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ORIGINAL ARTICLE

Dynamic analysis of a two-stage supply chain—a switched system theory approach

Hongwei Wang · Huixin Liu · Jian-bo Yang

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Abstract This paper investigates the dynamics of a two-stage supply chain consisting of one retailer and one distributor with order-up-to control policy. A discrete time state space model is developed to analyze the system. Due to the constraints of the distributor's capacity, the supply chain turns out to be an autonomous switched system corresponding to the shipment policy of the distributor. The stability of the system studied is analyzed using switched system theory and discrete system control theory for the practical perspective. This study shows that the system will exhibit stable, periodic, and divergent behavior with different order policy parameters due to the switching mechanism within the system. The relationship between dynamics and cost is investigated, and insights are obtained about the control of the supply chain.

Keywords Supply chain · Inventory control · Switched system · Nonlinear dynamics

H. Wang (✉) · J.-b. Yang
Institute of Systems Engineering and the State Key Laboratory of Image Processing and Intelligent Control,
Huazhong University of Science and Technology,
Wuhan 430074, People's Republic of China
e-mail: hwwang@mail.hust.edu.cn

J.-b. Yang
e-mail: jian-bo.yang@umist.ac.uk

H. Liu
Department of Management Engineering, Zheng Zhou University,
Zheng Zhou 450001, People's Republic of China
e-mail: zzu_lhx@zzu.edu.cn

J.-b. Yang
Manchester Business School (East),
The University of Manchester,
P.O. Box 88, Booth Street East,
Manchester M60 1QD, UK

1 Introduction

In today's global marketplace, industrial organizations no longer compete as independent entities with unique brand names but rather as integral parts of supply chains. As such, the success of a firm critically depends on the relationships and interdependencies among supply chain members; also, the competitiveness is tied to the dynamics of the supply chains in which the firm participates. Therefore, it becomes increasingly important to understand the dynamic behavior of a supply chain system in order to obtain a comprehensive understanding of the factors that have impact on system performance. In this paper, we discuss how the decision rules affect the stability of a two-stage supply chain and why a simple supply chain may exhibit complex behavior.

The dynamics of supply chains have been extensively studied over years since the first piece of work by Forrester [1] in the 1960s, in which the existence of 'demand amplification' in real-world supply chains was demonstrated with a simple linear supply chain. The related works can be classified into three broad approaches [2]: the control theory approach, the behavioral science approach, and the practitioner approach. The recent research progresses in these approaches are briefly reviewed below.

The focus of the control theory approach is to develop methodologies applying system control principles to dampen the dynamics within a supply chain. Classic control system engineering techniques are commonly used in the researches belonging to this category [3]. Burns and Sivazlian [4] used the servomechanism theory to study the inventory control problem in a supply chain. Popplewell and Bonney [5] analyzed production control systems, applying the discrete linear control theory and provided a sufficient condition for total system stability. Towill [6] designed inventory- and order-based production control

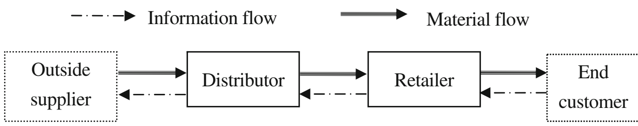

Fig. 1 The supply chain model

systems (IOBPCS) and analyzed single- or multi-loop systems using the linear control theory and system dynamics. Simon et al. [7] improved the IOBPCS to automatic pipeline inventory- and order-based production control system (APIOBPCS) by taking into consideration the work in progress (WIP) information. Disney and Towill [8] further analyzed the stability of a vendor-managed inventory supply chain, adopting APIOBPCS using the discrete linear control theory. Furthermore, Dejonckheere et al. [9–11] analyzed the bullwhip effect in supply chains applying control engineering techniques including z-transform, transfer function, frequency plot, and so on. Perea et al. [12] modeled a supply chain using differential equations to examine its dynamic behavior. Most recently, Lin [13] considered a supply chain as a linear discrete system with lead time and operation constraints, and a proportional integral controller was designed to suppress the bullwhip effect. The above-mentioned researches have suggested different methodologies to eliminate bullwhip effect and have provided a better understanding of the dynamics of multi-echelon supply chains. However, the systems studied were mostly linear without considering the complex interdependencies among organizational structures, which commonly exists in real world. Additionally, the developed models have a strong assumption that all the orders between echelons must be fulfilled without capacity constraints. The major limitation of these models results from the inherent lack of analyzing the influence each echelon possesses over the next higher echelon [14].

The second approach, termed by Wilding [2] as the behavioral science approach, is characterized by the development of business games, such as the well-known Beer Game, to demonstrate the bullwhip effect. The works in this category focus on the rules used by managers in operating hand simulations and investigate how human decision making generates uncertainties. Mosekilde and Larsen [15] described the occurrence of chaotic dynamics in a continuous-time model for a distribution chain. Thomsen et al. [16] showed how a distribution chain with realistic parameters can support hyperchaotic and higher-order hyperchaotic behavior. Larsen et al. [17] discussed the complex behavior of production–distribution systems caused by the changes of the order policy parameters.

For the third approach, the practitioner approach, a number of simulation models are developed, such as multi-agent models [18], system dynamics models [19], and Petri-net models [20–21], to study the dynamic behavior of supply chains

Although both the behavioral science approach and the practitioner approach have found that supply chains can exhibit complex behavior, they have not yet provided theoretical analyses for such behavior. On the other hand, under the control theory approach, supply chain systems are mostly modeled as linear systems rarely considering the complexity of complex system behavior. How, then, is such a problem to be solved?

