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Decision Support

The evidential reasoning approach for multiple attribute decision analysis using interval belief degrees

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Abstract

Multiple attribute decision analysis (MADA) problems having both quantitative and qualitative attributes under uncertainty can be modeled using the evidential reasoning (ER) approach. Several types of uncertainties such as ignorance and fuzziness can be modeled in the ER framework. In this paper, the ER approach will be extended to model new types of uncertainties including interval belief degrees and interval data that could be incurred in decision situations such as group decision making. The Dempster–Shafer (D–S) theory of evidence is first extended, which is one of the bases of the ER approach. The analytical ER algorithm is used to combine all evidence simultaneously. Two pairs of nonlinear optimization models are constructed to estimate the upper and lower bounds of the combined belief degrees and to compute the maximum and the minimum expected utilities of each alternative, respectively. Interval data are equivalently transformed to interval belief degrees and are incorporated into the nonlinear optimization models. A cargo ship selection problem is examined to show the implementation process of the proposed approach.

Keywords: Multiple attribute decision analysis; The evidential reasoning approach; Uncertainty modeling; Interval degrees of belief; Interval data; Nonlinear optimization

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1. Introduction

Many complex multiple attribute decision analysis (MADA) problems involve both quantitative and qualitative attributes as well as various types of uncertainties such as incomplete information, complete ignorance and fuzziness. Such complex MADA problems can always be modeled using the evidential reasoning (ER) approach (Yang and Sen, 1994; Yang and Singh, 1994; Yang, 2001; Yang and Xu, 2002a,b; Xu et al., in press; Wang et al., in press; Yang et al., in press). The ER approach models both quantitative and qualitative attributes using a distributed modeling framework, in which each attribute is characterized by a set of collectively exhaustive assessment grades, probabilistic uncertainty including incomplete information and complete ignorance by a belief structure, and fuzzy uncertainty by fuzzy linguistic variables.

In certain decision situations such as group decision making, however, a new type of interval uncertainty is likely encountered. For example, quantitative data may not be known precisely but may be estimated to belong to intervals with certain confidence levels. A decision maker (DM) may be unable to give precise judgement. In group decision analysis, different DMs may assign different degrees of belief to the same judgment. It will be very difficult to synthesize different degrees of belief to generate a precise point estimate if DMs cannot reach a consensus. Using interval belief degree may be a sensible option in such circumstances.

Xu et al. (in press) investigated the ER approach for MADA under interval uncertainties. They looked into another type of interval uncertainty caused by interval assessment grades. For instance, in real decision analyses, some alternative(s) may not be assigned to some definite assessment grade, say, *Excellent* or *Very Good* or *Good*. In this situation, DM may prefer to assign it/them to an interval assessment grade, say between *Excellent* and *Good*. It can be either *Excellent* or *Very Good* or *Good*. But the DM may not be sure which one.

The purpose of this paper is to investigate the first two types of interval uncertainties caused by interval data and interval belief degrees and to develop the ER approach for MADA with these two types of interval uncertainties. Due to the presence of interval belief degrees, the original Dempster–Shafer (D–S) theory of evidence, which is one of the bases of the ER approach, needs to be extended. The focus is on combining and normalizing interval evidence. This is because the combination and normalization process of interval evidence no longer preserves the associative property that the process does not depend on the order in which evidence is combined. To preserve the property, the analytical ER algorithm is used to combine all evidence simultaneously before the combined belief degrees are normalized which are of an interval nature as well. A pair of nonlinear optimization models is constructed to estimate the upper and lower bounds of the combined belief degrees. Interval data are equivalently transformed to interval belief degrees and are incorporated into the nonlinear optimization models.

The paper is organized as follows. In Section 2, the original D–S theory of evidence is first extended to combine interval evidence and the relevant theoretical issues are discussed. Section 3 develops the ER approach for MADA with interval data and interval belief degrees, where interval data are transformed into interval belief degrees, the analytical ER algorithm is utilized to combine all evidence simultaneously, two pairs of nonlinear optimization models are constructed to estimate the upper and lower bounds of the combined belief degrees and to compute the maximum and the minimum expected utilities of each alternative, respectively. Section 4 provides a real numerical example to illustrate the application and the detailed implementation process of the proposed approach. The paper is concluded in Section 5. A minimax regret approach (MRA) for ranking interval-valued expected utilities is provided in Appendix A.

2. The Dempster–Shafer theory of evidence and its extensions to interval belief structures

2.1. The D–S theory for combining deterministic evidence

The evidence theory was first developed by Dempster (1967) and was later extended and refined by Shafer (1976). Therefore, this theory is also called the Dempster–Shafer theory of evidence, or the D–S theory for short. So far, the D–S theory has found wide applications in many areas such as expert systems (Biswas et al., 1988; Wallery, 1996; Beynon et al., 2001), diagnosis and reasoning (Ishizuka et al., 1982; Chen, 1997; Rakar et al., 1999; Rakar and Juričić, 2002; Benferhat et al., 2000; Hullermeier, 2001; Jones et al., 2002), pattern classification (Denoeux, 1997, 1999, 2000a,b; Denoeux and Zouhal, 2001; Binaghi and Madella, 1999; Binaghi et al., 2000), information fusion (Telmoudi and Chakhar, 2004), audit risk assessment (Srivastava and Shafer, 1992; Srivastava, 1995, 1997; Krishnamoorthy et al., 1999; Bell and Carcello, 2000; Gillett, 2000; Gillett and Srivastava, 2000; Srivastava and Mock, 2000, 2002; Srivastava and Lu, 2002; Srivastava and Liu, 2003), multiple attribute decision analysis (MADA) (Yang and Sen, 1994; Yang and Singh, 1994; Bauer, 1997; Beynon et al., 2000, 2001; Beynon, 2002a,b; Yang, 2001; Yang and Xu, 2002a,b; Yang et al., in press), environmental impact assessment (EIA) (Wang et al., in press), contractor selection (Sönmez et al., 2001, 2002), organizational self-assessment (Siow et al., 2001; Yang et al., 2001), safety analysis (Wang and Yang, 2001; Liu et al., 2004), and regression analysis (Monney, 2003; Petit-Renaud and Denœux, 2004).

Let $H = \{H_1, \ldots, H_N\}$ be a collectively exhaustive and mutually exclusive set of hypotheses or propositions, which is called the frame of discernment. A basic probability assignment (bpa) (also called a belief structure) is a function $m : 2^H \to [0,1]$, which is called a mass function and satisfies:

$$m(\Phi) = 0 \quad \text{and} \quad \sum_{A \subseteq H} m(A) = 1, \tag{1}$$

where Φ is the null set, A is any subset of H, and 2^H is the power set of H, which consists of all the subsets of H, i.e.

$$2^H = \{\Phi, \{H_1\}, \ldots, \{H_N\}, \{H_1, H_2\}, \ldots, \{H_1, H_N\}, \ldots, H\}. \tag{2}$$

The assigned probability (also called probability mass) $m(A)$ measures the belief exactly assigned to A and represents how strongly the evidence supports A. All the assigned probabilities sum to unity and there is no belief in the empty set (Φ). The assigned probability to H, i.e. $m(H)$, is called the degree of ignorance. Each subset $\{A \subseteq H | m(A) > 0\}$ is called a focal element of m. All the related focal elements are collectively called the body of evidence.

A belief measure, Bel, and a plausibility measure, Pl, is associated with each bpa and they are both functions: $2^H \to [0, 1]$, defined by the following equations, respectively:

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B), \tag{3}$$

$$\mathrm{Pl}(A) = \sum_{A \cap B \neq \Phi} m(B), \tag{4}$$

where A and B are subsets of H. $\mathrm{Bel}(A)$ represents the exact support for A, i.e. the belief of the hypothesis of A being true; $\mathrm{Pl}(A)$ represents the possible support for A, i.e. the total amount of belief that could be potentially placed in A. $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$ constitutes the interval of support to A and can be seen as the lower and upper bounds of the probability to which A is supported. The two functions can be connected by the equation

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\overline{A}), \tag{5}$$

where $\overline{A}$ denotes the complement of A. The difference between the belief and the plausibility of a set A describes the ignorance of the assessment for the set A (Shafer, 1976).

Since $m(A)$, $\mathrm{Bel}(A)$ and $\mathrm{Pl}(A)$ are in one-to-one correspondence, they can be seen as three facets of the same piece of information. There are several other functions such as commonality and doubt functions, which can also be used to represent evidence and provide flexibility to match a variety of reasoning applications.

The core of the evidence theory is the Dempster's rule of combination by which evidence from different sources is combined or aggregated. The rule assumes that information sources are independent and it uses the so-called orthogonal sum to combine multiple belief structures:

$$m = m_1 \oplus m_2 \oplus \cdots \oplus m_K, \tag{6}$$

where $\oplus$ represents the operator of combination. With two belief structures m_1 and m_2, the Dempster's rule of combination is defined as follows:

$$[m_1 \oplus m_2](C) = \begin{cases} 0, & C = \Phi, \\ \frac{\sum_{A\cap B=C} m_1(A)m_2(B)}{1-\sum_{A\cap B=\Phi} m_1(A)m_2(B)}, & C \neq \Phi, \end{cases} \tag{7}$$

where A and B are both focal elements and $[m_1 \oplus m_2](C)$ itself is a bpa. The denominator, $1 - \sum_{A\cap B=\Phi} m_1(A)m_2(B)$, is denoted by k and called the normalization factor. $\sum_{A\cap B=\Phi} m_1(A)m_2(B)$ is called the degree of conflict and measures the conflict between the pieces of evidence (George and Pal, 1996). The division by k is called normalization.

The Dempster's rule of combination proved to be both commutative and associative (Shafer, 1976), i.e. $m_1 \oplus m_2 = m_2 \oplus m_1$ (commutativity) and $(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3)$ (associativity). These two properties show that evidence can be combined in any order. Therefore, in the case of multiple belief structures, evidence can be combined in a pairwise manner.

2.2. The D–S theory for combining interval evidence

The original D–S theory was developed to handle deterministic evidence. When evidence is uncertain, the D–S theory must be extended so that uncertain evidence can also be dealt with. By uncertain evidence we mean some or all probability masses assigned to focal elements are uncertain/imprecise. They may be expressed either by fuzzy numbers or by interval numbers. When evidence is expressed in the form of interval probability masses, we refer to it as the interval evidence. The corresponding belief structure (bpa) is called interval belief structure.

There have been several attempts to extend the D–S theory to uncertain evidence. The interested reader may refer to Lee and Zhu (1992), Denoeux (1999, 2000b), Lucas and Araabi (1999) and Yager (2001) for details. However, the theoretical issues involved in the combination and normalization of interval evidence are yet to be resolved. Existing approaches either ignore the normalization process or separate it from the evidential combination, leading to an irrational interval belief structure. A new logically correct approach was developed by Wang et al. (submitted for publication), where the combination and the normalization were conducted together in an optimization process. Below is a brief summary of the approach. We start with the definition of interval belief structure defined by Denoeux (1999).

Definition 1. Let $H = \{H_1, \ldots, H_N\}$ be the frame of discernment, $F_1, \ldots, F_n$ be n subsets of H and $[a_i, b_i]$ be n intervals with $0 \leqslant a_i \leqslant b_i \leqslant 1$ $(i = 1, \ldots, n)$. An interval belief structure is a set of belief structures on H such that

(1) $a_i \leqslant m(F_i) \leqslant b_i$, where $0 \leqslant a_i \leqslant b_i \leqslant 1$ for $i = 1, \ldots, n$;
(2) $\sum_{i=1}^{n} a_i \leqslant 1$ and $\sum_{i=1}^{n} b_i \geqslant 1$;
(3) $m(A) = 0 \ \forall A \notin \{F_1, \ldots, F_n\}$.

Remark 1. If $\sum_{i=1}^{n} a_i > 1$ or $\sum_{i=1}^{n} b_i < 1$, then the interval belief structure m is said to be invalid. Invalid interval belief structures need to be revised.

Remark 2. For a valid interval belief structure, we can always obtain a particular belief structure by selecting a value $m(F_i) \in [a_i, b_i]$ for each $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} m(F_i) = 1$.

Definition 2. Let m be a valid interval belief structure with interval probability masses $a_i \leqslant m(F_i) \leqslant b_i$ for $i = 1, \ldots, n$. If a_i and b_i satisfy

$$\sum_{j=1}^{n} b_j - (b_i - a_i) \geqslant 1 \quad \text{and} \quad \sum_{j=1}^{n} a_j + (b_i - a_i) \leqslant 1 \quad \text{for } \forall i \in \{1, \ldots, n\}, \tag{8}$$

then m is said to be a normalized interval belief structure.

Remark 3. Normalized interval belief structures are in fact the compact and equivalent form of valid interval belief structures. An interval belief structure may be valid, but may not necessarily be normalized. To illustrate, consider the two pieces of interval evidence in the following example.

$$m_1(\{P\}) = [0.5, 0.8], \quad m_1(\{L, K\}) = [0.3, 0.4], \quad m_1(H) = [0.2, 0.5],$$
$$m_2(\{P, L\}) = [0.4, 0.6], \quad m_2(\{L, K\}) = [0.3, 0.5], \quad m_2(H) = [0.3, 0.4].$$

Since $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i > 1$ hold for both of the interval belief structures, they are both valid. However, since neither of them satisfies (8), they are not normalized.

Remark 4. For a non-normalized interval-valued belief structure, it usually means that some intervals of probability masses are too wide to be reached. From the above interval-valued belief structures m_1 and m_2 we find that only the following belief structures are valid and all the others are infeasible:

$$m_1(\{P\}) = 0.5, \quad m_1(\{L, K\}) = 0.3, \quad m_1(H) = 0.2,$$
$$m_2(\{P, L\}) = 0.4, \quad m_2(\{L, K\}) = 0.3, \quad m_2(H) = 0.3.$$

Remark 5. For a valid interval belief structure that is not normalized, we need to normalize it by using the following formulas to screen all the infeasible belief structures:

$$\max\left[a_i, 1 - \sum_{j \neq i} b_j\right] \leqslant m(F_i) \leqslant \min\left[b_i, 1 - \sum_{j \neq i} a_j\right], \quad i = 1, \ldots, n, \tag{9}$$

where $[\max(a_i, 1 - \sum_{j \neq i} b_j), \min(b_i, 1 - \sum_{j \neq i} a_j)]$ $(i = 1, \ldots, n)$ form normalized interval probability masses.

