

# Risk evaluation in failure mode and effects analysis using fuzzy weighted geometric mean <sup>☆</sup>

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## Abstract

Failure mode and effects analysis (FMEA) has been extensively used for examining potential failures in products, processes, designs and services. An important issue of FMEA is the determination of risk priorities of the failure modes that have been identified. The traditional FMEA determines the risk priorities of failure modes using the so-called risk priority numbers (RPNs), which require the risk factors like the occurrence (O), severity (S) and detection (D) of each failure mode to be precisely evaluated. This may not be realistic in real applications. In this paper we treat the risk factors O, S and D as fuzzy variables and evaluate them using fuzzy linguistic terms and fuzzy ratings. As a result, fuzzy risk priority numbers (FRPNs) are proposed for prioritization of failure modes. The FRPNs are defined as fuzzy weighted geometric means of the fuzzy ratings for O, S and D, and can be computed using alpha-level sets and linear programming models. For ranking purpose, the FRPNs are defuzzified using centroid defuzzification method, in which a new centroid defuzzification formula based on alpha-level sets is derived. A numerical example is provided to illustrate the potential applications of the proposed fuzzy FMEA and the detailed computational process of the FRPNs.

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## 1. Introduction

Failure mode and effects analysis (FMEA) is a widely used engineering technique for defining, identifying and eliminating known and/or potential failures, problems, errors and so on from system, design, process, and/or service before they reach the customer (Stamatis, 1995). The so-called failure mode is defined as the manner in which a component, subsystem, system, process, etc. could potentially fail to meet the design intent. A failure mode in one

component can be the cause of a failure mode in another component. A failure cause is defined as a design weakness that may result in a failure. For each identified failure mode, their ultimate effects need to be determined, usually by a FMEA team. A failure effect is defined as the result of a failure mode on the function of the product/process as perceived by the customer.

A system, design, process, or service may usually have multiple failure modes or causes and effects. In this situation, each failure mode or cause needs to be assessed and prioritized in terms of their risks so that high risky (or most dangerous) failure modes can be corrected with top priority. The traditional FMEA determines the risk priorities of failure modes through the risk priority number (RPN), which is the product of the occurrence (O), severity (S) and detection (D) of a failure. That is

$$\text{RPN} = O \times S \times D, \quad (1)$$

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Table 1  
Crisp ratings for occurrence of a failure

Rating	Probability of occurrence	Failure probability
10	Very high: failure is almost inevitable	>1 in 2
9		1 in 3
8	High: repeated failures	1 in 8
7		1 in 20
6	Moderate: occasional failures	1 in 80
5		1 in 400
4		1 in 2000
3	Low: relatively few failures	1 in 15,000
2		1 in 150,000
1	Remote: failure is unlikely	<1 in 1,500,000

Table 3  
Crisp ratings for detection of a failure

Rating	Detection	Likelihood of detection by design control
10	Absolute uncertainty	Design control cannot detect potential cause/mechanism and subsequent failure mode
9	Very remote	Very remote chance the design control will detect potential cause/mechanism and subsequent failure mode
8	Remote	Remote chance the design control will detect potential cause/mechanism and subsequent failure mode
7	Very low	Very low chance the design control will detect potential cause/mechanism and subsequent failure mode
6	Low	Low chance the design control will detect potential cause/mechanism and subsequent failure mode
5	Moderate	Moderate chance the design control will detect potential cause/mechanism and subsequent failure mode
4	Moderately high	Moderately high chance the design control will detect potential cause/mechanism and subsequent failure mode
3	High	High chance the design control will detect potential cause/mechanism and subsequent failure mode
2	Very high	Very high chance the design control will detect potential cause/mechanism and subsequent failure mode
1	Almost certain	Design control will detect potential cause/mechanism and subsequent failure mode

where O and S are the frequency and seriousness (effects) of the failure, and D is the ability to detect the failure before it reaches the customer. The three risk factors are evaluated using the 10-point scale described in Tables 1–3. The failure modes with higher RPNs are assumed to be more important and will be given higher priorities for correction.

FMEA proves to be one of the most important early preventative actions in system, design, process or service which will prevent failures and errors from occurring and reaching the customer. However, the crisp RPNs have been considerably criticized for a variety of reasons (Ben-Daya & Raouf, 1996; Bowles, 2004; Braglia, Frosolini, & Montanari, 2003a; Chang, Liu, & Wei, 2001; Gilchrist, 1993; Pillay & Wang, 2003; Sankar & Prabhu, 2001). Significant criticisms include but are not limited to the following:

- Different combinations of O, S and D may produce exactly the same value of RPN, but their hidden risk implications may be totally different. For example, two different events with the values of 2, 3, 2 and 4, 1, 3 for O, S and D, respectively, have the same RPN value of 12. However, the hidden risk implications of the two

events may not necessarily be the same. This may cause a waste of resources and time or in some cases a high risky event unable to be noticed.

- The relative importance among O, S and D is not taken into consideration. The three risk factors are assumed to be equally important. This may not be the case when considering a practical application of FMEA.
- The three factors are difficult to be precisely estimated. Much information in FMEA can be expressed in a linguistic way such as *Likely*, *Important* or *Very high* and so on.

Table 2  
Crisp ratings for severity of a failure

Rating	Effect	Severity of effect
10	Hazardous without warning	Very high severity ranking when a potential failure mode effects safe system operation without warning
9	Hazardous with warning	Very high severity ranking when a potential failure mode affects safe system operation with warning
8	Very high	System inoperable with destructive failure without compromising safety
7	High	System inoperable with equipment damage
6	Moderate	System inoperable with minor damage
5	Low	System inoperable without damage
4	Very low	System operable with significant degradation of performance
3	Minor	System operable with some degradation of performance
2	Very minor	System operable with minimal interference
1	None	No effect

To overcome the above drawbacks, fuzzy logic has been widely applied in FMEA. This will be reviewed in the next section. Vast majority of fuzzy FMEA approaches employs fuzzy if–then rules for prioritization of failure modes. This requires a vast amount of expert knowledge and expertise. In particular, a complete if–then rule base may consist of hundreds of rules. It is absolutely not realistic to ask an expert to make hundreds of judgments. To avoid building a big if–then rule base, some fuzzy FMEA approaches utilize a reduced if–then rule base, which causes a number of new problems. First, if two if–then rules with different antecedents can be combined or reduced, then the consequences of the two rules must be the same. This shows the fact that the expert cannot differentiate the two different failure modes from each other. Second, different experts may have different knowledge and judgments. When their judgments

are inconsistent, it is nearly impossible to combine or reduce rules. Third, reduced rules will be incomplete if they are not reduced from a complete if–then rule base. Any inference from an incomplete rule base will be biased or even wrong because some knowledge can not be learned from such an incomplete rule base. In other words, if a rule base contains no some knowledge, then it can not be used for inference for such knowledge; otherwise, it will give wrong conclusions. Finally, if a complete if–then rule base can be built using expert knowledge, then failure modes should be prioritized into different priority categories rather than be given a full priority ranking.

Based upon the above analyses, we think it is inappropriate to use reduced if–then rules for prioritization of failure modes. Instead of using fuzzy if–then rules, we propose in this paper the use of fuzzy weighted geometric mean (FWGM) for risk evaluation and prioritization of failure modes in FMEA. This can overcome both the drawbacks of the crisp RPN and fuzzy if–then rules.

The paper is organized as follows. In Section 2, we give a literature review of the applications of fuzzy logic in FMEA and analyze their problems. In Section 3 we briefly review basic concepts of fuzzy logic and introduce FWGM and its computational procedure. In Section 4 we evaluate the risk factors O, S and D using fuzzy logic and develop a fuzzy risk priority number (FRPN) for prioritization of failure modes. An illustrative example is provided in Section 5 to illustrate the potential applications of the proposed fuzzy FMEA and the detailed computational process of the FRPNs. The paper is concluded in Section 6 with a brief summary.

## 2. Literature review of fuzzy FMEA

Significant efforts have been made in FMEA literature for overcoming the shortcomings of the traditional RPN. As a result, fuzzy logic has been extensively used for FMEA. For example, Braglia, Frosolini, and Montanari (2003b) proposed a multi-attribute decision-making approach called fuzzy TOPSIS approach for FMECA, which is a fuzzy version of the technique for order preference by similarity to ideal solution (TOPSIS). The proposed fuzzy TOPSIS approach allows for the risk factors O, S, and D and their relative importance weights to be evaluated using triangular fuzzy numbers. Chang, Wei, and Lee (1999) used fuzzy linguistic terms such as *Very Low*, *Low*, *Moderate*, *High* and *Very high* to evaluate O, S and D, and utilized grey relational analysis to determine the risk priorities of potential causes. Garcia, Schirru, Frutuoso, and Melo (2005) presented a fuzzy data envelopment analysis (DEA) approach for FMEA in which typical risk factors O, S, and D were modeled as fuzzy sets, and the fuzzy possibility DEA model developed by Lertworasirikul, Fang, Joines, and Nuttle (2003) was used for determining the ranking indices among failure modes. The proposed approach was applied to a pressurized water reactor (PWR) auxiliary feed-water system.

