as

$$\pi^1 = \left[-\lambda_{i_1}^1 \cdots -\lambda_{i_w}^1, -\lambda_{j_1}^1 \cdots -\lambda_{j_r}^1, -\lambda_{k_1}^1 \cdots -\lambda_{k_z}^1, -\mu_1 \cdots -\mu_m\right]^T. \tag{15}$$

Let $B(\epsilon^0)$ be the optimal basic matrix and $Cb(\epsilon^0)$ the corresponding basic objective coefficient vector to Xb^1; then

$$\begin{cases} Xb^1 = B(\epsilon^0)^{-1} \cdot b(\epsilon^0) \\ \pi^1 = Cb(\epsilon^0) \cdot B(\epsilon^0)^{-1} \end{cases} \tag{16}$$

where

$$b(\epsilon^0) = \left[F_{i_1}(X^0) \cdots F_{i_w}(X^0), F_{j_1}(X^0) \cdots F_{j_r}(X^0), \epsilon_{k_1}^0 \cdots \epsilon_{k_z}^0, 0 \cdots 0\right]^T. \tag{17}$$

Notice

$$\begin{aligned} u^1 &= Cb(\epsilon^0) \cdot Xb^1 = Cb(\epsilon^0) B(\epsilon^0)^{-1} b(\epsilon^0) \\ &= \pi^1 b(\epsilon^0) = \pi_e^1 b_e(\epsilon^0) + \pi_k^1 b_k(\epsilon^0) \end{aligned} \tag{18}$$

where

$$\pi_k^1 b_k(\epsilon^0) = \left[-\lambda_{k_1}^1 \cdots -\lambda_{k_z}^1\right]\left[\epsilon_{k_1}^1 \cdots \epsilon_{k_z}^1\right]^T \tag{19}$$

$$\pi_e^1 b_e(\epsilon^0) = \left[-\lambda_{i_1}^1 \cdots -\lambda_{i_w}^1\right]\left[F_{i_1}(X^0) \cdots F_{i_w}(X^0)\right]^T + \left[-\lambda_{j_1}^1 \cdots -\lambda_{j_r}^1\right]\left[F_{j_1}(X^0) \cdots F_{j_r}(X^0)\right]^T. \tag{20}$$

Then

$$-\lambda_k^1 = \left.\frac{\partial u(X)}{\partial \epsilon_k^0}\right|_{Xa = Xa^1} \qquad k \in Z. \tag{21}$$

TABLE I
SOO RESULTS

SOO Solution	Number			
	$F1(X1^*)$	$F2(X1^*)$	$\cdots$	$Fp(X1^*)$
$X1^*$	$F1(X1^*)$	$F2(X1^*)$	$\cdots$	$Fp(X1^*)$
$X2^*$	$F1(X2^*)$	$F2(X2^*)$	$\cdots$	$Fp(X2^*)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Xp^*	$F1(Xp^*)$	$F2(Xp^*)$	$\cdots$	$Fp(Xp^*)$

Through the similar discussion in Section III-B, we can get

$$\sum_{i \in W} \frac{1}{h_i} T_{i,k}(X^1, d^1) = -\lambda_k^1 \qquad k \in Z. \tag{22}$$

Equation (22) is similar to (13). The only difference is that $-\lambda_k^1$ in (22) represents the optimal simplex multipliers relative to the constraints $F_k(X) \geqslant \epsilon_k^0$.

IV. ISTM ALGORITHM

The algorithm consists of two basic parts. In the first part, an initial efficient solution X^0 is searched. Then an efficient solution set $\{X^0 X^1 X^2 \cdots\}$ of the MOP are generated, step by step, by solving $AP(\epsilon^{t-1})$ $(t = 1, 2, \cdots)$, which will approach an efficient solution preferred by the DM if the scopes of preferred objective values can be predetermined [26] by the DM.

A. Initial Efficient Solution X^0

A weighting ideal point method (WIP) is used to find X^0. First a single objective optimization (SOO) table is constructed. In Table I, X_ι^* represents the optimal solution obtained by only optimizing the lth objective. Let $M_\iota = F_\iota(X_\iota^*)$ and

$$N_\iota = \min_{l \leqslant i \leqslant p} F_\iota(X_i^*).$$

In (4), let $d_i = M_i$, $c_i = N_i$ $(i \in W)$. The weighting ideal point (WIP) problem of the MOP is then defined as

$$\begin{aligned} &\min d_\infty \\ &\text{s.t.}\, \pi_\iota(M_\iota - F_\iota(X)) - d_\infty \leqslant 0, \qquad l = 1, \cdots, p \\ &\quad X \in \Omega, \quad d_\infty \geqslant 0 \end{aligned} \tag{23}$$

where π_ι is a weighting coefficient, which is given by

$$\begin{aligned} &\pi_\iota = \mu_\iota / (M_\iota - N_\iota) \qquad (M_\iota > N_\iota) \\ &0 \leqslant \mu_\iota \leqslant 1, \qquad \sum_{\iota=1}^{p} \mu_\iota = 1. \end{aligned} \tag{24}$$