Definition 3. Let m be a normalized interval belief structure with interval probability masses $a_i \leqslant m(F_i) \leqslant b_i$ for $i = 1, \ldots, n$ and A be a subset of $H = \{H_1, \ldots, H_N\}$. The belief measure (Bel) and the plausibility measure (Pl) of A are the closed intervals defined respectively by

$$\mathrm{Bel}_m(A) = [\mathrm{Bel}_m^-(A), \mathrm{Bel}_m^+(A)],$$
$$\mathrm{Pl}_m(A) = [\mathrm{Pl}_m^-(A), \mathrm{Pl}_m^+(A)],$$

where

$$\mathrm{Bel}_m^-(A) = \min \sum_{F_i \subseteq A} m(F_i) = \max \left[\sum_{F_i \subseteq A} a_i, \left(1 - \sum_{F_j \not\subset A} b_j \right) \right], \tag{10}$$

$$\mathrm{Bel}_m^+(A) = \max \sum_{F_i \subseteq A} m(F_i) = \min \left[\sum_{F_i \subseteq A} b_i, \left(1 - \sum_{F_j \not\subset A} a_j \right) \right], \tag{11}$$

$$\mathrm{Pl}_m^-(A) = \min \sum_{F_i \cap A \neq \Phi} m(F_i) = \max \left[\sum_{F_i \cap A \neq \Phi} a_i, \left(1 - \sum_{F_i \cap A = \Phi} b_j \right) \right], \tag{12}$$

$$\mathrm{Pl}_m^+(A) = \max \sum_{F_i \cap A \neq \Phi} m(F_i) = \min \left[\sum_{F_i \cap A \neq \Phi} b_i, \left(1 - \sum_{F_i \cap A = \Phi} a_j \right) \right]. \tag{13}$$

Remark 6. Since the equation $\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\overline{A})$ holds for any deterministic belief structure, accordingly, we have the following equations for interval belief structures:

$$\mathrm{Pl}_m^-(A) = \min \mathrm{Pl}_m(A) = 1 - \max \mathrm{Bel}_m(\overline{A}) = 1 - \mathrm{Bel}_m^+(\overline{A}), \tag{14}$$

$$\mathrm{Pl}_m^+(A) = \max \mathrm{Pl}_m(A) = 1 - \min \mathrm{Bel}_m(\overline{A}) = 1 - \mathrm{Bel}_m^-(\overline{A}). \tag{15}$$

Definition 4. Let m_1 and m_2 be two interval belief structures with interval probability masses $m_1^-(A_i) \leqslant m_1(A_i) \leqslant m_1^+(A_i)$ for $i = 1, \ldots, n_1$ and $m_2^-(B_j) \leqslant m_2(B_j) \leqslant m_2^+(B_j)$ for $j = 1, \ldots, n_2$, respectively. Their combination, denoted by $m_1 \oplus m_2$, is also an interval belief structure defined by

$$[m_1 \oplus m_2](C) = \begin{cases} 0, & C = \Phi, \\ [(m_1 \oplus m_2)^-(C), (m_1 \oplus m_2)^+(C)], & C \neq \Phi, \end{cases} \tag{16}$$

where $(m_1 \oplus m_2)^-(C)$ and $(m_1 \oplus m_2)^+(C)$ are respectively the minimum and the maximum of the following optimization problem:

$$\begin{aligned} \text{Max/Min} \quad & [m_1 \oplus m_2](C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{1 - \sum_{A_i \cap B_j = \Phi} m_1(A_i) m_2(B_j)} \\ \text{s.t.} \quad & \sum_{i=1}^{n_1} m_1(A_i) = 1, \\ & \sum_{j=1}^{n_2} m_2(B_j) = 1, \\ & m_1^-(A_i) \leqslant m_1(A_i) \leqslant m_1^+(A_i), \quad i = 1, \ldots, n_1, \\ & m_2^-(B_j) \leqslant m_2(B_j) \leqslant m_2^+(B_j), \quad j = 1, \ldots, n_2. \end{aligned} \tag{17}$$

Note that quite different from the models proposed by Denoeux (1999, 2000), each of the above pair of models considers at the same time the combination and normalization of two pieces of interval evidence and optimizes them together rather than separately. The reason for doing so is to capture the true probability mass intervals of the combined focal elements. If the numerator and the denominator of (17) were optimised individually, the intrinsic relationships between them would be cut off and the results would be distorted. Besides, what we need is the normalized rather than non-normalized probability masses. So, we believe there is no need to optimize any probability masses before normalization.

It must be pointed out that the combination of interval evidence does not preserve the associativity property in the new approach defined by (16) and (17) where three or more pieces of evidence are combined recursively. This can be confirmed by the results shown in Table 1, where three pieces of interval evidence are combined in different orders. It is clear that $(m_1 \oplus m_2) \oplus m_3 \neq m_1 \oplus (m_2 \oplus m_3) \neq (m_1 \oplus m_3) \oplus m_2$.

In order that multiple interval-valued belief structures can be combined correctly and efficiently, they should be combined simultaneously and the optimization process should not be started until the end of the combination. The following definition shows how to combine them correctly.

Definition 5. Let $m_1, \ldots, m_n$ be n interval belief structures with interval probability masses $m_i^-(A_j^i) \leqslant m_i(A_j^i) \leqslant m_i^+(A_j^i)$ for $i = 1$ to n and $j = 1$ to n_i, where A_j^i represents the jth focal element of the ith interval belief structure. Their combination, denoted by $m_1 \oplus m_2 \oplus \cdots \oplus m_n$, is also an interval belief structure defined by

$$[m_1 \oplus m_2 \oplus \cdots \oplus m_n](C) = \begin{cases} 0, & C = \Phi, \\ [(m_1 \oplus \cdots \oplus m_2)^-(C), (m_1 \oplus \cdots \oplus m_2)^+(C)], & C \neq \Phi, \end{cases} \tag{18}$$

where $(m_1 \oplus m_2 \oplus \cdots \oplus m_n)^-(C)$ and $(m_1 \oplus m_2 \oplus \cdots \oplus m_n)^+(C)$ are respectively the minimum and the maximum of the following optimization problem:

$$\begin{aligned} \text{Max/Min} \quad & [m_1 \oplus m_2 \oplus \cdots \oplus m_n](C) = \frac{\sum_{A_{j_1}^1 \cap A_{j_2}^2 \cap \cdots \cap A_{j_n}^n = C} m_1(A_{j_1}^1) m_2(A_{j_2}^2) \cdots m_n(A_{j_n}^n)}{1 - \sum_{A_{j_1}^1 \cap A_{j_2}^2 \cap \cdots \cap A_{j_n}^n = \Phi} m_1(A_{j_1}^1) m_2(A_{j_2}^2) \cdots m_n(A_{j_n}^n)} \\ \text{s.t.} \quad & \sum_{j=1}^{n_i} m_i(A_j^i) = 1, \quad i = 1, \ldots, n, \\ & m_i^-(A_j^i) \leqslant m_i(A_j^i) \leqslant m_i^+(A_j^i), \qquad i = 1, \ldots, n; \quad j = 1, \ldots, n_i. \end{aligned} \tag{19}$$

In combining multiple interval belief structures, it is difficult to write the operational form of the objective function given in Eq. (19) directly. So we break the function down into several parts and write them one by one. To illustrate this process, we return to the example shown in Table 1.

Combining m_1 and m_2, we have the following formulas for the non-normalized probability masses:

$$\begin{aligned} \bar{m}_{1-2}(H_i) &= m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i), \quad i = 1, 2, 3, \\ \bar{m}_{1-2}(H) &= m_1(H)m_2(H), \\ \bar{m}_{1-2}(\Phi) &= \sum_{i=1}^{3} \sum_{j=1, j \neq i}^{3} m_1(H_i)m_2(H_j). \end{aligned}$$

Table 1
Non-associativity of the combination of interval belief structures

Interval Belief Structure	$\{H_1\}$	$\{H_2\}$	$\{H_3\}$	$\{H\}$
m_1	[0.2, 0.4]	[0.3, 0.5]	[0.1, 0.3]	[0, 0.4]
m_2	[0.3, 0.4]	[0.1, 0.2]	[0.2, 0.3]	[0.1, 0.4]
m_3	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]	[0, 0.1]
$m_1 \oplus m_2$	[0.2222, 0.5538]	[0.1875, 0.4808]	[0.0833, 0.3871]	[0, 0.2133]
$m_1 \oplus m_3$	[0.1111, 0.4000]	[0.2727, 0.6250]	[0.1250, 0.5208]	[0, 0.0635]
$m_2 \oplus m_3$	[0.2128, 0.3962]	[0.1915, 0.3846]	[0.3478, 0.5556]	[0, 0.0635]
$(m_1 \oplus m_2) \oplus m_3$	[0.1250, 0.5388]	[0.1661, 0.6148]	[0.1065, 0.5933]	[0, 0.0440]
$m_1 \oplus (m_2 \oplus m_3)$	[0.1197, 0.5246]	[0.1692, 0.6053]	[0.1041, 0.5852]	[0, 0.0430]
$(m_1 \oplus m_3) \oplus m_2$	[0.1199, 0.5422]	[0.1685, 0.6114]	[0.1042, 0.5952]	[0, 0.0443]
$m_1 \oplus m_2 \oplus m_3$	[0.1250, 0.5122]	[0.1698, 0.5952]	[0.1081, 0.5814]	[0, 0.0417]

The normalized probability masses will be

$$m_{1-2}(H_i) = \frac{\bar{m}_{1-2}(H_i)}{1-\bar{m}_{1-2}(\Phi)} = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_1(H_i)m_2(H_j)}, \quad i = 1,2,3,$$

$$m_{1-2}(H) = \frac{\bar{m}_{1-2}(H)}{1-\bar{m}_{1-2}(\Phi)} = \frac{m_1(H)m_2(H)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_1(H_i)m_2(H_j)}.$$

The above results can be viewed as a new piece of evidence, which is further combined with the evidence m_3. The combined results can be written out as follows.

$$m(H_i) = \frac{m_{1-2}(H_i)m_3(H_i) + m_{1-2}(H_i)m_3(H) + m_{1-2}(H)m_3(H_i)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_{1-2}(H_i)m_3(H_j)}, \quad i = 1,2,3,$$

$$m(H) = \frac{m_{1-2}(H)m_3(H)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_{1-2}(H_i)m_3(H_j)}.$$

Accordingly, models (19) can be expressed as

$$\begin{aligned}
\text{Max/Min} \quad & m(H_i) = \frac{m_{1-2}(H_i)m_3(H_i) + m_{1-2}(H_i)m_3(H) + m_{1-2}(H)m_3(H_i)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_{1-2}(H_i)m_3(H_j)} \\
\text{s.t.} \quad & m_{1-2}(H_i) = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_1(H_i)m_2(H_j)}, \quad i = 1,2,3, \\
& m_{1-2}(H) = \frac{m_1(H)m_2(H)}{1-\sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} m_1(H_i)m_2(H_j)}, \\
& m_1(H_1) + m_1(H_2) + m_1(H_3) + m_1(H) = 1, \\
& m_2(H_1) + m_2(H_2) + m_2(H_3) + m_2(H) = 1, \\
& m_3(H_1) + m_3(H_2) + m_3(H_3) + m_3(H) = 1, \\
& 0.2 \leqslant m_1(H_1) \leqslant 0.4, \\
& 0.3 \leqslant m_1(H_2) \leqslant 0.5, \\
& 0.1 \leqslant m_1(H_3) \leqslant 0.3, \\
& 0 \leqslant m_1(H) \leqslant 0.4, \\
& 0.3 \leqslant m_2(H_1) \leqslant 0.4, \\
& 0.1 \leqslant m_2(H_2) \leqslant 0.2, \\
& 0.2 \leqslant m_2(H_3) \leqslant 0.3, \\
& 0.1 \leqslant m_2(H) \leqslant 0.4, \\
& 0.2 \leqslant m_3(H_1) \leqslant 0.3, \\
& 0.3 \leqslant m_3(H_2) \leqslant 0.4, \\
& 0.4 \leqslant m_3(H_3) \leqslant 0.5, \\
& 0 \leqslant m_3(H) \leqslant 0.1.
\end{aligned}$$

These nonlinear programming models can be easily solved by LINGO software package. Similar models can also be constructed for $m(H)$. No matter how many interval belief structures are being combined, (19) can always be constructed in this way.

3. The ER approach for MADA using interval belief degrees

The ER algorithm is developed for aggregating multiple attributes based on a belief decision matrix and the evidence combination rule of D–S theory. Different from the traditional MADA approaches that describe a MADA problem using a decision matrix, the ER approach uses the belief decision matrix, in which each attribute of an alternative is described by a distribution assessment using a belief structure. The advantages of doing so lie in that using a distribution assessment can both model precise data and capture various types of uncertainties such as probabilities and vagueness in subjective judgments.

The existing ER approach models decision attributes, both quantitative and qualitative, using precise belief degrees and belief structures. As mentioned, it may be difficult to acquire precise belief degrees. Alternatively, interval belief degrees can better characterize DM's subjective judgments or assessments in this situation. Different from the existing ER approach, the current paper will allow alternatives to be assessed on each attribute using either precise or interval belief degrees. It will also allow the quantitative attributes to be modeled using interval data. As a consequence, the ER approach will be extended so that MADA problems can be modeled using interval belief structures.

