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Measuring the performances of decision-making units using geometric average efficiency

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The performances of decision-making units (DMUs) can be measured from two different points of view: optimistic or pessimistic, which leads to two different efficiencies for each DMU: the best relative efficiency and the worst relative efficiency. The conventional data envelopment analysis (DEA) considers only the best relative efficiency. It is argued that the two different efficiencies should be considered together and any approach considers only one of them is biased. This paper proposes to integrate the two different efficiencies into a geometric average efficiency, which measures the overall performance of each DMU. It is found that the geometric average efficiency has better discriminating power than either of the two efficiencies. This is illustrated by two numerical examples.
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Introduction

It has been known that the performances of decision-making units (DMUs) can be measured from different points of view. Data envelopment analysis (DEA), developed by Charnes *et al* (1978), measures the performances of DMUs from the optimistic point of view. The corresponding efficiencies are referred to as the best relative efficiencies or optimistic efficiencies, which are restricted to be less than or equal to one. If a DMU is evaluated to have the best relative efficiency of one, then it is said to be DEA efficient or optimistic efficient; otherwise, it is said to be optimistic non-efficient. Optimistic efficient DMUs are usually thought to perform better than optimistic non-efficient DMUs.

On the other hand, the performances of DMUs can also be measured from the pessimistic point of view. The efficiencies measured from the pessimistic viewpoint may be referred to as the worst relative efficiencies or pessimistic efficiencies, which are measured within the range of greater than or equal to one. Contrary to the best relative efficiencies that determine an efficiency frontier, the worst relative efficiencies of DMUs define an inefficiency frontier. If a DMU is evaluated to have the worst relative efficiency of one, then it is said to be pessimistic inefficient; otherwise, it is said to be pessimistic non-inefficient. Pessimistic inefficient DMUs are usually thought to perform worse than pessimistic non-inefficient DMUs.

The optimistic and pessimistic efficiencies measure the two extreme performances of each DMU. Any assessment approach considering only one of them is biased. The overall performance of each DMU should consider both of them at the same time. Doyle *et al* (1995) and Entani *et al* (2002) are the few persons, to the best of our knowledge, to consider and measure the efficiencies of DMUs from both the optimistic and the pessimistic points of view. Their models have a similar structure and in order to construct an interval of efficiency for each DMU, their models can only measure the pessimistic efficiency of each DMU under a single constraint condition such as the maximum optimistic efficiency being kept as one. This makes their models have a significant drawback, that is only one input and one output are utilized in calculating the pessimistic efficiency of each DMU, no matter how many inputs and outputs each DMU has. In addition, their models cannot identify DEA-inefficient DMUs very sufficiently. Wang and Luo (2006) measure the optimistic and the pessimistic efficiencies of DMUs by introducing two virtual DMUs: ideal DMU (IDMU) and anti-ideal DMU (ADMU), and integrate the two efficiencies into a relative closeness (RC) index, which servers as the basis for ranking DMUs. But in most cases, their models use fixed weights for all DMUs.

In this paper, we propose to integrate the two efficiencies, optimistic and pessimistic, into a geometric average efficiency, where the optimistic and pessimistic efficiencies of each DMU are measured using variable weights. The geometric average efficiency can be seen as an overall performance measure of each DMU. In comparison with the interval efficiencies proposed by Doyle *et al* (1995) and Entani *et al* (2002), the geometric average efficiency is much easier to compute and easier for ranking. More importantly, the pessimistic efficiency of each DMU is measured relative

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to all the other DMUs, and is computed using all the input and output information rather than only one input and one output. Differing from the RC index which is only for ranking but is not an efficiency measure, the geometric average efficiency is both an efficiency measure and a ranking index. Compared with the cross-efficiencies suggested by Sexton *et al* (1986), the geometric average efficiency is the geometric mean of the two efficiencies: optimistic and pessimistic, while the cross-efficiencies are those efficiencies calculated using the weights most favourable to each DMU, whose average (arithmetic or geometric) is called the average cross-efficiency. Obviously, the geometric average efficiency measure requires less computational effort than cross-efficiency or average cross-efficiency. Another difference between the geometric average efficiency and cross-efficiencies is that cross efficiencies can only involve either optimistic or pessimistic efficiency of each DMU, but cannot include both of them at the same time because the weights most favourable to one DMU do not mean they are the least favourable to the others or *vice versa*.

The rest of this paper is organized as follows. In the next section, we propose the geometric average efficiency measure and provide a theorem to show its validity. We then compare the geometric average efficiency measure with the Entani *et al*'s interval efficiency measure. The geometric average efficiency measure is examined with two numerical examples to illustrate its potential applications in performance measurement. Conclusions are offered in the last section. The proof of the theorem is provided in the Appendix.

