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On the combination and normalization of interval-valued belief structures ☆

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Abstract

This paper investigates the issues of combination and normalization of interval-valued belief structures within the framework of Dempster–Shafer theory (DST) of evidence. Existing approaches are reviewed, examined and critically analysed. They either ignore the normalization or separate it from the combination process, leading to irrational or suboptimal interval-valued belief structures. A new logically correct optimality approach is developed, where the combination and the normalization are optimised together rather than separately. Numerical examples are provided throughout the paper.

Keywords: Dempster–Shafer theory of evidence; Transferable belief model (TBM); Dezert–Smarandache theory (DSmT) of plausible and paradoxical reasoning; Interval-valued belief structures

1. Introduction

The Dempster–Shafer theory of evidence (D–S theory or DST for short), which was first developed by Dempster [12] and later extended and refined by Shafer [35], has so far found extensive applications in many areas such as expert systems [4,10,53], diagnosis and reasoning [3,11,24–26,31,33,34,50], pattern classification [8,9,13–17], information fusion [52], knowledge reduction [58], audit risk assessment [2,22,23,27,43–49], multiple attribute decision analysis (MADA) [1,4–7,57,59,62,64–68], environmental impact assessment (EIA) [56], contractor selection [41,42], organizational self-assessment [36,63], safety analysis [29,54] and regression analysis [30,32].

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The original D–S theory requires precise (crisp) belief degrees and belief structures. In many decision situations, however, precise belief structures are not always available due to the existence of uncertainty in human being's subjective judgments. For example, when assessing decision alternatives, decision maker (DM) may be unable to give a precise assessment if he/she is not definitely sure about his/her judgments. In this case, an interval-valued belief degree rather than a precise one may be provided. In group decision analysis, different DMs or experts may give different degrees of belief and it would be inappropriate to synthesize them into a precise point estimate. Using interval-valued belief degree might be a sensible option. This would preserve the different views of the DMs, thereby facilitating further discussion.

Uncertainty can be caused by the incompleteness or the lack of information, which results in partial or total ignorance. It can also be caused by linguistic ambiguity or vagueness. By ambiguity, we mean a word or phrase has more than one distinct meaning that make sense in context, while vagueness refers to a word or phrase with an imprecise meaning such as *good*, *young* and so on. Interval-valued belief structure represents the belief information which is known but not precise. It is not caused by linguistic ambiguity or vagueness and can therefore be seen as a partial ignorance, which can be well modeled using the D–S theory.

There have been several attempts to extend the D–S theory to interval-valued belief structures such as Lee and Zhu [28], Denoeux [14,16] and Yager [60]. However, the issues of combination and normalization of interval-valued belief structures have not been fully resolved. The purpose of this paper is to reinvestigate the issues and to develop a new logically correct optimality approach for combining and normalizing interval-valued belief structures.

The paper is organized as follows. Section 2 gives a brief description of the D–S theory of evidence. In Section 3, we review and critically analyse the existing approaches for combining and normalizing interval-valued belief structures and point out the irrationality or suboptimality of these methods. A new logically correct optimality approach for combining and normalizing interval-valued belief structures is developed, investigated and illustrated in Section 4. The paper is concluded in Section 5.

2. The D–S theory for combining deterministic evidence

Let $H = \{H_1, \ldots, H_N\}$ be a collectively exhaustive and mutually exclusive set of hypotheses or propositions, which is called the frame of discernment. A basic probability assignment (bpa) (also called a belief structure or a basic belief assignment) is a function $m : 2^H \to [0,1]$, which is called a mass function, satisfying:

$$m(\Phi) = 0 \quad \text{and} \quad \sum_{A \subseteq H} m(A) = 1, \tag{1}$$

where Φ is an empty set, A is any subset of H, and 2^H is the power set of H and consists of all the subsets of H, i.e.

$$2^H = \{\Phi, \{H_1\}, \ldots, \{H_N\}, \{H_1, H_2\}, \ldots, \{H_1, H_N\}, \ldots, H\}. \tag{2}$$

The assigned probability (also called probability mass) $m(A)$ measures the belief exactly assigned to A and represents how strongly the evidence supports A. All the assigned probabilities sum to unity and there is no belief in the empty set (Φ). The assigned probability to H, i.e. $m(H)$, is called the degree of ignorance. Each subset $A \subseteq H$ with $m(A) > 0$ is called a focal element of m. All the related focal elements are collectively called the body of evidence.

Belief measure, Bel, and plausibility measure, Pl, are the two functions: $2^H \to [0,1]$, associated with each bpa and defined by the following equations, respectively:

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B), \tag{3}$$

$$\mathrm{Pl}(A) = \sum_{A \cap B \neq \Phi} m(B), \tag{4}$$

where A and B are subsets of H. Bel(A) represents the exact support to A, i.e. the belief of the hypothesis A being true; Pl(A) represents the possible support to A, i.e. the total amount of belief that could be potentially placed in A. $[\mathrm{Bel}(A), \mathrm{Pl}(A)]$ constitutes the interval of support to A and can be seen as the lower and upper bounds of the probability to which A is supported. The two functions are connected by the equation

$$\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\overline{A}), \tag{5}$$

where $\overline{A}$ denotes the complement of A. The difference between the belief and the plausibility of a set A describes the ignorance of the assessment for the set A [35].

Since $m(A)$, $\mathrm{Bel}(A)$ and $\mathrm{Pl}(A)$ are in one-to-one correspondence, they can be seen as three facets of the same piece of information. There are several other functions such as commonality and doubt functions, which can also be used to represent evidence and provide flexibility to match a variety of reasoning applications.

The core of the evidence theory is the Dempster's rule of combination by which evidence from different sources is combined. The rule assumes that information sources are independent and uses the so-called orthogonal sum to combine multiple belief structures:

$$m = m_1 \oplus m_2 \oplus \cdots \oplus m_K, \tag{6}$$

where $\oplus$ represents the operator of combination. With two belief structures m_1 and m_2, the Dempster's rule of combination is defined as

$$[m_1 \oplus m_2](C) = \begin{cases} 0, & C = \Phi, \\ \frac{\sum_{A \cap B = C} m_1(A) m_2(B)}{1 - \sum_{A \cap B = \Phi} m_1(A) m_2(B)}, & C \neq \Phi, \end{cases} \tag{7}$$

where A and B are both focal elements and $[m_1 \oplus m_2](C)$ is a bpa. The denominator, $1 - \sum_{A \cap B = \Phi} m_1(A) m_2(B)$ denoted by k, is called the normalization factor, $\sum_{A \cap B = \Phi} m_1(A) m_2(B)$ is called the degree of conflict, which measures the conflict between the pieces of evidence [21] and the process of dividing by k is called normalization.

The Dempster's rule of combination proves to be commutative and associative [35], i.e. $m_1 \oplus m_2 = m_2 \oplus m_1$ (commutativity) and $(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3)$ (associativity). These two properties show that evidence can be combined in any order. Therefore, in the case of multiple belief structures, the combination of evidence can be carried out in a pairwise manner.

It must be pointed out that the Dempster's rule of combination requires the close-world assumption, i.e. one and only one hypothesis or proposition of H is true and no belief mass can be given to the empty set Φ. However, it is argued that in many problems, the belief mass given to the empty set may carry very useful information regarding the conflict between sources. So, basic belief assignments may not always be normalized. This is the so-called open-world assumption, which accepts that none of the elements of H could be true and allows a positive basic belief mass to be given to the empty set. Based on the open-world assumption, Smets [38,39] and Smets and Kennes [40] developed the transferable belief model (TBM), which combines two pieces of evidence in the following way:

$$[m_1 \oplus m_2](C) = \sum_{A \cap B = C} m_1(A) m_2(B), \quad \forall C \subseteq H. \tag{8}$$

In the TBM, $m(\Phi)$ may not be zero. By normalizing the result, the above TBM conjunctive combination rule becomes equivalent to the Dempster's rule of combination. For this reason, the TBM conjunctive combination rule can also be called non-normalized Dempster's rule of combination.