Bowles and Peláez (1995) described a fuzzy logic based approach for prioritizing failures in a system FMEA, which uses fuzzy linguistic terms to describe O, S, D, and the risks of failures. The relationships between the risks and O, S, D were characterized by fuzzy if–then rules extracted from expert knowledge and expertise. Crisp ratings for O, S, D were then fuzzified to match the premise of each possible if–then rule. All the rules that have any truth in their premises were fired to contribute to fuzzy conclusion. The fuzzy conclusion was finally defuzzified by the weighted mean of maximum method (WMoM) as the ranking value of risk priority. Based on the above described fuzzy logic approach, Xu, Tang, Xie, Ho, and Zhu (2002) developed a fuzzy FMEA assessment expert system for diesel engine's gas turbocharger, and Chin, Chan, and Yang (in press) developed a fuzzy FMEA based product design system called EPDS-1, which incorporates fuzzy logic and knowledge-based systems technologies into today's competitive product design and development with an emphasis on the design of high quality products at the conceptual design stage. The prototype system EPDS-1 contains 384 fuzzy if–then rules and can assist inexperienced users to perform the FMEA for quality and reliability improvement, alternative design evaluation, materials selection, and cost assessment, which could help to enhance robustness of new products at the conceptual design stage.

Pillay and Wang (2003) proposed a fuzzy rule base approach to avoid the use of traditional RPN. The first step of their approach is to set up the membership functions of the three risk factors O, S and D. Once these membership functions have been developed, FMEA is carried out in the traditional manner with the use of brainstorming techniques. Each failure mode is then assigned a linguistic term for each of the three risk factors. The three linguistic terms are integrated using the fuzzy rule base generated to produce a linguistic term representing the priority for attention. This linguistic term represents the risk ranking of the failure mode. Once a ranking has been established, the process then follows the traditional method of determining the corrective actions and generating the FMEA report. Differing from Bowles and Peláez's fuzzy logic approach which uses linguistic terms for building fuzzy if–then rules rather than for evaluating risk factors, Pillay and Wang's fuzzy rule approach evaluates risk factors using linguistic terms rather than crisp ratings. In our view, this should be more logical because the main purpose of using fuzzy logic is to avoid the difficulty in precisely assessing the risk factors. Another difference between the two approaches is that Pillay and Wang's fuzzy rule approach has no fuzzy inference system (FIS) because any failure mode can be precisely matched by one if–then rule in the rule base.

Building a fuzzy if–then rule base is thought to be tedious and critical to fuzzy FMEA. Braglia et al. (2003a) proposed a risk function which allows fuzzy if–then rules to be generated in an automatic way. The risk function links normalized RPN with linguistic terms for final failure risk evaluation, where normalized RPN is defined

as RPN/1000. Consider, for example, if the probability of a failure mode is *Moderate*, severity is *High*, and detectability is *Low*, then normalized RPN will be  $5 \times 7 \times 7/1000 = 0.245$ , where 5, 7, and 7 are mode values of the three membership functions for *Moderate*, *High* and *Low*, respectively. Suppose the normalized RPN of 0.245 corresponds to the fuzzy set *Moderate* in the risk function, then the fuzzy if–then rule will be generated as “IF the probability of a failure mode is *Moderate*, severity is *High*, and detectability is *Low*, THEN the failure risk is *Moderate*.” The proposed risk function and fuzzy FMEA were applied to a failure analysis concerning an Italian process plant in milling field for human consumption flour. Obviously, the shortcoming of this approach is the need of defining the risk function, which may be affected by decision makers’ attitude towards risk.

Tay and Lim (2006) argued that it might be not true to assume fuzzy if–then rules to be certain and of equal importance. They therefore proposed the use of weighted fuzzy production rules in fuzzy inference system of FMEA, which allows a global weight to be attached to each if–then rule. A fuzzy if–then rule with a global weight can be expressed in the following way:

IF Occurrence is *Very high* and Severity is *Very high* and Detection is *Very low*, THEN RPN is *High* (weight 0.95).

IF Occurrence is *Very high* and Severity is *Very high* and Detection is *Very low*, THEN RPN is *High Medium* (weight 0.05).

The above if–then rule allows the same antecedent to be mapped to two different consequences: *High* with 95% confidence and *High Medium* with 5% confidence. The authors also argued in Tay and Lim (2006) that not all rules were actually required in the fuzzy RPN model and eliminating some of the rules did not necessarily lead to a significant change in the model output, however, some of the rules were vitally important and could not be ignored. They therefore proposed a guided rules reduction system (GRRS) to provide guidelines to the users which rules are required and which can be eliminated. The effectiveness of the proposed GRRS was investigated using three real-world case studies in a semiconductor manufacturing process.

Rule reduction has been applied by many researches to reduce the size of a fuzzy if–then rule base. In Pillay and Wang’s (2003) illustrative application to an ocean going fishing vessel, a total of 125 fuzzy if–then rules were generated, but were further combined and reduced to 35 rules. A typical rule reduction is recalled as follows:

Rule 1: If probability of occurrence is *Moderate*, severity is *Low*, and detectability is *High*, then priority for attention is 0.66 *Moderate* and 0.94 *Fairly high*.

Rule 2: If probability of occurrence is *Low*, severity is *Moderate*, and detectability is *High*, then priority for attention is 0.66 *Moderate* and 0.94 *Fairly high*.

Rule 3: If probability of occurrence is *Moderate*, severity is *High*, and detectability is *Low*, then priority for attention is 0.66 *Moderate* and 0.94 *Fairly high*.

Rules 1, 2 and 3 were then combined as “If probability of occurrence is *Moderate*, severity is *Low*, and detectability is *High* or any combinations of the three linguistic terms assigned to the three risk factors, then priority for attention is 0.66 *Moderate* and 0.94 *Fairly high*.”

The above rule reduction apparently implies that the three risk factors are of equal importance; otherwise, their different combinations should not lead to the same consequence. Sharma, Kumar, and Kumar (2005) employed 27 fuzzy if–then rules in their fuzzy FMEA for the feeding system in a paper mill. In their applications of fuzzy FMEA to an auxiliary feed-water system of a two-loop PWR, the Chemical and Volume Control System (CVCS) of a PWR, and the Hazard and Operability Study (HAZOP), Guimarães and Lapa (2006), Guimarães and Lapa (2004), Guimarães and Lapa (2004) reduced a total of 125 fuzzy if–then rules to 6, 14, and 16 rules, respectively, as shown in Table 4. It can be observed from Table 4 that the consequences of the first 6 rules contain no *Low* risk and the middle 14 rules no *Moderate* and *Fairly high* risks. In other words, *Low* risk cannot be learned or inferred from the first six rules and *Moderate* and *Fairly high* risks cannot be learned or inferred from the middle 14 rules. Any inferences related to these risks will be wrong unless the analyzed systems do not contain such risks.

Besides, it is also observed that the last six rules in Table 4 have the same consequence, which means the expert(s) cannot differentiate the six rules from one another. In another word, the failure modes expressed by these six rules are unable to be ranked or prioritized by the expert(s). Similar rule reduction was also applied by Guimarães and Lapa (2007) in their fuzzy FMEA application to a standard four-loop PWR containment cooling system (CCS), but it was not clear how many rules they utilized in this application.

In our view, rule reduction does not mean a complete if–then rule base is not necessary. On the contrary, it can only be done on the basis of a full rule base; otherwise the reduced rule base will be incomplete and contain ignorance information. Any inference from an incomplete rule base will be inaccurate and should be avoided.

