ELSEVIER

European Journal of Operational Research 95 (1996) 115–138

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Preference modelling by estimating local utility functions for multiobjective optimization

Jian-Bo Yang [a,*], Pratyush Sen [b]

[a] *Operational Research Group, School of Manufacturing and Mechanical Engineering, University of Birmingham, Birmingham B15 2TT, UK*

[b] *Engineering Design Centre, University of Newcastle upon Tyne, Newcastle upon Tyne, NE1 7RU, UK*

Received May 1993; revised September 1994

Abstract

This paper is intended to design goal programming models for capturing the decision maker's (DM's) preference information and for supporting the search for the best compromise solutions in multiobjective optimization. At first, a linear goal programming model is built to estimate piecewise linear local utility functions based on pairwise comparisons of efficient solutions as well as objectives. The interactive step trade-off method (ISTM) is employed to generate a typical subset of efficient solutions of a multiobjective problem. Another general goal programming model is then constructed to embed the estimated utility functions in the original multiobjective problem for utility optimization using ordinary nonlinear programming algorithms. This technique, consisting of the ISTM method and the newly investigated search process, facilitates the identification and elimination of possible inconsistent information which may exist in the DM's preferences. It also provides various ways to carry out post-optimality analysis to test the robustness of the obtained best solutions. A modified nonlinear multiobjective management problem is taken as example to demonstrate the technique.

Keywords: Multiobjective optimization; Preference modelling; Interactive methods; Utility functions

1. Introduction

Multiobjective optimization problems are common in practice [3,4,11,21]. In many decision situations, interactive methods are suitable for solving such problems [1,2,13,14,18–20]. At a preliminary design stage of a large engineering product, for example, interactive methods allow the solution to progress towards a preferred solution through an evolving or an adaptive approach. This mirrors the common engineering design processes of evolving design and adaptive design.

As a learning-oriented interactive technique, the ISTM method supports the DM to freely search for favourable efficient solutions, which leads to a progressive, implicit articulation of DM priorities [15,16,23–25]. The DM can discover his true preferences by such implicit trade-off studies as is often done in real-life non-technical decision situations. This is one of the features of the ISTM method that has been found to be

* Corresponding author.

SSDI 0377-2217(96)00300-9

attractive to the users of an integrated multiple criteria decision support system (IMC-DSS) [15,25], which is under constant development and currently consists of eight multiple objective decision making (MODM) and multiple attribute decision making (MADM) methods with ISTM being one of the MODM methods.

It needs to be pointed out that other interactive methods such as the ε-constraint method [3], the STEM method and other minimax formulation related methods [7,15,19,20,25] may also provide good procedures for interactive preference learning. It has been well known that no single interactive method could be absolutely better than others in multiobjective optimization. In fact, any interactive method has its own features as well as weak points. Which method should be used to a large extent depends upon the decision situation where the method may be applied and the persons who may use it. This is one of the main reasons why multiple MODM and MADM methods have been elaborately selected for the IMC-DSS.

In its present version of ISTM, the DM is expected to directly terminate the interactive process when he recognizes that his best compromise solution has been evolved. Such a termination strategy may be implemented when the DM no longer wishes to improve an objective at the expense of any other objectives. As the DM is only able to examine a limited number of solutions in such a pure learning process, this non-directed termination strategy can suffer from depending on too much guesswork on the part of the DM [2], without any explicit evidence on whether the best compromise solution generated is precisely the solution which the DM really favours. In other words, other favourable (perhaps better) solutions may still be available but remain unexamined. Furthermore, such a strategy lacks a mechanism for checking the consistency of the preferences that the DM provides.

In this paper, the original version of ISTM is used for generating favourable efficient solutions with each solution being comparable with others. A search process investigated in this paper is then incorporated into ISTM. This search process is composed of the estimation and optimization of a set of utility functions on the basis of the preferences provided by the DM. The preferences are acquired in terms of the pairwise comparisons of the generated efficient solutions and maybe the objectives as well. A utility estimation model is designed so that any inconsistency in the preferences can be identified and eliminated whenever necessary. A search model is then constructed to facilitate the use of existing nonlinear programming algorithms for the search of the best compromise solutions. The basic concept of the above framework originates from the UTA method [8,10].