There are also other combination rules such as disjunctive combination rule [39], Yager's combination rule [61], which transfers the mass of the empty set Φ to the whole set H, Dubious and Prade's combination rule [19,20], hybrid DSm (Dezert–Smarandache) rule of combination [18,37], and so on. Interested readers may refer to Smets [39], where different combination rules were compared and justified. This paper is not intended to discuss different combination rules, but focuses on the investigation of the issues of combination and normalization of interval-valued belief structures within the framework of DST.

3. Existing approaches for combining interval evidence

The original D–S theory was developed to handle deterministic evidence and is not originally suited to handling uncertain evidence, by which we mean some or all probability masses assigned to focal elements are uncertain/imprecise. Fuzzy or interval numbers may be used to represent such uncertainty. Evidence expressed in the form of interval-valued probability masses is referred to as interval evidence in this paper. The corresponding belief structure (bpa) is called interval-valued belief structure or interval belief structure for short.

Lee and Zhu [28] are probably the first to investigate the combination of interval evidence. In order to combine multiple pieces of interval evidence, they defined a class of generalized summation and multiplication operations, which are shown below:

$$\begin{aligned}&\text{Generalized summation}: \quad [a,b]+[c,d]=[u(a,c),u(b,d)],\\&\text{Generalized multiplication}: \quad [a,b]\times[c,d]=[i(a,c),i(b,d)],\end{aligned}$$

where

$$\begin{aligned}&u(a,b)=\min[1,(a^w+b^w)^{1/w}],\\&i(a,b)=1-\min[1,((1-a)^w+(1-b)^w)^{1/w}],\end{aligned}$$

where $w\in(0,\infty)$ is a parameter. Based on the generalized summation and multiplication operations, they expressed the combination of two pieces of interval evidence as

$$[m_1\oplus m_2](C)=\sum_{A\cap B=C} m_1(A)\times m_2(B), \tag{9}$$

where m_1 and m_2 are two pieces of interval evidence, respectively, A and B are their focal elements, $m_1\oplus m_2$ is the combined interval evidence, and $[m_1\oplus m_2](C)$ is the combined probability mass of focal element C.

It is believed that Lee and Zhu's approach has at least the following two drawbacks. One is the selection of parameter w, which is highly subjective and arbitrary. Different choices may lead to different results. The other is the use of non-normalized interval evidence both before and after combination, which they claimed to be the advantage of their approach because normalization was not needed in their approach. In fact, their approach is problematic. Take for example their numerical example of pneumonia diagnosis, in which there are three possible organisms causing the pneumonia:

$$\{\text{Pneumococcus, Legionella, Klebsiella}\}.$$

For simplicity, the frame of discernment is denoted as $H=\{P,L,K\}$. The two pieces of interval evidence they utilized are recorded below:

$$\begin{aligned}&m_1(\{P\})=[0.5,0.8],\quad m_1(\{L,K\})=[0.3,0.4],\quad m_1(H)=[0.2,0.5],\\&m_2(\{P,L\})=[0.4,0.6],\quad m_2(\{L,K\})=[0.3,0.5],\quad m_2(H)=[0.3,0.4].\end{aligned}$$

Obviously, these two pieces of interval evidence do not meet the requirement of normalization. Since probability masses represent the probabilities assigned to each focal element and their sum in each piece of evidence should not exceed one, the above two pieces of interval evidence are both invalid except for the following specific situation:

$$\begin{aligned}&m_1(\{P\})=0.5,\quad m_1(\{L,K\})=0.3,\quad m_1(H)=0.2,\\&m_2(\{P,L\})=0.4,\quad m_2(\{L,K\})=0.3,\quad m_2(H)=0.3,\end{aligned}$$

which are not interval but deterministic (point) evidence. Therefore, the above two pieces of interval evidence should not lead to a new piece of interval evidence but to a piece of combined deterministic evidence. Besides, the intersection of $\{P\}$ and $\{L,K\}$ is an empty set. The basic probability mass assigned to this intersection should not be ignored. The above analysis shows that Lee and Zhu's approach without the consideration of normalization is in fact wrong and unacceptable.

Denoeux [14,16] systematically studied the issues of combination and normalization of interval evidence. He found that interval arithmetic was not a good option to compute the combined probability masses of two pieces of interval evidence. The so-called interval arithmetic means the interval operation rules below:

$$\begin{aligned}
&\text{Addition}: \quad [a,b]+[c,d]=[a+c,b+d],\\
&\text{Subtraction}: \quad [a,b]-[c,d]=[a-d,b-c],\\
&\text{Multiplication}: \quad [a,b]\times[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)],\\
&\text{Division}: \quad [a,b]\div[c,d]=[a,b]\times[1/d,1/c]=\left[\min\left(\frac{a}{d},\frac{a}{c},\frac{b}{c},\frac{b}{d}\right),\max\left(\frac{a}{d},\frac{a}{c},\frac{b}{c},\frac{b}{d}\right)\right] \quad \text{with } c\neq 0 \text{ and}\\
&\quad d\neq 0.
\end{aligned}$$

When $[a,b]$ and $[c,d]$ are both positive interval numbers, the above multiplication and division can be simplified as

$$\begin{aligned}
&\text{Multiplication}: \quad [a,b]\times[c,d]=[ac,bd],\\
&\text{Division}: \quad [a,b]\div[c,d]=[a/d,b/c].
\end{aligned}$$

Instead of the use of interval arithmetic, Denoeux constructed the following pair of quadratic programming models to calculate the combined probability masses of two pieces of interval evidence:

$$\begin{aligned}
\text{Max/Min} \quad & [m_1\oplus m_2](C)=\sum_{A\cap B=C} m_1(A)m_2(B)\\
\text{s.t.} \quad & \sum_{A\in F(m_1)} m_1(A)=1,\\
& \sum_{B\in F(m_2)} m_2(B)=1,\\
& m_1^-(A)\leqslant m_1(A)\leqslant m_1^+(A), \quad \forall A\in F(m_1),\\
& m_2^-(B)\leqslant m_2(B)\leqslant m_2^+(B), \quad \forall B\in F(m_2),
\end{aligned} \tag{10}$$

where m_1 and m_2 are respectively two pieces of interval evidence, A and B are their focal elements, $m_1(A)\in\left[m_1^-(A),m_1^+(A)\right]$ and $m_2(B)\in\left[m_2^-(B),m_2^+(B)\right]$ are interval-valued probability masses, $F(m_1)=\{A\subseteq H|m_1(A)>0\}$ and $F(m_2)=\{B\subseteq H|m_2(B)>0\}$ are respectively the bodies of interval evidence m_1 and m_2 (i.e. the sets of focal elements), and $[m_1\oplus m_2](C)$ is the combined but non-normalized interval-valued probability mass of focal element C.