It is clear from the above literature review that the development of a fuzzy if–then rule base is not an easy task which requires experts to make a vast number of judgments and will be highly costly and time-consuming even if not impossible. The bigger the number of the rules, the more judgments the experts have to make. Another significant drawback of using fuzzy if–then rules for FMEA is that the fuzzy if–then rules with the same consequence but different antecedents are unable to be distinguished from one another. As a result, the failure modes characterized by these fuzzy if–then rules will be unable to be prioritized or ranked. Beside, the use of fuzzy if–then rules has no way to incorporate the relative importance of risk factors into the fuzzy inference system. These drawbacks make the use of fuzzy if–then rules for FMEA far from perfect. There is a clear need to develop a new fuzzy logic approach for FMEA which can take advantage of the benefits of

Table 4  
Reduced fuzzy if–then rules for FMEA by Guimarães and Lapa

Reference	Rule	IF			THEN
		O	S	D	
Guimarães and Lapa (2004)	1	Low	Moderate	Remote	Fairly low
	2	Low	High	Remote	Moderate
	3	Moderate	Low	Low	Fairly high
	4	Low	Very high	Low	High
	5	Moderate	Moderate	Low	High
	6	Low	Remote	High	Fairly low
Guimarães and Lapa (2004)	1	Moderate	Very high	Very high	High
	2	Moderate	High	Moderate	High
	3	Moderate	Moderate	Moderate	Fairly low
	4	Low	Very high	Moderate	Fairly low
	5	Low	High	High	Fairly low
	6	Moderate	Moderate	Low	Low
	7	Moderate	High	Low	Low
	8	Low	Moderate	Moderate	Low
	9	Low	Moderate	Moderate	Fairly low
	10	Low	Remote	Low	Lowly
	11	Low	Low	Low	Lowly
	12	Low	Moderate	Low	Low
	13	Remote	Moderate	Moderate	Lowly
	14	Remote	Very high	Moderate	Fairly low
Guimarães and Lapa (2006)	1	Impossible	None	None	Not_exist
	2	Remote	Moderate	Low	Fairly high
	3	Very remote	Very high	Remote	Fairly low
	4	Low	Moderate	Remote	Moderate
	5	Remote	Moderate	Low	Moderate
	6	Very remote	Very high	Remote	Fairly low
	7	Remote	Moderate	Moderate	Fairly high
	8	Remote	Very high	Remote	Fairly high
	9	Very remote	Remote	Moderate	Low
	10	Very remote	Very high	Low	Moderate
	11	Low	Moderate	Moderate	High
	12	Moderate	Moderate	Moderate	High
	13	Moderate	High	Moderate	High
	14	Moderate	Very high	Moderate	High
	15	Moderate	Very high	High	High
	16	Moderate	Very high	Very high	High

fuzzy logic without the need of asking experts too much. This is main motivation of this paper.

### 3. Fuzzy logic and fuzzy weighted geometric mean

A fuzzy set is a collection of elements in a universe of information where the boundary of the set contained in

the universe is ambiguous, vague and otherwise fuzzy. It is specified by a membership function, which assigns a value within the unit interval  $[0, 1]$  to each element in the universe of discourse. The assigned value is called membership degree, which specifies the extent to which a given element belongs to the fuzzy set. If the assigned value is 0, then the given element does not belong to the set. If the assigned value is 1, then the element totally belongs to the set. If the value lies within the interval  $(0, 1)$ , then the element only partially belongs to the set. Therefore, any fuzzy set can be uniquely determined by its membership function.

Fuzzy sets can also be represented by intervals, which are called  $\alpha$ -level sets. Let  $\tilde{A}$  be a fuzzy set on the universe of discourse  $X$ . Then the  $\alpha$ -level sets of  $\tilde{A}$  are defined as

$$A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\} = [\min\{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}, \max\{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}]. \quad (2)$$

According to Zadeh’s extension principle (Dubois & Prade, 1980; Zadeh, 1965), the fuzzy set  $\tilde{A}$  can be equivalently expressed as

$$\tilde{A} = \cup_{\alpha} \alpha A_\alpha, \quad 0 < \alpha \leq 1. \quad (3)$$

Fuzzy numbers are special cases of fuzzy sets. A fuzzy number is a convex fuzzy set characterized by a given interval of real numbers, each with a membership degree between 0 and 1. The membership functions of fuzzy numbers are piecewise continuous and satisfy the following conditions:

- (a)  $\mu_{\tilde{A}}(x) = 0$  for each  $x \notin [a, d]$ ;
- (b)  $\mu_{\tilde{A}}(x)$  is non-decreasing (monotonic increasing) on  $[a, b]$  and non-increasing (monotonic decreasing) on  $[c, d]$ ;
- (c)  $\mu_{\tilde{A}}(x) = 1$  for each  $x \in [b, c]$ ,

where  $a \leq b \leq c \leq d$  are real numbers in the real line  $R = (-\infty, +\infty)$ .

The most commonly used fuzzy numbers are triangular and trapezoidal fuzzy numbers, whose membership functions are respectively defined as

$$\mu_{A_1}(x) = \begin{cases} (x - a)/(b - a), & a \leq x \leq b, \\ (d - x)/(d - b), & b \leq x \leq d, \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

$$\mu_{A_2}(x) = \begin{cases} (x - a)/(b - a), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ (d - x)/(d - c), & c \leq x \leq d, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

For brevity, triangular and trapezoidal fuzzy numbers are often denoted as  $(a, b, d)$  and  $(a, b, c, d)$ . Obviously, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers with  $b = c$ .

Let  $\tilde{A} = (a_1, a_2, a_3)$  and  $\tilde{B} = (b_1, b_2, b_3)$  be two positive triangular fuzzy numbers. Then basic fuzzy arithmetic operations on these fuzzy numbers are defined as (Dubois & Prade, 1980)

Addition :  $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3);$

Subtraction :  $\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1);$

Multiplication :  $\tilde{A} \times \tilde{B} \approx (a_1 b_1, a_2 b_2, a_3 b_3);$

Division :  $\tilde{A} \div \tilde{B} \approx \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right).$

Fuzzy numbers often need to be transformed into crisp numbers for comparison or ranking purpose. Such a transformation process is called defuzzification, which can be conducted in many different ways. The most extensively used defuzzification approach is the centroid defuzzification, which is also known as the center of gravity or center of area defuzzification. The centroid defuzzification method defines the centroid of a fuzzy number  $\tilde{A}$  as its defuzzified value, as shown below (Yager, 1981):

$$\bar{x}_0(\tilde{A}) = \frac{\int_a^d x \mu_{\tilde{A}}(x) dx}{\int_a^d \mu_{\tilde{A}}(x) dx}, \tag{6}$$

where  $\bar{x}_0(\tilde{A})$  is the defuzzified value. For a triangular fuzzy number  $A = (a, b, d)$ , its defuzzified centroid turns out to be

$$\bar{x}_0(\tilde{A}) = \frac{a + b + d}{3}. \tag{7}$$

When a fuzzy number  $\tilde{A}$  is expressed by its  $\alpha$ -level sets, i.e.  $\tilde{A} = \cup_{\alpha} \alpha \cdot (A)_{\alpha} = \cup_{\alpha} \alpha \cdot [(x)_{\alpha}^L, (x)_{\alpha}^U] (0 < \alpha \leq 1)$ , its defuzzified centroid can be determined by the following equations:

$$\int_a^d \mu_{\tilde{A}}(x) dx = \frac{1}{2} \left[ (x)_{\alpha_n}^U - (x)_{\alpha_n}^L - \sum_{i=1}^{n-1} \alpha_i ((x)_{\alpha_{i+1}}^U - (x)_{\alpha_{i+1}}^L) + \sum_{i=0}^{n-1} \alpha_{i+1} ((x)_{\alpha_i}^U - (x)_{\alpha_i}^L) \right], \tag{8}$$

$$\int_a^d x \mu_{\tilde{A}}(x) dx = \frac{1}{6} \left[ (x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L} - \sum_{i=1}^{n-1} \alpha_i ((x)_{\alpha_{i+1}}^{2U} - (x)_{\alpha_{i+1}}^{2L}) + \sum_{i=0}^{n-1} \alpha_{i+1} ((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}) \right] + \frac{1}{6} \sum_{i=0}^{n-1} \Delta \alpha_i ((x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L). \tag{9}$$

In particular, when  $\Delta \alpha_i \equiv \frac{1}{n}$  and  $\alpha_i = \frac{i}{n}, i = 0, \dots, n$ , the above equations are simplified as

$$\int_a^d \mu_{\tilde{A}}(x) dx = \frac{1}{2n} \left[ ((x)_{\alpha_0}^U - (x)_{\alpha_0}^L) + ((x)_{\alpha_n}^U - (x)_{\alpha_n}^L) + 2 \sum_{i=1}^{n-1} ((x)_{\alpha_i}^U - (x)_{\alpha_i}^L) \right], \tag{10}$$

$$\int_a^d x \mu_{\tilde{A}}(x) dx = \frac{1}{6n} \left[ ((x)_{\alpha_0}^{2U} - (x)_{\alpha_0}^{2L}) + ((x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L}) + 2 \sum_{i=1}^{n-1} ((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}) \right] + \frac{1}{6n} \sum_{i=0}^{n-1} ((x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L). \tag{11}$$

The derivations of these equations are provided in Appendix A.