3.1. The ER distributed modeling framework for MADA—interval belief structures

Suppose that a MADA problem has M alternatives a_l $(l = 1, \ldots, M)$, one upper level attribute, which is also referred to as general attribute, and L lower level attributes e_i $(i = 1, \ldots, L)$, which are called basic attributes. The relative weights of L basic attributes are denoted by $w = (w_1, \ldots, w_L)$, which are known or given and satisfy the following condition:

$$0 \leqslant w_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{L} w_i = 1. \tag{20}$$

In order to use the ER approach to conduct decision analysis for the MADA problem, it is necessary to define a set of assessment grades for each attribute. The defined assessment grades may be crisp (precise) or fuzzy numbers. Different sets of assessment grades can be defined for different attributes to facilitate data collection but need to be transformed into a unified set of assessment grades before implementing the ER algorithm. The transformation techniques can be found in Yang (2001) and Yang et al. (in press). The interested reader may refer to them for details. For convenience and simplicity, only one set of crisp evaluation grades is defined for all the attributes in this paper.

Suppose M alternatives are all assessed using N crisp assessment grades H_n $(n = 1, \ldots, N)$, which are mutually exclusive and collectively exhaustive. The N assessment grades form the frame of discernment $H = \{H_1, \ldots, H_N\}$ in the D–S theory of evidence. If alternative a_l is assessed on an attribute e_i to a grade H_n with an interval belief degree of $[\beta_{n,i}^-, \beta_{n,i}^+]$, we denote this by $S(e_i(a_l)) = \{(H_n, [\beta_{n,i}^-(a_l), \beta_{n,i}^+(a_l)]), n = 1, \ldots, N\}$, which is an interval-valued distribution assessment vector, where $\beta_{n,i}^-(a_l) \geqslant 0$. Note that precise belief degree is a special case of interval belief degree with $\beta_{n,i}^- \equiv \beta_{n,i}^+$.

With regards to the above interval-valued distribution assessment vector, we have the following definitions:

Definition 6. Let $S(e_i(a_l)) = \{(H_n, [\beta_{n,i}^-(a_l), \beta_{n,i}^+(a_l)]), n = 1, \ldots, N\}$ be an interval-valued distribution assessment vector. If the interval belief degrees $[\beta_{n,i}^-(a_l), \beta_{n,i}^+(a_l)]$ satisfy $\sum_{n=1}^{N} \beta_{n,i}^-(a_l) \leqslant 1$, then $S(e_i(a_l))$ is said to be valid; otherwise, it is invalid.

For an invalid interval-valued distribution assessment vector, it needs to be revised or adjusted before it can be used to conduct decision analysis.

Definition 7. Let $S(e_i(a_l)) = \{(H_n, [\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)]), n = 1, \ldots, N\}$ be a valid interval-valued distribution assessment vector. If the interval belief degrees $[\beta^-_{n,i}, \beta^+_{n,i}]$ always satisfy $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1$ in any circumstances, where $\beta_{n,i}(a_l) \in [\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)]$ for each $n = 1$ to N, then $S(e_i(a_l))$ is said to be a complete interval-valued distribution assessment vector; otherwise, it is incomplete. Especially, when $\sum_{n=1}^{N} \beta^+_{n,i}(a_l) = 0$, the assessment is said to be totally ignorant.

For a complete interval-valued distribution assessment, alternative a_l is assessed for sure to one or more of the defined assessment grades and there is no remaining belief degree assigned to the whole set H. However, if an interval-valued distribution assessment is incomplete, then there might exist an interval belief degree that is unassigned to any of the defined assessment grades. This unassigned interval belief degree should be assigned to the whole set H. The following definition shows how to capture this unassigned interval belief degree.

Definition 8. Let $S(e_i(a_l)) = \{(H_n, [\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)]), n = 1, \ldots, N\}$ be an incomplete interval-valued distribution assessment vector. The belief degree, $\beta_{H,i}(a_l)$, assigned to the whole set, H, is an interval defined by $[\beta^-_{H,i}(a_l), \beta^+_{H,i}(a_l)]$, where

$$\beta^-_{H,i}(a_l) = \max\left(0, 1 - \sum_{n=1}^{N} \beta^+_{n,i}(a_l)\right) \quad \text{and} \quad \beta^+_{H,i}(a_l) = 1 - \sum_{n=1}^{N} \beta^-_{n,i}(a_l). \tag{21}$$

For example, given the frame of discernment $H = \{H_1, H_2, H_3\}$ and an interval-valued distribution assessment vector $S(e_i(a_l)) = \{(H_1, [0.3, 0.4]); (H_2, [0.2, 0.3]); (H_3, [0.4, 0.5])\}$, we have

$$\beta^-_{H,i}(a_l) = \max\left(0, 1 - \sum_{n=1}^{N} \beta^+_{n,i}(a_l)\right) = \max(0, 1 - 0.4 - 0.3 - 0.5) = 0,$$

$$\beta^+_{H,i}(a_l) = 1 - \sum_{n=1}^{N} \beta^-_{n,i}(a_l) = 1 - 0.3 - 0.2 - 0.4 = 0.1.$$

So, the belief degree assigned to H will be the interval $[0, 0.1]$.

After M alternatives are assessed on L basic attributes, we get the following decision matrix:

$$D_g = (S(e_i(a_l)))_{L \times M}, \tag{22}$$

which is called interval-valued belief decision matrix, based on which and the evidence combination rule of the D–S theory, a recursive or an analytical ER algorithm and a pair of nonlinear optimization models can be developed to aggregate the L basic attributes. Before developing the ER algorithm and models, we first discuss how to model quantitative attributes using belief or interval belief structures.

3.2. Modeling quantitative attributes using belief or interval belief structures

3.2.1. Modeling precise data using belief structure

Qualitative attributes can be easily modeled by using the interval-valued distribution assessment vector introduced above. Quantitative attributes, however, are normally measured by numerical data. To use the ER approach to conduct decision analysis for MADA problems, all numerical data need to be transformed into the format of distribution assessment. This is logical as the assessment of a quantitative attribute can also be properly characterized by using the defined assessment grades.

In order to model quantitative attributes using the defined assessment grades, we need to know DM's utilities (or values) about each assessment grade. Utilities can be elicited from DM in different ways (Keeney and Raiffa, 1976; Zeleny, 1982; Winston, 1994). Suppose DM's utilities for each assessment grade are known and denoted by $u(H_n)$ $(n = 1, \ldots, N)$, which are called grade utilities. Without loss of generality, we

assume that H_1 is the least preferred assessment grade and H_N the most preferred one. Their utilities are set to be $u(H_1) = 0$ and $u(H_N) = 1$. For all the other grade utilities we have $0 = u(H_1) < u(H_2) < \cdots < u(H_N) = 1$.

Quantitative attributes may be classified into two categories: one is benefit attribute (or called maximization attribute) and the other is cost attribute (or called minimization attribute). For benefit attributes, their values can be normalized as

$$u(e_i(y)) = \frac{y - y_i^{\min}}{y_i^{\max} - y_i^{\min}}, \quad i \in \Omega_{\mathrm{b}}, \tag{23}$$

where $u(e_i(y))$ stands for the normalized value of y, $y_i^{\min}$ and $y_i^{\max}$ are the minimum and the maximum of the attribute e_i, respectively, and Ω_{b} is the set of quantitative benefit attributes.

Similarly, for cost attributes, we have the following expressions for their normalized valus:

$$u(e_i(y)) = \frac{y_i^{\max} - y}{y_i^{\max} - y_i^{\min}}, \quad i \in \Omega_{\mathrm{c}}, \tag{24}$$

where Ω_{c} is the set of quantitative cost attributes.

According to the principle of utility (value) equivalence (Yang, 2001), we let

$$u(e_i(y)) = u(H_n), \quad n = 1, \ldots, N. \tag{25}$$

Substituting (23) or (24) into (25) and finding y for each $n = 1$ to N, we get N non-decreasing or non-increasing numerical values: $y_{1,i}, \ldots, y_{N,i}$, which can be seen as N crisp assessment grades for quantitative attribute e_i.

Since (23) and (24) are both linear utility functions, they may be inconsistent with DM's preferences. In this situation, the actual assessment grades for each quantitative attribute still have to be properly adjusted or revised by DM's real utilities.

Let $Y_{1,i}, \ldots, Y_{N,i}$ be adjusted assessment grades for quantitative attribute e_i. For any precisely known attribute value y_i, it must lie between two adjacent assessment grades, say $Y_{n,i} \leqslant y_i \leqslant Y_{n+1,i}$, where $n \in \{1, \ldots, N-1\}$. That means we can use these two assessment grades to characterize the attribute value y_i. It is obvious that the closer y_i is to $Y_{n,i}$ or $Y_{n+1,i}$, the larger the belief degree to which y_i is assessed to $Y_{n,i}$ or $Y_{n+1,i}$. Let $\beta_{n,i}$ and $\beta_{n+1,i}$ characterize the belief degrees to which y_i is assessed to the grades $Y_{n,i}$ and $Y_{n+1,i}$, respectively, calculated by

$$\beta_{n,i} = \frac{Y_{n+1,i} - y_i}{Y_{n+1,i} - Y_{n,i}} \quad \text{and} \quad \beta_{n+1,i} = \frac{y_i - Y_{n,i}}{Y_{n+1,i} - Y_{n,i}}. \tag{26}$$

Then, the precise attribute value y_i can be equivalently expressed in the form of belief structure as follows:

$$y_i \Longleftrightarrow \{(H_n, \beta_{n,i}); (H_{n+1}, \beta_{n+1,i})\}, \tag{27}$$

where the symbol '$\Longleftrightarrow$' means 'be equivalent to'. Note that the above equivalent transformation is unique. From the attribute value y_i, the distribution assessment vector, $\{(H_n, \beta_{n,i}); (H_{n+1}, \beta_{n+1,i})\}$, can be uniquely determined and vice versa.

3.2.2. Modeling interval data using interval belief structure

Since interval data may span several assessment grades, their modeling is not as easy as precise data. We begin with the simplest situation where an interval value is totally included by two adjacent assessment grades. Let $y_i \in [y_i^-, y_i^+]$ be an interval number, which contains no assessment grade and is totally included by two adjacent assessment grades, say $Y_{n,i}$ and $Y_{n+1,i}$ (see Fig. 1(a)). Evidently, the belief degrees to which

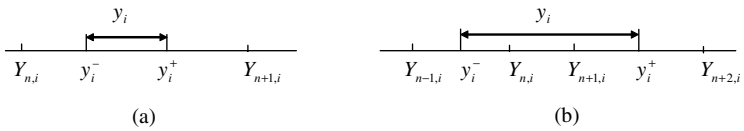

Fig. 1. The relationships among interval data and assessment grades. (a) y_i is included by $Y_{n,i}$ and $Y_{n+1,i}$ and (b) y_i includes $Y_{n,i}$ and $Y_{n+1,i}$.

y_i is assessed to $Y_{n,i}$ and $Y_{n+1,i}$, respectively, are both intervals, which are denoted by $[\beta_{n,i}^-, \beta_{n,i}^+]$ and $[\beta_{n+1,i}^-, \beta_{n+1,i}^+]$, respectively, and determined by

$$\beta_{n,i}^- = \frac{Y_{n+1,i} - y_i^+}{Y_{n+1,i} - Y_{n,i}} \quad \text{and} \quad \beta_{n,i}^+ = \frac{Y_{n+1,i} - y_i^-}{Y_{n+1,i} - Y_{n,i}}, \tag{28}$$

$$\beta_{n+1,i}^- = \frac{y_i^- - Y_{n,i}}{Y_{n+1,i} - Y_{n,i}} \quad \text{and} \quad \beta_{n+1,i}^+ = \frac{y_i^+ - Y_{n,i}}{Y_{n+1,i} - Y_{n,i}}. \tag{29}$$

Note that the above interval belief degrees $\beta_{n,i} \in [\beta_{n,i}^-, \beta_{n,i}^+]$ and $\beta_{n+1,i} \in [\beta_{n+1,i}^-, \beta_{n+1,i}^+]$ are not independent. They have to satisfy $\beta_{n,i} + \beta_{n+1,i} = 1$. Using the interval belief degrees, the interval value $y_i \in [y_i^-, y_i^+]$ can be equivalently expressed as

$$y_i \in [y_i^-, y_i^+] \Longleftrightarrow \{(H_n, [\beta_{n,i}^-, \beta_{n,i}^+]); (H_{n+1}, [\beta_{n+1,i}^-, \beta_{n+1,i}^+])\} \quad \text{with } \beta_{n,i} + \beta_{n+1,i} = 1. \tag{30}$$

According to Definition 7, $\{(H_n, [\beta_{n,i}^-, \beta_{n,i}^+]); (H_{n+1}, [\beta_{n+1,i}^-, \beta_{n+1,i}^+])\}$ is a complete interval-valued distribution assessment vector.

Now let us investigate the situation where the interval value $y_i \in [y_i^-, y_i^+]$ contains one or more assessment grades. Without loss of generality, we consider the situation where two assessment grades are included in the interval $[y_i^-, y_i^+]$. For other situations, $y_i \in [y_i^-, y_i^+]$ can be modeled in the same way.

Suppose $Y_{n-1,i}$ and $Y_{n+2,i}$ are the two closest grades including the interval $[y_i^-, y_i^+]$ (see Fig. 1(b)). It is obvious that if y_i lies within $[y_i^-, Y_{n,i})$, then it will be assessed to $Y_{n-1,i}$ and $Y_{n,i}$ with different interval belief degrees, respectively; if y_i happens to be equal to $Y_{n,i}$, then it will be assessed to $Y_{n,i}$ for sure; if y_i lies within $(Y_{n,i}, Y_{n+1,i})$, then it should be assessed to $Y_{n,i}$ and $Y_{n+1,i}$ with different interval belief degrees, respectively; if y_i happens to be identical with $Y_{n+1,i}$, then it will be 100 percent assessed to $Y_{n+1,i}$; if y_i lies between $Y_{n+1,i}$ and y_i^+, then it will be assessed to $Y_{n+1,i}$ and $Y_{n+2,i}$ with different interval belief degrees, respectively. From the above analyses we know that y_i should be assessed to either $Y_{n-1,i}$ and $Y_{n,i}$ or $Y_{n,i}$ and $Y_{n+1,i}$ or $Y_{n+1,i}$ and $Y_{n+2,i}$. It should not be assessed to three or more assessment grades simultaneously.