This paper investigates the dynamics of a two-stage supply chain consisting of one retailer and one distributor with order-up-to control policy. A discrete time state space model is developed to analyze the system. Due to the constraints of the distributor's capacity, the supply chain turns out to be an autonomous switched system corresponding to the shipment policy of the distributor. The stability of the considered system is analyzed using switched system theory and discrete system control theory for the practical perspective. This paper will answer why the system can exhibit stable, periodic, and divergent behavior, and how to choose proper order parameters to guarantee the stability of supply chains.

The rest of the paper is organized as follows: Section 2 gives a brief introduction of the switched system; it is followed by the description of the difference equations of a two-stage supply chain in Section 3; in Section 4, a switched system model of the supply chain is presented, and each of its subsystems is specified; in Section 5, stability of the system is analyzed; Section 6 describes the simulation study showing the diverse behavior of the system; Section 7 concludes this study.

2 Introduction on switched systems

A switched system is a hybrid system that is composed of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between the subsystems [22]. At any given time, a particular subsystem may be chosen by some 'higher process,' such as a controller, computer, or human operator, in which case, we say that the system is *controlled*. It may also be a function of time or state, or both, in which case we say that the system is *autonomous*. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear, time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems.

Switched systems arise in varied contexts in manufacturing, communication networks, computer synchronization, traffic control, chemical processes, automotive engine control, auto pilot design, and so on. Switched server

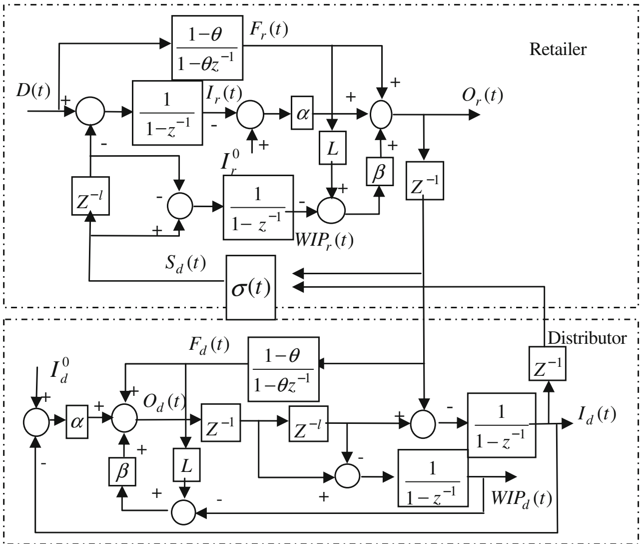

Fig. 2 The block diagram of the supply chain

systems and switched arrival systems are some of the generalized models [23–27]. A more detailed description of such applications could be found in Morse [28].

The stability of a switched system is a basic problem, which is affected by two factors: the stability of its subsystems and the switching rule. The supply chain system investigated in this paper is considered as an *autonomous switched system*, and its stability analysis is decomposed into three parts: analysis of the dynamics of the linear subsystems, analysis of switching rules, and analysis of the stability of supply chain under the switching rules. This analysis method can more easily handle the stability and dynamics of the system than using the nonlinear system theory.

3 Basic dynamic model

3.1 System description

The system under study is a typical cascaded production–distribution system consisting of two sectors: a retailer and a distributor (see Fig. 1). Customer demand is exogenous. Given that stocks are available, each sector must supply the amount requested by the next sector. If the stocks are insufficient, surplus orders must be kept in backlog until deliveries can be made. With the assumption that the system is a periodic-reviewed supply chain, the sequence of events during a period is, first, the member observes and satisfies consumer demand if there is no backlog and then receives goods; next, the member observes the new inventory level and finally places an order on its supplier. Based on this arrangement, there is a period from the time the order is sent until the supplier receives it. Suppose it takes the distributor l periods to transport goods to the retailer. The total lead time is then given by $L=l+1$. To simplify the model without loss of generality, we assume that the outside supplier's lead time is L as well. It is assumed that the outside supplier has infinite inventory so that the distributor's order can be fully satisfied. Capacity of the distributor is assumed to be limited. Therefore, he will satisfy the retailer's demand based on his inventory level.

3.2 Block diagram and difference equations of the system

Model notation Hereafter, let r, d, and s denote the retailer's state variables, the distributor's state variables, and the outside supplier's state variables, respectively.

t	time period, t=1,2,...;
$D(t)$	the outside demand at time period t
$I_i(t)$	inventory level of member i at the end of time period t, $i\in\{r,d\}$, and similarly below
$\mathrm{BL}_i(t)$	backup orders of member i at the end of time period t
$\mathrm{O}_i(t)$	order quantity of member i at the end of time period t
$R_i(t)$	quantity of goods that member i received at the beginning of time period t
$S_i(t)$	shipments of member i to its downstream member at time period t
$S_s(t)$	shipments of outside supplier to the distributor at time period t
$\mathrm{WIP}_i(t)$	work in progress of member i at time period t
$F_i(t)$	forecasted demand of member i in period t