μ_ι can be determined according to the relative importance of the objectives. Normally, let $\mu_\iota = 1/p)(l = 1, \cdots, p)$. Suppose $\bar{X}$ is the optimal solution of the WIP, then $\bar{X}$ must be an efficient (weakly efficient) solution of the MOP [4], [9]. Let $X^0 = \bar{X}$.

B. Algorithm

Considering the preceding discussion, we can now construct the algorithm of our ISTM.

Step 1: Construct the single objective optimization table.

Step 2: Find M_ι and N_ι in Table I $(l = 1, \cdots, p)$, and compute weighting coefficients π_ι $(l = 1, \cdots, p)$ and h_i $(i \in W)$.

Step 3: Solve the WIP to obtain the initial efficient solution X^0 and the corresponding objective vector $F(X^0)$. If $F(X^0)$ is preferred by the DM, X^0 is the preferred solution. Otherwise, let $t = 1$, and continue.

Step 4: Construct the first decision analysis, Table II, according to the current values of the objectives. The DM is required to determine W^{t-1}, R^{t-1} and Z^{t-1} in the last row of Table II.

TABLE II
DECISION ANALYSIS I (DA I)

OF Level	OF Number			
	1	2	$\cdots$	p
Ideal values	$M1$	$M2$	$\cdots$	Mp
Lower bounds	$N1$	$N2$	$\cdots$	Np
Current values	$F1(X^{t-1})$	$F2(X^{t-1})$	$\cdots$	$Fp(X^{t-1})$
Index subset				

Step 5: Construct the second decision analysis, Table III, in light of Table II and the trade-off rates. The DM is required to select the values of $dF_k(X^{t-1})$ $(K \in Z)$ by answering Q_k^i $(i \in W,\ k \in Z)$. If the DM selects $dF_k(X^{t-1}) = 0$ $(k \in Z)$, the interactive procedure will be stopped and the current efficient solution X^{t-1} is the preferred solution, because no objectives can be sacrificed according to the DM's opinion.

Step 6: Design a new auxiliary problem $AP(\epsilon^{t-1})$ on the basis of Tables II and III.

Step 7: Solve $AP(\epsilon^{t-1})$ to obtain a new efficient solution X^t, the corresponding Kuhn–Tucker (simplex for linear case) multipliers λ_k^t $(k \in Z)$, and the new values of the objectives $F(X^t)$.

Step 8: If the DM is satisfied with $F(X^t)$, the interactive procedure is stopped and the corresponding efficient solution X^t is the preferred solution. Otherwise let $t = t + 1$, and go to Step 4.

In the preceding steps, the convergence issue of the designed interactive procedure has not been dealt with, which depends on the DM's preference structures. The conventional utility theory or surrogate functions may be used to treat the problem [3], [4], [13]. In [26] and [29], however, we have proposed a hierarchical analysis model and a corresponding analysis method to express the DM's preference structure, based on the concept of satisfiability degree. The basic idea is to project the objective function space of the MOP onto a membership function space, according to the DM's satisfiability degrees, about some objective functions' values. Then a preferred subset is defined, and the interaction is executed in the membership function space. If the membership function distribution at an efficient solution X^t generated in the interaction is located in the subset, when X^t is a preferred solution. If the subset is empty over the efficient solution face of the MOP, additional preference information is required from the DM to revise the definition of the subset. More details and an application can be found in references [26], [27], and [29].

VI. EXAMPLE

Consider the following multiobjective linear programming [18]:

$$\max F(X) = \left\{ \begin{aligned} F1(X) &= x1 - x2 + x3 \\ F2(X) &= -x1 + 2x2 + 3x3 \\ F3(X) &= x1 + 4x2 - x3 \end{aligned} \right\}$$

$$\text{s.t.}\, X \in \Omega \qquad X = [x_1 x_2 x_3]^T$$

$$\Omega \triangleq \left\{ X \left| \begin{aligned} &2x1 + x2 + x3 \leqslant 1 \\ &x1 + 3x2 + x3 \leqslant 1 \quad x1x2x3 \geqslant 0 \\ &x1 + x2 + 4x3 \leqslant 1 \end{aligned} \right. \right\}.$$

A. Look for the Initial Efficient Solution

By optimizing $F_\iota(X)$ $(l = 1,2,3)$ individually, we get $X_1^* = [0.4286, 0, 0.1429]^T$, $X_2^* = [0, 0.2727, 0.1818]^T$, $X_3^* = [0, 0.3333, 0]^T$.