The geometric average efficiency measure

Optimistic efficiency—the best relative efficiency

Assume that there are n DMUs to be evaluated, each DMU with m inputs and s outputs. We denote by x_{ij} $(i=1,\ldots,m)$ and y_{rj} $(r=1,\ldots,s)$ the values of inputs and outputs of DMU_j $(j=1,\ldots,n)$, which are known and positive. The efficiency of DMU_j is defined as

$$\theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, \tag{1}$$

where u_r and v_i are output and input weights assigned to the rth output and the ith input, respectively. To determine the efficiency of DMU_j relative to the other DMUs, Charnes *et al* (1978) developed the following well-known CCR model, which measures the best relative efficiencies of DMUs:

$$\begin{aligned} \text{Maximize} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\ \text{Subject to} \quad & \theta_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1 \quad j=1,\ldots,n \\ & u_r, v_i \geqslant \varepsilon \quad r=1,\ldots,s; \quad i=1,\ldots,m \end{aligned} \tag{2}$$

where the subscript zero represents the DMU under evaluation, u_r and v_i are decision variables and ε is the non-Archimedean infinitesimal. Through Charnes and Cooper's (1963) transformation the above fractional programming can be transformed into the following equivalent linear programming (LP) model:

$$\begin{aligned} \text{Maximize} \quad & \theta_0 = \sum_{r=1}^{s} u_r y_{r0} \\ \text{subject to} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0 \quad j=1,\ldots,n \\ & \sum_{i=1}^{m} v_i x_{i0} = 1 \quad u_r, v_i \geqslant \varepsilon \quad r=1,\ldots,s; \\ & i=1,\ldots,m \end{aligned} \tag{3}$$

If there exists a set of positive weights that makes $\theta_0^* = 1$, then DMU_0 is referred to as DEA efficient (optimistic efficient); otherwise, we refer to it as optimistic non-efficient. For n different DMUs, there are a total of n LP models to be solved. Accordingly, there are n different sets of weights available, which are the basis for calculating the cross-efficiency matrix (Sexton *et al*, 1986; Doyle and Green, 1994, 1995). All the optimistic efficient units determine an efficiency frontier.

Pessimistic efficiency—the worst relative efficiency

Efficiency is a relative measure and can be measured within different ranges. The CCR model measures the optimistic efficiency of each DMU by maximization within the range of less than or equal to one. If the efficiency of a DMU is measured by minimization within the range of greater than or equal to one, then we have so-called pessimistic efficiency or called the worst relative efficiency (Parkan and Wang, 2000; Paradi *et al*, 2004). The pessimistic efficiency of DMU_0 can be measured by the following pessimistic DEA model (Parkan and Wang, 2000):

$$\begin{aligned} \text{Minimize} \quad & \psi_0 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\ \text{subject to} \quad & \psi_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geqslant 1 \quad j=1,\ldots,n \\ & u_r, v_i \geqslant \varepsilon \quad r=1,\ldots,s; \quad i=1,\ldots,m \end{aligned} \tag{4}$$

which can be further converted into the following equivalent LP model:

$$\begin{aligned} \text{Minimize} \quad & \psi_0 = \sum_{r=1}^{s} u_r y_{r0} \\ \text{subject to} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geqslant 0 \quad j=1,\ldots,n \\ & \sum_{i=1}^{m} v_i x_{i0} = 1 \quad u_r, v_i \geqslant \varepsilon \quad r=1,\ldots,s; \\ & i=1,\ldots,m \end{aligned} \tag{5}$$

If there exists a set of positive weights making $\psi_0^* = 1$, then DMU_0 is referred to as pessimistic inefficient; otherwise, we refer it to as pessimistic non-inefficient. It is obvious that optimistic non-efficient does not necessarily mean pessimistic inefficient. As such, pessimistic non-inefficient does not necessarily mean optimistic efficient. All the pessimistic inefficient units determine an inefficiency frontier.

Contrary to the CCR models (2) and (3), which may be referred to as optimistic DEA models, the pessimistic DEA models (4) and (5) seek the least favourable weights for each DMU.

Geometric average efficiency—the integration of optimistic and pessimistic efficiencies

Theoretically, optimistic and pessimistic efficiencies should form an interval. To do so, the pessimistic efficiency needs to be adjusted. Let α be an adjustment coefficient ($0<\alpha<1$). Then, adjusted pessimistic efficiency can be written as $\tilde{\psi}_j^* = \alpha\psi_j^*$, which should satisfy $\tilde{\psi}_j^* = \alpha\psi_j^* < \theta_j^*$ for $j = 1, \ldots, n$. That is, $\alpha < \min_{j\in\{1,\ldots,n\}} \left\{\theta_j^*/\psi_j^*\right\}$. Accordingly, the efficiency interval for DMU_j can be expressed as $[\alpha\psi_j^*, \theta_j^*]$ ($j = 1, \ldots, n$). It has been known that the comparison of interval numbers often comes down to the comparison of their midpoints. The midpoints of the above efficiency intervals are given by $(\alpha\psi_j^* + \theta_j^*)/2$, $j = 1, \ldots, n$. Owing to the presence of the parameter α, the rankings among the n DMUs will be affected by the magnitude of α. To eliminate the impact of the parameter α on the rankings of DMUs and avoid the difficulty in determining the parameter α, we calculate the midpoints of efficiency intervals using geometric average instead of arithmetic average. The geometric averages of the above efficiency intervals are given by $\sqrt{\alpha\psi_j^*\theta_j^*} = \sqrt{\alpha}\sqrt{\psi_j^*\theta_j^*}$, $j = 1, \ldots, n$. Let

$$\phi_j^* = \sqrt{\psi_j^*\theta_j^*}, \quad j = 1, \ldots, n \tag{6}$$

We refer to ϕ_j^* as the geometric average efficiency of DMU_j ($j = 1, \ldots, n$). It is evident that when the midpoints of efficiency intervals are computed using their geometric averages, the rankings among the n DMUs are only affected by their geometric average efficiencies, but not affected by the magnitude of α. This good property enables us not to worry about how to determine the parameter α. We can therefore leave it alone and compare directly the geometric average efficiencies of the n DMUs to determine their performances. The geometric average efficiency measures the overall performance of each DMU and is therefore more comprehensive than either of the optimistic and pessimistic efficiencies.