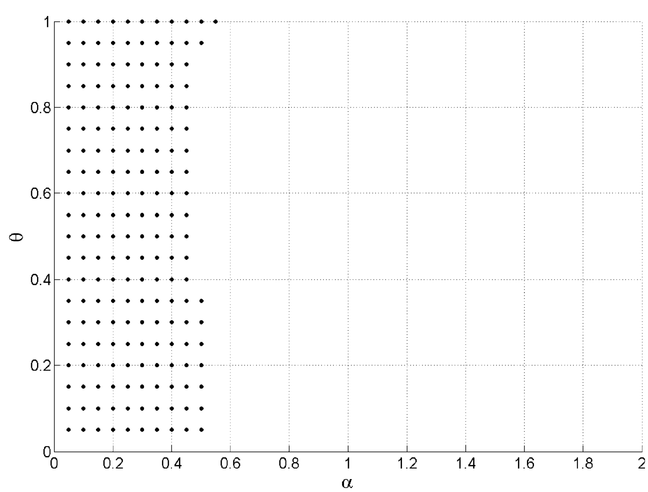

Fig. 3 The states of the eigenvalues of subsystem 15

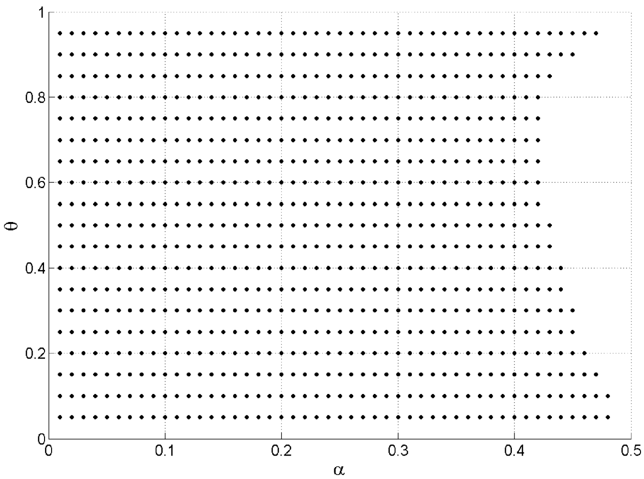

Fig. 4 The solutions of inequality 24 when A_1 and A_2 are stable

3.2.1 Demand policy

Exponential smoothing is used to forecast demand following formula 1 [9]:

$$F_i(t) = \theta F_i(t-1) + (1-\theta)D_i(t), 0 < \theta < 1,\ i \in \{r, d\} \quad (1)$$

where $D_r = D(t)$ and $D_d = O_r(t-1)$ are the demand that the retailer and the distributor faced, respectively. Note that ordering decision is made at the end of a period, and current demand can be used in forecast.

3.2.2 Shipment policy

At the beginning of period t, demand is realized and fulfilled with the available inventory of period $t-1$. Under the assumption that the outside demand is positive, if the demand is less than the inventory, it can be fully met; otherwise, the available stock is sent. Therefore,

$$S_r(t) = \begin{cases} D(t) - BL_r(t-1), & I_r(t-1) \geq D(t) - BL_r(t-1) \\ I_r(t-1), & 0 \leq I_r(t-1) < D(t) - BL_r(t-1) \\ 0, & I_r(t-1) < 0 \end{cases}. \quad (2)$$

Shipment of the distributor in time period t has the same structure as Eq. 2. Due to the one-time period order transfer delay, in time period t, the demand the distributor needs to meet is the order the retailer placed at time period $t-1$. Hence,

$$S_d(t) = \begin{cases} O_r(t-1) - \mathrm{BL}_d(t-1), & I_d(t-1) \geq O_r(t-1) - \mathrm{BL}_d(t-1) \geq 0 \\ I_d(t-1), & 0 \leq I_d(t-1) > O_r(t-1) - \mathrm{BL}_d(t-1) \\ 0, & I_d(t-1) > 0 \,\&\, O_r(t-1) - \mathrm{BL}_d(t-1) \geq 0 \\ O_r(t-1) - \mathrm{BL}_d(t-1), & O_r(t-1) - \mathrm{BL}_d(t-1) > 0 \end{cases}. \quad (3)$$

3.2.3 Inventory and pipeline policy

Given that there is no loss of goods on passage, the material the retailer received in period t is the shipment from the distributor in the previous l periods (see Eq. 4), while the goods the distributor received is shown in Eq. 5 based on the assumption that the outside supplier has infinite stock.

$$R_r(t) = S_d(t-l) \quad (4)$$

$$R_d(t) = O_d(t-L). \quad (5)$$

The on-hand inventory at the end of period t is given by Eq. 6:

$$I_i(t) = I_i(t-1) + R_i(t) - S_i(t), i \in \{r, d\}. \quad (6)$$

Equation 6 indicates that the current inventory level is equal to the previous inventory level plus the difference between the demand and the received goods. The balance of the backlogged order is given by Eq. 7:

$$\mathrm{BL}_i(t) = \mathrm{BL}_i(t-1) + S_i(t) - D_i(t), i \in \{r, d\}. \quad (7)$$

There are significant advantages to incorporate WIP information into the production control system as outlined by Simon et al. [7]. This makes stable and faster responses possible. In our model, WIP can be calculated by Eq. 8:

$$\mathrm{WIP}_i(t) = \mathrm{WIP}_i(t-1) + S_j(t) - R_i(t), i \in \{r, d\}, \quad (8)$$

$$j \in \{d, s\}.$$

It is clear that $S_s(t) = O_d(t-1)$.

3.2.4 Order policy

An order-up-to policy is optimal to bring the inventory position up to a predetermined target level with minimum expected holding and shortage costs. In practice, production is possibly very inflexible. In this circumstance, frequently ramping up and down production levels might incur significant costs. As a result, the classical order-up-to policies will always generate a bullwhip effect when demand has to be forecasted. In order to avoid the bullwhip effect, production planning and inventory control should be considered together. Therefore, the production control model named APIOBPCS is often used in supply chain systems [6–11]. APIOBPCS can be expressed in such words as ‘let the production targets be equal to the sum of an exponentially smoothed representation of the perceived demand, plus a fraction of the inventory error in stock, plus a fraction of the WIP error’ [7]. The key idea is to design replenishment rules based on what we call fractional adjustments in order to generate smooth order patterns and dampen order fluctuations.