Denoeux also found that the combination of interval evidence has no property of associativity, namely, $(m_1\oplus m_2)\oplus m_3\neq m_1\oplus(m_2\oplus m_3)$ in general. This lack of associativity is obviously a drawback and it makes the result of the combination of several interval-valued belief structures dependent on the order in which they are combined. In order to overcome this drawback, he suggested combining all interval evidence in one step. After the combination of interval evidence in one step, the following normalization approach was developed to normalize all the combined but non-normalized interval-valued probability masses:

$$m_d^{*-}(A)=\min\frac{m(A)}{1-m(\Phi)}=\frac{m^-(A)}{1-\max\left[m^-(\Phi),1-\sum_{B\neq A,B\neq\Phi}m^+(B)-m^-(A)\right]}, \tag{11}$$

$$m_d^{*+}(A)=\max\frac{m(A)}{1-m(\Phi)}=\frac{m^+(A)}{1-\min\left[m^+(\Phi),1-\sum_{B\neq A,B\neq\Phi}m^-(B)-m^+(A)\right]}, \tag{12}$$

where $m(A)\in[m^-(A),m^+(A)]$, $m(B)\in[m^-(B),m^+(B)]$ and $m(\Phi)\in[m^-(\Phi),m^+(\Phi)]$ are non-normalized interval-valued probability masses, and $m_d^*(A)\in\left[m_d^{*-}(A),m_d^{*+}(A)\right]$ is the normalized interval-valued probability mass of $m(A)\in[m^-(A),m^+(A)]$. All the normalized interval-valued probability masses form a piece of interval evidence, also called an interval-valued belief structure or an interval-valued bpa.

Due to the fact that Denoeux worked in the framework of the TMB, he considered not only normalized combination, but also non-normalized combination. For the latter he gave an efficient solution procedure, which solves a quadratic programming problem using an alternate direction scheme. However, the normalized combination was not conducted in a precise way. The combination and the normalization of interval evidence were separated from each other and optimized individually, which makes the normalized interval evidence suboptimal and the normalized interval-valued probability masses more uncertain than their true intervals. To show this point clearly, let us take a look at the following example.

Suppose $H = \{H_1, H_2, H_3\}$ and two pieces of interval evidence are given by

$$m_1(H_1) \in [0.2, 0.4], \quad m_1(H_2) \in [0.3, 0.5], \quad m_1(H_3) \in [0.1, 0.3], \quad m_1(H) \in [0, 0.4],$$
$$m_2(H_1) \in [0.3, 0.4], \quad m_2(H_2) \in [0.1, 0.2], \quad m_2(H_3) \in [0.2, 0.3], \quad m_2(H) \in [0.1, 0.4].$$

By solving the quadratic programming model (10) we obtain the following non-normalized interval-valued probability masses, which are the results of the combination of the two pieces of interval evidence m_1 and m_2:

$$\begin{aligned}
m(H_1) &= m_1(H_1)m_2(H_1) + m_1(H_1)m_2(H) + m_1(H)m_2(H_1) \in [0.1, 0.36],\\
m(H_2) &= m_1(H_2)m_2(H_2) + m_1(H_2)m_2(H) + m_1(H)m_2(H_2) \in [0.09, 0.29],\\
m(H_3) &= m_1(H_3)m_2(H_3) + m_1(H_3)m_2(H) + m_1(H)m_2(H_3) \in [0.04, 0.24],\\
m(H) &= m_1(H)m_2(H) \in [0, 0.16],\\
m(\Phi) &= m_1(H_1)m_2(H_2) + m_1(H_1)m_2(H_3) + m_1(H_2)m_2(H_1)\\
&\quad + m_1(H_2)m_2(H_3) + m_1(H_3)m_2(H_1) + m_1(H_3)m_2(H_2) \in [0.25, 0.63].
\end{aligned}$$

It is evident that

$$\begin{aligned}
&m(H_1) \in [0.1, 0.36] \neq [0.2, 0.4] \times [0.3, 0.4] + [0.2, 0.4] \times [0.1, 0.4] + [0, 0.4] \times [0.3, 0.4] = [0.08, 0.48],\\
&m(H_2) \in [0.09, 0.29] \neq [0.3, 0.5] \times [0.1, 0.2] + [0.3, 0.5] \times [0.1, 0.4] + [0, 0.4] \times [0.1, 0.2] = [0.06, 0.38],\\
&m(H_3) \in [0.04, 0.24] \neq [0.1, 0.3] \times [0.2, 0.3] + [0.1, 0.3] \times [0.1, 0.4] + [0, 0.4] \times [0.2, 0.3] = [0.03, 0.33],\\
&m(\Phi) \in [0.25, 0.63] \neq [0.2, 0.4] \times [0.1, 0.2] + [0.2, 0.4] \times [0.2, 0.3] + [0.3, 0.5] \times [0.3, 0.4] + [0.3, 0.5] \times [0.2, 0.3]\\
&\quad + [0.1, 0.3] \times [0.3, 0.4] + [0.1, 0.3] \times [0.1, 0.2] = [0.25, 0.73].
\end{aligned}$$

This shows that interval arithmetic is not a good option to compute the combined probability masses of two interval-valued belief structures. Due to the presence of the probability mass assigned to the empty set Φ, the above interval-valued probability masses have to be normalized. Using the formulae (11) and (12) to normalize them, we get the following normalized interval-valued belief structure:

$$m_d^*(H_1) \in [0.13, 0.73], \quad m_d^*(H_2) \in [0.12, 0.67], \quad m_d^*(H_3) \in [0.05, 0.56] \quad \text{and} \quad m_d^*(H) \in [0, 0.43].$$

In fact, these normalized interval-valued probability masses are too wide to be true. The true ones should be as follows:

$$m(H_1) \in [0.22, 0.55], \quad m(H_2) \in [0.19, 0.48], \quad m(H_3) \in [0.08, 0.39] \quad \text{and} \quad m(H) \in [0, 0.21].$$

It will be discussed in the next section how to generate these true intervals. Evidently, Denoeux's approach fails to capture the true intervals for the combined probability masses of two interval-valued belief structures.

Yager [60] also explored the issues of the combination and normalization of interval evidence. His approach is summarized below.

Suppose m_1 and m_2 are two belief structures with focal elements A_i for $i = 1$ to n_1 and B_j for $j = 1$ to n_2, respectively. Associated with each A_i is an interval-valued probability $[L_{1i}, U_{1i}]$, $i = 1$ to n_1 and associated with B_j is an interval-valued probability $[L_{2j}, U_{2j}]$, $j = 1$ to n_2. The combined interval-valued belief structure $m_1 \oplus m_2$ is determined as follows:

(1) For each pair A_i and B_j, form $A_i \cap B_j = E_{ij}$ and calculate $R_{ij} = L_{1i}L_{2j}$ and $S_{ij} = U_{1i}U_{2j}$;
(2) The combined belief structure, $m_1 \oplus m_2$, has focal elements $F_{ij} = E_{ij}$ if $E_{ij} \neq \Phi$;
(3) The focal elements F_{ij} have interval-valued probabilities $[L_{ij}, U_{ij}]$ such that

$$L_{ij} = R_{ij}\left(1 + \frac{\sum_{E_{ij}=\Phi} R_{ij}}{\sum_{E_{ij}\neq\Phi} R_{ij}}\right) \quad \text{and} \quad U_{ij} = S_{ij}\left(1 + \frac{\sum_{E_{ij}=\Phi} S_{ij}}{\sum_{E_{ij}\neq\Phi} S_{ij}}\right). \tag{13}$$

Yager's approach has been found to have the following two fundamental faults. One is the use of interval arithmetic in computing the combined probability masses of two pieces of interval evidence. The other is the infeasibility of using (13) to normalize the combined interval-valued probability masses $[R_{ij}, S_{ij}]$. Formula (13) does not capture the true probability intervals of focal elements F_{ij}. It may happen that the upper bound probability generated by (13) is less than the lower bound probability. To make these points clear, we re-examine Yager's numerical example as follows:

Let $H = \{a,b,c,d,e,f\}$. $A_1 = \{a,b,c\}$, $A_2 = \{c,d\}$, $B_1 = \{a,e\}$ and $B_2 = \{b,c,f\}$ are all the subsets of H. Suppose m_1 and m_2 are the two interval-valued belief structures shown below:

$$m_1(A_1) = [0.2, 0.4], \quad m_1(A_2) = [0.4, 1], \quad m(H) = [0, 0.3],$$
$$m_2(B_1) = [0.5, 1], \quad m_2(B_2) = [0, 0.5].$$