A theorem about the geometric average efficiency

As is known, DEA always finds the most favourable weights for each DMU. For n DMUs, there exist n sets of favourable weights, which are denoted by u_{rk}^* and v_{ik}^*, $k = 1, \ldots, n$, where u_{rk}^* and v_{ik}^* are the most favourable weights for DMU_k. Let

$$E_{jk} = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rj}}{\sum_{i=1}^{m} v_{ik}^* x_{ij}}, \quad j, k = 1, \ldots, n \tag{7}$$

Then, E_{jk} ($k = 1, \ldots, n$) are referred to as the cross-efficiencies of DMU_j, which are calculated using the most favourable weights of DMU_k. In particular, E_{jj} is the optimistic efficiency of DMU_j, that is, $E_{jj} = \theta_j^*$.

The concept of cross-efficiency was first proposed by Sexton *et al* (1986) and further investigated by Doyle and Green (1994, 1995). The above evaluation is referred to as cross-evaluation or peer evaluation (Doyle and Green, 1995) and the matrix $E = (E_{jk})_{n\times n}$ is called the cross-efficiency matrix. For each DMU, say, DMU_j, there exists n cross-efficiencies: $E_{j1}, \ldots, E_{jn}$, whose arithmetic or geometric average is referred to as the average cross-efficiency of DMU_j. That is

$$\bar{E}_j = \frac{1}{n}\sum_{k=1}^{n} E_{jk}, \quad j = 1, \ldots, n \tag{8}$$

or

$$\bar{E}_j = \left(\prod_{k=1}^{n} E_{jk}\right)^{1/n}, \quad j = 1, \ldots, n \tag{9}$$

In DEA literature, cross-efficiencies are always defined using the most favourable weights of each DMU. But in fact, based on the pessimistic DEA models (4) or (5), cross-efficiencies can also be defined using the least favourable weights as

$$H_{jk} = \frac{\sum_{r=1}^{s} \tilde{u}_{rk}^* y_{rj}}{\sum_{i=1}^{m} \tilde{v}_{ik}^* x_{ij}}, \quad j, k = 1, \ldots, n \tag{10}$$

where $\tilde{u}_{rk}^*$ and $\tilde{v}_{ik}^*$ are the least favourable weights of DMU_k. Particularly, H_{jj} is the pessimistic efficiency of DMU_j, that is, $H_{jj} = \psi_j^*$. Accordingly, the average cross-efficiency of DMU_j can be defined as

$$\bar{H}_j = \frac{1}{n}\sum_{k=1}^{n} H_{jk}, \quad j = 1, \ldots, n \tag{11}$$

or

$$\bar{H}_j = \left(\prod_{k=1}^{n} H_{jk}\right)^{1/n}, \quad j = 1, \ldots, n \tag{12}$$

Note that $\bar{E}_j \leqslant 1$ but $\bar{H}_j \geqslant 1$ due to the fact that $E_{jk} \leqslant 1$ and $H_{jk} \geqslant 1$ for any $k \in \{1, \ldots, n\}$. Clearly, no matter what kinds of weights, favourable or unfavourable, are utilized to define the cross-efficiencies, they can only involve either optimistic or pessimistic efficiency of each DMU, but cannot include both of them at the same time.

Based on the above definitions of cross-efficiency, we have the following theorem for the geometric average efficiency.

Theorem 1 *Let θ_j^* be the optimistic efficiency of DMU_j determined by Models* (2) *or* (3), *ψ_j^* be its pessimistic efficiency determined by Models* (4) *or* (5), *E_{jk} and H_{jk} be its cross-efficiencies determined by Equations* (7) *and* (10), *respectively.*

If $0 < \alpha \leqslant \min_{j\in\{1,\dots,n\}}\{\theta_j^*/H_j^{\max}, E_j^{\min}/\psi_j^*\}$ *where* $E_j^{\min} = \min_{k\in\{1,\dots,n\}}\{E_{jk}\}$ *and* $H_j^{\max} = \max_{k\in\{1,\dots,n\}}\{H_{jk}\}$, *then* $[\alpha\psi_j^*, \theta_j^*]$ *is the optimal efficiency interval of* DMU_j *determined by the following upper and lower bounds fractional programming models*

$$\begin{aligned} \text{Max/Min} \quad & \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_r x_{i0} \\ \text{subject to} \quad & \alpha \leqslant \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \leqslant 1 \quad j = 1, \dots, n \\ & u_r, v_i \geqslant \varepsilon \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{13}$$

The proof of Theorem 1 is provided in the Appendix.

Seemingly, θ_j^* and ψ_j^* are two efficiencies measured using different constraints and could not be integrated together. However, the above Theorem 1 has clearly shown that θ_j^* and $\alpha\psi_j^*$ are in fact measured using the same interval constraints. Therefore, they can be combined in the way of geometric aggregation. The geometric average efficiency defined by (6) just eliminates the impact of α on efficiency measure which is the same for all DMUs.