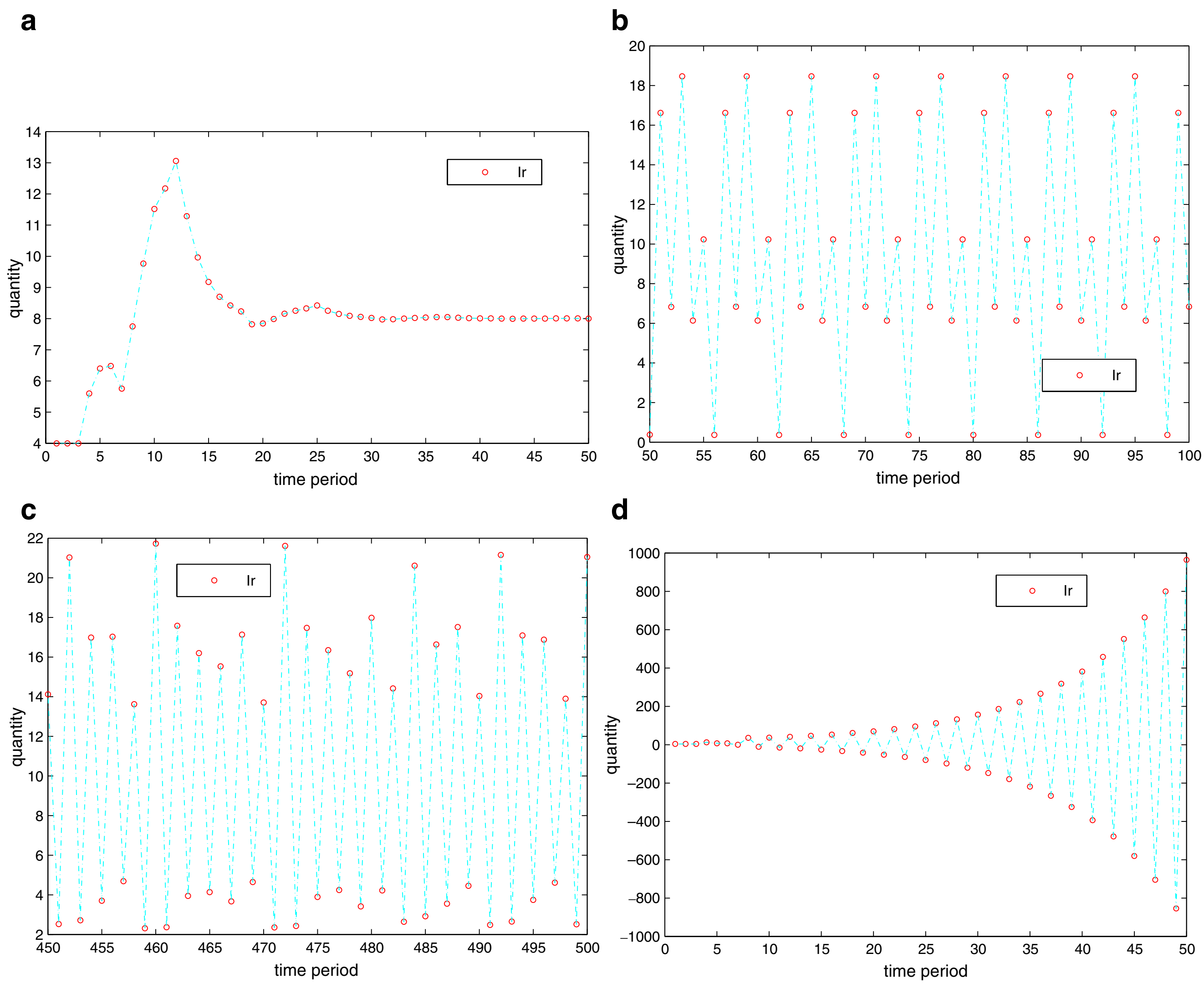

Fig. 5 The dynamic behaviors of the presented supply chain, $\left(I_r^0, I_d^0\right) = (8, 8)$

The replenishment rule is formulated as Eq. 9:

$$O_i(t) = F_i(t) + \alpha\left(I_i^0 - I_i(t)\right) + \beta\left(\mathrm{WIP}_i^0 - \mathrm{WIP}_i(t)\right). \quad (9)$$

where I_i^0 is a constant that specifies the desired or target inventory level to meet a certain customer service level; WIP_i^0 is the target WIP level; α is a parameter that characterizes the fraction of the inventory error that is corrected; β is a parameter that characterizes the fraction of the WIP error that is corrected. Since the speed of inventory (WIP) adjustment is slow if $\alpha(\beta)$ is small, even a large inventory imbalance has only a small impact on ordering and vice versa. Parameters α and β are the key parameters of the decision rule. Ordering quantities are set equal to the sum of forecasted demand, a fraction α of the discrepancy of finished goods net stock, and a fraction β of our on-order position discrepancy [7]. Parameter I_i^0 is the target inventory level, which is similar to the safety stock in order-up-to policies. It is manually updated by members to regulate the dynamics of the supply chain according to the new demand forecast. The level of WIP_i^0 is updated every period as well and is calculated as Eq. 10:

$$WIP_i^0 = L \bullet F_i(t), \quad (10)$$

where L is the lead time, meaning that the desired on-transit inventory should cover the demands over the whole lead time.

To simplify the model, identical parameters are used for the ordering rule including α and β, while different desired inventory levels are used including I_i^0 and WIP_i^0. The block diagram of the supply chain is shown in Fig. 2. The aim of this paper is to investigate how these parameters of order policy affect the behavior of the supply chain.