Combining these two interval-valued belief structures, Yager generated the following E_{ij} with an associated interval probability $[R_{ij}, S_{ij}]$:

$$A_1 \cap B_1 = \{a\} \quad \text{with } [0.1, 0.4],$$
$$A_1 \cap B_2 = \{b, c\} \quad \text{with } [0, 0.2],$$
$$A_2 \cap B_1 = \Phi \quad \text{with } [0.2, 1],$$
$$A_2 \cap B_2 = \{c\} \quad \text{with } [0, 0.5],$$
$$H \cap B_1 = \{a, e\} \quad \text{with } [0, 0.3],$$
$$H \cap B_2 = \{b, c, f\} \quad \text{with } [0, 0.15].$$

Since $A_2 \cap B_1 = \Phi$, its probability mass has to be reassigned to the other focal elements. Using (13) to normalize the above interval-valued probabilities leads to the following combined interval-valued belief structure:

$$m(A_1 \cap B_1) = m(\{a\}) = [0.3, 0.66], \quad m(A_1 \cap B_2) = m(\{b, c\}) = [0, 0.33],$$
$$m(A_2 \cap B_2) = m(\{c\}) = [0, 0.82], \quad m(H \cap B_1) = m(\{a, e\}) = [0, 0.49],$$
$$m(H \cap B_2) = m(\{b, c, f\}) = [0, 0.25].$$

Now, let us examine a few cases to show whether or not the above results are rational. First, if we take the values $m_1(A_1) = 0.2 \in [0.2, 0.4]$, $m_2(A_2) = 0.8 \in [0.4, 1]$ and $m_1(H) = 0$ as well as $m_2(B_1) = 1 \in [0.5, 1]$ and $m_2(B_2) = 0 \in [0, 0.5]$, then through combining these two deterministic pieces of evidence, we have

$$m(A_1 \cap B_1) = 0.2, \quad m(A_1 \cap B_2) = 0, \quad m(A_2 \cap B_1) = 0.8,$$
$$m(A_2 \cap B_2) = 0, \quad m(H \cap B_1) = 0, \quad m(H \cap B_2) = 0.$$

Since $A_2 \cap B_1 = \Phi$, the above combined probability assignment has to be normalized. After normalization, we get

$$m(A_1 \cap B_1) = 1, \quad m(A_1 \cap B_2) = 0, \quad m(A_2 \cap B_2) = 0, \quad m(H \cap B_1) = 0, \quad m(H \cap B_2) = 0.$$

It is obvious that $m(A_1 \cap B_1) = 1 > 0.66$, which means that (13) fails to capture the true upper bound probability of the focal element $A_1 \cap B_1 = \{a\}$.

Secondly, if we take the values $m_1(A_1) = 0.2 \in [0.2, 0.4]$, $m_1(A_2) = 0.5 \in [0.4, 1]$ and $m_1(H) = 0.3 \in [0, 0.3]$ as well as $m_2(B_1) = 0.5 \in [0.5, 1]$ and $m_2(B_2) = 0.5 \in [0, 0.5]$. By combining these two deterministic pieces of evidence, we get

$$m(A_1 \cap B_1) = 0.1, \quad m(A_1 \cap B_2) = 0.1, \quad m(A_2 \cap B_1) = 0.25,$$
$$m(A_2 \cap B_2) = 0.25, \quad m(H \cap B_1) = 0.15, \quad m(H \cap B_2) = 0.15.$$

After normalization, we have

$$m(A_1 \cap B_1) = 0.13, \quad m(A_1 \cap B_2) = 0.13, \quad m(A_2 \cap B_2) = 0.33,$$
$$m(H \cap B_1) = 0.2, \quad m(H \cap B_2) = 0.2.$$

Obviously, $m(A_1 \cap B_1) = 0.13 < 0.3$, which means that (13) fails to capture the true lower bound probability of the focal element $A_1 \cap B_1 = \{a\}$.

If we slightly change the original two interval-valued belief structures into the following:

$$m_1(A_1) = [0.15, 0.25], \quad m_1(A_2) = [0.6, 1], \quad m(H) = [0, 0.25],$$
$$m_2(B_1) = [0.8, 1], \quad m_2(B_2) = [0, 0.2].$$

Then we have

$$A_1 \cap B_1 = \{a\} \quad \text{with } [0.12, 0.25],$$
$$A_1 \cap B_2 = \{b, c\} \quad \text{with } [0, 0.05],$$
$$A_2 \cap B_1 = \Phi \quad \text{with } [0.48, 1],$$
$$A_2 \cap B_2 = \{c\} \quad \text{with } [0, 0.2],$$
$$H \cap B_1 = \{a, e\} \quad \text{with } [0, 0.25],$$
$$H \cap B_2 = \{b, c, f\} \quad \text{with } [0, 0.05].$$

After normalization using (13), we get

$$m(A_1 \cap B_1) = m(\{a\}) = [0.6, 0.56], \quad m(A_1 \cap B_2) = m(\{b, c\}) = [0, 0.11],$$
$$m(A_2 \cap B_2) = m(\{c\}) = [0, 0.45], \quad m(H \cap B_1) = m(\{a, e\}) = [0, 0.56],$$
$$m(H \cap B_2) = m(\{b, c, f\}) = [0, 0.11].$$

It is clear in the above results that the upper bound probability of $m(A_1 \cap B_1)$ is less than its lower bound probability. Such a normalized interval-valued belief structure is obviously irrational and unacceptable.

As can be seen from the above literature review, the issues of combination and normalization of interval evidence have not been fully resolved so far. Therefore, there is a clear need to reinvestigate them. In the next section, we will develop an optimality approach for combining and normalizing interval-valued belief structures, where the combination and the normalization are optimized together rather than separately. It will be shown that the optimality approach can generate the true interval for each focal element.

4. The optimality approach for combining and normalizing interval-valued belief structures

Without loss of generality, we first give the following definitions. Definitions 1 and 3 are mainly based on Denoeux's published work [14].

Definition 1. Let $H = \{H_1, \dots, H_N\}$ be the frame of discernment, $F_1, \dots, F_n$ be n subsets of H and $[a_i, b_i]$ be n intervals with $0 \leqslant a_i \leqslant b_i \leqslant 1$ $(i = 1, \dots, n)$. An interval-valued belief structure is the belief structures on H such that

(1) $a_i \leqslant m(F_i) \leqslant b_i$, where $0 \leqslant a_i \leqslant b_i \leqslant 1$ for $i = 1, \dots, n$;
(2) $\sum_{i=1}^{n} a_i \leqslant 1$ and $\sum_{i=1}^{n} b_i \geqslant 1$;
(3) $m(A) = 0$, $\forall A \notin \{F_1, \dots, F_n\}$.

If $\sum_{i=1}^{n} a_i > 1$ or $\sum_{i=1}^{n} b_i < 1$, then the interval-valued belief structure m is said to be invalid. Invalid interval-valued belief structures cannot be interpreted as probability and thus need to be revised or adjusted.

For a valid interval-valued belief structure, we can always obtain a particular belief structure by selecting a value $m(F_i) \in [a_i, b_i]$ for each $i = 1$ to n such that $\sum_{i=1}^{n} m(F_i) = 1$. The above condition (2) ensures that there exists at least one belief structure.