The fuzzy weighted average of  $n$  fuzzy numbers is often referred to as the fuzzy weighted average (FWA) (Dong & Wong, 1987; Guh, Hon, & Lee, 2001; Kao & Liu, 2001) in the literature. Similarly, we refer to the fuzzy weighted geometric mean of  $n$  fuzzy numbers as the fuzzy weighted geometric mean (FWGM), which can be expressed as

$$\begin{aligned} \tilde{y}_G &= f_G(\tilde{x}_1, \dots, \tilde{x}_n; \tilde{w}_1, \dots, \tilde{w}_n) \\ &= (\tilde{x}_1)^{\frac{\tilde{w}_1}{\tilde{w}_1 + \tilde{w}_2 + \dots + \tilde{w}_n}} (\tilde{x}_2)^{\frac{\tilde{w}_2}{\tilde{w}_1 + \tilde{w}_2 + \dots + \tilde{w}_n}} \dots (\tilde{x}_n)^{\frac{\tilde{w}_n}{\tilde{w}_1 + \tilde{w}_2 + \dots + \tilde{w}_n}} \\ &= \prod_{i=1}^n (\tilde{x}_i)^{\frac{\tilde{w}_i}{\sum_{j=1}^n \tilde{w}_j}}, \end{aligned} \tag{12}$$

where  $\tilde{x}_1, \dots, \tilde{x}_n$  are the  $n$  positive fuzzy numbers to be weighted and  $\tilde{w}_1, \dots, \tilde{w}_n$  are their fuzzy weights. Obviously,  $\tilde{y}_G$  is also a fuzzy number and can be computed using  $\alpha$ -level sets and the extension principle.

Let  $(y_G)_{\alpha} = [(y_G)_{\alpha}^L, (y_G)_{\alpha}^U]$  be an  $\alpha$ -level set of  $\tilde{y}_G$ . Then it can be determined by the following mathematical models:

$$(y_G)_{\alpha}^L = \text{Min} \prod_{i=1}^n (x_i)^{\sum_{j=1}^n w_j^{\alpha}} \tag{13}$$

s.t.  $(w_i)_{\alpha}^L \leq w_i \leq (w_i)_{\alpha}^U, \quad i = 1, \dots, n,$   
 $(x_i)_{\alpha}^L \leq x_i \leq (x_i)_{\alpha}^U, \quad i = 1, \dots, n,$

$$(y_G)_{\alpha}^U = \text{Max} \prod_{i=1}^n (x_i)^{\sum_{j=1}^n w_j^{\alpha}} \tag{14}$$

s.t.  $(w_i)_{\alpha}^L \leq w_i \leq (w_i)_{\alpha}^U, \quad i = 1, \dots, n,$   
 $(x_i)_{\alpha}^L \leq x_i \leq (x_i)_{\alpha}^U, \quad i = 1, \dots, n.$

Due to the fact that

$$f_G(x_1, \dots, x_n; w_1, \dots, w_n) = \prod_{i=1}^n (x_i)^{\sum_{j=1}^n w_j^{\alpha}} \tag{15}$$

is an increasing function of variables  $x_i (i = 1, \dots, n)$ , the above mathematical models can therefore be equivalently rewritten as:

$$(y_G)_{\alpha}^L = \text{Min} \exp \left( \frac{\sum_{i=1}^n w_i \ln(x_i)_{\alpha}^L}{\sum_{i=1}^n w_i} \right) \tag{16}$$

s.t.  $(w_i)_{\alpha}^L \leq w_i \leq (w_i)_{\alpha}^U, \quad i = 1, \dots, n.$

$$(y_G)_{\alpha}^U = \text{Max} \exp \left( \frac{\sum_{i=1}^n w_i \ln(x_i)_{\alpha}^U}{\sum_{i=1}^n w_i} \right) \tag{17}$$

s.t.  $(w_i)_{\alpha}^L \leq w_i \leq (w_i)_{\alpha}^U, \quad i = 1, \dots, n,$

where  $\exp()$  is the exponential function.

Based upon the transformation:  $z = 1/\sum_{i=1}^n w_i$  and  $u_i = zw_i$  for  $i = 1, \dots, n$ , models (16) and (17) can be transformed into the following:

$$\text{Min } z_1 = \sum_{i=1}^n u_i \ln(x_i)_\alpha^L \tag{18}$$

$$\begin{aligned} \text{s.t. } & u_1 + u_2 + \dots + u_n = 1, \\ & (w_i)_\alpha^L \cdot z \leq u_i \leq (w_i)_\alpha^U \cdot z, \quad i = 1, \dots, n, \\ & z \geq 0, \end{aligned}$$

$$\text{Max } z_2 = \sum_{i=1}^n u_i \ln(x_i)_\alpha^U \tag{19}$$

$$\begin{aligned} \text{s.t. } & u_1 + u_2 + \dots + u_n = 1, \\ & (w_i)_\alpha^L \cdot z \leq u_i \leq (w_i)_\alpha^U \cdot z, \quad i = 1, \dots, n, \\ & z \geq 0. \end{aligned}$$

These are linear programming (LP) models and easy to be solved using MS Excel Solver. Let  $z_1^*$  and  $z_2^*$  be the optimal objective function values of the above models (18) and (19), respectively. Then  $(y_G)_\alpha^L = \exp(z_1^*)$  and  $(y_G)_\alpha^U = \exp(z_2^*)$ . By setting different  $\alpha$  levels, different  $\alpha$ -level sets of  $\tilde{y}_G$  can be generated, based on which  $\tilde{y}_G$  can be expressed as

$$\tilde{y}_G = \cup_\alpha \alpha \cdot (y_G)_\alpha = \cup_\alpha \alpha [(y_G)_\alpha^L, (y_G)_\alpha^U], \quad 0 < \alpha \leq 1. \tag{20}$$

Table 5  
Fuzzy ratings for occurrence of a failure

Rating	Probability of occurrence	Fuzzy number
Very high (VH)	Failure is almost inevitable	(8, 9, 10, 10)
High (H)	Repeated failures	(6, 7, 8, 9)
Moderate (M)	Occasional failures	(3, 4, 6, 7)
Low (L)	Relatively few failures	(1, 2, 3, 4)
Remote (R)	Failure is unlikely	(1, 1, 2)

Table 6  
Fuzzy ratings for severity of a failure

Rating	Severity of effect	Fuzzy number
Hazardous without warning (HWOW)	Very high severity ranking without warning	(9, 10, 10)
Hazardous with warning (HWW)	Very high severity ranking with warning	(8, 9, 10)
Very high (VH)	System inoperable with destructive failure	(7, 8, 9)
High (H)	System inoperable with equipment damage	(6, 7, 8)
Moderate (M)	System inoperable with minor damage	(5, 6, 7)
Low (L)	System inoperable without damage	(4, 5, 6)
Very low (VL)	System operable with significant degradation of performance	(3, 4, 5)
Minor (MR)	System operable with some degradation of performance	(2, 3, 4)
Very minor (VMR)	System operable with minimal interference	(1, 2, 3)
None (N)	No effect	(1, 1, 2)

#### 4. Fuzzy risk priority numbers for FMEA

It has been extensively argued that the risk factors O, S and D are not easy to be precisely evaluated. Significant efforts have thus been made to evaluate them in a linguistic way. Tables 5–7 show the linguistic terms and their fuzzy numbers used for evaluating the risk factors in this paper. These linguistic terms are perfectly consistent with those defined by the traditional FMEA, but they are treated as trapezoidal and triangular fuzzy numbers in this paper rather than precise numerical values. Figs. 1–3 show their membership functions for the sake of visualization.

Table 7  
Fuzzy ratings for detection of a failure

Rating	Likelihood of detection	Fuzzy number
Absolute uncertainty (AU)	No chance	(9, 10, 10)
Very remote (VR)	Very remote chance	(8, 9, 10)
Remote (R)	Remote chance	(7, 8, 9)
Very low (VL)	Very low chance	(6, 7, 8)
Low (L)	Low chance	(5, 6, 7)
Moderate (M)	Moderate chance	(4, 5, 6)
Moderately high (MH)	Moderately high chance	(3, 4, 5)
High (H)	High chance	(2, 3, 4)
Very high (VH)	Very high chance	(1, 2, 3)
Almost certain (AC)	Almost certainty	(1, 1, 2)

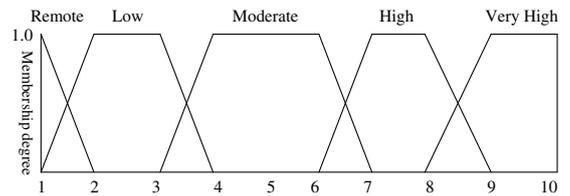


Fig. 1. Fuzzy ratings for occurrence assessment of failures and their membership functions.

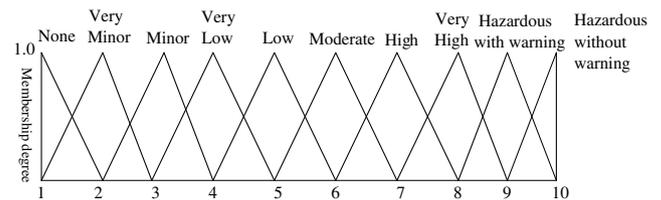


Fig. 2. Fuzzy ratings for severity assessment of failures and their membership functions.

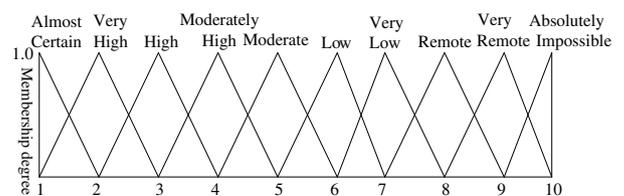


Fig. 3. Fuzzy ratings for detection assessment of failures and their membership functions.