Let $\beta_{n-1,i} \in [\beta_{n-1,i}^-, \beta_{n-1,i}^+], \beta_{n,i} \in [\beta_{n,i}^-, \beta_{n,i}^+], \beta_{n+1,i} \in [\beta_{n+1,i}^-, \beta_{n+1,i}^+]$ and $\beta_{n+2,i} \in [\beta_{n+2,i}^-, \beta_{n+2,i}^+]$ be the interval belief degrees to which $y_i \in [y_i^-, y_i^+]$ may possibly be assessed to $Y_{n-1,i}$, $Y_{n,i}$, $Y_{n+1,i}$ and $Y_{n+2,i}$. Ideally, these interval belief degrees may be determined by the following formulas:

$$\beta_{n-1,i}^- = 0 \quad \text{and} \quad \beta_{n-1,i}^+ = \frac{y_i^- - Y_{n-1,i}}{Y_{n,i} - Y_{n-1,i}}, \tag{31}$$

$$\beta_{n,i}^- = 0 \quad \text{and} \quad \beta_{n,i}^+ = 1, \tag{32}$$

$$\beta_{n+1,i}^- = 0 \quad \text{and} \quad \beta_{n+1,i}^+ = 1, \tag{33}$$

$$\beta_{n+2,i}^- = 0 \quad \text{and} \quad \beta_{n+2,i}^+ = \frac{y_i^+ - Y_{n+1,i}}{Y_{n+2,i} - Y_{n+1,i}}, \tag{34}$$

Since these belief degrees cannot hold at the same time. They still need to be properly revised. To do so, we introduce the following 0–1 integer variables:

$$I_{k-1,k} = \begin{cases} 1, & \text{if } y_i \text{ lies between } Y_{k-1,i} \text{ and } Y_{k,i}, \\ 0, & \text{otherwise,} \end{cases} \quad k = n, n+1, n+2.$$

Due to the fact that y_i can only lie within one of the three intervals $[y_i^-, Y_{n,i})$, $[Y_{n,i}, Y_{n+1,i}]$ and $(Y_{n+1,i}, y_i^+]$, only one of the above 0–1 integer variables will be non-zero. So, we have

$$I_{n-1,n} + I_{n,n+1} + I_{n+1,n+2} = 1. \tag{35}$$

Using the above 0–1 integer variables, (31)–(34) can be revised as follows:

$$\beta_{n-1,i}^- = 0 \quad \text{and} \quad \beta_{n-1,i}^+ = I_{n-1,n} \cdot \frac{y_i^- - Y_{n-1,i}}{Y_{n,i} - Y_{n-1,i}}, \tag{36}$$

$$\beta_{n,i}^- = 0 \quad \text{and} \quad \beta_{n,i}^+ = I_{n-1,n} + I_{n,n+1}, \tag{37}$$

$$\beta_{n+1,i}^- = 0 \quad \text{and} \quad \beta_{n+1,i}^+ = I_{n,n+1} + I_{n+1,n+2}, \tag{38}$$

$$\beta_{n+2,i}^- = 0 \quad \text{and} \quad \beta_{n+2,i}^+ = I_{n+1,n+2} \cdot \frac{y_i^+ - Y_{n+1,i}}{Y_{n+2,i} - Y_{n+1,i}}. \tag{39}$$

It is clear from (36)–(39) that if $I_{n-1,n} \neq 0$, then y_i will be assessed to $Y_{n-1,i}$ and $Y_{n,i}$; if $I_{n,n+1} \neq 0$, then y_i will be assessed to $Y_{n,i}$ and $Y_{n+1,i}$; if $I_{n+1,n+2} \neq 0$, then y_i will be assessed to $Y_{n+1,i}$ and $Y_{n+2,i}$. So, (36)–(39) correctly characterize the interval belief degrees to which y_i will be assessed to those relevant assessment grades.

Note that $\beta_{n-1,i}$,$\beta_{n,i}$,$\beta_{n+1,i}$ and $\beta_{n+2,i}$ are not independent interval belief structures. They have to meet the requirement of normalization, namely, $\beta_{n-1,i}+\beta_{n,i} + \beta_{n+1,i} + \beta_{n+2,i} = 1$. Using interval belief structures, y_i can be equivalently expressed as follows:

$$\begin{aligned} y_i \in [y_i^-, y_i^+] \iff & \{(H_{n-1}, [\beta_{n-1,i}^-, \beta_{n-1,i}^+]); (H_n, [\beta_{n,i}^-, \beta_{n,i}^+]); (H_{n+1}, [\beta_{n+1,i}^-, \beta_{n+1,i}^+]); (H_{n+2}, [\beta_{n+2,i}^-, \beta_{n+2,i}^+])\} \\ & \text{with } \beta_{n-1,i} + \beta_{n,i} + \beta_{n+1,i} + \beta_{n+2,i} = 1 \quad \text{and } I_{n-1,n} + I_{n,n+1} + I_{n+1,n+2} = 1, \end{aligned} \tag{40}$$

where $\beta_{n-1,i} \in [\beta_{n-1,i}^-, \beta_{n-1,i}^+]$, $\beta_{n,i} \in [\beta_{n,i}^-, \beta_{n,i}^+]$, $\beta_{n+1,i} \in [\beta_{n+1,i}^-, \beta_{n+1,i}^+]$ and $\beta_{n+2,i} \in [\beta_{n+2,i}^-, \beta_{n+2,i}^+]$ are determined by (36)–(39). All interval data can be modeled using interval belief structure in the way as shown in (28)–(30) and (36)–(40). This will be illustrated through a numerical example in Section 4.

3.3. The ER analytical algorithm and models for attribute aggregation

3.3.1. The ER analytical algorithm for aggregating multiple belief structures

The ER approach is a nonlinear aggregation approach in nature. It provides two equivalent algorithms: the recursive algorithm and the analytical algorithm. Here we briefly introduce the ER analytical algorithm to pave the way for our later optimization models.

The ER algorithm first transforms the original belief degrees into basic probability masses by combining the relative weights and the belief degrees using the following equations:

$$m_{n,i} = m_i(H_n) = w_i \beta_{n,i}(a_l), \quad n = 1, \ldots, N; \quad i = 1, \ldots, L, \tag{41}$$

$$m_{H,i} = m_i(H) = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - w_i \sum_{n=1}^{N} \beta_{n,i}(a_l), \quad i = 1, \ldots, L, \tag{42}$$

$$\bar{m}_{H,i} = \bar{m}_i(H) = 1 - w_i, \quad i = 1, \ldots, L, \tag{43}$$

$$\tilde{m}_{H,i} = \tilde{m}_i(H) = w_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}(a_l) \right), \quad i = 1, \ldots, L, \tag{44}$$

$$\text{with } m_{H,i} = \bar{m}_{H,i} + \tilde{m}_{H,i} \quad \text{and} \quad \sum_{i=1}^{L} w_i = 1.$$

Note that the probability mass assigned to the whole set H, $m_{H,i}$, which is currently unassigned to any individual grades, is split into two parts: $\bar{m}_{H,i}$ and $\tilde{m}_{H,i}$, where $\bar{m}_{H,i}$ is caused by the relative importance of attribute e_i and $\tilde{m}_{H,i}$ by the incompleteness of the assessment on attribute e_i for a_l.

Next, the basic probability masses on L basic attributes are combined into an aggregated basic probability assignment by using the following analytical formulas (Wang et al., in press):

$$\{H_n\} : m_n = k\left[\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right], \quad n = 1, \ldots, N, \tag{45}$$

$$\{H\} : \tilde{m}_H = k\left[\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{46}$$

$$\{H\} : \bar{m}_H = k\left[\prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{47}$$

where

$$k = \left[\sum_{n=1}^{N}\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - (N-1)\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right]^{-1}. \tag{48}$$

Finally, the aggregated probability assignments are normalized into overall belief degrees by using the following equations:

$$\{H_n\} : \beta_n = \frac{m_n}{1 - \bar{m}_H}, \quad n = 1, \ldots, N, \tag{49}$$

$$\{H\} : \beta_H = \frac{\tilde{m}_H}{1 - \bar{m}_H}, \tag{50}$$

where β_n and β_H represent the overall belief degrees of the aggregated assessment. The aggregated assessment is also a distribution assessment vector, which can be denoted by $S(y(a_l)) = \{(H_n, \beta_n(a_l)), n = 1, \ldots, N\}$.

The above formulas (41)–(50) together constitute a complete ER analytical algorithm. Compared with the evidence combination rule of D–S theory, the ER algorithm has at least the following features: (1) taking into account the relative importance of evidence; (2) modeling ignorance clearly by breaking down unassigned probability mass into two parts and treating them differently; (3) generating rational conclusions in the combination of multiple pieces of conflict evidence (Murphy, 2000). To show these features clearly, examine the following two pieces of evidence in conflict:

$$m_1(A) = 0.99, \quad m_1(B) = 0.01, \quad m_1(C) = 0,$$
$$m_2(A) = 0, \quad m_2(B) = 0.01, \quad m_2(C) = 0.99.$$

Before combination, the two pieces of evidence show that B is an unlikely event as only a probability of 1% is assigned to it. After the implementation of the original D–S combination rule, B becomes a certain event and is assigned a probability of 100%, which does not make sense. Table 2 shows the results generated by employing the original D–S combination rule.

Different from the D–S combination rule, the ER algorithm considers the original evidence as belief structures, assigns a relative weight to each piece of evidence, and then combines them to generate a combined basic probability assignment, which will finally be normalized into the belief structure again. Suppose the relative weights of the two pieces of evidence are w_1 and w_2 satisfying $w_1 + w_2 = 1$. Table 3 shows the results generated using the ER algorithm. The combined belief degrees depend to a large extent on the assignment of the relative weights to the two pieces of evidence. Different weights lead to different belief

Table 2
The combination of conflict evidence by the D–S combination rule

Belief structure	A	B	C
m_1	0.99	0.01	0
m_2	0	0.01	0.99
$m_1 \oplus m_2$ (before normalization)	0	0.0001	0
$m_1 \oplus m_2$ (after normalization)	0	1	0

Table 3
The combination of conflict evidence by the ER algorithm

$m_1 \oplus m_2$	m_1			
	$m_1(A) = 0.99w_1$	$m_1(B) = 0.01w_1$	$m_1(C) = 0$	$m_1(\overline{H}) = 1 - w_1$
m_2				
$m_2(A) = 0$	0	0	0	0
$m_2(B) = 0.01w_2$	$0.0099w_1w_2$	$0.0001w_1w_2$	0	$0.01(1-w_1)w_2$
$m_2(C) = 0.99w_2$	$0.9801w_1w_2$	$0.0099w_1w_2$	0	$0.99(1-w_1)w_2$
$m_2(\overline{H}) = 1 - w_2$	$0.99w_1(1-w_2)$	$0.01w_1(1-w_2)$	0	$(1-w_1)(1-w_2)$
Combined basic probability assignment (before normalization)	$0.99w_1\ (1-w_2)$	0.01 - $0.0199w_1w_2$	$0.99(1-w_1)w_2$	$(1-w_1)(1-w_2)$
Combined basic probability assignment (after normalization)	$\frac{0.99w_1(1-w_2)}{1-0.9999w_1w_2}$	$\frac{0.01-0.0199w_1w_2}{1-0.9999w_1w_2}$	$\frac{0.99(1-w_1)w_2}{1-0.9999w_1w_2}$	$\frac{(1-w_1)(1-w_2)}{1-0.9999w_1w_2}$
Combined belief degrees	$\frac{0.99w_1(1-w_2)}{1-1.9999w_1w_2}$	0.01	$\frac{0.99(1-w_1)w_2}{1-1.9999w_1w_2}$	0

Note: w_1 and w_2 are the relative weights of the evidence m_1 and m_2 satisfying $w_1 + w_2 = 1$.

Table 4
The combined belief degrees under different sets of relative weights

Relative weights		$(m_1 \oplus m_2)(A)$	$(m_1 \oplus m_2)(B)$	$(m_1 \oplus m_2)(C)$
$w_1 = 0$,	$w_2 = 1$	0	0.01	0.99
$w_1 = 0.2$,	$w_2 = 0.8$	0.058	0.01	0.932
$w_1 = 0.4$,	$w_2 = 0.6$	0.305	0.01	0.685
$w_1 = 0.5$,	$w_2 = 0.5$	0.495	0.01	0.495
$w_1 = 0.6$,	$w_2 = 0.4$	0.685	0.01	0.305
$w_1 = 0.8$,	$w_2 = 0.2$	0.932	0.01	0.058
$w_1 = 1$,	$w_2 = 0$	0.99	0.01	0

degrees, which are recorded in Table 4, from which it is clear that if the evidence m_1 is assigned a zero weight, then the combined belief degrees are exactly the same as those in the evidence m_2; if the evidence m_1 is assigned a weight of one, then the combined belief degrees are exactly the same as those in m_1; if m_1 and m_2 are both assigned an equal weight 0.5, then the combined belief degrees will be $(m_1 \oplus m_2)(A) = (m_1 \oplus m_2)(C) = 0.495$. It is obvious that such conclusions produced by the ER algorithm make sense.

3.3.2. The ER nonlinear optimization models for aggregating multiple interval belief structures

If the original decision matrix, $D_g = (S(e_i(a_l)))_{L\times M}$, contains interval belief degrees, we cannot simply incorporate the interval arithmetic into the ER analytical algorithm and use them to aggregate multiple interval belief structures. That will lead to wrong results. In order to aggregate multiple interval belief structures correctly, the ER nonlinear optimization models are developed.