From $a_i \leqslant m(F_i) \leqslant b_i$ and $\sum_{i=1}^{n} m(F_i) = 1$, we have

$$1 - \sum_{j \neq i} b_j \leqslant m(F_i) = 1 - \sum_{j \neq i} m(F_j) \leqslant 1 - \sum_{j \neq i} a_j, \quad i = 1, \dots, n.$$

Since $m(F_i)$ has to satisfy $a_i \leqslant m(F_i) \leqslant b_i$, it follows that

$$\max\left[a_i, 1 - \sum_{j\neq i} b_j\right] \leqslant m(F_i) \leqslant \min\left[b_i, 1 - \sum_{j\neq i} a_j\right], \quad i = 1,\ldots,n. \tag{14}$$

It is obvious that if $a_i \geqslant 1 - \sum_{j\neq i} b_j$ and $b_i \leqslant 1 - \sum_{j\neq i} a_j$, then (14) is reduced to $a_i \leqslant m(F_i) \leqslant b_i$. For this type of interval-valued belief structure, we have the following definition:

Definition 2. Let m be a valid interval-valued belief structure with interval-valued probability masses $a_i \leqslant m(F_i) \leqslant b_i$ for $i = 1,\ldots,n$. If a_i and b_i satisfy

$$\sum_{j=1}^{n} b_j - (b_i - a_i) \geqslant 1 \quad \text{and} \quad \sum_{j=1}^{n} a_j + (b_i - a_i) \leqslant 1 \qquad \text{for } \forall i \in \{1,\ldots,n\}, \tag{15}$$

then m is said to be a normalized interval-valued belief structure [51,55].

Normalized interval-valued belief structures are in fact the compact and equivalent form of valid interval-valued belief structures. An interval-valued belief structure may be valid, but may not necessarily be normalized. Take the previous pneumonia diagnosis for example. The two interval-valued belief structures are re-recorded below:

$$m_1(\{P\}) = [0.5, 0.8], \quad m_1(\{L,K\}) = [0.3, 0.4], \quad m_1(H) = [0.2, 0.5],$$
$$m_2(\{P,L\}) = [0.4, 0.6], \quad m_2(\{L,K\}) = [0.3, 0.5], \quad m_2(H) = [0.3, 0.4].$$

Since $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i > 1$ hold for the both interval-valued belief structures, they are both considered to be valid. However, neither of them can meet the requirements of (15). So, they are both non-normalized.

For a non-normalized interval-valued belief structure, it usually means that some intervals of probability masses are too wide to be reached. From the above interval-valued belief structures m_1 and m_2 we find that only the following belief structures are valid and all the others are infeasible:

$$m_1(\{P\}) = 0.5, \quad m_1(\{L,K\}) = 0.3, \quad m_1(H) = 0.2,$$
$$m_2(\{P,L\}) = 0.4, \quad m_2(\{L,K\}) = 0.3, \quad m_2(H) = 0.3.$$

So, for a valid but not normalized interval-valued belief structure, what we need to do is normalize it using (14) so that all infeasible belief structures can be screened out from it. In what follows, we assume that interval-valued belief structures are all normalized.

Let A be a subset of $H = \{H_1,\ldots,H_N\}$, which divides n focal elements, $F_1,\ldots,F_n$, into two parts: either $F_i \subseteq A$ or $F_i \not\subseteq A$. Accordingly, $\sum_{i=1}^{n} m(F_i) = 1$ can be broken down into

$$\sum_{F_i\subseteq A} m(F_i) + \sum_{F_j\not\subseteq A} m(F_j) = 1,$$

which can be further expressed as

$$1 - \sum_{F_j\not\subseteq A} b_j \leqslant \sum_{F_i\subseteq A} m(F_i) = 1 - \sum_{F_j\not\subseteq A} m(F_j) \leqslant 1 - \sum_{F_j\not\subseteq A} a_j.$$

Since $\sum_{F_i\subseteq A} m(F_i)$ is still subject to the constraint of $\sum_{F_i\subseteq A} a_i \leqslant \sum_{F_i\subseteq A} m(F_i) \leqslant \sum_{F_i\subseteq A} b_i$, it follows that

$$\max\left[\sum_{F_i\subseteq A} a_i, \left(1 - \sum_{F_j\not\subseteq A} b_j\right)\right] \leqslant \sum_{F_i\subseteq A} m(F_i) \leqslant \min\left[\sum_{F_i\subseteq A} b_i, \left(1 - \sum_{F_j\not\subseteq A} a_j\right)\right]. \tag{16}$$

Moreover, the subset A may also divide the focal elements, $F_1,\ldots,F_n$, into either $F_i \cap A \neq \Phi$ or $F_i \cap A = \Phi$. Accordingly, $\sum_{i=1}^{n} m(F_i) = 1$ can be decomposed into

$$\sum_{F_i\cap A\neq\Phi} m(F_i) + \sum_{F_j\cap A=\Phi} m(F_j) = 1,$$

which can be further rewritten as

$$1-\sum_{F_j\cap A=\Phi} b_j \leqslant \sum_{F_i\cap A\neq\Phi} m(F_i) = 1-\sum_{F_j\cap A=\Phi} m(F_j) \leqslant 1-\sum_{F_j\cap A=\Phi} a_j.$$

Since $\sum_{F_i\cap A\neq\Phi} m(F_i)$ satisfies $\sum_{F_i\cap A\neq\Phi} a_i \leqslant \sum_{F_i\cap A\neq\Phi} m(F_i) \leqslant \sum_{F_i\cap A\neq\Phi} b_i$, it follows that

$$\max\left[\sum_{F_i\cap A\neq\Phi} a_i, \left(1-\sum_{F_j\cap A=\Phi} b_j\right)\right] \leqslant \sum_{F_i\cap A\neq\Phi} m(F_i) \leqslant \min\left[\sum_{F_i\cap A\neq\Phi} b_i, \left(1-\sum_{F_j\cap A=\Phi} a_j\right)\right]. \tag{17}$$

Based on the inequalities (16) and (17), we have the following definitions of belief and plausibility measures in an interval environment.

Definition 3. Let m be a normalized interval-valued belief structure with interval-valued probability masses $a_i \leqslant m(F_i) \leqslant b_i$ for $i=1,\ldots,n$ and A be a subset of $H=\{H_1,\ldots,H_N\}$. The belief measure (Bel) and the plausibility measure (Pl) of A are both the closed intervals defined respectively by

$$\mathrm{Bel}_m(A) = [\mathrm{Bel}_m^-(A), \mathrm{Bel}_m^+(A)],$$
$$\mathrm{Pl}_m(A) = [\mathrm{Pl}_m^-(A), \mathrm{Pl}_m^+(A)],$$

where

$$\mathrm{Bel}_m^-(A) = \min\sum_{F_i\subseteq A} m(F_i) = \max\left[\sum_{F_i\subseteq A} a_i, \left(1-\sum_{F_j\not\subset A} b_j\right)\right], \tag{18}$$

$$\mathrm{Bel}_m^+(A) = \max\sum_{F_i\subseteq A} m(F_i) = \min\left[\sum_{F_i\subseteq A} b_i, \left(1-\sum_{F_j\not\subset A} a_j\right)\right], \tag{19}$$

$$\mathrm{Pl}_m^-(A) = \min\sum_{F_i\cap A\neq\Phi} m(F_i) = \max\left[\sum_{F_i\cap A\neq\Phi} a_i, \left(1-\sum_{F_i\cap A=\Phi} b_j\right)\right], \tag{20}$$

$$\mathrm{Pl}_m^+(A) = \max\sum_{F_i\cap A\neq\Phi} m(F_i) = \min\left[\sum_{F_i\cap A\neq\Phi} b_i, \left(1-\sum_{F_i\cap A=\Phi} a_j\right)\right]. \tag{21}$$

Table 1 shows an illustrative example and the results of belief and plausibility measures in an interval-valued belief environment. Since the equation $\mathrm{Pl}(A) = 1-\mathrm{Bel}(\overline{A})$ holds for any deterministic belief structure, accordingly, we have the following equations in an interval-valued belief environment:

$$\mathrm{Pl}_m^-(A) = \min \mathrm{Pl}_m(A) = 1-\max \mathrm{Bel}_m(\overline{A}) = 1-\mathrm{Bel}_m^+(\overline{A}), \tag{22}$$
$$\mathrm{Pl}_m^+(A) = \max \mathrm{Pl}_m(A) = 1-\min \mathrm{Bel}_m(\overline{A}) = 1-\mathrm{Bel}_m^-(\overline{A}), \tag{23}$$

where $\overline{A}$ is the complement of A.