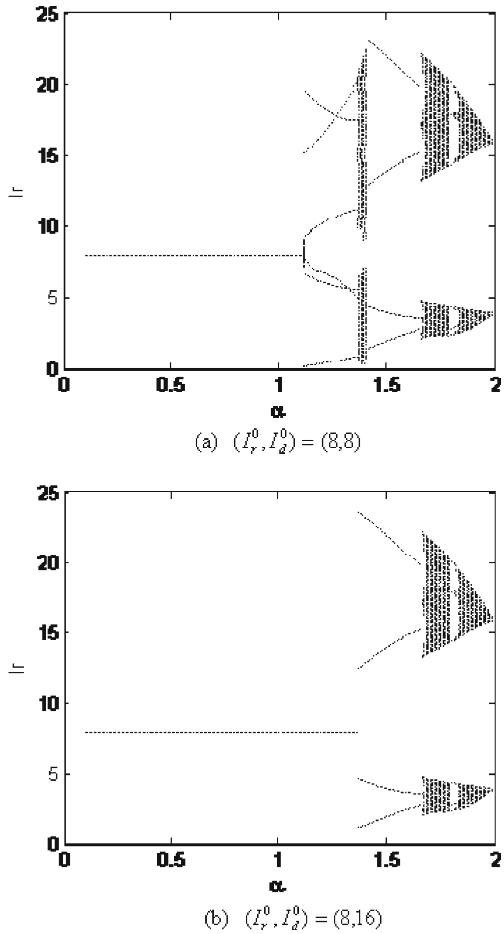

Fig. 6 Bifurcation diagram, taking α as bifurcation parameter

3.3 System state space equation

To construct the state space equation of the supply chain, we choose $I_i(t)$, $BL_i(t)$, $F_i(t)$, and $WIP_i(t)$ as the state variables, I_i^0 and $D(t)$ as the outside input and disturbance, respectively, and the net stock $I_d(t)+BL_d(t)$ as the output of the system.

Since the shipment of the retailer has no other effect on the system state than to show the actual customer service level, one can incorporate the backup orders (Eq. 7) into the on-hand inventory (Eq. 6) represented by $I_r(t)$. Thus,

$$I_r(t) = I_r(t-1) + R_r(t) - D(t). \tag{11}$$

The state space function of the system can be given as:

$$\begin{cases} I_r(t) = I_r(t-1) + S_d(t-L+1) - D(t) \\ F_r(t) = \theta F_r(t-1) + (1-\theta)D(t) \\ \mathrm{WIP}_r(t) = \mathrm{WIP}_r(t-1) + S_d(t) - S_d(t-L+1) \\ I_d(t) = I_d(t-1) + O_d(t-L) - S_d(t) \\ \mathrm{BL}_d(t) = \mathrm{BL}_d(t-1) + S_d(t) - O_r(t-1) \\ F_d(t) = \theta F_d(t-1) + (1-\theta)O_r(t-1) \\ \mathrm{WIP}_d(t) = \mathrm{WIP}_d(t-1) + O_d(t-1) - O_d(t-L) \end{cases} \tag{12}$$

Equation 3 shows that the expression of $S_d(\bullet)$ is not unique and is decided by the distributor's net stock and the demand he faced. Therefore, it is necessary to analyze the different expressions for $S_d(\bullet)$. For the purpose of simplicity, we only consider systems with a lead time of L=1. Thus, we only need to take into account the various expressions of $S_d(t)$. For L>1, the analysis of the system can be conducted following a similar yet somewhat more complicated procedure.

Note that the lead time L=1 means that there is no production delay, and as a result, there is no work in progress. Hence, the state space equation of the system can be formulated as follows:

$$\begin{cases} I_r(t) = I_r(t-1) + S_d(t) - D(t) \\ F_r(t) = \theta F_r(t-1) + (1-\theta)D(t) \\ I_d(t) = I_d(t-1) + O_d(t-1) - S_d(t) \\ \mathrm{BL}_d(t) = \mathrm{BL}_d(t-1) + S_d(t) - O_r(t-1) \\ F_d(t) = \theta F_d(t-1) + (1-\theta)O_r(t-1) \end{cases}. \tag{13}$$

In the next section, we will analyze the specific expression of the system.

4 Switched system model of the supply chain

In this section, different submodels for the system will be investigated with respect to different expressions for the shipment of the distributor ($S_d(\bullet)$). Hereafter, let

$$X(t) = [I_r(t) \quad F_r(t) \quad I_d(t) \quad \mathrm{BL}_d(t) \quad F_d(t)]^T, r(t)$$

$$= \begin{bmatrix} I_r^0 & I_d^0 & D(t) \end{bmatrix}^T.$$

4.1 Case when the distributor has enough stock to satisfy the demand

In this case, the shipment of the distributor is expressed as $S_d(t) = O_r(t-1) - BL_d(t-1)$, which means that the current order of the retailer and backlogged orders of the distributor can be fully satisfied. Submitting $S_d(t)$ and Eq. 9 into Eq. 13, the state space function of the system can be expressed as Eq. 14.

$$X(t) = A_1 X(t-1) + B_1 r(t), \tag{14}$$

$$\text{where } A_1 = \begin{bmatrix} (1-\alpha) & 1 & 0 & -1 & 0 \\ 0 & \theta & 0 & 0 & 0 \\ \alpha & -1 & (1-\alpha) & (1-\alpha) & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -\alpha(1-\theta) & (1-\theta) & 0 & 0 & \theta \end{bmatrix}$$

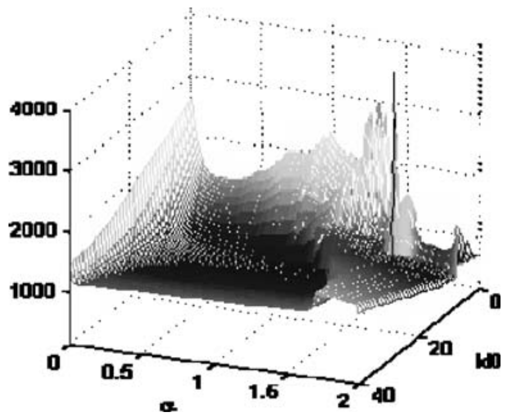

Fig. 7 Total cost of the supply chain

$$\text{and } B_1 = \begin{bmatrix} \alpha & 0 & -1 \\ 0 & 0 & 1-\theta \\ -\alpha & \alpha & 0 \\ 0 & 0 & 0 \\ \alpha(1-\theta) & 0 & 0 \end{bmatrix}.$$