Comparison with Entani *et al*'s interval efficiency measure

In this section, we provide some comparisons of the geometric average efficiency measure with Entani *et al*'s interval efficiency measure. To generate an interval of efficiency for each DMU, Entani *et al* (2002) construct the following upper and lower bounds mathematical programming models for DMU_0

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\max_j \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}} \\ \text{subject to} \quad & u_r, v_i \geqslant 0 \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{14}$$

The upper bound model can be transformed into the model below, which is equivalent to the standard CCR Model (2) and can be solved through Model (3)

$$\begin{aligned} \text{Maximize} \quad & \theta_0^U = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0} \\ \text{subject to} \quad & \max_j \left\{ \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \right\} = 1 \\ & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{15}$$

The lower bound model can be converted into

$$\begin{aligned} \text{Minimize} \quad & \theta_0^L = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0} \\ \text{subject to} \quad & \max_j \left\{ \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} \right\} = 1 \\ & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{16}$$

which cannot be replaced with an equivalent LP model. By assuming that $\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij} = 1$ for each DEA efficient unit (optimistic efficient DMU), Entani *et al* divided Model (16) into the following n_1 sub-optimization problems $(j = J_1, \dots, J_{n_1})$, where n_1 is the number of optimistic efficient DMUs and $J_1, \dots, J_{n_1}$ are the DMUs which are optimistic efficient

$$\begin{aligned} \text{Minimize} \quad & \theta_{0j}^L = \sum_{r=1}^{s} u_r y_{r0} \Big/ \sum_{i=1}^{m} v_i x_{i0} \\ \text{subject to} \quad & \sum_{r=1}^{s} u_r y_{rj} \Big/ \sum_{i=1}^{m} v_i x_{ij} = 1 \\ & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{17}$$

which can be further simplified as the n_1 LP models below

$$\begin{aligned} \text{Minimize} \quad & \theta_{0j}^L = \sum_{r=1}^{s} u_r y_{r0} \\ \text{subject to} \quad & \sum_{i=1}^{m} v_i x_{i0} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} = 0 \\ & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{18}$$

Let θ_{0j}^{L*} be the optimum objective function value of the above LP model (18). Obviously, when $j = 0$, $\theta_{0j}^{L*} = 1$. So, the lower-bound efficiency of DMU_0 was finally determined by

$$\theta_0^{L*} = 1 \wedge \min_{j \neq 0} \{\theta_{0j}^{L*}\}, \tag{19}$$

where $a \wedge b = \min\{a, b\}$. Accordingly, the efficiency interval of DMU_0 is denoted as $[\theta_0^{L*}, \theta_0^{U*}]$, where θ_0^{U*} is the optimum objective function value of the upper bound model (15).

Before Entani *et al*, Doyle *et al* (1995) developed the following three pairs of upper and lower bounds evaluation models

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\max_{j \neq 0} \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}} \\ \text{subject to} \quad & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{20}$$

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\frac{1}{n-1} \sum_{j \neq 0}^{n} \left(\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\right)} \\ \text{subject to} \quad & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{21}$$

$$\begin{aligned} \text{Max/Min} \quad & \theta_0 = \frac{\sum_{r=1}^{s} u_r y_{r0} / \sum_{i=1}^{m} v_i x_{i0}}{\min_{j \neq 0} \{\sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{m} v_i x_{ij}\}} \\ \text{subject to} \quad & u_r, v_i \geqslant 0, \quad r = 1, \dots, s; \quad i = 1, \dots, m \end{aligned} \tag{22}$$

They all have a similar structure to Model (14) and can be solved in a similar way.

As can be seen from Model (17), the model has only one fractional constraint. So, only two decision variables can be nonzero no matter how many inputs and outputs are involved in the problem under discussion. One is for some input weight and the other is for some output weight. That is the reason why we say Entani *et al*'s approach measures the pessimistic efficiency of each DMU using only one input and one output information. This is obviously a drawback of their approach.

In comparison with model (17), Models (4) and (13) contain *n* fractional constraints and can therefore make the most of all inputs and outputs information. In addition, Models (4) and (13) can exactly identify DEA inefficient units and inefficiency frontier, while Model (11) cannot. This will be illustrated in the next section by a numerical example.

Numerical examples

In this section, we examine two numerical examples using the geometric average efficiency measure to illustrate its applications, advantages and good discriminating power. All models are implemented in MS-Excel worksheets and are solved using the MS-Excel Solver. The non-Archimedean infinitesimal was set as $\varepsilon = 10^{-10}$.

Example 1 Consider a performance measurement problem with 10 DMUs, each DMU with one input and two outputs. The data set is taken from Entani *et al* (2002) and is shown in Table 1, where all inputs are normalized to one for simplicity.

For this example, Table 2 shows the interval efficiency obtained by Entani *et al*, optimistic efficiency, pessimistic efficiency, and geometric average efficiency of each DMU as well as the rankings among the 10 DMUs by their geometric average efficiencies.

It is clear from Table 2 that three DMUs, that is, $\mathrm{DMU_A}$, $\mathrm{DMU_E}$ and $\mathrm{DMU_J}$, are evaluated to be DEA efficient by the optimistic efficiency model (3). These three DEA efficient units together determine an efficiency frontier AEJ, which is shown in Figure 1. The performances of the three DEA efficient units are traditionally thought to be better than those of the other seven units which are evaluated to be optimistic non-efficient. The ranking of the 10 DMUs by their optimistic efficiencies are $\mathrm{DMU_A} \sim \mathrm{DMU_E} \sim \mathrm{DMU_J} \succ \mathrm{DMU_G} \sim \mathrm{DMU_I} \succ \mathrm{DMU_H} \succ \mathrm{DMU_C} \succ \mathrm{DMU_F} \succ \mathrm{DMU_D} \succ \mathrm{DMU_B}$, where the symbol '$\sim$' represents indifference and '$\succ$' means 'superior to'.