Table 8  
Fuzzy weights for the relative importance of risk factors

Linguistic term	Fuzzy number
Very low (VL)	(0, 0, 0.25)
Low (L)	(0, 0.25, 0.5)
Medium (M)	(0.25, 0.5, 0.75)
High (H)	(0.5, 0.75, 1)
Very high (VH)	(0.75, 1, 1)

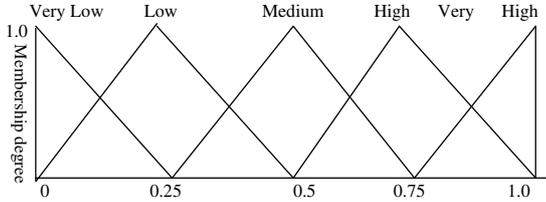


Fig. 4. Membership functions of fuzzy weights.

The traditional FMEA has been largely criticized for the reason that it takes no account of the relative importance of the risk factors and treats them equally. To overcome this drawback, the relative importance weights of the risk factors are considered in this paper, but they are not easy to be precisely determined due to the same reason as O, S and D. They will be assessed using the linguistic terms in Table 8, whose membership functions are visualized in Fig. 4.

Suppose there are  $n$  failure modes,  $FM_i (i = 1, \dots, n)$ , to be evaluated and prioritized by a FMEA team consisting of  $m$  cross-functional team members,  $TM_j (j = 1, \dots, m)$ . Let  $\tilde{R}_{ij}^O = (R_{ijL}^O, R_{ijM_1}^O, R_{ijM_2}^O, R_{ijU}^O)$ ,  $\tilde{R}_{ij}^S = (R_{ijL}^S, R_{ijM}^S, R_{ijU}^S)$  and  $\tilde{R}_{ij}^D = (R_{ijL}^D, R_{ijM}^D, R_{ijU}^D)$  be the fuzzy ratings of the  $i$ th failure mode on the risk factors O, S and D,  $\tilde{w}_j^O = (w_{jL}^O, w_{jM}^O, w_{jU}^O)$ ,  $\tilde{w}_j^S = (w_{jL}^S, w_{jM}^S, w_{jU}^S)$  and  $\tilde{w}_j^D = (w_{jL}^D, w_{jM}^D, w_{jU}^D)$  be the fuzzy weights of the three risk factors provided by the  $j$ th FMEA team member ( $TM_j$ ), and  $h_j (j = 1, \dots, m)$  be the relative importance weights of the  $m$  team members, satisfying  $\sum_{j=1}^m h_j = 1$  and  $h_j > 0$  for  $j = 1, \dots, m$ . Based upon these assumptions, the  $n$  failure modes can be prioritized by the following steps.

**Step 1.** Aggregate the FMEA team members' subjective opinions by Eqs. (21)–(26).

$$\tilde{R}_i^O = \sum_{j=1}^m h_j \tilde{R}_{ij}^O = \left( \sum_{j=1}^m h_j R_{ijL}^O, \sum_{j=1}^m h_j R_{ijM_1}^O, \sum_{j=1}^m h_j R_{ijM_2}^O, \sum_{j=1}^m h_j R_{ijU}^O \right), \quad i = 1, \dots, n, \quad (21)$$

$$\tilde{R}_i^S = \sum_{j=1}^m h_j \tilde{R}_{ij}^S = \left( \sum_{j=1}^m h_j R_{ijL}^S, \sum_{j=1}^m h_j R_{ijM}^S, \sum_{j=1}^m h_j R_{ijU}^S \right), \quad i = 1, \dots, n, \quad (22)$$

$$\tilde{R}_i^D = \sum_{j=1}^m h_j \tilde{R}_{ij}^D = \left( \sum_{j=1}^m h_j R_{ijL}^D, \sum_{j=1}^m h_j R_{ijM}^D, \sum_{j=1}^m h_j R_{ijU}^D \right), \quad i = 1, \dots, n, \quad (23)$$

$$\tilde{w}^O = \sum_{j=1}^m h_j \tilde{w}_j^O = \left( \sum_{j=1}^m h_j w_{jL}^O, \sum_{j=1}^m h_j w_{jM}^O, \sum_{j=1}^m h_j w_{jU}^O \right), \quad (24)$$

$$\tilde{w}^S = \sum_{j=1}^m h_j \tilde{w}_j^S = \left( \sum_{j=1}^m h_j w_{jL}^S, \sum_{j=1}^m h_j w_{jM}^S, \sum_{j=1}^m h_j w_{jU}^S \right), \quad (25)$$

$$\tilde{w}^D = \sum_{j=1}^m h_j \tilde{w}_j^D = \left( \sum_{j=1}^m h_j w_{jL}^D, \sum_{j=1}^m h_j w_{jM}^D, \sum_{j=1}^m h_j w_{jU}^D \right), \quad (26)$$

where  $\tilde{R}_i^O = (R_{iL}^O, R_{iM_1}^O, R_{iM_2}^O, R_{iU}^O)$ ,  $\tilde{R}_i^S = (R_{iL}^S, R_{iM}^S, R_{iU}^S)$  and  $\tilde{R}_i^D = (R_{iL}^D, R_{iM}^D, R_{iU}^D)$  are aggregated occurrence, severity and detection ratings for failure mode  $FM_i$  and  $\tilde{w}^O = (w_L^O, w_M^O, w_U^O)$ ,  $\tilde{w}^S = (w_L^S, w_M^S, w_U^S)$  and  $\tilde{w}^D = (w_L^D, w_M^D, w_U^D)$  are aggregated fuzzy weights for the three risk factors O, S and D, respectively.

**Step 2.** Define the fuzzy risk priority number (FRPN) of each failure mode as

$$FRPN_i = (\tilde{R}_i^O)^{\frac{w_U^O}{w_O^O + w_S^O + w_D^O}} \times (\tilde{R}_i^S)^{\frac{w_U^S}{w_O^S + w_S^S + w_D^S}} \times (\tilde{R}_i^D)^{\frac{w_U^D}{w_O^D + w_S^D + w_D^D}}, \quad i = 1, \dots, n. \quad (27)$$

Differing from the traditional FMEA which defines the RPNs as the simple product of O, S and D without considering their relative importance weights, the FRPN is defined as the fuzzy weighted geometric mean of the three risk factors. This overcomes the drawback that the three risk factors are treated equally. Since the FRPNs are fuzzy numbers, they can be computed using  $\alpha$ -level sets.

**Step 3.** Calculate the  $\alpha$ -level sets of the FRPN of each failure mode by the solution of the following LP models.

$$\text{Min } z_1 = u_1 \ln(R_i^O)_\alpha^L + u_2 \ln(R_i^S)_\alpha^L + u_3 \ln(R_i^D)_\alpha^L \quad (28)$$

$$\text{s.t. } \begin{aligned} u_1 + u_2 + u_3 &= 1, \\ (w^O)_\alpha^L \cdot z &\leq u_1 \leq (w^O)_\alpha^U \cdot z, \\ (w^S)_\alpha^L \cdot z &\leq u_2 \leq (w^S)_\alpha^U \cdot z, \\ (w^D)_\alpha^L \cdot z &\leq u_3 \leq (w^D)_\alpha^U \cdot z, \\ z &\geq 0, \end{aligned}$$

$$\text{Max } z_2 = u_1 \ln(R_i^O)_\alpha^U + u_2 \ln(R_i^S)_\alpha^U + u_3 \ln(R_i^D)_\alpha^U \quad (29)$$

$$\begin{aligned}
 \text{s.t. } & u_1 + u_2 + u_3 = 1, \\
 & (w^O)_\alpha^L \cdot z \leq u_1 \leq (w^O)_\alpha^U \cdot z, \\
 & (w^S)_\alpha^L \cdot z \leq u_2 \leq (w^S)_\alpha^U \cdot z, \\
 & (w^D)_\alpha^L \cdot z \leq u_3 \leq (w^D)_\alpha^U \cdot z, \\
 & z \geq 0,
 \end{aligned}$$

where  $[\ln(R_i^O)_\alpha^L, \ln(R_i^O)_\alpha^U], [\ln(R_i^S)_\alpha^L, \ln(R_i^S)_\alpha^U]$  and  $[\ln(R_i^D)_\alpha^L, \ln(R_i^D)_\alpha^U]$  are the logarithms of  $[(R_i^O)_\alpha^L, (R_i^O)_\alpha^U], [(R_i^S)_\alpha^L, (R_i^S)_\alpha^U]$  and  $[(R_i^D)_\alpha^L, (R_i^D)_\alpha^U]$  that are the  $\alpha$ -level sets of the aggregated occurrence, severity and detection ratings  $\tilde{R}_i^O, \tilde{R}_i^S$  and  $\tilde{R}_i^D$  for the failure mode  $FM_i$ , and  $[(w^O)_\alpha^L, (w^O)_\alpha^U], [(w^S)_\alpha^L, (w^S)_\alpha^U]$  and  $[(w^D)_\alpha^L, (w^D)_\alpha^U]$  are the  $\alpha$ -level sets of the aggregated risk factor weights  $\tilde{w}^O, \tilde{w}^S$  and  $\tilde{w}^D$ , respectively.