TABLE III
DECISION ANALYSIS II (DA II)

Z Rates	W Rates				
	$F_{i_1}(X^{t-1})$	$F_{i_2}(X^{t-1})$	$\cdots$	$F_{i_w}(X^{t-1})$	$dF_k(X^{t-1})$
$F_{k_1}(X^{t-1})$	$-T_{i_1,k_1}$	$-T_{i_2,k_1}$	$\cdots$	$-T_{i_w,k_1}$	
$F_{k_2}(X^{t-1})$	$-T_{i_1,k_2}$	$-T_{i_2,k_2}$	$\cdots$	$-T_{i_w,k_2}$	
$\vdots$	$\vdots$	$\vdots$			
$F_{k_z}(X^{t-1})$	$-T_{i_1,k_z}$	$-T_{i_2,k_z}$	$\cdots$	$-T_{i_2,k_z}$	

TABLE IV
SOO RESULTS

	$F1(X1^*)$	$F2(X1^*)$	$F3(X1^*)$
$X1^*$	0.5714	0	0.2857
$X2^*$	0.0909	1.0909	0.9090
$X3^*$	−0.3333	0.6666	1.3333

The single objective optimization results are shown in Table IV. From Table IV, we obtain

$$(M1M2M3) = (0.5714, 1.0909, 1.3333),$$

$$(N1N2N3) = (-0.3333, 0, 0.2857),$$

and then $(h1, h2, h3) = (0.9047, 1.0909, 1.0476)$, $(\pi_1, \pi_2, \pi_3) = (0.3316, 0.3667, 0.2804)$, given $(\mu_1, \mu_2, \mu_3) = (0.3, 0.4, 0.3)$. Therefore, WIP can be written as

$$\text{WIP}\begin{cases}\min d_\infty \\ \text{s.t. } X_w \in \Omega_w, \quad Xw = [x1x2x3d_\infty]^T\end{cases}$$

$$\Omega_w = \left\{ Xw \,\middle|\, \begin{array}{l} x1 - x2 + x3 + 3.0157 d_\infty \geqslant 0.5714 \\ -x1 + 2x2 + 3x3 + 2.727 d_\infty \geqslant 1.0909 \\ x1 + 4x2 - x3 + 3.4916 d_\infty \geqslant 1.3333 \\ d_\infty \geqslant 0, \quad X \in \Omega \end{array} \right\}.$$

Having solved WIP, we get $X^0 = [0.1761, 0.2008, 0.1558]^T$, $d_\infty = 0.146$, and $F(X^0) = [0.1311, 0.6929, 0.8235]^T$.

B. The First Interaction

Construct the Table DA I where W, Z, Z are determined by the DM.

DA I

$F1(X^0)$	$F2(X^0)$	$F3(X^0)$
0.1311	0.6929	0.8235
w	z	z

The DM selects $dF2(X^0) = dF3(X^0) = 0.1$. $AP(\epsilon^0)$ is given by

$$AP(\epsilon^0)\begin{cases}\max u1 \\ \text{s.t. } Xa \in \Omega a\end{cases}$$

$$Xa = [x1x2x3u1]^T$$

$$\Omega_a = \left\{ Xa \,\middle|\, \begin{array}{l} x1 - x2 + x3 - 0.9047u1 \geqslant 0.1311 \\ -x1 + 2x2 + 3x3 \geqslant 0.5929 \\ x1 + 4x2 - x3 \geqslant 0.7235 \\ u1 \geqslant 0, \quad X \in \Omega \end{array} \right\}.$$

Solving $AP(\epsilon^0)$, we get

$$Xa^1 = [0.2094, 0.1675, 0.1558, 0.07362]^T,$$

$$F(X^1) = [0.1977, 0.5929, 0.7235]^T.$$

C. The Second Interaction

Construct the Tables DA I and DA II.