From the angle of pessimistic efficiency, four DMUs: $\mathrm{DMU_A}$, $\mathrm{DMU_B}$, $\mathrm{DMU_F}$ and $\mathrm{DMU_J}$ are evaluated to be pessimistic inefficient. They together define an inefficiency frontier ABFJ, which is also shown in Figure 1. The performances of these four pessimistic inefficient units are thought to be poorer than those of the other six units which are evaluated to be pessimistic non-inefficient. The performances of the 10 DMUs are ranked as $\mathrm{DMU_G} \succ \mathrm{DMU_E} \succ \mathrm{DMU_C} \succ \mathrm{DMU_I} \succ \mathrm{DMU_D} \succ \mathrm{DMU_H} \succ \mathrm{DMU_A} \sim \mathrm{DMU_B} \sim \mathrm{DMU_F} \sim \mathrm{DMU_J}$ in terms of their pessimistic efficiencies.

The above assessments are based on different points of view and may therefore be different. For example, for $\mathrm{DMU_A}$ and $\mathrm{DMU_J}$, when they are evaluated from the optimistic point of view, they are evaluated to be optimistic efficient, which means they perform better than any other DMUs. However, when they are evaluated from the pessimistic point of view, they are both evaluated to be pessimistic inefficient, which means they perform worse than any other DMUs. Such two assessment results are obviously conflict with each other. Any assessment conclusion considering only one point of view is apparently one-sided, unrealistic, and unconvincing.

Table 1 Data for 10 DMUs with one input and two outputs

DMU	*Input* (X_1)	*Output 1* (Y_1)	*Output 2* (Y_2)
A	1	1	8
B	1	2	3
C	1	2	6
D	1	3	3
E	1	3	7
F	1	4	2
G	1	4	5
H	1	5	2
I	1	6	2
J	1	7	1

Table 2 Efficiency ratings for the ten DMUs

DMU	*Interval efficiency by Entani* et al	*Optimistic efficiency*	*Pessimistic efficiency*	*Geometric average efficiency*	*Ranking*
A	[0.1429, 1.0000]	1	1	1.0000	5
B	[0.2857, 0.5217]	0.5217	1	0.7223	10
C	[0.2857, 0.8235]	0.8235	1.2308	1.0068	4
D	[0.3750, 0.6522]	0.6522	1.1250	0.8566	8
E	[0.4286, 1.0000]	1	1.6923	1.3009	1
F	[0.2500, 0.6957]	0.6957	1	0.8341	9
G	[0.5714, 0.9565]	0.9565	1.7500	1.2938	2
H	[0.2500, 0.8261]	0.8261	1.1000	0.9533	7
I	[0.2500, 0.9565]	0.9565	1.2000	1.0714	3
J	[0.1250, 1.0000]	1	1	1.0000	5

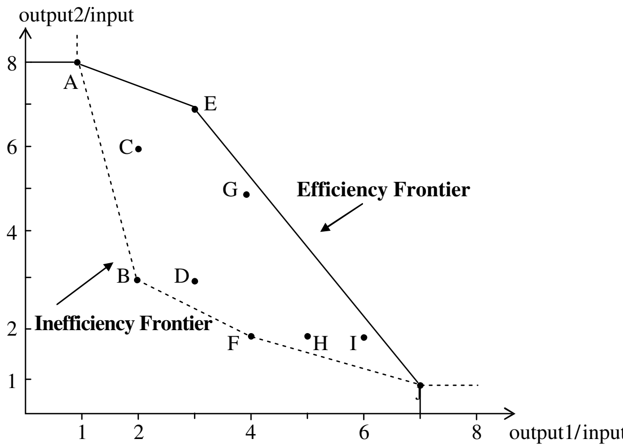

Figure 1 Efficiency and inefficiency frontiers of 10 DMUs.

The geometric average efficiency considers not only the optimistic efficiency, but also the pessimistic efficiency and is therefore comprehensive and can be used as an overall efficiency measure of each DMU. According to their geometric average efficiencies, the 10 DMUs are ranked as $DMU_E \succ DMU_G \succ DMU_I \succ DMU_C \succ DMU_J \sim DMU_A \succ DMU_H \succ DMU_D \succ DMU_F \succ DMU_B$. Although DMU_A and DMU_J are evaluated to be optimistic efficient, they are also evaluated to be pessimistic inefficient. Therefore, their overall performances are not better than those of DMU_G, DMU_I and DMU_C, which are neither optimistic efficient nor pessimistic inefficient, but have better overall performances.