First, the interval belief degrees need to be converted into interval probability masses by combining the relative weights and the interval belief degrees using the following equations:

$$m_{n,i} = m_i(H_n) \in [m_{n,i}^-, m_{n,i}^+] = [w_i\beta_{n,i}^-(a_l), w_i\beta_{n,i}^+(a_l)], \quad n = 1, \ldots, N; \quad i = 1, \ldots, L, \tag{51}$$

$$\bar{m}_{H,i} = \bar{m}_i(H) = 1 - w_i, \quad i = 1, \cdots, L, \tag{52}$$

$$\tilde{m}_{H,i} = \tilde{m}_i(H) \in [\tilde{m}_{H,i}^-, \tilde{m}_{H,i}^+] = [w_i\beta_{H,i}^-(a_l), w_i\beta_{H,i}^+(a_l)], \quad i = 1, \ldots, L, \tag{53}$$

$$\text{with} \quad \sum_{n=1}^{N} m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i} = 1 \quad \text{for } i = 1 \text{ to } L \quad \text{and} \quad \sum_{i=1}^{L} w_i = 1.$$

Note that if an interval-valued distribution assessment vector $S(e_i(a_l)) = \{(H_n, [\beta_{n,i}^-(a_l), \beta_{n,i}^+(a_l)]), n = 1, \ldots, N\}$ is complete (see Definition 7), then $\beta_{H,i}^-(a_l) = \beta_{H,i}^+(a_l) \equiv 0$; otherwise, $\beta_{H,i}^-(a_l)$ and $\beta_{H,i}^+(a_l)$ should be determined by (21), namely, $\beta_{H,i}^-(a_l) = \max(0, 1 - \sum_{n=1}^{N}\beta_{n,i}^+(a_l))$ and $\beta_{H,i}^+(a_l) = 1 - \sum_{n=1}^{N}\beta_{n,i}^-(a_l)$. All interval-valued distribution assessment vectors transformed from interval data are complete. Besides, interval-valued distribution assessment vectors transformed from interval data may have to include such constraints with integer variables as in (35).

Next, the interval probability masses on L basic attributes are aggregated and transformed into the overall interval belief degrees by solving the following pair of nonlinear optimization models for each $n = 1$ to N:

$$\text{Max/Min} \quad \beta_n(a_l) = \frac{m_n}{1 - \bar{m}_H} \tag{54}$$

$$\text{s.t.} \quad m_n = k\left[\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right], \quad n = 1, \ldots, N, \tag{55}$$

$$\tilde{m}_H = k\left[\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{56}$$

$$\bar{m}_H = k\left[\prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{57}$$

$$k = \left[\sum_{n=1}^{N}\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - (N-1)\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right]^{-1}, \tag{58}$$

$$m_{n,i}^- \leqslant m_{n,i} \leqslant m_{n,i}^+, \quad n = 1, \ldots, N; \quad i = 1, \ldots, L, \tag{59}$$

$$\bar{m}_{H,i} = 1 - w_i \quad \text{and} \quad \tilde{m}_{H,i}^- \leqslant \tilde{m}_{H,i} \leqslant \tilde{m}_{H,i}^+, \quad i = 1, \ldots, L, \tag{60}$$

$$\sum_{n=1}^{N} m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i} = 1, \quad i = 1, \ldots, L. \tag{61}$$

Let $\beta_n^+(a_l)$ and $\beta_n^-(a_l)$ be the optimal objective function values of the above pair of models. They form an overall interval belief degree $[\beta_n^-(a_l), \beta_n^+(a_l)]$, which represents the aggregated assessment of alternative a_l assessed to the grade H_n. The aggregated assessment is also an interval-valued distribution assessment vector, which we denote by $S(y(a_l)) = \{(H_n, [\beta_n^-(a_l), \beta_n^+(a_l)]), n = 1, \ldots, N\}$.

Moreover, using $\beta_H(a_l) = \frac{\tilde{m}_H}{1-\bar{m}_H}$ instead of $\beta_n(a_l) = \frac{m_n}{1-\bar{m}_H}$ in (54) and then solving the above pair of models, we can get the interval belief degree for $\beta_H(a_l)$, i.e. $[\beta_H^-(a_l), \beta_H^+(a_l)]$.

Note that the interval probability masses on L basic attributes can be aggregated into combined interval probability assignment, which can be captured by solving the above models using m_n instead of $\beta_n(a_l)$ in (54). The models (54)–(61), however, optimize $\beta_n(a_l)$ rather than m_n. This is mainly because the aggregated interval probability assignment is not what we are concerned about. What we are really concerned about is the overall interval belief degrees.

Special attentions must be paid to the interval-valued distribution assessment vectors transformed from interval data. Some of them may involve the constraints on 0–1 integer variables. In this case, they should be included in the above pair of nonlinear optimization models. We will show this feature in Section 4 using a numerical example.

3.4. The ER nonlinear optimization models for computing expected utilities

Different from the other MADA approaches, the ER approach provides an aggregated distribution assessment for each alternative. For the aggregated distribution assessments, the ER approach employs expected utilities to compare or rank them. The expected utility is defined as:

$$u(S(y(a_l))) = \sum_{n=1}^{N} u(H_n)\beta_n(a_l), \quad l = 1, \ldots, M, \tag{62}$$

where $u(S(y(a_l)))$ is the expected utility of the aggregated distribution assessment $S(y(a_l))$, $u(H_n)$ is the grade utility of H_n and $\beta_n(a_l)$ is the overall belief degree to which a_l is assessed to H_n. For convenience and simplicity, $u(S(y(a_l)))$ is usually referred to as the expected utility of a_l for short. If the aggregated distribution assessment is incomplete, then $\beta_H(a_l) \neq 0$, which may be assigned to any assessment grade. When it is assigned to the most preferred assessment grade H_N, $u(S(y(a_l)))$ achieves its maximum, which is defined by

$$u_{\max}(a_l) = \sum_{n=1}^{N-1} \beta_n(a_l)u(H_n) + (\beta_N(a_l) + \beta_H(a_l))u(H_N), \quad l = 1, \ldots, M. \tag{63}$$

If $\beta_H(a_l)$ is assigned to the least preferred assessment grade H_1, then $u(S(y(a_l)))$ achieves its minimum defined by

$$u_{\min}(a_l) = (\beta_1(a_l) + \beta_H(a_l))u(H_1) + \sum_{n=2}^{N} \beta_n(a_l)u(H_n), \quad l = 1, \ldots, M. \tag{64}$$

So, $u(S(y(a_l)))$ is in fact an interval if $\beta_H(a_l) \neq 0$. The midpoint of the interval can be expressed as

$$u_{\text{aver}}(a_l) = \frac{u_{\max}(a_l) + u_{\min}(a_l)}{2}, \quad l = 1, \ldots, M, \tag{65}$$

which is referred to as the average expected utility.

However, for the aggregated interval-valued distribution assessments, the calculation of expected utilities will be more complicated. Due to interval belief degrees, the expected utilities of the aggregated interval-valued distribution assessments will be intervals as well, whose maximums and minimums can be determined by solving the following pair of nonlinear optimization models for $l = 1$ to M, respectively:

$$\text{Max} \quad u_{\max}(a_l) = \sum_{n=1}^{N-1} u(H_n)\beta_n(a_l) + u(H_N)(\beta_N(a_l) + \beta_H(a_l)) \tag{66}$$

$$\text{s.t.} \quad \beta_n(a_l) = \frac{m_n}{1 - \bar{m}_H}, \quad n = 1, \ldots, N, \tag{67}$$

$$\beta_H(a_l) = \frac{\tilde{m}_H}{1 - \bar{m}_H}, \tag{68}$$

$$m_n = k\left[\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right], \quad n = 1, \ldots, N, \tag{69}$$

$$\tilde{m}_H = k\left[\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{70}$$

$$\bar{m}_H = k\left[od_{i=1}^{L}\bar{m}_{H,i}\right], \tag{71}$$

$$k = \left[\sum_{n=1}^{N}\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - (N-1)\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right]^{-1}, \tag{72}$$

$$m_{n,i}^{-} \leqslant m_{n,i} \leqslant m_{n,i}^{+}, \quad n = 1,\ldots,N; \quad i = 1,\ldots,L, \tag{73}$$

$$\bar{m}_{H,i} = 1 - w_i \quad \text{and} \quad \tilde{m}_{H,i}^{-} \leqslant \tilde{m}_{H,i} \leqslant \tilde{m}_{H,i}^{+}, \quad i = 1,\ldots,L, \tag{74}$$

$$\sum_{n=1}^{N} m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i} = 1, \quad i = 1,\ldots,L \tag{75}$$

and

$$\text{Min} \quad u_{\min}(a_l) = u(H_1)(\beta_1(a_l) + \beta_H(a_l)) + \sum_{n=2}^{N} u(H_n)\beta_n(a_l) \tag{76}$$

$$\text{s.t.} \quad \beta_n(a_l) = \frac{m_n}{1 - \bar{m}_H}, \quad n = 1,\ldots,N, \tag{77}$$

$$\beta_H(a_l) = \frac{\tilde{m}_H}{1 - \bar{m}_H}, \tag{78}$$

$$m_n = k\left[\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right], \quad n = 1,\ldots,N, \tag{79}$$

$$\tilde{m}_H = k\left[\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i}) - \prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{80}$$

$$\bar{m}_H = k\left[\prod_{i=1}^{L}\bar{m}_{H,i}\right], \tag{81}$$

$$k = \left[\sum_{n=1}^{N}\prod_{i=1}^{L}(m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i}) - (N-1)\prod_{i=1}^{L}(\bar{m}_{H,i} + \tilde{m}_{H,i})\right]^{-1}, \tag{82}$$

$$m_{n,i}^{-} \leqslant m_{n,i} \leqslant m_{n,i}^{+}, \quad n = 1,\ldots,N; \quad i = 1,\ldots,L, \tag{83}$$

$$\bar{m}_{H,i} = 1 - w_i \quad \text{and} \quad \tilde{m}_{H,i}^{-} \leqslant \tilde{m}_{H,i} \leqslant \tilde{m}_{H,i}^{+}, \quad i = 1,\ldots,L, \tag{84}$$

$$\sum_{n=1}^{N} m_{n,i} + \bar{m}_{H,i} + \tilde{m}_{H,i} = 1, \quad i = 1,\ldots,L. \tag{85}$$

The computation of the average expected utility for each alternative is still the same as (65). If some of the interval-valued distribution assessment vectors transformed from interval data contain the constraints on 0–1 integer variables, then they should be included in the above pair of nonlinear optimization models.

Since the ER approach characterizes the expected utility of each alternative as an interval, a simple yet practical minimax regret ranking approach is developed (Wang et al., 2005), which can be used to compare and rank the interval-valued expected utilities of alternatives even if they are equi-centred but different in

widths. The approach is detailed in Appendix A and will be used for the numerical study reported in the next section.

4. Application of the ER approach to a cargo ship selection problem

In this section, a cargo ship selection problem is examined to demonstrate the application of the ER approach in modeling MADA problems using interval belief degrees.

4.1. The description of a cargo ship selection problem

Consider a cargo ship selection problem with six competing cargo ship designs, which are compared on the basis of nine basic attributes shown in Table 5, where *Load factor* and *Effective load factor* are two qualitative attributes and *Bale capacity*, *Deadweight*, *Speed*, *Capital investment*, *Annual M&R and manning costs*, *Sea fuel consumption* and *Off-hire* are seven quantitative attributes. Among the nine basic attributes, the first five are benefit attributes and the others are cost attributes. For the purpose of illustration, two quantitative attributes, i.e. *Sea fuel consumption* and *Off-hire*, are assumed to be interval-valued attributes, whose exact values are not known, but can be estimated using interval numbers. The relative weights of the nine basic attributes are given as: (12, 15, 10, 10, 12, 15, 10, 10, 6), which are normalized as (0.12, 0.15, 0.1, 0.1, 0.12, 0.15, 0.1, 0.1, 0.06).

Suppose a cargo ship can be assessed by using the following assessment grades: *Excellent* (E), *Very Good* (V), *Good* (G), *Average* (A) and *Poor* (P). They form the set of assessment grades:

$$H = \{H_n, n = 1, \ldots, 5\} = \{Poor, Average, Good, VeryGood, Excellent\} = \{P, A, G, V, E\},$$

which is called the frame of discernment. Two qualitative attributes, *Load factor* and *Effective load factor*, are both assessed by the above set of assessment grades. However, when DM is asked the question 'To what degree can you assess each cargo ship to the five assessment grades?', he feels difficult to give a precise assessment. Instead of giving a precise belief structure, he is asked to give an interval belief structure for each cargo ship. The interval belief structures provided by the DM together with the other original quantitative data are presented in Table 5.

Table 5
Original assessment data for six cargo ships

Attributes	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5	Ship 6
Load factor (12)	{(V, [0.2, 0.4]), (E, [0.7, 0.8])}	{(A, [0.4, 0.5]), (G, [0.55, 0.65])}	{(G, [0.3, 0.5]), (V, [0.5, 0.6])}	{(V, [0.1, 0.15]), (E, [0.8, 0.95])}	{(A, [0.3, 0.4]), (G, [0.5, 0.6])}	{(P, [0.5, 0.65]), (A, [0.4, 0.5])}
Bale capacity (M^3)(15)	21,070	28,000	31,587	26,718	28,874	33,117
Deadweight (t) (10)	17,200	23,200	22,351	20,621	22,233	25,500
Speed (kn) (10)	14.3	14.8	17.7	15.0	18.2	17.6
Effective load factor (%) (12)	{(V, [0.1, 0.2]), (E, [0.8, 0.9])}	{(G, [0.2, 0.4]), (V, [0.6, 0.7])}	{(A, [0.4, 0.5]), (G, [0.5, 0.6])}	{(V, [0.15, 0.2]), (E, [0.8, 0.85])}	{(A, [0.5, 0.6]), (G, [0.4, 0.5])}	{(A, [0.3, 0.5]), (P, [0.6, 0.7])}
Capital investment (pound M) (15)	16.66	20.00	31.33	16.66	25.00	25.00
Annual M&R and manning costs (pound M) (10)	0.95	1.005	1.045	0.97	1.10	1.09
Sea fuel consumption (tons per day) (10)	[18, 20.1]	[23, 25.3]	[37, 39]	[20.2, 22.1]	[38, 40]	[38.5, 40.5]
Off-hire (days per year) (6)	[15, 17]	[16, 18]	[18, 20]	[18, 21]	[19, 22]	[17, 19]

4.2. Modeling quantitative data using belief or interval belief structures

In order to model the quantitative data in Table 5 using belief or interval belief structures, we first need to know the assessment grades of each quantitative attribute. Suppose the DM's utilities of the five assessment grades are

$$u(P) = 0, \quad u(A) = 0.4, \quad u(G) = 0.6, \quad u(V) = 0.8 \quad \text{and } u(E) = 1.$$

By the principle of utility equivalence and the formulas (23)–(25), we can find out the approximate estimates of the assessment grades of each quantitative attribute. Table 6 shows the assessment grades of each quantitative attribute, which are revised or adjusted by the DM.