DA I			DA II		
$F1(X^1)$	$F2(X^1)$	$F3(X^1)$		$F1(X^1)$	
0.1977	0.5929	0.7235	$F2(X^1)$	0.5278	$dF2(X^1) = 0.2$
w	z	z	$F3(X^1)$	0.1389	$dF3(X^1) = 0.1$

Then $AP(\epsilon^1)$ can be written as

$$AP(\epsilon^1)\begin{cases}\max u1 \\ \text{s.t. } Xa \in \Omega a\end{cases}$$

$$\Omega_a = \left\{ Xa \,\middle|\, \begin{array}{l} x1 - x2 + x3 - 0.9047u1 \geqslant 0.1977 \\ -x1 + 2x2 + 3x3 \geqslant 0.3929 \\ x1 + 4x2 - x3 \geqslant 0.6235 \\ u1 \geqslant 0, \quad X \in \Omega \end{array} \right\}.$$

Solving $AP(\epsilon^1)$, we have

$$Xa^2 = [0.29, 0.1202, 0.1475, 0.132]^T$$

and

$$F(X^2) = [0.3173, 0.3929, 0.6236]^T.$$

D. The Third Interaction

Similarly, construct tables DA I and DA II.

DA I			DA II		
$F1(X^2)$	$F2(X^2)$	$F3(X^2)$		$F1(X^2)$	
0.3173	0.3929	0.6236	$F2(X^2)$	0.5278	$dF2(X^2) = 0.0929$
w	z	z	$F3(X^2)$	0.1389	$dF3(X^2) = 0.0236$

We formulate $AP(\epsilon^2)$ as

$$AP(\epsilon^2)\begin{cases}\max u1 \\ \text{s.t. } Xa \in \Omega a\end{cases}$$

$$\Omega_a \triangleq \left\{ Xa \,\middle|\, \begin{array}{l} x1 - x2 + x3 - 0.9047u1 \geqslant 0.3173 \\ -x1 + 2x2 + 3x3 \geqslant 0.3 \\ x1 + 4x2 - x3 \geqslant 0.6 \\ u1 \geqslant 0, \quad X \in \Omega \end{array} \right\}.$$

We obtain from $AP(\epsilon^2)$, $Xa^3 = [0.3306, 0.1028, 0.141, 0.05764]^T$ and $F(X^3) = [0.37, 0.3, 0.6]^T$. The values of objective vector $F(X^3)$ are all in the scope of the DM's preferred objective values. So we obtain the preferred solution $X^* = X^3 = [0.3306, 0.1028, 0.141]^T$.

VII. CONCLUSION

A newly developed interactive method (ISTM) for solving the MOP has been proposed. In this method, the trade-off rate and the classification of objectives' three subsets (W, R, and Z) are clear and direct for decision analysis; tables DA I and DA II are designed to help the DM make decisions step by step; and the computation for efficient solutions and local trade-off information is done simply by the auxiliary problem $AP(\epsilon^{t-1})$. It is these characteristics of the ISTM that enable the DM to be actively involved in the decisionmaking process. The extensions and applications of the ISTM are dealt with in other papers [26]–[31].

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Methods of Digraph Representation and Cluster Analysis for Analyzing Free Association

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***Abstract* — A method for constructing two measures of association between a pair of words that distribute over a sequence is developed. The association measures are used for digraph representation and cluster analysis. In particular, study of a measure for cluster analysis leads to a new algorithm of hierarchical agglomerative clustering. The digraph representation and the cluster analysis are applied to data of free (psychological) association obtained from a questionnaire survey on living environment of local residents. The two association measures are interpreted as estimates of probabilistic parameters. Hence methods of hypothesis testing are developed for showing differences of structures of the free associations between two different populations. The result of analysis on the association data is summarized into figures of digraphs and clusters that show structures of free associations of groups of people.**

I. INTRODUCTION

Methods developed for structural modeling [1], [2] have frequently been applied to represent, and to help in the understanding of, the structures of human cognition related to complex systems. When these methods are applied to represent cognitive structures of a group of people, the problem of aggregating the individual structures into a whole structure should be studied. This means that a method of structural modeling should include a feature of statistical analysis to deal with such a problem. Since important problems in social studies require the analysis and representation of a structure of cognition for a large number of people, a method of structural modeling that includes statistical analysis is important as a tool of analysis in these problems.

The authors studied a method of digraph representation with cluster analysis and applied it to the cognition by local residents of living environment [3]. The previous paper [3] introduced a symmetric measure and an asymmetric measure of association between a pair of words. Then a family of four methods for analyzing psychological associations based on the two measures is developed: 1) cluster analysis based on symmetric measure, 2) digraph representation based on asymmetric measure, and 3) two methods of statistical hypothesis testing based on both measures.

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