In contrast, we present in the second column of Table 2 the interval efficiencies of the 10 DMUs obtained by Entani *et al*'s models (14). Owing to the fact that three DMUs, DMU_A, DMU_E and DMU_J, are identified to be optimistic efficient, in order to determine the lower bound efficiencies of the 10 DMUs, each DMU needs to solve three LP models. Take DMU_A for example. The following three LP models need to be solved to determine its lower-bound efficiency

$$(\text{LP1}):\quad \theta_{AA}^{L*} = \text{Minimize}\quad u_1 + 8u_2 \qquad \text{subject to}\quad \begin{cases} v_1 = 1 \\ u_1 + 8u_2 - v_1 = 0 \\ u_1, u_2, v_1 \geqslant 0 \end{cases}$$

$$(\text{LP2}):\quad \theta_{AE}^{L*} = \text{Minimize}\quad u_1 + 8u_2 \qquad \text{subject to}\quad \begin{cases} v_1 = 1 \\ 3u_1 + 7u_2 - v_1 = 0 \\ u_1, u_2, v_1 \geqslant 0 \end{cases}$$

$$(\text{LP3}):\quad \theta_{AJ}^{L*} = \text{Minimize}\quad u_1 + 8u_2 \qquad \text{subject to}\quad \begin{cases} v_1 = 1 \\ 7u_1 + u_2 - v_1 = 0 \\ u_1, u_2, v_1 \geqslant 0 \end{cases}$$

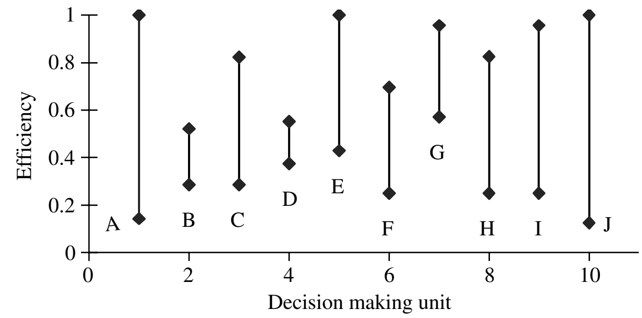

Figure 2 Interval efficiencies of the 10 DMUs by Entani *et al*'s approach.

Table 3 Input and output weights obtained by Entani *et al*'s lower-bound model and the pessimistic efficiency model

	Entani et al*'s model (16)*			*Pessimistic DEA model (5)*		
DMU	v_1	u_1	u_2	v_1	u_1	u_2
A	1	0.1429	0	1	0.3846	0.0769
B	1	0.1429	0	1	0.3846	0.0769
C	1	0.1429	0	1	0.3846	0.0769
D	1	0	0.1250	1	0.1250	0.2500
E	1	0.1429	0	1	0.3846	0.0769
F	1	0	0.1250	1	0.1000	0.3000
G	1	0.1429	0	1	0.1250	0.2500
H	1	0	0.1250	1	0.1000	0.3000
I	1	0	0.1250	1	0.1000	0.3000
J	1	0	0.1250	1	0.1000	0.3000

Each of the three LP models keeps only one of the three optimistic efficient units unchanged. The solutions to the above three LP models are as follows:

$$\theta_{AA}^{L*} = 1,\quad u_1^* = 0,\quad u_2^* = 1/8 \quad \text{and} \quad v_1^* = 1$$

$$\theta_{AE}^{L*} = 1/3,\quad u_1^* = 1/3,\quad u_2^* = 0 \quad \text{and} \quad v_1^* = 1$$

$$\theta_{AJ}^{L*} = 1/7,\quad u_1^* = 1/7,\quad u_2^* = 0 \quad \text{and} \quad v_1^* = 1$$

So, the final lower-bound efficiency of DMU_A is given by

$$\theta_A^{L*} = \min\{1, 1/3, 1/7\} = 0.1429$$

The lower-bound efficiencies of the other nine DMUs are computed in the same way. Figure 2 shows the interval efficiencies of the 10 DMUs, from which it is found that DMU_J has the smallest lower-bound efficiency, followed by DMU_A. So, DMU_J and DMU_A can be seen as pessimistic inefficient units. However, neither of DMU_B and DMU_F can be identified to be pessimistic inefficient. In other words, Entani *et al.*'s lower-bound efficiency model only identifies two of the four pessimistic inefficient units. As a result, their approach fails to determine the inefficiency frontier ABFJ shown in Figure 1.

It is also very clear from the above three sets of input and output weights that only one output (either output 1 or

Table 4 Data for 31 provinces, municipalities and autonomous regions of China in year 2000

	Inputs			*Output*
DMU	*Original value of fixed assets*	*Current assets*	*Number of staff and workers at year end*	*Gross industrial output value*
Beijing	2402.79	2005.63	113.13	2565.38
Tianjing	2488.60	1787.41	120.19	2606.38
Hebei	3532.84	2000.19	269.75	3426.05
Liaoning	5372.79	3155.9	295.18	4249.46
Shanghai	5373.06	4370.38	204.94	6204.52
Jiangsu	6181.57	5499.34	518.19	10452.87
Zhejiang	3753.68	3377.81	323.22	6603.65
Fujian	2032.18	1401.27	155.55	2616.12
Shandong	6297.37	4076.79	522.37	8311.53
Guangdong	8005.77	6891.49	572.79	12480.93
Guangxi	1296.41	684.22	91.25	1003.24
Hainan	264.43	156.70	12.00	202.87
Shanxi	2170.68	1221.05	183.56	1216.86
Neimenggu	1385.74	603.7	85.34	748.97
Jilin	1833.04	1291.93	134.85	1679.91
Heilongjiang	3233.51	1567.07	195.17	2460.88
Anhui	1880.95	1212.57	162.61	1661.44
Jiangxi	1154.45	730.33	108.85	932.21
Henan	3447.01	2216.51	345.20	3494.96
Hubei	2989.31	1941.97	230.36	3064.43
Hunan	1947.23	1107.19	166.71	1627.94
Chongqing	1151.58	865.97	90.79	962.32
Sichuang	2917.04	1845.51	208.00	2076.96
Guizhou	913.15	676.07	68.34	631.64
Yunnan	1409.92	812.81	77.07	1063.36
Tibet	59.58	25.62	2.92	16.43
Shanxi	1730.35	1084.49	124.98	1184.58
Gansu	1165.68	713.65	91.25	840.58
Qinghai	505.81	223.06	15.87	196.08
Ningxia	362.21	212.73	22.41	239.11
Sinkiang	1387.56	578.79	46.52	852.01