The UTA method adopted additive utility functions for representing DM preferences [8]. In UTA, a class of linear regression models were discussed for assessing a set of additive piecewise linear utility functions by using pairwise comparisons of a subset of solutions. UTA was initially used for dealing with MADM problems [8]. It was then extended to treat MODM problems [9][10]. One of the important contributions made by the UTA methods may be that a clear two step procedure was designed for multiobjective optimization, that is, preference learning and utility optimization for the best compromise solution. In such a procedure, the DM is allowed to conduct post-optimality analysis in an explicit manner, which may be favoured by practitioners. The research as reported in this paper is therefore to a large extent based on this procedure and the linear utility estimation models of UTA.

From our experience of using UTA [15,25], we greatly value its underlying principle and abilities to solve decision problems, especially MADM problems. UTA has actually been selected as one of the MADM methods in the IMC-DSS. However, it is also based on our experience and studies that its limitations have been discovered for example in identifying inconsistent preferences and in using existing ordinary linear or nonlinear programming algorithms for utility optimization. It is to address such problems that this research was initialized. First of all, the utility estimation model proposed in UTA does not clearly define estimation errors in the sense that model approximation errors and preference inconsistency errors are not distinguished. Consequently, when errors do occur, clear indication may not be given as to whether the inconsistency exists or the model structure is inappropriate. A new linear goal programming model explored in this paper distinguishes estimation errors into inconsistency errors and approximation errors. Such separate treatment of the errors is especially necessary in an interactive articulation of preferences as it may be easier to capture inconsistent information in an interactive process of evaluating a lot of solutions.

The justification of an additive utility function depends on a fundamental hypothesis that objectives satisfy a mutual preferential independence condition [5,8,11]. In many situations, this hypothesis can be satisfied at least in a sub-space (or local region) of the feasible region if not in the whole region. Based on such recognition, the new preference estimation model is only intended to estimate a set of additive local utility functions with marginal utility functions being of a piecewise linear structure. By 'local' in this paper, it is meant that the estimated utility functions only represent the preferences which the DM has already provided and that the functions are only valid within a local region which should be defined by the DM.

The search model is elaborately constructed as a linear or nonlinear goal programming problem, depending upon the type of the original problem. In this model, the estimated utility functions are optimized in the defined feasible local region to locate the best compromise solutions. The technique suggested in this paper for expressing the additive piecewise linear local utility functions allows the optimization to be implemented using existing software for ordinary nonlinear optimization.

The following section briefly introduces the ISTM method and discusses problems related to the definition of a local region. In Section 3, a linear goal programming model is investigated for assessing additive piecewise linear local utility functions. A search model is then constructed to search for the best compromise solution. A numerical example about a modified nonlinear multiobjective management problem is fully examined to demonstrate the new technique.

2. An interactive preference learning process

2.1. Multiobjective optimization and local utility functions

A multiobjective optimization problem (MOP) may generally be represented as follows:

(MOP)

$$\begin{aligned} \max \quad & F(X) = \{f_1(X), \dots, f_i(X), \dots, f_K(X)\} \\ \text{s.t.} \quad & X \in \Omega, \end{aligned} \tag{1}$$

where

$$\Omega = \left\{ X \left| \begin{array}{ll} g_i(X) \le 0, & i = 1, \dots, m_1 \\ h_j(X) = 0, & j = 1, \dots, m_2 \\ X = [x_1, \dots, x_n]^T & \end{array} \right. \right\},$$

and where x_i is a decision variable, X denotes a solution, $f_i(X)$ is generally a nonlinear objective function, and $g_i(X)$ and $h_j(X)$ are nonlinear inequality and equality constraint functions. These objectives are usually incommensurate and conflicting with one another. Therefore there normally exist infinite number of efficient (non-dominated or Pareto-optimal) solutions in the MOP. The problem is how to search for a best compromise solution with these multiple objectives being considered simultaneously.