Table 1
An interval-valued belief structure and the corresponding belief and plausibility measures

A	$m(A)$	Normalized $m(A)$	$\mathrm{Bel}_m(A)$	$\mathrm{Pl}_m(A)$
$\{a\}$	[0.05, 0.10]	[0.05, 0.10]	[0.05, 0.10]	[0.50, 0.55]
$\{b\}$	[0.10, 0.20]	[0.10, 0.15]	[0.10, 0.15]	[0.60, 0.65]
$\{c\}$	[0.10, 0.25]	[0.10, 0.15]	[0.10, 0.15]	[0.65, 0.70]
$\{a,b\}$	[0.15, 0.25]	[0.15, 0.20]	[0.30, 0.35]	[0.85, 0.90]
$\{a,c\}$	[0.20, 0.30]	[0.20, 0.25]	[0.35, 0.40]	[0.85, 0.90]
$\{b,c\}$	[0.25, 0.35]	[0.25, 0.30]	[0.45, 0.50]	[0.90, 0.95]
$\{a,b,c\}$	[0.10, 0.20]	[0.10, 0.15]	[1.00, 1.00]	[1.00, 1.00]

Definition 4. Let m_1 and m_2 be two interval-valued belief structures with interval-valued probability masses $m_1^-(A_i) \leqslant m_1(A_i) \leqslant m_1^+(A_i)$ for $i = 1$ to n_1 and $m_2^-(B_j) \leqslant m_2(B_j) \leqslant m_2^+(B_j)$ for $j = 1$ to n_2, respectively. Their combination, denoted by $m_1 \oplus m_2$, is also an interval-valued belief structure defined by

$$[m_1 \oplus m_2](C) = \begin{cases} 0, & C = \Phi, \\ \left[(m_1 \oplus m_2)^-(C), (m_1 \oplus m_2)^+(C)\right], & C \neq \Phi, \end{cases} \tag{24}$$

where $(m_1 \oplus m_2)^-(C)$ and $(m_1 \oplus m_2)^+(C)$ are respectively the minimum and the maximum of the following pair of optimization problems:

$$\begin{aligned} \text{Max/Min} \quad & [m_1 \oplus m_2](C) = \frac{\sum_{A_i \cap B_j = C} m_1(A_i) m_2(B_j)}{1 - \sum_{A_i \cap B_j = \Phi} m_1(A_i) m_2(B_j)} \\ \text{s.t.} \quad & \sum_{i=1}^{n_1} m_1(A_i) = 1, \\ & \sum_{j=1}^{n_2} m_2(B_j) = 1, \\ & m_1^-(A_i) \leqslant m_1(A_i) \leqslant m_1^+(A_i), \quad i = 1, \ldots, n_1, \\ & m_2^-(B_j) \leqslant m_2(B_j) \leqslant m_2^+(B_j), \quad j = 1, \ldots, n_2. \end{aligned} \tag{25}$$

Note that quite different from models (10)–(12), each of the above pair of models considers at the same time the combination and normalization of two pieces of interval evidence and optimizes them together rather than separately. The reason for doing so is to capture the true probability mass intervals of the combined focal elements. If the numerator and the denominator of (25) were optimized individually, the intrinsic relationships between them would be cut off and the results would be distorted. Besides, what we need in the framework of D–S theory is the normalized rather than non-normalized probability masses. So, we believe there is no need to optimize any probability masses before normalization within the framework of D–S theory.

To illustrate the implementation process of (25), we re-examine the following example:

$$\begin{aligned} & m_1(H_1) \in [0.2, 0.4], \quad m_1(H_2) \in [0.3, 0.5], \quad m_1(H_3) \in [0.1, 0.3], \quad m_1(H) \in [0, 0.4], \\ & m_2(H_1) \in [0.3, 0.4], \quad m_2(H_2) \in [0.1, 0.2], \quad m_2(H_3) \in [0.2, 0.3], \quad m_2(H) \in [0.1, 0.4], \end{aligned}$$

where m_1 and m_2 are two interval-valued belief structures and $H = \{H_1, H_2, H_3\}$ is the frame of discernment. Combining these two pieces of interval evidence, we have the following formulas for the non-normalized probability masses:

$$\begin{aligned} & \bar{m}(H_i) = m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i), \quad i = 1, 2, 3, \\ & \bar{m}(H) = m_1(H)m_2(H), \\ & \bar{m}(\Phi) = \sum_{i=1}^{3} \sum_{j=1, j \neq i}^{3} m_1(H_i) m_2(H_j). \end{aligned}$$

The normalized probability masses will be given by

$$\begin{aligned} & m(H_i) = \frac{\bar{m}(H_i)}{1 - \bar{m}(\Phi)} = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1 - \sum_{k=1}^{3} \sum_{j=1, j \neq k}^{3} m_1(H_k) m_2(H_j)}, \quad i = 1, 2, 3, \\ & m(H) = \frac{\bar{m}(H)}{1 - \bar{m}(\Phi)} = \frac{m_1(H) m_2(H)}{1 - \sum_{k=1}^{3} \sum_{j=1, j \neq k}^{3} m_1(H_k) m_2(H_j)}. \end{aligned}$$

In order to determine the normalized probability mass interval for each focal element, we need to solve the following pairs of optimization models:

$$\begin{aligned}
\text{Max/Min} \quad & m(H_i) = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1 - \sum_{k=1}^{3}\sum_{j=1,j\neq k}^{3} m_1(H_k)m_2(H_j)} \\
\text{s.t.} \quad & m_1(H_1) + m_1(H_2) + m_1(H_3) + m_1(H) = 1, \\
& m_2(H_1) + m_2(H_2) + m_2(H_3) + m_2(H) = 1, \\
& 0.2 \leqslant m_1(H_1) \leqslant 0.4, \\
& 0.3 \leqslant m_1(H_2) \leqslant 0.5, \\
& 0.1 \leqslant m_1(H_3) \leqslant 0.3, \\
& 0 \leqslant m_1(H) \leqslant 0.4, \\
& 0.3 \leqslant m_2(H_1) \leqslant 0.4, \\
& 0.1 \leqslant m_2(H_2) \leqslant 0.2, \\
& 0.2 \leqslant m_2(H_3) \leqslant 0.3, \\
& 0.1 \leqslant m_2(H) \leqslant 0.4.
\end{aligned}$$

With the help of LINGO software package and solving the above pair of models for $i = 1, 2, 3$, respectively, we get the normalized probability mass intervals as follows:

$$m(H_1) \in [0.22, 0.55], \quad m(H_2) \in [0.19, 0.48], \quad m(H_3) \in [0.08, 0.39] \quad \text{and} \quad m(H) \in [0, 0.21],$$

where the interval for $m(H)$ is obtained by solving the models with $m_1(H)m_2(H)$ used as the numerator of the objective function.

From the previous calculations we know that when Denoeux's approach was used to combine m_1 and m_2, the normalized probability mass intervals were given by

$$m_d^*(H_1) \in [0.13, 0.73], \quad m_d^*(H_2) \in [0.12, 0.67], \quad m_d^*(H_3) \in [0.05, 0.56] \quad \text{and} \quad m_d^*(H) \in [0, 0.43].$$

It is obvious that the probability mass intervals obtained by Denoeux's approach are much wider than those by our optimality approach. This is because the true probability mass intervals are exaggerated by Denoeux's approach.