Data source: China Industrial Economy Statistical Yearbook 2001; China Statistical Press, Beijing, 2001.

output 2) is involved in the computation of lower-bound efficiency. The reason for this has been analysed in the previous section. Table 3 shows all the input and output weights of the 10 DMUs for computing their lower-bound efficiencies and pessimistic efficiencies. It is obvious that the pessimistic efficiency DEA model makes use of both outputs to compute pessimistic efficiency, while Entani *et al*'s lower-bound model does not.

Example 2 Consider a real efficiency measurement problem of the industrial economy of China, where 31 provinces, municipalities and autonomous regions (ie DMUs) of China are evaluated in terms of their efficiencies of industrial economy in the past two decades. For illustrative purpose, only the data of inputs and outputs in Year 2000 are considered here and are presented in Table 4, where *Original value of fixed assets*, *Current assets* and *Gross industrial output value* are measured in 100 million RMB (Chinese monetary unit) and calculated at current prices, *Number of staff and workers at year end* is expressed in units of 10 thousand persons; DMU_{13} and DMU_{27} are two different Chinese Provinces with the same English name.

For this example, Table 5 shows the optimistic, pessimistic and geometric average efficiencies of the 31 DMUs in year 2000 and their rankings in terms of geometric average efficiency. As can be seen from Table 5, from the angle of optimistic efficiency, Shanghai Municipality, Zhejiang and Shandong Provinces are evaluated as optimistic efficient units and cannot be differentiated further. Jiangsu, Guangdong and Fujian Provinces are evaluated to have higher optimistic efficiencies than the other DMUs except for the three optimistic efficient units. Tibet Autonomous Region is rated to be the worst, followed by Guizhou Province.

From the pessimistic point of view, Zhejiang Province is rated to have the highest pessimistic efficiency, followed by Jiangsu, Fujian, Shandong and Guandong Provinces. Tibet is again rated to be the worst and is also the only pessimistic inefficient unit. So, no matter what point of view efficiency is measured from, Zhejiang is certainly the best unit and Tibet

Table 5 Geometric average efficiencies of the 31 provinces, municipalities and autonomous regions of China and their rankings

DMU	*Optimistic efficiency*	*Pessimistic efficiency*	*Geometric average efficiency*	*Ranking*
Beijing	0.8409	1.9945	1.2951	12
Tianjing	0.8876	2.2738	1.4206	7
Hebei	0.8402	2.2572	1.3771	8
Liaoning	0.6972	2.0997	1.2099	14
Shanghai	1	2.2138	1.4879	6
Jiangsu	0.9982	2.9639	1.7200	2
Zhejiang	1	3.0485	1.7460	1
Fujian	0.9360	2.9112	1.6507	5
Shandong	1	2.8278	1.6816	3
Guangdong	0.9967	2.8241	1.6777	4
Guangxi	0.7192	1.9540	1.1855	15
Hainan	0.7418	2.0188	1.2237	13
Shanxi	0.4888	1.1782	0.7589	30
Neimenggu	0.6085	1.5598	0.9742	25
Jilin	0.6576	2.0276	1.1547	18
Heilongjiang	0.7738	2.2409	1.3168	11
Anhui	0.6721	1.8159	1.1047	20
Jiangxi	0.6261	1.5221	0.9762	24
Henan	0.7766	1.7994	1.1821	16
Hubei	0.7834	2.3642	1.3609	10
Hunan	0.7212	1.7355	1.1188	19
Chongqing	0.5616	1.7328	0.9865	23
Sichuang	0.5630	1.7549	0.9940	22
Guizhou	0.4745	1.4569	0.8314	29
Yunnan	0.6725	2.0400	1.1713	17
Tibet	0.3203	1	0.5660	31
Shanxi	0.5447	1.6845	0.9579	27
Gansu	0.5779	1.6372	0.9727	26
Qinghai	0.5215	1.3707	0.8455	28
Ningxia	0.5677	1.7527	0.9975	21
Sinkiang	0.8238	2.2954	1.3751	9

is without doubt the worst unit. The overall performances of the other DMUs can only be judged in terms of their geometric average efficiencies.

It is clear from the geometric average efficiencies in Table 5 that Jiangsu Province has better overall performance than Shandong and is ranked in the second place although Jiangsu is not rated as an optimistic efficient unit. This is mainly because Jiangsu has better pessimistic efficiency than Shandong province. This is also true for Guangdong and Fujian Provinces, which are not evaluated as optimistic efficient units, but have better pessimistic efficiencies than Shanghai. So, Guangdong and Fujian are both ranked ahead of Shanghai, which is ranked only in the sixth place. The overall rankings of the other DMUs are shown in the last column of Table 5. All the 31 DMUs are distinguished by their geometric average efficiencies.