The above LP models are an application of models (18) and (19) in fuzzy FMEA and should be solved for all  $\alpha$  levels and every failure mode. Let  $z_1^*$  and  $z_2^*$  be the optimal objective function values of models (28) and (29), respectively. Then  $(FRPN_i)_\alpha^L = \exp(z_1^*)$  and  $(FRPN_i)_\alpha^U = \exp(z_2^*)$ . By setting different  $\alpha$  levels, different  $\alpha$ -level sets of  $FRPN_i$  can be generated, based on which  $FRPN_i$  can be expressed as

$$FRPN_i = \bigcup_{\alpha} \alpha \cdot [(FRPN_i)_\alpha^L, (FRPN_i)_\alpha^U], \quad 0 < \alpha \leq 1. \tag{30}$$

**Step 4.** Defuzzify the FRPNs by the centroid defuzzification method.

Since the FRPNs are characterized by  $\alpha$ -level sets, their defuzzified centroids should be determined by Eqs. (8) and (9). In particular, when the unit interval  $[0, 1]$  is equally divided by different  $\alpha$  levels, the defuzzified centroids can be determined by Eqs. (10) and (11).

**Step 5.** Prioritize the failure modes by the defuzzified centroid values of their FRPNs.

The bigger the defuzzified centroid value, the bigger the overall risk, and the higher the risk priority. All the failure modes can be prioritized or ranked in terms of the defuzzified centroid values of their FRPNs.

In the above computations, the relative importance weights of FMEA team members are assumed to be crisp values. This is mainly because they are relatively easier to be determined than the weights of risk factors. If they are also difficult to be precisely determined, they can be assessed using the linguistic terms in Table 8. This leads to a fuzzy weight for each FMEA team member. As a result, the aggregation of FMEA team members' subjective opinions in Step 1 will become a fuzzy weighted average (FWA) problem (Dong & Wong, 1987). Theoretically, any FWA can be solved using  $\alpha$ -level sets and by linear programming solver (Guh et al., 2001; Kao & Liu, 2001), but this apparently increases the complexity of problem solving. To simplify the computation of FRPNs, the fuzzy weights of FMEA team members can be defuzzified using Eq. (7) and the defuzzified values can then be normalized as the relative importance weights of the FMEA team members. This will to a great extent facilitate the calculation of FRPNs.

### 5. An illustrative example

In this section, we provide a numerical example to illustrate the potential applications of the proposed fuzzy FMEA and particularly the potentials of fuzzy risk priority numbers in prioritization of failure modes.

A FMEA team consisting of five cross-functional team members identifies seven potential failure modes in a system and needs to prioritize them in terms of their failure risks such as probability of occurrence, severity and detectability so that high risky failure modes can be corrected

Table 9  
Assessment information on seven failure modes by five FMEA team members

Risk factors	FMEA team members	Factor weights	Failure modes						
			1	2	3	4	5	6	7
Occurrence	TM <sub>1</sub> (15%)	M	L	M	H	H	VH	R	M
	TM <sub>2</sub> (20%)	H	L	M	M	H	H	L	M
	TM <sub>3</sub> (30%)	M	M	M	M	H	H	L	H
	TM <sub>4</sub> (25%)	VH	H	L	H	M	M	M	M
	TM <sub>5</sub> (10%)	M	L	M	VH	H	H	M	VH
Severity	TM <sub>1</sub> (15%)	VH	L	VL	H	VH	M	VL	MR
	TM <sub>2</sub> (20%)	H	VL	M	M	H	VH	L	VL
	TM <sub>3</sub> (30%)	VH	L	L	H	M	H	VL	L
	TM <sub>4</sub> (25%)	H	M	H	VH	H	M	M	L
	TM <sub>5</sub> (10%)	H	L	M	H	M	H	M	VMR
Detection	TM <sub>1</sub> (15%)	L	VR	R	VL	MH	MH	H	M
	TM <sub>2</sub> (20%)	L	AI	VR	R	VL	VL	L	M
	TM <sub>3</sub> (30%)	L	R	VL	R	L	M	L	L
	TM <sub>4</sub> (25%)	M	L	VL	MH	H	VR	R	H
	TM <sub>5</sub> (10%)	L	VL	R	VR	MH	MH	L	VL

Table 10

Aggregated fuzzy assessment information for the seven failure modes and the relative importance weights of three risk factors

Failure mode	Occurrence	Severity	Detection
1	(2.85, 3.85, 5.15, 6.15)	(4.05, 5.05, 6.05)	(6.95, 7.95, 8.75)
2	(2.5, 3.5, 5.25, 6.25)	(4.65, 5.65, 6.65)	(6.65, 7.65, 8.65)
3	(4.7, 5.7, 7.2, 8.1)	(6.05, 7.05, 8.05)	(5.95, 6.95, 7.95)
4	(5.25, 6.25, 7.5, 8.5)	(5.75, 6.75, 7.75)	(3.95, 4.95, 5.95)
5	(5.55, 6.55, 7.8, 8.65)	(5.8, 6.8, 7.8)	(5.15, 6.15, 7.15)
6	(1.7, 2.55, 3.75, 4.75)	(3.9, 4.9, 5.9)	(5.05, 6.05, 7.05)
7	(4.4, 5.4, 7, 7.9)	(3.2, 4.2, 5.2)	(4, 5, 6)
Importance weights	(0.425, 0.675, 0.8625)	(0.6125, 0.8625, 1)	(0.0625, 0.2875, 0.5375)

Table 11

Alpha-level sets of the fuzzy risk priority numbers for the seven failure modes

$\alpha$	Failure modes						
	1	2	3	4	5	6	7
0	[3.3992, 6.8923]	[3.3310, 7.1536]	[5.2474, 8.0739]	[4.9358, 8.0750]	[5.5036, 8.2368]	[2.4736, 5.9134]	[3.5378, 6.6133]
0.1	[3.5538, 6.7441]	[3.4993, 7.0083]	[5.3766, 7.9768]	[5.0709, 7.9200]	[5.6147, 8.1156]	[2.6257, 5.7800]	[3.6612, 6.4671]
0.2	[3.7075, 6.5972]	[3.6667, 6.8647]	[5.5048, 7.8793]	[5.2052, 7.7848]	[5.7254, 7.9948]	[2.7776, 5.6489]	[3.7838, 6.3225]
0.3	[3.8605, 6.4515]	[3.8330, 6.7227]	[5.6322, 7.7815]	[5.3384, 7.6423]	[5.8359, 7.8744]	[2.9293, 5.5174]	[3.9055, 6.1795]
0.4	[4.0127, 6.3071]	[3.9982, 6.5822]	[5.7588, 7.6834]	[5.4708, 7.5013]	[5.9460, 7.7544]	[3.0806, 5.3856]	[4.0264, 6.0379]
0.5	[4.1639, 6.1638]	[4.1623, 6.4431]	[5.8845, 7.5850]	[5.6023, 7.3618]	[6.0552, 7.6348]	[3.2316, 5.2535]	[4.1466, 5.8977]
0.6	[4.3143, 6.0216]	[4.3251, 6.3055]	[6.0095, 7.4864]	[5.7330, 7.2237]	[6.1640, 7.5155]	[3.3821, 5.1211]	[4.2661, 5.7587]
0.7	[4.4637, 5.8806]	[4.4868, 6.1691]	[6.1338, 7.3874]	[5.8628, 7.0869]	[6.2730, 7.3966]	[3.5320, 4.9884]	[4.3849, 5.6210]
0.8	[4.6122, 5.7406]	[4.6472, 6.0340]	[6.2573, 7.2882]	[5.9919, 6.9513]	[6.3822, 7.2780]	[3.6814, 4.8553]	[4.5030, 5.4845]
0.9	[4.7597, 5.6018]	[4.8064, 5.9002]	[6.3801, 7.1888]	[6.1202, 6.8176]	[6.4916, 7.1597]	[3.8302, 4.7220]	[4.6206, 5.3491]
1.0	[4.9063, 5.4637]	[4.9643, 5.7675]	[6.5023, 7.0891]	[6.2477, 6.6836]	[6.6011, 7.0416]	[3.9784, 4.5883]	[4.7375, 5.2147]
Centroid	5.1835	5.3200	6.7561	6.5147	6.8837	4.2514	5.0592
Priority ranking	5	4	2	3	1	7	6

with top priorities. Due to the difficulty in precisely assessing the risk factors and their relative importance weights, the FMEA team members reach a consensus to evaluate them using the linguistic terms defined in Tables 5–8. The assessment information of the seven failure modes on each risk factor and the risk factor weights provided by the five team members is presented in Table 9. The five team members from different departments are assumed to be of different importance because of their different domain knowledge and expertise. To reflect their differences in performing FMEA, the five team members are assigned the following relative weights: 15%, 20%, 30%, 25% and 10%, which are shown in the second column of Table 9.