Similarly, for Yager's numerical example discussed in the previous section, we have the following normalized interval-valued belief structure:

$$\begin{aligned}
& m(A_1 \cap B_1) = m(\{a\}) = [0.13, 1], \quad m(A_1 \cap B_2) = m(\{b, c\}) = [0, 0.29], \\
& m(A_2 \cap B_2) = m(\{c\}) = [0, 0.67], \quad m(H \cap B_1) = m(\{a, e\}) = [0, 0.6], \\
& m(H \cap B_2) = m(\{b, c, f\}) = [0, 0.2],
\end{aligned}$$

which are quite different from the results obtained by Yager's approach.

It must be pointed out that the combination of interval evidence does not preserve the associativity property in the new approach as defined by (24) and (25) where three or more pieces of evidence are combined recursively. This can be confirmed by the results in Table 2, where three pieces of interval evidence are combined in different orders, respectively. It is clear that $(m_1 \oplus m_2) \oplus m_3 \neq m_1 \oplus (m_2 \oplus m_3) \neq (m_1 \oplus m_3) \oplus m_2$.

In order that multiple interval-valued belief structures can be combined correctly and efficiently, they should be combined simultaneously and the optimization process should not be started until the end of the combination. The following definition shows how to combine them correctly.

Definition 5. Let $m_1, \ldots, m_n$ be n interval-valued belief structures with interval-valued probability masses $m_i^-(A_j^i) \leqslant m_i(A_j^i) \leqslant m_i^+(A_j^i)$ for $i = 1$ to n and $j = 1$ to n_i, where A_j^i represents the jth focal element of ith interval-valued belief structure. Their combination, denoted by $m_1 \oplus m_2 \oplus \cdots \oplus m_n$, is also an interval-valued belief structure defined by

$$[m_1 \oplus m_2 \oplus \cdots \oplus m_n](C) = \begin{cases} 0, & C = \Phi, \\ \left[(m_1 \oplus \cdots \oplus m_2)^-(C), (m_1 \oplus \cdots \oplus m_2)^+(C)\right], & C \neq \Phi, \end{cases} \tag{26}$$

Table 2
Non-associativity of the combination of interval-valued belief structures

Interval-valued belief structure	$\{H_1\}$	$\{H_2\}$	$\{H_3\}$	$\{H\}$
m_1	[0.2, 0.4]	[0.3, 0.5]	[0.1, 0.3]	[0, 0.4]
m_2	[0.3, 0.4]	[0.1, 0.2]	[0.2, 0.3]	[0.1, 0.4]
m_3	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]	[0, 0.1]
$m_1 \oplus m_2$	[0.22, 0.55]	[0.19, 0.48]	[0.08, 0.39]	[0, 0.21]
$m_1 \oplus m_3$	[0.11, 0.40]	[0.27, 0.63]	[0.13, 0.52]	[0, 0.06]
$m_2 \oplus m_3$	[0.21, 0.40]	[0.19, 0.38]	[0.35, 0.56]	[0, 0.06]
$(m_1 \oplus m_2) \oplus m_3$	[0.13, 0.54]	[0.17, 0.61]	[0.11, 0.59]	[0, 0.04]
$m_1 \oplus (m_2 \oplus m_3)$	[0.12, 0.52]	[0.17, 0.61]	[0.10, 0.59]	[0, 0.04]
$(m_1 \oplus m_3) \oplus m_2$	[0.12, 0.54]	[0.17, 0.61]	[0.10, 0.60]	[0, 0.04]
$m_1 \oplus m_2 \oplus m_3$	[0.13, 0.53]	[0.17, 0.60]	[0.11, 0.58]	[0, 0.04]

where $(m_1 \oplus m_2 \oplus \cdots \oplus m_n)^-(C)$ and $(m_1 \oplus m_2 \oplus \cdots \oplus m_n)^+(C)$ are respectively the minimum and the maximum of the following pair of optimization problems:

$$\begin{aligned}
&\text{Max/Min} \quad [m_1 \oplus m_2 \oplus \cdots \oplus m_n](C) = \frac{\sum_{A^1_{j_1} \cap A^2_{j_2} \cap \cdots \cap A^n_{j_n} = C} m_1\left(A^1_{j_1}\right) m_2\left(A^2_{j_2}\right) \cdots m_n(A^n_{j_n})}{1 - \sum_{A^1_{j_1} \cap A^2_{j_2} \cap \cdots \cap A^n_{j_n} = \Phi} m_1(A^1_{j_1}) m_2(A^2_{j_2}) \cdots m_n(A^n_{j_n})} \\
&\text{s.t.} \quad \sum_{j=1}^{n_i} m_i(A^i_j) = 1, \quad i = 1, \ldots, n, \\
&\qquad m_i^-(A^i_j) \leqslant m_i(A^i_j) \leqslant m_i^+(A^i_j), \quad i = 1, \ldots, n; \ j = 1, \ldots, n_i.
\end{aligned} \tag{27}$$

In combining multiple interval-valued belief structures, it is difficult to write the expression of (27) in general. It is therefore useful to break it down into several parts and write them one by one. To show this process clearly, let us examine the example in Table 2.

From the previous discussion we have already known that the results of the combination of m_1 and m_2 are

$$m_{1\text{–}2}(H_i) = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_1(H_k)m_2(H_j)}, \quad i = 1, 2, 3,$$

$$m_{1\text{–}2}(H) = \frac{m_1(H)m_2(H)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_1(H_k)m_2(H_j)}.$$

The above results can be viewed as a new piece of evidence, which is further combined with the evidence m_3. The combined results can be written as

$$m(H_i) = \frac{m_{1\text{–}2}(H_i)m_3(H_i) + m_{1\text{–}2}(H_i)m_3(H) + m_{1\text{–}2}(H)m_3(H_i)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_{1\text{–}2}(H_k)m_3(H_j)}, \quad i = 1, 2, 3,$$

$$m(H) = \frac{m_{1\text{–}2}(H)m_3(H)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_{1\text{–}2}(H_k)m_3(H_j)}.$$

Accordingly, models (27) are transformed into the following:

$$\begin{aligned}
&\text{Max/Min} \quad m(H_i) = \frac{m_{1\text{–}2}(H_i)m_3(H_i) + m_{1\text{–}2}(H_i)m_3(H) + m_{1\text{–}2}(H)m_3(H_i)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_{1\text{–}2}(H_k)m_3(H_j)} \\
&\text{s.t.} \quad m_{1\text{–}2}(H_i) = \frac{m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_1(H_k)m_2(H_j)}, \quad i = 1, 2, 3, \\
&\qquad m_{1\text{–}2}(H) = \frac{m_1(H)m_2(H)}{1 - \sum_{k=1}^3 \sum_{j=1, j \neq k}^3 m_1(H_k)m_2(H_j)},
\end{aligned}$$

$$
\begin{aligned}
&m_1(H_1)+m_1(H_2)+m_1(H_3)+m_1(H)=1,\\
&m_2(H_1)+m_2(H_2)+m_2(H_3)+m_2(H)=1,\\
&m_3(H_1)+m_3(H_2)+m_3(H_3)+m_3(H)=1,\\
&0.2\leqslant m_1(H_1)\leqslant 0.4,\\
&0.3\leqslant m_1(H_2)\leqslant 0.5,\\
&0.1\leqslant m_1(H_3)\leqslant 0.3,\\
&0\leqslant m_1(H)\leqslant 0.4,\\
&0.3\leqslant m_2(H_1)\leqslant 0.4,\\
&0.1\leqslant m_2(H_2)\leqslant 0.2,\\
&0.2\leqslant m_2(H_3)\leqslant 0.3,\\
&0.1\leqslant m_2(H)\leqslant 0.4,\\
&0.2\leqslant m_3(H_1)\leqslant 0.3,\\
&0.3\leqslant m_3(H_2)\leqslant 0.4,\\
&0.4\leqslant m_3(H_3)\leqslant 0.5,\\
&0\leqslant m_3(H)\leqslant 0.1.
\end{aligned}
$$

Such non-linear programming models can be easily implemented by using existing optimization packages such as LINGO software package or MATLAB optimization tool box. Similar models can also be constructed for $m(H)$. The optimized results are recorded in the last row of Table 2. No matter how many interval-valued belief structures are combined, expression (27) can always be written in this way if H_i and H are the only focal elements.