It is also found that those DMUs with the best overall performances such as Zhejiang, Jiangsu, Shandong, Guangdong, Fujian and Shanghai are all in the east costal areas of China, while those DMUs with the worst overall performances such as Tibet, Shanxi, Guizhou, Qinghai, Gansu, Neimenggu etc. are all in the western inland areas of China. So, it can be concluded that China's industrial economy develops better in the east costal areas than in the western inland areas. This has been supported by the evidence from the efficiency assessments in other years.

Conclusions

The performances of DMUs can be measured from different points of view. As a result, the conclusions may be quite confusing even conflicting. So, there is a clear need to integrate different performance measures to give an overall assessment of the performance of each DMU.

To meet such a requirement, we proposed in this paper a geometric average efficiency measure to gauge the overall performance of each DMU. The geometric average efficiency integrates both the optimistic and the pessimistic efficiencies of each DMU and is therefore more comprehensive than either of them. We also developed a theorem to show the validity of the geometric average efficiency measure and defined the cross-efficiency using the least favourable weights of each DMU. Two numerical examples were assessed using the geometric average efficiency measure. It is shown that the geometric average efficiency measure has better discriminating power than the traditional DEA efficiency and pessimistic efficiency measures and that both the optimistic efficient and pessimistic inefficient units as well as efficiency and inefficiency frontiers can all be precisely identified using the optimistic and pessimistic efficiency measures, which are the two inseparable parts of the geometric average efficiency measure. It can be expected that the geometric average efficiency measure will find more potential applications in performance assessment in the future.

Finally, we point out that the optimistic, pessimistic and geometric average efficiencies discussed in this paper are all based on the so-called CCR model, but they can also be extended to the so-called BCC model or additive DEA model. The interested reader may refer to Jahanshahloo and Afzalinejad (2006) and Parkan and Wang (2000) for discussions about the extension of pessimistic efficiency to BCC model and additive DEA model.

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Appendix

Proof of Theorem 1

- $\alpha \leqslant E_j^{\min}$ and $H_j^{\max} \leqslant \theta_j^*/\alpha$ for $j = 1, \ldots, n$.
  They can be directly derived from $0 < \alpha \leqslant \min_{j\in\{1,\ldots,n\}} \left\{\theta_j^*/H_j^{\max}, E_j^{\min}/\psi_j^*\right\}$.
- θ_0^* is the maximum of model (13).
  Let u_r^* and v_i^* be the optimal weights of DMU_0 obtained from model (2). Then,

$$E_j = \sum_{r=1}^{s} u_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} v_i^* x_{ij} \geqslant E_j^{\min} \geqslant \alpha, \quad j = 1, \ldots, n \tag{A.1}$$

$$E_j = \sum_{r=1}^{s} u_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} v_i^* x_{ij} \leqslant 1, \quad j = 1, \ldots, n \tag{A.2}$$

  So, u_r^* and v_i^* are also the optimal solution to the upper bound model of (13).
- $\alpha\psi_0^*$ is the minimum of model (13).
  Let $\tilde{u}_r^*$ and $\tilde{v}_i^*$ be the least favourable weights of DMU_0 obtained from model (4) and $\hat{u}_r^* = \alpha\tilde{u}_r^*$ for $r = 1, \ldots, s$. Then,

$$\psi_j = \sum_{r=1}^{s} \tilde{u}_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} \tilde{v}_i^* x_{ij} = \frac{1}{\alpha}\left(\sum_{r=1}^{s} \hat{u}_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} \tilde{v}_i^* x_{ij}\right) \geqslant 1, \quad j = 1, \ldots, n \tag{A.3}$$

$$\psi_j = \sum_{r=1}^{s} \tilde{u}_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} \tilde{v}_i^* x_{ij} = \frac{1}{\alpha}\left(\sum_{r=1}^{s} \hat{u}_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} \tilde{v}_i^* x_{ij}\right) \leqslant H_j^{\max} \leqslant \theta_j^*/\alpha, \quad j = 1, \ldots, n \tag{A.4}$$

  which are equivalent to

$$\alpha \leqslant \sum_{r=1}^{s} \hat{u}_r^* y_{rj} \Bigg/ \sum_{i=1}^{m} \tilde{v}_i^* x_{ij} \leqslant \theta_j^* \leqslant 1 \quad j = 1, \ldots, n \tag{A.5}$$

  So, $\hat{u}_r^*$ and $\tilde{v}_i^*$ are the feasible solution of model (13). Owing to the fact that $\psi_0^* = \sum_{r=1}^{s} \tilde{u}_r^* y_{r0} / \sum_{i=1}^{m} \tilde{v}_i^* x_{i0} = (1/\alpha)(\sum_{r=1}^{s} \hat{u}_r^* y_{r0} / \sum_{i=1}^{m} \tilde{v}_i^* x_{i0})$ is the minimum of model (4), it can be concluded that $\theta_0^{L*} = \sum_{r=1}^{s} \hat{u}_r^* y_{r0} / \sum_{i=1}^{m} \tilde{v}_i^* x_{i0}$ is the minimum of model (13), which can be expressed as $\theta_0^{L*} = \alpha\psi_0^*$.
- $[\alpha\psi_0^*,\ \theta_0^*]$ is the optimal efficiency interval of DMU_0 determined by model (13).

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