Based upon the information in Table 9, the five team members' assessment information is first aggregated by Eqs. (21)–(26). The results are provided in Table 10. Since the risk ratings and the risk factor weights are all fuzzy numbers, the overall risk of each failure mode will be a fuzzy number either, which we refer to as the fuzzy risk priority number, as defined by Eq. (27).

To calculate the fuzzy risk priority numbers of the seven failure modes, we solve the LP models (28) and (29) for each of the seven failure modes and all  $\alpha$  levels, where the  $\alpha$  levels are set as 0, 0.1, 0.2, ..., 1.0. The results are provided in Table 11 and depicted in Fig. 5.

As can be seen from Fig. 5, FM6 is apparently the failure mode with the least overall risk and should be given the lowest risk priority, while FM5 is without doubt the failure

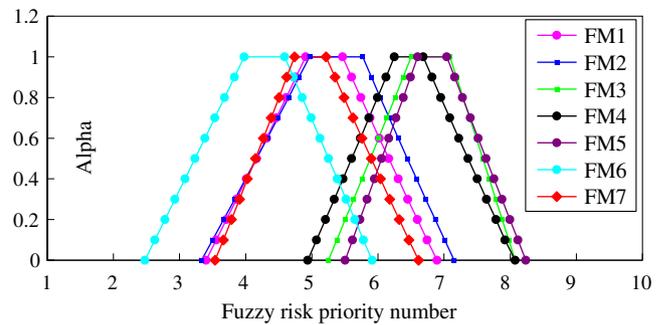


Fig. 5. Fuzzy risk priority numbers of the seven failure modes (FMs).

mode with the maximum overall risk and should be given the top risk priority, followed by FM3, FM4, FM2, FM1 and FM7. This seems a particular example in which the seven failure modes can be prioritized intuitively, but this is not always the case in many FMEA applications. A more generic method for prioritizing failure modes is to defuzzify their fuzzy risk priority numbers by Eqs. (8), (9) or (10), (11) and prioritize them by their defuzzified centroid values.

Since the unit interval [0, 1] is equally divided by eleven  $\alpha$  levels into eleven subintervals, Eqs. (10),(11) are therefore used to compute the centroids of the seven FRPNs. The results are shown in the last but one row of Table 11. The defuzzified centroid values of the seven FRPNs give the priority ranking of the seven failure modes

as FM5 > FM3 > FM4 > FM2 > FM1 > FM7 > FM6, which is perfectly consistent with the ranking achieved by our former intuitive analysis.

So, the final conclusion for this example is that failure model 5 is given the top priority for correction, followed by failure modes 3, 4, 2, 1, 7, and 6.

**6. Conclusions**

FMEA is a very important safety and reliability analysis tool which has been widely used in many areas and industries. In view of its difficulty in acquiring precise assessment information on failure risks such as probability of occurrence, severity and detectability and the difficulty in building a complete fuzzy if–then rule base, this paper proposed a new fuzzy FMEA which allows the risk factors and their relative weights to be evaluated in a linguistic manner rather than in a precise way and a fuzzy RPN rather than a crisp RPN or fuzzy if–then rules to be defined for prioritization of failure modes. The fuzzy RPN or FRPN for short was defined as the fuzzy weighted geometric mean of the risk factors and can be exactly solved by using  $\alpha$ -level sets and the fuzzy extension principle. The  $\alpha$ -level sets of FRPNs are easy to be generated by solving a series of linear programming models. The potential applications of the proposed fuzzy FMEA and the detailed computational process of FRPNs were examined and illustrated with a numerical example. It was shown that the proposed fuzzy FMEA provided a useful, practical, effective and flexible way for risk evaluation in FMEA. In particular, the defined FRPNs offered a new way for prioritizing failure modes in FMEA.

Compared with the traditional RPN and its various fuzzy improvements, the proposed fuzzy FMEA has the following advantages:

- The relative importance among the risk factors O, S and D is taken into consideration in the process of prioritization of failure modes, which makes the proposed fuzzy FMEA more realistic, more practical and more flexible.
- Risk factors and their relative importance weights are evaluated in a linguistic manner rather than in precise numerical values. This makes the assessment easier to be carried out.
- Different combinations of O, S and D produce different FRPNs unless the relative weights among the O, S and D are exactly the same, which enables the proposed fuzzy FMEA to fully prioritize failure modes and distinguish them from one another.
- There is no need to build any if–then rule base which proves to be highly subjective, costly and time-consuming.
- More risk factors can be incorporated into the FRPNs if necessary. The proposed fuzzy FMEA is not limited to O, S and D, but applicable to any number of risk factors.
- The derived centroid formula based on  $\alpha$ -level sets is new and different from existing formulas (Yager, 1981;

Yager & Filev, 1999; Uehara & Hirota, 1998). It provides useful decision support to the comparison of FRPNs and those fuzzy numbers whose membership functions are not known, but their  $\alpha$ -level sets are available.

- The fuzzy weighted geometric mean proposed in this paper is new and has never appeared in the literature before, which makes the FRPNs new to FMEA.

**Appendix A. Derivations of Eqs. (8) and (9)**

For a fuzzy number expressed by its  $\alpha$ -level sets:  $\tilde{A} = \cup_{\alpha} \alpha \cdot (A)_{\alpha} = \cup_{\alpha} \alpha \cdot [(x)_{\alpha}^L, (x)_{\alpha}^U] (0 < \alpha \leq 1)$ , we define its membership function as

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < (x)_{\alpha_0}^L \text{ or } x > (x)_{\alpha_n}^U \\ \alpha_i + \frac{\Delta\alpha_i(x - (x)_{\alpha_i}^L)}{(x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L}, & (x)_{\alpha_i}^L \leq x \leq (x)_{\alpha_{i+1}}^L, \quad i = 0, 1, \dots, n-1 \\ 1, & (x)_{\alpha_n}^L \leq x \leq (x)_{\alpha_n}^U, \\ \alpha_i + \frac{\Delta\alpha_i((x)_{\alpha_i}^U - x)}{(x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U}, & (x)_{\alpha_{i+1}}^U \leq x \leq (x)_{\alpha_i}^U, \quad i = 0, 1, \dots, n-1 \end{cases} \quad (31)$$

where  $\Delta\alpha_i = \alpha_{i+1} - \alpha_i, i = 0, 1, \dots, n-1$  and  $0 = \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \alpha_n = 1$ . Fig. 6 shows the graphical representation of the above piecewise linear membership function.

Under the assumption of piecewise linearity, we have

$$\int_a^d \mu_{\tilde{A}}(x) dx = \int_{(x)_{\alpha_0}^L}^{(x)_{\alpha_1}^L} \mu_{\tilde{A}}(x) dx + \dots + \int_{(x)_{\alpha_{n-1}}^L}^{(x)_{\alpha_n}^L} \mu_{\tilde{A}}(x) dx + \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} \mu_{\tilde{A}}(x) dx + \int_{(x)_{\alpha_n}^U}^{(x)_{\alpha_{n-1}}^U} \mu_{\tilde{A}}(x) dx + \dots + \int_{(x)_{\alpha_1}^U}^{(x)_{\alpha_0}^U} \mu_{\tilde{A}}(x) dx, \quad (32)$$

$$\int_a^d x \mu_{\tilde{A}}(x) dx = \int_{(x)_{\alpha_0}^L}^{(x)_{\alpha_1}^L} x \mu_{\tilde{A}}(x) dx + \dots + \int_{(x)_{\alpha_{n-1}}^L}^{(x)_{\alpha_n}^L} x \mu_{\tilde{A}}(x) dx + \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} x \mu_{\tilde{A}}(x) dx + \int_{(x)_{\alpha_n}^U}^{(x)_{\alpha_{n-1}}^U} x \mu_{\tilde{A}}(x) dx + \dots + \int_{(x)_{\alpha_1}^U}^{(x)_{\alpha_0}^U} x \mu_{\tilde{A}}(x) dx. \quad (33)$$

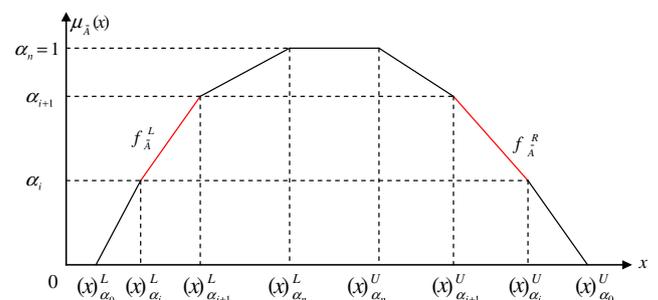


Fig. 6. Piecewise linear membership function represented by  $\alpha$ -level sets.