With the help of the above assessment grades, all the quantitative data shown in Table 5 can be properly modeled by using belief or interval belief structures. Take Cargo Ship 1 for example. The Bale *capacity* of Ship 1 is 21,070 M^3, which lies between the assessment grades P and A. So, it can be fully characterized by using these two assessment grades. The belief degrees, to which Ship 1 is assessed to P and A on *Bale capacity*, can be calculated as follows:

$$\beta_{1,2}(a_1) = \frac{26000 - 21070}{26000 - 20000} = 0.8217 \quad \text{and} \quad \beta_{2,2}(a_1) = \frac{21070 - 20000}{26000 - 20000} = 0.1783.$$

Thus, The *Bale capacity* of Ship 1 can be equivalently expressed in the form of belief structure as {(P, 0.8217), (A, 0.1783)}. As far as the *Capital investment* attribute is concerned, the value of Ship 1 is 16.66 (pound M), which lies between the assessment grades V and E. The belief degrees, to which Ship 1 is assessed to V and E on *Capital investment*, are given by

$$\beta_{4,6}(a_1) = \frac{15 - 16.66}{15 - 18.4} = 0.4882 \quad \text{and} \quad \beta_{5,6}(a_1) = \frac{16.66 - 18}{15 - 18.4} = 0.5118.$$

Thus, the *Capital investment* of Ship 1 can be modeled by using the belief structure {(V, 0.4882), (E, 0.5118)}. All the other precise data can be modeled in the same way.

The S*ea fuel consumption* of Ship 1 is an interval value $[18, 20.1]$, which lies between the assessment grades V and E. So, it can be entirely characterized by these two assessment grades. The belief degrees, to which Ship 1 is assessed to V and E on *Sea fuel consumption*, are both interval numbers and can be calculated as follows:

$$\beta^-_{4,8}(a_1) = \frac{18 - 18}{18 - 22.6} = 0 \quad \text{and} \quad \beta^+_{4,8}(a_1) = \frac{18 - 20.1}{18 - 22.6} = 0.4565,$$

$$\beta^-_{5,8}(a_1) = \frac{20.1 - 22.6}{18 - 22.6} = 0.5435 \quad \text{and} \quad \beta^+_{5,8}(a_1) = \frac{18 - 22.6}{18 - 22.6} = 1.$$

Table 6
Assessment grades for seven quantitative attributes

Assessment grades	P	A	G	V	E
Grade utilities	0	0.4	0.6	0.8	1
Bale capacity (M^3)	20,000	26,000	29,000	32,000	35,000
Deadweight (t)	17,000	20,600	22,400	24,200	26,000
Speed (kn)	14	16	17	18	19
Capital investment (pound M)	32	25.2	21.8	18.4	15
Annual M&R and manning costs (pound M)	1.15	1.07	1.03	0.99	0.95
Sea fuel consumption (tons per day)	41	31.8	27.2	22.6	18
Off-hire (days per year)	22	19	18	16	15

Therefore, the S*ea fuel consumption* of Ship 1 can be modeled by using interval belief structure as

$$\{(V,[0,0.4565]),(E,[0.5435,1])\} \quad \text{with } \beta_{4,8}(a_1)+\beta_{5,8}(a_1)=1.$$

Note that this is a complete interval-valued distribution assessment. The normalization constraint $\beta_{4,8}(a_1)+\beta_{5,8}(a_1)=1$ should always be applied.

As far as the *Off-hire* attribute is concerned, the *Off-hire* time of Ship 1 is [15, 17] days per year. Since the assessment grade V is included in the interval [15, 17], which lies between the assessment grades G and E. In order to fully characterize the *Off-hire* time of Ship 1, the assessment grades G, V and E are all needed, though the *Off-hire* time of Ship 1 can only be assessed to either G and V or V and E. The belief degrees, to which Ship 1 may probably be assessed on *Off-hire* to G, V and E, are all interval numbers, which can be calculated respectively as follows:

$$\beta^-_{3,9}(a_1)=\frac{16-16}{16-18}=0 \quad \text{and} \quad \beta^+_{3,9}(a_1)=\frac{16-17}{16-18}=0.5,$$

$$\beta^-_{4,9}(a_1)=0 \quad \text{and} \quad \beta^+_{4,9}(a_1)=1,$$

$$\beta^-_{5,9}(a_1)=0 \quad \text{and} \quad \beta^+_{5,9}(a_1)=\frac{15-16}{15-16}=1.$$

In order to express that the *Off-hire* time of Ship 1 can only be assessed to either G and V or V and E, two 0–1 integer variables I_1 and I_2 are introduced. Accordingly, the *Off-hire* time of Ship 1 can be modeled by using interval belief structure as follows:

$$\{(G,[0,0.5I_1]),(V,[0,I_1+I_2]),(E,[0,I_2])\} \quad \text{with } I_1+I_2=1.$$

Note that this is a complete interval-valued distribution assessment. The normalization constraint condition $\beta_{3,9}(a_1)+\beta_{4,9}(a_1)+\beta_{5,9}(a_1)=1$ has to be included. All the quantitative data for the other five cargo ships can be modeled in the same way as for Ship 1. Table 7 shows all the belief or interval belief structures for the six cargo ships.

It must be pointed out that in order to show clearly which interval-valued distribution assessments are incomplete and which are complete, the interval belief degrees for the whole set H are also calculated and presented in Table 7. The interval-valued distribution assessments with $\beta_{H,i}\in[\beta^-_{H,i},\beta^+_{H,i}]\neq 0$ are incomplete and the others are complete.

4.3. Aggregating multiple interval belief structures using the ER nonlinear optimization models

In order to aggregate nine belief or interval belief structures for each cargo ship, we first need to transform the belief or interval belief degrees shown in Table 7 into basic or interval probability masses using formulas (41)–(44) or (51)–(53). Take Cargo Ship 1 for example.

For attribute 1 (*Load factor*), we have

$$m_{1,1}=m_{2,1}=m_{3,1}=0,$$

$$m_{4,1}=m_1(V)\in[m^-_{4,1},m^+_{4,1}]=w_1\times[\beta^-_{4,1}(a_1),\beta^+_{4,1}(a_1)]=0.12\times[0.2,0.4]=[0.024,0.048],$$

$$m_{5,1}=m_1(E)\in[m^-_{5,1},m^+_{5,1}]=w_1\times[\beta^-_{5,1}(a_1),\beta^+_{5,1}(a_1)]=0.12\times[0.7,0.8]=[0.084,0.096],$$

$$\bar{m}_{H,1}=\bar{m}_1(H)=1-w_1=1-0.12=0.88,$$

$$\tilde{m}_{H,1}=\tilde{m}_1(H)\in[\tilde{m}^-_{H,1},\tilde{m}^+_{H,1}]=w_1\times[\beta^-_{H,1}(a_1),\beta^+_{H,1}(a_1)]=0.12\times[0,0.1]=[0,0.012],$$

with $m_{4,1}+m_{5,1}+\bar{m}_{H,1}+\tilde{m}_{H,1}=1$.

Table 7
Distribution assessment matrix for the six cargo ships

Attributes	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5	Ship 6
Load factor (12)	$\{(V, [0.2, 0.4]), (E, [0.7, 0.8]), (H, [0, 0.1])\}$	$\{(A, [0.4, 0.5]), (G, [0.55, 0.65]), (H, [0, 0.05])\}$	$\{(G, [0.3, 0.5]), (V, [0.5, 0.6]), (H, [0, 0.2])\}$	$\{(V, [0.1, 0.15]), (E, [0.8, 0.95]), (H, [0, 0.1])\}$	$\{(A, [0.3, 0.4]), (G, [0.5, 0.6]), (H, [0, 0.2])\}$	$\{(P, [0.5, 0.65]), (A, [0.4, 0.5]), (H, [0, 0.1])\}$
Bale capacity (M^3) (15)	$\{(P, 0.8217), (A, 0.1783)\}$	$\{(A, 0.3333), (G, 0.6667)\}$	$\{(G, 0.1377), (V, 0.8623)\}$	$\{(A, 0.7607), (G, 0.2393)\}$	$\{(A, 0.042), (G, 0.958)\}$	$\{(V, 0.6277), (E, 0.3723)\}$
Deadweight (t) (10)	$\{(P, 0.9444), (A, 0.0556)\}$	$\{(G, 0.5556), (V, 0.4444)\}$	$\{(A, 0.0272), (G, 0.9728)\}$	$\{(A, 0.9883), (G, 0.0117)\}$	$\{(A, 0.0928), (G, 0.9072)\}$	$\{(V, 0.2778), (E, 0.7222)\}$
Speed (kn) (10)	$((P, 0.85), (A, 0.15))$	$\{(P, 0.6), (A, 0.4)\}$	$\{(G, 0.3), (V, 0.7)\}$	$\{(P, 0.5), (A, 0.5)\}$	$\{(V, 0.8), (E, 0.2)\}$	$\{(G, 0.4), (V, 0.6)\}$
Effective load factor (%) (12)	$\{(V, [0.1, 0.2]), (E, [0.8, 0.9]), (H, [0, 0.1])\}$	$\{(G, [0.2, 0.4]), (V, [0.6, 0.7]), (H, [0, 0.2])\}$	$\{(A, [0.4, 0.5]), (G, [0.5, 0.6]), (H, [0, 0.1])\}$	$\{(V, [0.15, 0.2]), (E, [0.8, 0.85]), (H, [0, 0.05])\}$	$\{(A, [0.5, 0.6]), (G, [0.4, 0.5]), (H, [0, 0.1])\}$	$\{(P, [0.6, 0.7]), (A, [0.3, 0.5]), (H, [0, 0.1])\}$
Capital investment (pound M) (15)	$\{(V, 0.4882), (E, 0.5118)\}$	$\{(G, 0.4706), (V, 0.5294)\}$	$\{(P, 0.9015), (A, 0.0985)\}$	$\{(V, 0.4882), (E, 0.5118)\}$	$\{(A, 0.9412), (G, 0.0588)\}$	$\{(A, 0.9412), (G, 0.0588)\}$
Annual M&R and manning costs (pound M) (10)	$\{(E, 1.0)\}$	$\{(G, 0.375), (V, 0.625)\}$	$\{(A, 0.375), (G, 0.625)\}$	$\{(V, 0.5), (E, 0.5)\}$	$\{(P, 0.375), (A, 0.625)\}$	$\{(P, 0.25), (A, 0.75)\}$
Sea fuel consumption (tons per day) (10)	$\{(V, [0, 0.4565]), (E, [0.5435, 1])\}$	$\{(G, [0.087, 0.587]), (V, [0.413, 0.913])\}$	$\{(P, [0.5652, 0.7826]), (A, [0.2174, 0.4348])\}$	$\{(V, [0.4783, 0.8913]), (E, [0.1087, 0.5217])\}$	$\{(P, [0.6739, 0.8913]), (A, [0.1087, 0.3261])\}$	$\{(P, [0.7283, 0.9457]), (A, [0.0543, 0.2717])\}$
Off-hire (days per year) (6)	$\{(G, [0, 0.5I_1]), (V, [0, I_1 + I_2]), (E, [0, I_2])\}$	$\{(G, [0, 1]), (V, [0, 1])\}$	$\{(P, [0, 0.3333I_1]), (A, [0, I_1 + I_2]), (G, [0, I_2])\}$	$\{(P, [0, 0.6667I_1]), (A, [0, I_1 + I_2]), (G, [0, I_2])\}$	$\{(P, [0, 1]), (A, [0, 1])\}$	$\{(A, [0, I_1]), (G, [0, I_1 + I_2]), (V, [0, 0.5I_2])\}$

Note: $I_1 + I_2 = 1$ and $I_1, I_2 = 0$ or 1.

For attribute 2 (*Bale capacity*), we have

$$m_{1,2} = m_2(P) = w_2 \times \beta_{1,2}(a_1) = 0.15 \times 0.8217 = 0.123255,$$

$$m_{2,2} = m_2(A) = w_2 \times \beta_{2,2}(a_1) = 0.15 \times 0.1783 = 0.026745,$$

$$m_{3,2} = m_{4,2} = m_{5,2} = 0,$$

$$\bar{m}_{H,2} = \bar{m}_2(H) = 1 - w_2 = 1 - 0.15 = 0.85,$$

$$\tilde{m}_{H,2} = \tilde{m}_2(H) = 0.$$

Similarly, for attribute 3 (*Deadweight*), we have $m_{1,3} = 0.09444$, $m_{2,3} = 0.00556$, $m_{3,3} = m_{4,3} = m_{5,3} = 0$, $\bar{m}_{H,3} = \bar{m}_3(H) = 0.9$, $\tilde{m}_{H,3} = 0$.

For attribute 4 (*Speed*), we have $m_{1,4} = 0.085$, $m_{2,4} = 0.015$, $m_{3,4} = m_{4,4} = m_{5,4} = 0$, $\bar{m}_{H,4} = \bar{m}_4(H) = 1 - w_4 = 1 - 0.1 = 0.9$, $\bar{m}_{H,4} = \bar{m}_4(H) = 0.9$, $\tilde{m}_{H,4} = \tilde{m}_4(H) = 0$.