Moreover, due to the fact that

$$
1-\sum_{A^1_{j_1}\cap A^2_{j_2}\cap\cdots\cap A^n_{j_n}=\Phi} m_1(A^1_{j_1})m_2(A^2_{j_2})\cdots m_n(A^n_{j_n})=\sum_{C\neq\Phi}\ \sum_{A^1_{j_1}\cap A^2_{j_2}\cap\cdots\cap A^n_{j_n}=C} m_1(A^1_{j_1})m_2(A^2_{j_2})\cdots m_n(A^n_{j_n}),
$$

(27) can also be equivalently expressed as

$$
\begin{aligned}
\text{Max/Min}\quad & [m_1\oplus m_2\oplus\cdots\oplus m_n](C)=\frac{\sum_{A^1_{j_1}\cap A^2_{j_2}\cap\cdots\cap A^n_{j_n}=C} m_1\left(A^1_{j_1}\right)m_2(A^2_{j_2})\cdots m_n\left(A^n_{j_n}\right)}{\sum_{C\neq\Phi}\sum_{A^1_{j_1}\cap A^2_{j_2}\cap\cdots\cap A^n_{j_n}=C} m_1\left(A^1_{j_1}\right)m_2(A^2_{j_2})\cdots m_n\left(A^n_{j_n}\right)}\\
\text{s.t.}\quad & \sum_{j=1}^{n_i} m_i(A^i_j)=1,\quad i=1,\ldots,n,\\
& m_i^-(A^i_j)\leqslant m_i(A^i_j)\leqslant m_i^+(A^i_j),\quad i=1,\ldots,n;\ j=1,\ldots,n_i.
\end{aligned}
\tag{28}
$$

The main advantage of models (28) is that they take no account of the probability mass assigned to the empty set, $\{\Phi\}$, in the process of combining multiple interval-valued belief structures. Therefore, there is no need for normalization to be conducted in the intermediate process of the combination. It is done at the end of the combination. In general, it is more convenient to solve (28) than (27).

Considering again the example in Table 2, we have the following non-normalized probability masses from the combination of m_1 and m_2:

$$
\begin{aligned}
&\bar{m}_{1\text{-}2}(H_i)=m_1(H_i)m_2(H_i)+m_1(H_i)m_2(H)+m_1(H)m_2(H_i),\quad i=1,2,3,\\
&\bar{m}_{1\text{-}2}(H)=m_1(H)m_2(H),
\end{aligned}
$$

which are further combined with the evidence m_3, leading to the results below:

$$
\begin{aligned}
&\bar{m}(H_i)=\bar{m}_{1\text{-}2}(H_i)m_3(H_i)+\bar{m}_{1\text{-}2}(H_i)m_3(H)+\bar{m}_{1\text{-}2}(H)m_3(H_i),\quad i=1,2,3,\\
&\bar{m}(H)=\bar{m}_{1\text{-}2}(H)m_3(H)=m_1(H)m_2(H)m_3(H).
\end{aligned}
$$

After normalizing the above results, it follows that

$$m(H_i) = \frac{\bar{m}(H_i)}{\sum_{j=1}^{3}\bar{m}(H_j) + \bar{m}(H)}, \quad i = 1, 2, 3,$$
$$m(H) = \frac{\bar{m}(H)}{\sum_{j=1}^{3}\bar{m}(H_j) + \bar{m}(H)}.$$

Accordingly, (28) becomes

$$\begin{aligned}
\text{Max/Min} \quad & m(H_i) = \frac{\bar{m}(H_i)}{\sum_{j=1}^{3}\bar{m}(H_j) + \bar{m}(H)} \\
\text{s.t.} \quad & \bar{m}(H_i) = \bar{m}_{1\text{-}2}(H_i)m_3(H_i) + \bar{m}_{1\text{-}2}(H_i)m_3(H) + \bar{m}_{1\text{-}2}(H)m_3(H_i), \quad i = 1, 2, 3, \\
& \bar{m}(H) = \bar{m}_{1\text{-}2}(H)m_3(H), \\
& \bar{m}_{1\text{-}2}(H_i) = m_1(H_i)m_2(H_i) + m_1(H_i)m_2(H) + m_1(H)m_2(H_i), \quad i = 1, 2, 3, \\
& \bar{m}_{1\text{-}2}(H) = m_1(H)m_2(H), \\
& m_1(H_1) + m_1(H_2) + m_1(H_3) + m_1(H) = 1, \\
& m_2(H_1) + m_2(H_2) + m_2(H_3) + m_2(H) = 1, \\
& m_3(H_1) + m_3(H_2) + m_3(H_3) + m_3(H) = 1, \\
& 0.2 \leqslant m_1(H_1) \leqslant 0.4, \\
& 0.3 \leqslant m_1(H_2) \leqslant 0.5, \\
& 0.1 \leqslant m_1(H_3) \leqslant 0.3, \\
& 0 \leqslant m_1(H) \leqslant 0.4, \\
& 0.3 \leqslant m_2(H_1) \leqslant 0.4, \\
& 0.1 \leqslant m_2(H_2) \leqslant 0.2, \\
& 0.2 \leqslant m_2(H_3) \leqslant 0.3, \\
& 0.1 \leqslant m_2(H) \leqslant 0.4, \\
& 0.2 \leqslant m_3(H_1) \leqslant 0.3, \\
& 0.3 \leqslant m_3(H_2) \leqslant 0.4, \\
& 0.4 \leqslant m_3(H_3) \leqslant 0.5, \\
& 0 \leqslant m_3(H) \leqslant 0.1.
\end{aligned}$$

Similar models can be written for $m(H)$. The results are the same as those in the last row of Table 2.

5. Concluding remarks

In this paper we have reinvestigated the issues of combination and normalization of interval-valued belief structures. The three existing approaches for combining and normalizing interval evidence are reviewed and thoroughly examined. The irrationality or suboptimality of these methods has been pointed out and an optimality approach has been developed and illustrated with numerical examples. The optimality approach has been successfully applied to a cargo ship selection problem, which is a complex multiple criteria decision making problem involving six decision alternatives and nine decision attributes, two of which are qualitative and assessed using interval-valued belief degrees and the others are quantitative and assessed using either precise or interval data. The findings are reported in Wang et al. [57].

Finally, we point out the issue of computational complexity of the optimality approach raised by a referee. The optimality approach aims at combining and normalizing interval evidence in one step to obtain a precise solution rather than separately to achieve a suboptimal solution and involves the solution of a number of nonlinear programming models. There is no doubt that it is computationally complicated than Denoeux's approach [14]. However, according to our observations from the calculations of Table 2 and the applications in [57], where nine pieces of interval evidence with six focal elements were combined to make a decision for six cargo ships, the computing time for the optimality approach is trivial. Each interval-valued belief degree can be obtained within just 1 or 2 s when LINGO software package is used to solve their models. This fact shows that computational complexity is not a problem for the optimality approach.

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