Let

$$Q_{iL} = \int_{(x)_{\alpha_i}^L}^{(x)_{\alpha_{i+1}}^L} \mu_{\tilde{A}}(x) dx, \quad i = 0, 1, \dots, n-1 \tag{34}$$

$$Q_m = \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} \mu_{\tilde{A}}(x) dx, \tag{35}$$

$$Q_{iU} = \int_{(x)_{\alpha_{i+1}}^U}^{(x)_{\alpha_i}^U} \mu_{\tilde{A}}(x) dx, \quad i = 0, 1, \dots, n-1 \tag{36}$$

$$R_{iL} = \int_{(x)_{\alpha_i}^L}^{(x)_{\alpha_{i+1}}^L} x \mu_{\tilde{A}}(x) dx, \quad i = 0, 1, \dots, n-1 \tag{37}$$

$$R_m = \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} x \mu_{\tilde{A}}(x) dx, \tag{38}$$

$$R_{iU} = \int_{(x)_{\alpha_{i+1}}^U}^{(x)_{\alpha_i}^U} x \mu_{\tilde{A}}(x) dx, \quad i = 0, 1, \dots, n-1 \tag{39}$$

Then we have

$$\begin{aligned} Q_{iL} &= \int_{(x)_{\alpha_i}^L}^{(x)_{\alpha_{i+1}}^L} \left[ \alpha_i + \frac{\Delta\alpha_i(x - (x)_{\alpha_i}^L)}{(x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L} \right] dx \\ &= \alpha_i((x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L) + \frac{\Delta\alpha_i}{(x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L} \\ &\quad \times \left[ \frac{1}{2}((x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L}) - (x)_{\alpha_i}^L((x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L) \right] \\ &= \left( \alpha_i + \frac{1}{2} \Delta\alpha_i \right) ((x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L), \quad i = 0, 1, \dots, n-1 \end{aligned} \tag{40}$$

$$Q_m = \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} \mu_{\tilde{A}}(x) dx = \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} dx = (x)_{\alpha_n}^U - (x)_{\alpha_n}^L, \tag{41}$$

$$\begin{aligned} Q_{iU} &= \int_{(x)_{\alpha_{i+1}}^U}^{(x)_{\alpha_i}^U} \left[ \alpha_i + \frac{\Delta\alpha_i((x)_{\alpha_i}^U - x)}{(x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U} \right] dx \\ &= \alpha_i((x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U) + \frac{\Delta\alpha_i}{(x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U} \\ &\quad \times \left[ (x)_{\alpha_i}^U((x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U) - \frac{1}{2}((x)_{\alpha_i}^{2U} - (x)_{\alpha_{i+1}}^{2U}) \right] \\ &= \left( \alpha_i + \frac{1}{2} \Delta\alpha_i \right) ((x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U), \quad i = 0, 1, \dots, n-1 \end{aligned} \tag{42}$$

$$\begin{aligned} R_{iL} &= \int_{(x)_{\alpha_i}^L}^{(x)_{\alpha_{i+1}}^L} x \left[ \alpha_i + \frac{\Delta\alpha_i(x - (x)_{\alpha_i}^L)}{(x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L} \right] dx \\ &= \frac{\alpha_i}{2}((x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L}) + \frac{\Delta\alpha_i}{(x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L} \\ &\quad \times \left[ \frac{1}{3}((x)_{\alpha_{i+1}}^{3L} - (x)_{\alpha_i}^{3L}) - \frac{1}{2}(x)_{\alpha_i}^L((x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L}) \right] \\ &= \frac{\alpha_i}{2}((x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L}) + \frac{\Delta\alpha_i}{6}(2(x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L} - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L), \\ &\quad i = 0, 1, \dots, n-1 \end{aligned} \tag{43}$$

$$R_m = \int_{(x)_{\alpha_n}^L}^{(x)_{\alpha_n}^U} x dx = \frac{1}{2}[(x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L}], \tag{44}$$

$$\begin{aligned} R_{iU} &= \int_{(x)_{\alpha_{i+1}}^U}^{(x)_{\alpha_i}^U} x \left[ \alpha_i + \frac{\Delta\alpha_i((x)_{\alpha_i}^U - x)}{(x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U} \right] dx \\ &= \frac{\alpha_i}{2}((x)_{\alpha_i}^{2U} - (x)_{\alpha_{i+1}}^{2U}) + \frac{\Delta\alpha_i}{(x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U} \\ &\quad \times \left[ \frac{1}{2}(x)_{\alpha_i}^U((x)_{\alpha_i}^{2U} - (x)_{\alpha_{i+1}}^{2U}) - \frac{1}{3}((x)_{\alpha_i}^{3U} - (x)_{\alpha_{i+1}}^{3U}) \right] \\ &= \frac{\alpha_i}{2}((x)_{\alpha_i}^{2U} - (x)_{\alpha_{i+1}}^{2U}) + \frac{\Delta\alpha_i}{6}((x)_{\alpha_i}^{2U} + (x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U - 2(x)_{\alpha_{i+1}}^{2U}), \\ &\quad i = 0, 1, \dots, n-1 \end{aligned} \tag{45}$$

Further, we have

$$\begin{aligned} \int_a^d \mu_{\tilde{A}}(x) dx &= Q_m + \sum_{i=0}^{n-1} (Q_{iL} + Q_{iU}) = (x)_{\alpha_n}^U - (x)_{\alpha_n}^L \\ &\quad + \sum_{i=0}^{n-1} \left[ \left( \alpha_i + \frac{1}{2} \Delta\alpha_i \right) ((x)_{\alpha_{i+1}}^L - (x)_{\alpha_i}^L) \right. \\ &\quad \left. + \left( \alpha_i + \frac{1}{2} \Delta\alpha_i \right) ((x)_{\alpha_i}^U - (x)_{\alpha_{i+1}}^U) \right] \\ &= (x)_{\alpha_n}^U - (x)_{\alpha_n}^L + \sum_{i=0}^{n-1} \left( \alpha_i + \frac{1}{2} \Delta\alpha_i \right) \\ &\quad \times [((x)_{\alpha_i}^U - (x)_{\alpha_i}^L) - ((x)_{\alpha_{i+1}}^U - (x)_{\alpha_{i+1}}^L)] \\ &= \frac{1}{2} \left[ (x)_{\alpha_n}^U - (x)_{\alpha_n}^L - \sum_{i=1}^{n-1} \alpha_i ((x)_{\alpha_{i+1}}^U \right. \\ &\quad \left. - (x)_{\alpha_{i+1}}^L) + \sum_{i=0}^{n-1} \alpha_{i+1} ((x)_{\alpha_i}^U - (x)_{\alpha_i}^L) \right] \end{aligned} \tag{46}$$

$$\begin{aligned} \int_a^d x \mu_{\tilde{A}}(x) dx &= R_m + \sum_{i=0}^{n-1} (R_{iL} + R_{iU}) = \frac{1}{2}[(x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L}] \\ &\quad + \sum_{i=0}^{n-1} \left[ \frac{\alpha_i}{2}((x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L}) + \frac{\Delta\alpha_i}{6}(2(x)_{\alpha_{i+1}}^{2L} - (x)_{\alpha_i}^{2L} \right. \\ &\quad \left. - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L) \right] + \sum_{i=0}^{n-1} \left[ \frac{\alpha_i}{2}((x)_{\alpha_i}^{2U} - (x)_{\alpha_{i+1}}^{2U}) + \frac{\Delta\alpha_i}{6}((x)_{\alpha_i}^{2U} \right. \\ &\quad \left. + (x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U - 2(x)_{\alpha_{i+1}}^{2U}) \right] = \frac{1}{2}[(x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L}] \\ &\quad + \frac{1}{2} \sum_{i=0}^{n-1} \alpha_i [((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}) - ((x)_{\alpha_{i+1}}^{2U} - (x)_{\alpha_{i+1}}^{2L})] \\ &\quad + \frac{1}{6} \sum_{i=0}^{n-1} \Delta\alpha_i [((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}) + ((x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U \\ &\quad - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L) - 2((x)_{\alpha_{i+1}}^{2U} - (x)_{\alpha_{i+1}}^{2L})] = \frac{1}{6} \left[ (x)_{\alpha_n}^{2U} - (x)_{\alpha_n}^{2L} \right. \\ &\quad \left. - \sum_{i=1}^{n-1} \alpha_i ((x)_{\alpha_{i+1}}^{2U} - (x)_{\alpha_{i+1}}^{2L}) + \sum_{i=0}^{n-1} \alpha_{i+1} ((x)_{\alpha_i}^{2U} - (x)_{\alpha_i}^{2L}) \right] \\ &\quad + \frac{1}{6} \sum_{i=0}^{n-1} \Delta\alpha_i ((x)_{\alpha_i}^U \cdot (x)_{\alpha_{i+1}}^U - (x)_{\alpha_i}^L \cdot (x)_{\alpha_{i+1}}^L). \end{aligned} \tag{47}$$

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