4.2 Case when the distributor supplies the retailer with available stock but the demand is not fulfilled

In this circumstance, $S_d(t) = I_d(t-1)$ and the function is expressed as follows:

$$X(t) = A_2X(t-1) + B_2r(t), \tag{15}$$

$$\text{where } A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & \theta & 0 & 0 & 0 \\ 0 & 0 & -\alpha & -\alpha & 1 \\ \alpha & -1 & 1 & 1 & 0 \\ -\alpha(1-\theta) & (1-\theta) & 0 & 0 & \theta \end{bmatrix}$$

$$\text{and } B_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1-\theta \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ \alpha(1-\theta) & 0 & 0 \end{bmatrix}.$$

4.3 Case when the distributor has no available stock

Nothing can be sent to the retailer under this condition, namely $S_d(t)=0$. Therefore, the system equation is given as follows:

$$X(t) = A_3X(t-1) + B_3r(t), \tag{16}$$

$$\text{where } A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & -\alpha & 1 \\ \alpha & -1 & 0 & 1 & 0 \\ -\alpha(1-\theta) & (1-\theta) & 0 & 0 & \theta \end{bmatrix}$$

$$\text{and } B_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1-\theta \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ \alpha(1-\theta) & 0 & 0 \end{bmatrix}.$$

4.4 Switched system model

Based on the above analysis, we deduce that the two-stage supply chain system can be regarded as a *switched system* due to its shipment policy, which switches among the three modes as shown by Eqs. 14, 15, and 16. The system is an autonomous discrete switched system as the shipment policy is an internal mechanism. In terms of switched system theory, we hereafter call the ‘mode’ a ‘subsystem.’

Since all the individual subsystems (corresponding to the three modes by Eqs. 14, 15, and 16) are linear, the supply chain system can be modeled as a linear switched system by Eq. 17. According to Eq. 3, the switching functions that orchestrate the switch actions of the system are expressed by Eqs. 18, 19, and 20.

$$X(t) = A_iX(t-1) + Br(t), i \in \{1,2,3\} \tag{17}$$

$$\sigma_1(t) = sign(I_d(t-1)) \tag{18}$$

$$\sigma_2(t) = \mathrm{sign}(I_d(t-1) - F_r(t-1) - \alpha\big(I_r^0 - I_r(t-1)\big) + \mathrm{BL}_d(t-1) \tag{19}$$

$$\sigma_3(t) = \mathrm{sign}(F_r(t-1) + \alpha\big(I_r^0 - I_r(t-1)\big) - \mathrm{BL}_d(t-1) \tag{20}$$

The inventory state of the distributor is evaluated by $\sigma_1(t)$. With $\sigma_1(t) \geq 0$, the distributor has on-hand stock, which means that the distributor will have something to ship to the retailer and that the supply chain will stay out of the subsystem A_3. The order state of the retailer is evaluated by $\sigma_3(t)$. With $\sigma_3(t) \geq 0$, the demand that the distributor needs to meet is larger than zero. A demand of less than zero for the distributor means that the retailer wants to return goods to the distributor. Function $\sigma_2(t)$ is to show whether the distributor's inventory is enough to meet the demand of the retailer; $\sigma_2(t) \geq 0$ means that the on-hand stock of the distributor can fully satisfy the demand he faces and vice versa.

In summary, there are following instances:

C1. If $\sigma_1(t) \geq 0$ and $\sigma_2(t) \geq 0$, then the system switches to subsystem A_1.

C2. If $\sigma_1(t)\geq 0$ and $\sigma_2(t)<0$, then the system switches to subsystem A_2.

C3. If $\sigma_1(t)<0$ and $\sigma_3(t)\geq 0$, then the system switches to subsystem A_3.

C4. If $\sigma_1(t)<0$ and $\sigma_3(t)<0$, then the system switches to subsystem A_1.

5 Stability analysis

A supply chain system is always regarded as a control system, and it is necessary to study its dynamic behaviors. The system can become stable or unstable. For a supply chain system, instability means that any sudden change in the outside demand will result in uncontrollable oscillations of the inventory and order rate of the members. Switching production levels up and down frequently may be very expensive in practice. Beside these production-switching costs, inventory costs and punishment cost due to the unsatisfied demands are also very large. This will consequently affect many supply chain performance measures such as fill rate, delivery performance, lead time, flexibility of the supply chain, and so on. It is significant to understand how the supply chain system responds to any change in decisions, especially under a fluctuating market circumstance. Does the response result in increasing amplitude oscillations and chaos in general, or does the response appear controllable and damped? To answer these questions, it is essential to identify under what conditions the system is stable or unstable. We utilize the stability property of the switched system represented by Eq. 17 to discuss these issues.

5.1 Stability criterion of the subsystems

For the sake of clarity, whenever we say that a subsystem is stable, we mean it is Schur stable. A linear time–invariant discrete time system $x[k+1]=Ax[k]$ (or simply the matrix **A**) is called Schur stable if all eigenvalues of matrix **A** are in the open unit circle; otherwise, it is not Schur stable[21].