For attribute 5 (*Effective load factor*), we have $m_{1,5} = m_{2,5} = m_{3,5} = 0$, $m_{4,5} = m_5(V) \in [0.012, 0.024]$, $m_{5,5} = m_5(E) \in [0.096, 0.108]$, $\bar{m}_{H,5} = \bar{m}_5(H) = 0.88$, $\tilde{m}_{H,5} = \tilde{m}_5(H) \in [0, 0.012]$ with $m_{4,5} + m_{5,5} + \bar{m}_{H,5} + \tilde{m}_{H,5} = 1$.

For attribute 6 (*Capital investment*), we have $m_{1,6} = m_{2,6} = m_{3,6} = 0$, $m_{4,6} = 0.07323$, $m_{5,6} = 0.07677$, $\bar{m}_{H,6} = 0.85$, $\tilde{m}_{H,6} = \tilde{m}_6(H) = 0$.

For attribute 7 (*Annual M&R and manning costs*), we have $m_{1,7} = m_{2,7} = m_{3,7} = m_{4,7} = 0$, $m_{5,7} = 0.1$, $\bar{m}_{H,7} = 0.9$, $\tilde{m}_{H,7} = \tilde{m}_7(H) = 0$.

For attribute 8 (*Sea fuel consumption*), we have

$$m_{1,8} = m_{2,8} = m_{3,8} = 0,$$

$$m_{4,8} = m_8(V) \in [m_{4,8}^-, m_{4,8}^+] = w_8 \times [\beta_{4,8}^-(a_1), \beta_{4,8}^+(a_1)] = 0.1 \times [0, 0.4565] = [0, 0.04565],$$

$$m_{5,8} = m_8(E) \in [m_{5,8}^-, m_{5,8}^+] = w_8 \times [\beta_{5,8}^-(a_1), \beta_{5,8}^+(a_1)] = 0.1 \times [0.5435, 1] = [0.05435, 0.1],$$

$$\bar{m}_{H,8} = \bar{m}_8(H) = 1 - w_8 = 1 - 0.1 = 0.9,$$

$$\tilde{m}_{H,8} = \tilde{m}_8(H) = 0,$$

with $m_{4,8} + m_{5,8} + \bar{m}_{H,8} = 1$.

For attribute 9 (*Off-hire*), we have

$$m_{1,9} = m_{2,9} = 0,$$

$$m_{3,9} = m_9(G) \in [m_{3,9}^-, m_{3,9}^+] = w_9 \times [\beta_{3,9}^-(a_1), \beta_{3,9}^+(a_1)] = 0.06 \times [0, 0.5I_1] = [0, 0.03I_1],$$

$$m_{4,9} = m_9(V) \in [m_{4,9}^-, m_{4,9}^+] = w_9 \times [\beta_{4,9}^-(a_1), \beta_{4,9}^+(a_1)] = 0.06 \times [0, I_1 + I_2] = [0, 0.06(I_1 + I_2)],$$

$$m_{5,9} = m_9(E) \in [m_{5,9}^-, m_{5,9}^+] = w_9 \times [\beta_{5,9}^-(a_1), \beta_{5,9}^+(a_1)] = 0.06 \times [0, I_2] = [0, 0.06I_2],$$

$$\bar{m}_{H,9} = \bar{m}_9(H) = 1 - w_9 = 1 - 0.06 = 0.94,$$

$$\tilde{m}_{H,9} = \tilde{m}_9(H) = 0,$$

with $m_{3,9} + m_{4,9} + m_{5,9} + \bar{m}_{H,9} = 1$ and $I_1 + I_2 = 1$, where I_1 and $I_2 = 0$ or 1.

Next, based on the above basic and interval probability masses, we need to solve the ER mixed nonlinear optimization models (54)–(61) to generate an aggregated interval-valued distribution assessment for each cargo ship. Since the above interval probability masses involve 0–1 variables I_1 and I_2, the ER nonlinear optimization models (54)–(61) still have to involve the constraint condition $I_1 + I_2 = 1$, where I_1 and

$I_2 = 0$ or 1. The aggregated interval-valued distribution assessment for each cargo ship is shown in Table 8 and plotted in Fig. 2.

Table 8
The aggregated interval belief degrees for each cargo ship

Cargo ship	*P*	*A*	*G*	*V*	*E*	*H*
Ship 1	[0.2841, 0.3003]	[0.0407, 0.0430]	[0.0000, 0.0249]	[0.0968, 0.2296]	[0. 4318, 0.5784]	[0, 0.0209]
Ship 2	[0.0493, 0.0509]	[0.1229, 0.1325]	[0.3731, 0.5405]	[0.2873, 0.4352]	0	[0, 0.0253]
Ship 3	[0.1818, 0.2315]	[0.1563, 0.2144]	[0.3155, 0.4249]	[0.2592, 0.2754]	0	[0, 0.0321]
Ship 4	[0.0432, 0.0801]	[0.2591, 0.3267]	[0.0339, 0.0883]	[0.1934, 0.2558]	[0.3395, 0.4141]	[0, 0.0161]
Ship 5	[0.0913, 0.1736]	[0.3364, 0.4568]	[0.3557, 0.3942]	[0.0678, 0.0708]	[0.0170, 0.0177]	[0, 0.0316]
Ship 6	[0.2232, 0.2799]	[0.3182, 0.4402]	[0.0435, 0.1000]	[0.1742, 0.2116]	[0.1189, 0.1234]	[0, 0.0218]

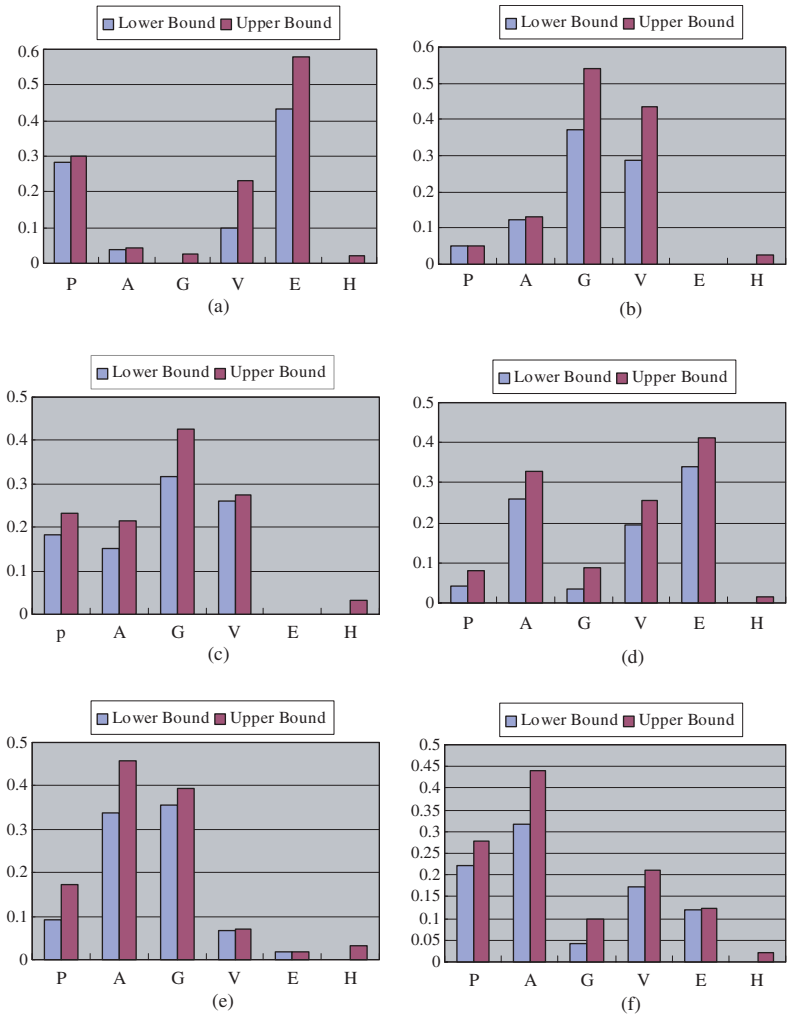

Fig. 2. The aggregated interval belief degrees for the six cargo ships. (a) Cargo Ship 1, (b) Cargo Ship 2, (c) Cargo Ship 3, (d) Cargo Ship 4, (e) Cargo Ship 5 and (f) Cargo Ship 6.

4.4. Computing expected utilities using the ER nonlinear optimization models

In spite of the fact that Fig. 2 provides DM with rich information, it is not easy to compare the aggregated interval-valued distribution assessments of the six cargo ships directly. In order to compare or rank them, it is necessary to compute their expected utilities. The ER nonlinear optimization models discussed in Section 3.4 can be used to derive the expected utilities, which are shown in Table 9 and plotted in Fig. 3.

4.5. Ranking the expected utilities using the minimax regret approach (MRA)

As long as there exists uncertainty in modeling MADA problems, the expected utilities would always be characterized by intervals whose lower and upper bounds are the minimum and the maximum expected utilities, respectively. Due to the presence of interval belief degrees and interval data in the cargo ship selection problem, the expected utilities of the six cargo ships are all interval numbers (as shown in Table 9 and Fig. 3). In order to compare and rank them, we use the minimax regret approach (MRA) to calculate the maximum loss of expected utility each cargo ship may suffer:

$$R(\text{Ship1}) = \max[\max(0.6386, 0.5254, 0.7268, 0.4912, 0.4940) - 0.6078, 0] = 0.7268 - 0.6078 = 0.1190,$$
$$R(\text{Ship2}) = \max[\max(0.6721, 0.5254, 0.7268, 0.4912, 0.4940) - 0.5880, 0] = 0.7268 - 0.5880 = 0.1388,$$
$$R(\text{Ship3}) = \max[\max(0.6721, 0.6386, 0.7268, 0.4912, 0.4940) - 0.4626, 0] = 0.7268 - 0.4626 = 0.2642,$$
$$R(\text{Ship4}) = \max[\max(0.6721, 0.6386, 0.5254, 0.4912, 0.4940) - 0.6708, 0] = 0.6721 - 0.6708 = 0.0013,$$
$$R(\text{Ship5}) = \max[\max(0.6721, 0.6386, 0.5254, 0.7268, 0.4940) - 0.4304, 0] = 0.7268 - 0.4304 = 0.2964,$$
$$R(\text{Ship6}) = \max[\max(0.6721, 0.6386, 0.5254, 0.7268, 0.4912) - 0.4413, 0] = 0.7268 - 0.4413 = 0.2855.$$

Table 9
The expected utilities of the six cargo ships and their ranking order

Expected utilities	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5	Ship 6
Minimum expected utility	0.6078	0.5880	0.4626	0.6708	0.4304	0.4413
Maximum expected utility	0.6721	0.6386	0.5254	0.7268	0.4912	0.4940
Average expected utility	0.6400	0.6133	0.4940	0.6988	0.4608	0.4677
Rank	2	3	4	1	6	5

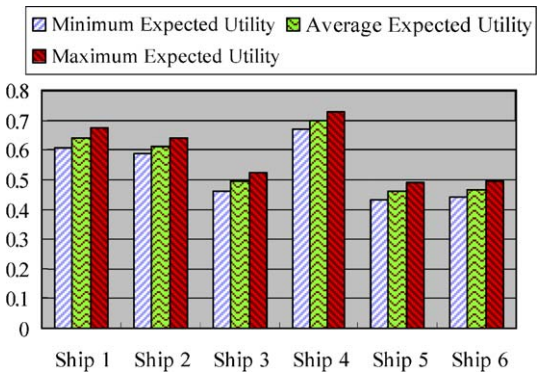

Fig. 3. The expected utilities of the six cargo ships.

Ship 4 has the smallest maximum loss of expected utility. So, Ship 4 is selected as the best cargo ship. If the purpose of decision analysis is only to choose a best cargo ship from among the six cargo ships, then the process of decision analysis is terminated. If the purpose of decision analysis is not only to choose a best cargo ship, but also to give a ranking order for the six cargo ships, then Ship 4 should be eliminated from further consideration. Thus, the remaining cargo ships are Ships 1, 2, 3, 5 and 6, whose maximum losses of expected utilities are recalculated as follows:

$$R(\text{Ship1}) = \max[\max(0.6386, 0.5254, 0.4912, 0.4940) - 0.6078, 0] = 0.6386 - 0.6078 = 0.0308,$$

$$R(\text{Ship2}) = \max[\max(0.6721, 0.5254, 0.4912, 0.4940) - 0.5880, 0] = 0.6721 - 0.5880 = 0.0841,$$

$$R(\text{Ship3}) = \max[\max(0.6721, 0.6386, 0.4912, 0.4940) - 0.4626, 0] = 0.6721 - 0.4626 = 0.2095,$$

$$R(\text{Ship5}) = \max[\max(0.6721, 0.6386, 0.5254, 0.4940) - 0.4304, 0] = 0.6721 - 0.4304 = 0.2417,$$

$$R(\text{Ship6}) = \max[\max(0.6721, 0.6386, 0.5254, 0.4912) - 0.4413, 0] = 0.6721 - 0.4413 = 0.2308.$$

Among the above regrets, the maximum loss of expected utility of Ship 1 is the smallest. So, Ship 1 is selected as the second best cargo ship and eliminated from further consideration. The remaining cargo ships are Ships 2, 3, 5 and 6, whose maximum losses of expected utilities are re-computed and shown below:

$$R(\text{Ship2}) = \max[\max(0.5254, 0.4912, 0.4940) - 0.5880, 0] = \max[0.5254 - 0.5880, 0] = 0,$$

$$R(\text{Ship3}) = \max[\max(0.6386, 0.4912, 0.4940) - 0.4626, 0] = 0.6386 - 0.4626 = 0.1760,$$

$$R(\text{Ship5}) = \max[\max(0.6386, 0.5254, 0.4940) - 0.4304, 0] = 0.6386 - 0.4304 = 0.2082,$$

$$R(\text{Ship6}) = \max[\max(0.6386, 0.5254, 0.4912) - 0.4413, 0] = 0.6386 - 0.4413 = 0.1973.$$

Among the above regrets, Ship 2 has the smallest maximum loss of expected utility. So, Ship 2 is selected as the third best cargo ship and eliminated from further consideration. The remaining cargo ships are now Ships 3, 5 and 6, whose maximum losses of expected utilities need recalculating:

$$R(\text{Ship3}) = \max[\max(0.4912, 0.4940) - 0.4626, 0] = 0.4940 - 0.4626 = 0.0314,$$

$$R(\text{Ship5}) = \max[\max(0.5254, 0.4940) - 0.4304, 0] = 0.5254 - 0.4304 = 0.0950,$$

$$R(\text{Ship6}) = \max[\max(0.5254, 0.4912) - 0.4413, 0] = 0.5254 - 0.4413 = 0.0841.$$

Since Ship 3 has the smallest maximum loss of expected utility, it is selected as the best among the three cargo ships. Repeating the above process, we finally get the ranking order of the six cargo ships as Ship 4 $\succ$ Ship 1 $\succ$ Ship 2 $\succ$ Ship 3 $\succ$ Ship 6 $\succ$ Ship 5, which is shown in the last row of Table 9.