5.1.1 Subsystem 14

To specify the eigenvalues of A_1, we first give the characteristic equation of A_1:

$$|\lambda I - A_1| = \begin{vmatrix} \lambda-1+\alpha & -1 & 0 & 1 & 0 \\ 0 & \lambda-\theta & 0 & 0 & 0 \\ -\alpha & 1 & \lambda-1+\alpha & -1+\alpha & -1 \\ 0 & 0 & 0 & \lambda & 0 \\ \alpha(1-\theta) & -1+\theta & 0 & 0 & \lambda-\theta \end{vmatrix} \quad (21)$$

$$= \lambda(\lambda-\theta)^2(\lambda-1+\alpha)^2 = 0.$$

The roots of Eq. 21 are $\lambda_1=0, \lambda_{2,3}=\theta, \lambda_{4,5}=1-\alpha$. The parameter stability region is given by $0<\theta<1$ and $0<\alpha<2$. The result shows that the supply chain members can choose an adjustment ratio from scope (0, 2) to compensate his inventory imbalance without inducing the system to be unstable if the supplier has enough stocks t. In this case, the members can choose a larger α to make the system react faster to the outside change, or choose a smaller α to restrain the effect of outside changes on the order rate of the members.

5.1.2 Subsystem 15

The characteristic equation of system 15 is given by:

$$|\lambda I - A_2| = (\lambda-\theta)(\lambda^4+(\alpha-2-\theta)\lambda^3+(1-\alpha+2\theta-\theta\alpha)\lambda^2 + (\alpha^2+\alpha-\theta)\lambda-\theta\alpha^2-\alpha+\alpha\theta)=0 \quad (22)$$

Obviously, finding the analytical expression of all the roots of Eq. 22 is a hopeless task. To determine the stability region of this subsystem, we look for a numerical approximation by evaluating the largest roots of Eq. 22 using tools such as the control toolbox provided in Matlab 6.5. Assuming that α varies between 0 and 2, and θ varies between 0 and 1, the distribution of the roots of Eq. 22 can be obtained, as shown in Fig. 3, where the dots denote the largest eigenvalues that are less than 1. Hence, the approximate stability regions of parameters α and θ are given by:

$$\begin{cases} 0<\theta<1, 0<\alpha<0.5 \\ \alpha=0.5, \theta>0.9 \text{ or } \theta<0.3 \end{cases}.$$

The result shows that the supply chain members can only choose an adjustment ratio from a smaller scope (0, 0.5) to compensate his inventory imbalance without inducing the system to be unstable if the supplier's stocks are not sufficient, that is, large reactions to outside changes will make the system unstable.

5.1.3 Subsystem 16

Similarly, the characteristic equation of system 16 can be expressed as:

$$|\lambda I - A_3| = (\lambda-1)^2(\lambda-\theta)^2(\lambda-1+\alpha)=0. \quad (23)$$

The roots of Eq. 23 are $\lambda_{1,2}=1, \lambda_{3,4}=\theta, \lambda_5=1-\alpha$. None of the eigenvalues of system 16 lies outside the unit circle for $0<\alpha<2$, $0<\theta<1$. But it is necessary to point out that the system is unstable because there are two eigenvalues equal to 1, which means that the supply chain system will be unstable whatever the order parameters are, that is, the inventory level of the supply chain will always go out of control.

5.2 Stability of the supply chain system

In this subsection, we analyze the stability properties of the switched system expressed by Eq. 17. From the above analysis, subsystem A_3 is always unstable. Hence, we mainly analyze the different combinations of the states of subsystems A_1 and A_2 in the following.

5.2.1 Case when both A_1 and A_2 are stable

The stability of a switched system with stable as well as unstable subsystems is difficult to analyze. From C1 and C2, we can see that if the distributor has on-hand stock, the system will switch between two stable subsystems $A_1$and A_2. In this paper, the inventory control system of the distributor can be considered as a relatively single-stage system, so that given $0<\alpha<2$, $0<\theta<1$, its inventory level can be kept at a desired level of above zero. Thus, the condition $\sigma_1(t)>0$ can be satisfied. We only need to inspect the case where the system switches between stable subsystems.

Theorem 2 in [24] proves that if there exist two symmetric matrices, P_1 and P_2, satisfying

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0, \forall (i,j) \in \{1,2\} \times \{1,2\}, \tag{24}$$

there exists a Lyapunov function governing the asymptotic stability of the switched system under arbitrary switching between the subsystems A_1 and A_2.

Inequality 24 is a linear matrix inequality (LMI) problem, the solution of which can be found using the LMI toolbox in Matlab6.5. By changing α and θ within the common stability region of A_1 and A_2 given in Sections 4.1 and 4.2, the solution plot of inequality 24 is drawn in Fig. 4, where the dots denote that inequality 24 is solvable.

Obviously, system 17 will be stable if both A_1 and A_2 are stable, given that the on-hand inventory of the distributor does not go below zero, otherwise its stability will be uncertain. From Fig. 2, the transfer function of the distributor is given by:

$$G_1 = \frac{I_d}{I_0} = \frac{\alpha}{z - (1 - \alpha)} \tag{25}$$

$$G_2 = \frac{I_d}{O_r} = \frac{z(z-1)}{(z-\theta)(z-1+\alpha))} \tag{26}$$

If $0<\alpha<1$ and $0<\theta<1$, the poles of the transfer function will lie on the right real axis. The smaller α is, the larger the poles and the better the system's dynamic performance. In practice, a small α means a slow speed of inventory adjustment, so that even a large inventory imbalance has only a small impact on ordering. This can help the system to suppress the oscillation of the retailer's order.

5.2.2 Case when neither A_1 nor A_2 is stable

Under this circumstance, the system switches between three unstable subsystems according to C1–C4. The parameter θ of a simple exponential smoothing policy is always set within (0,1), and this setting will not give rise to unstable behavior of the system. Therefore, we mainly need to pay attention to the effect of α.

Note that if α is larger than 2.0 under any switched mode, the pole of the transfer functions G1 and G2 will lie outside the unit disc, which means that the inventory and order rate of the distributor will diverge exponentially for any order patterns of the retailer. In other words, the order quantity and the inventory level of the retailer oscillate, and thus, the demand the distributor faces will inversely affect its inventory level. This will cause the instability of the supply chain. In fact, a large α means that the speed of inventory adjustment is fast and that the system will be apt to overreact to the changes in outside demand, so it is reasonable to say that the supply chain is unstable under this condition. It is necessary to point out that the forecasted demands of the retailer and the distributor are stable since they are out of the control of α. The simulation in the following section supports this argument.

5.2.3 Case when subsystem A_1 is stable and A_2 is unstable

Liberzon and Morse [29] stated that the stability of a switched system can usually be ensured if each stable subsystem is activated for a long enough time to allow the transient effects to dissipate. For a switched system where both Schur stable and unstable subsystems co-exist and none of subsystem matrices is commutative pairwise, Theorem 3 in [23] proves that the system is exponentially stable if the chosen average dwell time is sufficiently large and the total activation time ratio between Schur stable and unstable subsystems is not smaller than a specified constant. In this study, the dwell time is the interval between any two consecutive switching times.

The switching functions of system 17 are functions of the system's states, and thus, it is difficult to calculate the dwell time and the total activation time of each subsystem. However, some qualitative results may be obtained by evaluating the switching functions. It can be seen from

Eqs. 18, 19, and 20 that the switching functions are affected by the corrected ratio of inventory error α and the desired inventory levels I_r^0 and I_d^0. According to [25], the parameters that make subsystem A_1 stable will also make the inventory control system of the distributor stable, and thus, the inventory level of the distributor will mainly be decided by the desired inventory level I_d^0. A large I_d^0will lead to the condition $\sigma_1(t)>0$, and the system will not be switched to unstable subsystem A_3. Given that α is constant, a small retailer's desired inventory level I_r^0 results in a large $\sigma_2(t)$, so that the total activation time of a stable subsystem will be large and the system will be apt to be stable.

In the next section, we will show by simulation that with α getting large, the system will evolve from stable to periodic behavior given that I_r^0 is constant.

6 Simulation study

In this section, simulation study is carried out to prove the analytical results of Section 4.2 and to show the relationship between dynamics and cost in the two-stage production–distribution system.

We consider a supply chain system consisting of one retailer, one distributor, and an external supplier. The supplier and the retailer review their inventory level with a period of 1 week. At the end of the week, the retailer and the distributor each issue an order to its supplier and, in the next week, the goods will arrive. Both the retailer and the distributor have an initial inventory level of 12 units, and both of them also have an order of four units to be issued. In the first 4 weeks, the demand of the retailer is four units per week. In week 5, the demand increases to eight units per week and then stays constant at eight units per week for the rest of the simulation. For clarity, in Figs. 5, 6, and 7, 'Ir' denotes the inventory of the retailer, and 'Id' denotes the inventory of the distributor.

Figure 5 shows the dynamics of the supply chain with different inventory adjustment rates. According to Section 4.2, for $\alpha=0.4$, both subsystems 14 and 15 are stable, and the switched system is stable as shown in Fig. 5a. Figure 5d shows the unstable case of the system where $\alpha=2.1$. Figure 5b and d indicates that the system becomes unstable if α is increased to a certain value. By plotting the bifurcation diagram of inventory level of the retailer (as shown in Fig. 6), the diversification of the supply chain states is more clearly shown. Comparing Fig. 6a with b, it is indicated that increasing the distributor's desired inventory level can stabilize the supply chain. Hence, it is concluded that the switching mechanism within the two-stage supply chain is the main cause of such behavior.

Figure 7 shows the total cost of the supply chain over 500 weeks, where inventory cost is taken to be \$0.50 per unit per week, and out-of-stock cost is taken to be \$2.00 per unit per week. The relation between cost and stable/unstable behavior of the system can be discovered in Fig. 7. In particular, the chaotic behavior gives rise to higher cost, and the stable behavior induce much lower cost.

Insights can be obtained by studying Figs. 6 and 7 together. On one hand, either a too small or a too large inventory adjustment rate α will induce high cost. For a too small α, the system react more slack to the outside demand change, although the system is stable, and a long time out-of-stock results in high cost too. On the contrary, for a too large α, the order policy reacts faster to inventory imbalances so as to make the inventory level run up and down, and both of the inventory cost and out-of-stock cost will be higher. On the other hand, Fig. 7 also indicates that the desired inventory level I_d^0 should be set cautiously. A higher I_d^0 can help the members to suppress the vibration of the supply chain state and reduce the cost to some extent. But, the cost will arise too if it is too high, or the system runs into chaos.

7 Conclusions

This paper reports a study analyzing the stability and dynamic behavior of a two-stage supply chain. It is found that the shipment policy and the limited inventory level of the distributor lead to an autonomously switched supply chain system. We develop the stability criterion for the subsystems of the supply chain using linear discrete system control theory. Since the system might be of high dimension and the switching functions are functions of states, it is difficult to analyze its stability, and no theory is available to handle such systems. From a practical perspective, we analyze the stability of such systems using switched system theory and discrete control system theory. Theoretic analysis and simulation study indicate that too aggressive inventory management (i.e., a large α) can induce the system to chaos and that either too aggressive inventory management or too slack inventory control (i.e., a small α) can result in high cost. Setting a high desired inventory level can restrain the oscillations of inventory and order, but a too large stock brings high cost naturally.

The dynamics of the supply chain are analyzed based on the switched system theory in this study with a number of assumptions in the model. It is assumed in our study that the return of goods is allowed. In other words, orders can be negative, which might lead to a negative on-hand inventory (beside the backlogged orders). Another assumption is that there is no transport delay. These assumptions are helpful for analyzing the system but are a little unpractical. Our further research will focus on models without these assumptions.

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