4.6. Analysing the effects of weight uncertainty on final assessment and ranking order

The previous analysis focuses on the assumption that the relative weights of attributes are precisely known. In real applications, however, DM may be unable to provide precise weights. As such, uncertain weights such as interval or preference weights may be provided.

In this subsection, we examine whether weight uncertainty has any significant impact on the above assessments and ranking order. Two cases are considered as follows:

Case 1: $w_1 \in [0.1, 0.15]$, $w_2 \in [0.15, 0.2]$, $w_3 \in [0.08, 0.12]$, $w_4 \in [0.08, 0.12]$, $w_5 \in [0.1, 0.15]$, $w_6 \in [0.15, 0.2]$, $w_7 \in [0.08, 0.12]$, $w_8 \in [0.08, 0.12]$, $w_9 \in [0.05, 0.08]$ with $\sum_{i=1}^{9} w_i = 1$.

Case 2: $w_1 \sim w_5 \in [0.1, 0.15]$, $w_2 \sim w_6 \in [0.15, 0.2]$, $w_3 \sim w_4 \sim w_7 \sim w_8 \in [0.08, 0.12]$, $w_9 \in [0.05, 0.08]$ with $w_1 \geqslant w_3$ and $\sum_{i=1}^{9} w_i = 1$.

In case 1, the DM can only provide a set of interval weights, but in case 2, the DM can provide extra weight preferences in addition to the interval weights. For convenience, we refer to the case 1 as interval weights and the case 2 as preference weights. Obviously, the precise weights used in Sections 4.1–4.5 are the special case of the above preference weights.

The ER approach can model MADA problems having both certain and uncertain weights. In the case of uncertain weights, the relative weights will be regarded as decision variables, which are incorporated into the ER nonlinear optimization models and optimized together with the other decision variables. The aggregated interval belief degrees for each cargo ship under two different sets of uncertain weights are shown in Tables 10 and 12. The corresponding expected utilities and the ranking order are presented in Tables 11 and 13.

As can been seen from Tables 10–13, as the weight uncertainty changes from interval to preference weights, the uncertainty in both the aggregated interval belief degrees and the expected utilities reduce as the intervals become narrower. When the precise weights are provided, the uncertainty caused by the weights no longer exists. So, the intervals shown in Tables 8 and 9 are the smallest compared with their counterparts in Tables 10–13.

It is also clear that the three different sets of relative weights, both precise and imprecise (interval and preferential), all lead to the same decision conclusion and the same ranking order for the six cargo ships.

Table 10
The aggregated interval belief degrees for each cargo ship under interval weights

Cargo ship	*P*	*A*	*G*	*V*	*E*	*H*
Ship 1	[0.2403, 0.4012]	[0.0363, 0.0573]	[0.0000, 0.0342]	[0.0896, 0.2846]	[0.3500, 0.6325]	[0, 0.0268]
Ship 2	[0.0376, 0.0631]	[0.1047, 0.1733]	[0.3385, 0.5831]	[0.2341, 0.4896]	0	[0, 0.0327]
Ship 3	[0.1670, 0.3173]	[0.0932, 0.2614]	[0.2493, 0.4699]	[0.2256, 0.3634]	0	[0, 0.0412]
Ship 4	[0.0332, 0.1043]	[0.2190, 0.4214]	[0.0329, 0.1229]	[0.1646, 0.3104]	[0.2805, 0.4743]	[0, 0.0207]
Ship 5	[0.0691, 0.2254]	[0.2987, 0.5296]	[0.3083, 0.4805]	[0.0516, 0.0879]	[0.0129, 0.0220]	[0, 0.0402]
Ship 6	[0.1723, 0.3364]	[0.2785, 0.5211]	[0.0352, 0.1325]	[0.1512, 0.2766]	[0.1016, 0.1638]	[0, 0.0279]

Table 11
The expected utilities of the six cargo ships and their ranking order under interval weights

Expected utilities	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5	Ship 6
Minimum expected utility	0.5088	0.5652	0.4058	0.6149	0.3950	0.3988
Maximum expected utility	0.7200	0.6610	0.5580	0.7653	0.5180	0.5585
Average expected utility	0.6144	0.6131	0.4819	0.6901	0.4565	0.4787
Rank	2	3	4	1	6	5

Table 12
The aggregated interval belief degrees for each cargo ship under preference weights

Cargo ship	*P*	*A*	*G*	*V*	*E*	*H*
Ship 1	[0.2403, 0.3257]	[0.0363, 0.0502]	[0.0000, 0.0342]	[0.0918, 0.2590]	[0.3868, 0.6187]	[0, 0.0268]
Ship 2	[0.0376, 0.0556]	[0.1174, 0.1441]	[0.3606, 0.5732]	[0.2664, 0.4538]	0	[0, 0.0324]
Ship 3	[0.1676, 0.2764]	[0.1002, 0.2353]	[0.2562, 0.4441]	[0.2485, 0.3088]	0	[0, 0.0412]
Ship 4	[0.0332, 0.0953]	[0.2227, 0.3611]	[0.0333, 0.1210]	[0.1791, 0.2643]	[0.3060, 0.4643]	[0, 0.0207]
Ship 5	[0.0691, 0.2006]	[0.3249, 0.4890]	[0.3410, 0.4291]	[0.0517, 0.0775]	[0.0129, 0.0194]	[0, 0.0401]
Ship 6	[0.1723, 0.3092]	[0.3077, 0.4739]	[0.0352, 0.1222]	[0.1524, 0.2409]	[0.1026, 0.1353]	[0, 0.0279]

Table 13
The expected utilities of the six cargo ships and their ranking order under preference weights

Expected utilities	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5	Ship 6
Minimum expected utility	0.5836	0.5861	0.4451	0.6479	0.4202	0.4140
Maximum expected utility	0.7171	0.6465	0.5380	0.7562	0.5015	0.5283
Average expected utility	0.6504	0.6163	0.4916	0.7021	0.4609	0.4712
Rank	2	3	4	1	6	5

This means that selecting Cargo ship 4 as the best decision alternative and ranking the cargo ships as Ship 4 $\succ$ Ship 1 $\succ$ Ship 2 $\succ$ Ship 3 $\succ$ Ship 6 $\succ$ Ship 5 are robust given the change of the relative weights of the nine attributes. Similarly, sensitivity analyses for the other parameters such as belief degrees and the number of assessment grades may also be conducted.

5. Concluding remarks

In this paper we investigated two new types of interval uncertainties, interval belief degrees and interval data, and showed how these two types of interval uncertainties could be properly modeled using the ER approach which was originally developed to model MADA problems with both quantitative and qualitative attributes and probabilistic and fuzzy uncertainties using precise belief structures.

In order to allow MADA problems to be modeled using interval belief degrees, the traditional D–S theory of evidence and the ER approach were both extended. The focus of the investigation was on the issues of combining and normalizing interval belief structures. Two pairs of nonlinear optimization models were developed to aggregate multiple interval belief structures and generate expected utilities. Interval data were transformed into interval belief structures in terms of utility equivalence and incorporated into the ER nonlinear optimization models. A numerical study about a cargo ship selection problem was conducted using the proposed interval ER approach. The study shows that the interval ER approach provides a flexible way to model complex MADA problems. It allows MADA problems to be modeled using both precise and interval belief degrees as well as precise and imprecise relative weights.

Compared with Monte Carlo (MC) simulation, the interval ER approach requires no assumption about the distribution of belief degrees within their intervals and is more efficient in computation and more accurate in results because there is no iterative process that needs to be carried out, which saves lots of time and increases accuracy.

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Appendix A. The minimax regret approach (MRA) for ranking interval-valued expected utilities

Let $u_i = [u_i^-, u_i^+] =< c_i, d_i > (i = 1, \ldots, M)$ be M interval-valued expected utilities, where $c_i = \frac{1}{2}(u_i^+ + u_i^-)$ and $d_i = \frac{1}{2}(u_i^+ - u_i^-)$ are their centres (midpoints) and widths. Without loss of generality, suppose

$u_i = [u_i^-, u_i^+]$ is chosen as the best interval-valued expected utility. Let $v = \max_{j \neq i}\{u_j^+\}$. Obviously, if $u_i^- < v$, DM may suffer the loss of expected utility (also called the loss of opportunity or regret) and feels regret. The maximum loss of expected utility he/she might suffer is given by

$$\max(r_i) = v - u_i^- = \max_{j \neq i}\{u_j^+\} - u_i^-.$$

If $u_i^- \geqslant v$, DM will definitely suffer no loss of expected utility and produce no regret. In this situation, his/her regret is defined to be zero, i.e. $r_i = 0$. Combining the above two situations, we have

$$\max(r_i) = \max[\max_{j \neq i}(u_j^+) - u_i^-, 0].$$

Thus, the minimax regret criterion will choose the interval-valued expected utility satisfying the following condition as the best (most desirable) one:

$$\min_i\{\max(r_i)\} = \min_i\{\max[\max_{j \neq i}(u_j^+) - u_i^-, 0]\}.$$

Based on the analysis above, Wang et al. (2005) gave the following definition for comparing and ranking interval-valued expected utilities.

Definition 9. Let $u_i = [u_i^-, u_i^+] =< c_i, d_i >$ $(i = 1, \ldots, M)$ be M interval-valued expected utilities. The maximum loss of expected utility (also called maximum regret) of each interval-valued expected utility u_i is defined as

$$R(A_i) = \max[\max_{j \neq i}(u_j^+) - u_i^-, 0] = \max[\max_{j \neq i}(c_j + d_j) - (c_i - d_i), 0], \quad i = 1, \cdots, M.$$

It is evident that the interval-valued expected utility with the smallest maximum loss of expected utility is the most desirable one. In order to generate a complete ranking order for a set of interval-valued expected utilities, the following eliminating steps were suggested by Wang et al.

Step 1: Calculate the maximum loss of expected utility of each interval-valued expected utility and choose a most desirable interval-valued expected utility that has the smallest maximum loss of expected utility (regret). Suppose u_{i_1} is selected, where $1 \leqslant i_1 \leqslant M$.

Step 2: Eliminate u_{i_1} from the consideration, recalculate the maximum loss of expected utility of every interval-valued expected utility and determine a most desirable interval-valued expected utility from the remaining $(M-1)$ interval-valued expected utilities. Suppose u_{i_2} is chosen, where $1 \leqslant i_2 \leqslant M$ but $i_2 \neq i_1$.

Step 3: Eliminate u_{i_2} from the further consideration, re-compute the maximum loss of expected utility of each interval-valued expected utility and determine a most desirable interval-valued expected utility u_{i_3} from the remaining $(M-2)$ interval-valued expected utilities.

Step 4: Repeat the above eliminating process until only one interval-valued expected utility u_{i_M} is left. The final ranking is $u_{i_1} \succ u_{i_2} \succ \cdots \succ u_{i_M}$, where the symbol '$\succ$' means 'is superior to'.

The above ranking approach is referred to as the minimax regret approach (MRA), which has the following properties.

Property 1. *Let $u_A = [u_A^-, u_A^+]$ and $u_B = [u_B^-, u_B^+]$ be two interval-valued expected utilities. If $u_A^- \leqslant u_B^-$ and $u_A^+ \leqslant u_B^+$, then $R(u_A) \geqslant R(u_B)$.*

Property 2. *Let $u_A = [u_A^-, u_A^+] =< c_A, d_A >$ and $u_B = [u_B^-, u_B^+] =< c_B, d_B >$ be two interval-valued expected utilities. If u_A is included in u_B, i.e. $u_A^- \geqslant u_B^-$ but $u_A^+ \leqslant u_B^+$, then*

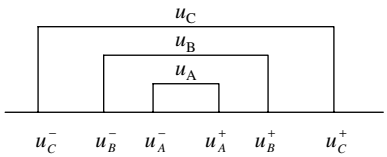

Fig. 4. Three euqi-centred interval-valued expected utilities.

(1) $R(u_A) > R(u_B)$ *if* $c_A < c_B$;
(2) $R(u_A) = R(u_B)$ *if* $c_A = c_B$;
(3) $R(u_A) < R(u_B)$ *if* $c_A > c_B$.

Property 3. *Let* $u_A = [u_A^-, u_A^+] =< c_A, d_A >$, $u_B = [u_B^-, u_B^+] =< c_B, d_B >$ *and* $u_C = [u_C^-, u_C^+] =< c_C, d_C >$ *be three equi-centred interval-valued expected utilities. If* $d_A < d_B < d_C$ (*see Fig.* 4), *then* $R(u_A) < R(u_B)$ *and* $R(u_A) < R(u_C)$.

Property 1 shows that for two non-nested interval-valued expected utilities, the one with bigger lower and upper bounds is superior to the other. Property 2 shows how the MRA compares and ranks two interval-valued expected utilities if one is included in another. In this situation, the order relationship generated by the MRA depends only on their centres if they are not the same. But if they are equi-centred, the MRA needs further to use property 3 to compare and rank them. Property 3 shows that the interval-valued expected utility with the same centre but smallest width is the most desirable.

Note that the MRA takes no account of DM's attitude towards risk. If DM's attitude towards risk needs to be considered in ranking interval-valued utilities, then Hurwice decision criterion should be used.

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