The best compromise solution should be such an efficient solution that can best satisfy the DM's preferences. If preferences are modeled by a utility function aggregating all objective functions into one criterion, denoted by

$$u(F(X)) = u(f_1(X), \dots, f_i(X), \dots, f_K(X)) \quad \text{for } X \in \Omega, \tag{2}$$

then the best compromise solution may be defined as a solution that maximizes the utility function $u(F(X))$ for all $X \in \Omega$.

$u(F(X))$ defined by (2) may be referred to as an overall utility function as it represents the DM's overall preferences and it is valid in the whole feasible decision space. Such a function could be assessed when *a priori* overall preference information is available from the DM. In many cases, such as at the preliminary design stage of a new engineering product, however, such *a priori* overall knowledge may not be available. On the other hand, a utility function may be iteratively assessed if local preference information can be elicited in an interactive manner. In the latter case, the structure of u needs to be assumed.

The simplest utility function is an additive one, which requires the satisfaction of the preferential independence by the objectives. The preferential independence condition essentially means that the tradeoffs or evaluations for any subset of objectives are independent of the achievement levels of other objectives. This condition could be locally satisfied in most situations by introducing a standard level for each objective. Such standard levels could be used to set up a limit to dichotomize feasible solutions into acceptable and not acceptable categories. The best compromise solution may then be expected to fall into the acceptable category. Let f_i^0 and f_i^* be the standard level and the best value of the i-th objective $f_i(X)$, respectively. Then, $[f_i^0 \; f_i^*]$ is defined as the acceptable interval of $f_i(X)$, and

$$\Omega_a = \left\{X \mid f_i(X) \in \left[f_i^0 \; f_i^*\right],\ i = 1, \ldots, K;\ X \in \Omega\right\} \tag{3}$$

as the acceptable decision space of Ω, or a local region of Ω ($\Omega_a \subseteq \Omega$). It is fundamental to assume that f_i^0 ($i = 1, \ldots, K$) are set up so that Ω_a is at least not empty if $\Omega \neq \emptyset$. Any $X \in \Omega_a$ is thus called an acceptable solution or an acceptable efficient solution if it is also Pareto-optimal.

If the tradeoff analysis for objectives can be conducted independently in the local region Ω_a, then an additive local utility function aggregating the objective functions for $K \geq 3$ can be defined by [12]

$$u(F(X)) = \sum_{i=1}^{K} u_i(f_i(X)) \quad \text{for } X \in \Omega_a, \tag{4}$$

where $u_i(f_i(X))$ is generally a nonlinear marginal utility function of $f_i(X)$. By 'local' in this paper, it is meant that $u(F(X))$ is estimated using the DM's local preferences articulated on the basis of a sub-set of acceptable solutions and that $u(F(X))$ is only valid in Ω_a.

To enable the DM to provide his preferences, it is necessary to generate a representative sub-set of acceptable efficient solutions. This is because the DM may only be able to clearly know what solutions he really favours after he has been shown what solutions are available. As the standard levels are assigned and may be modified by the DM in an interactive manner, this may necessarily lead to an interactive generation of the acceptable solutions.

2.2. A preference learning process based on ISTM

A preference learning process based on ISTM is developed to support the DM to freely conduct implicit trade-off analysis along the acceptable efficient frontier of problem (1) in a natural manner, so that a representative sub-set of acceptable efficient solutions could be evolved. Suppose $\{X^0, \ldots, X^h, \ldots, X^l, \ldots, X^{t-1}\}$ is a sub-set of acceptable efficient solutions already generated with each of the solutions being comparable with at least one of the others. Let P be the strict preference relation and I the indifference relation between two solutions. The relation that X^l is preferred to X^h is then denoted by $X^l P X^h$, and that X^l is indifferent to X^h by $X^l I X^h$. The ISTM method provides a mechanism for generating from a current solution X^{l-1} a new efficient solution, denoted by X^l, which is pairwisely comparable with other solutions already generated. This process is composed of three main steps. The first two steps are used for implicit trade-off analysis among the objectives and the third step is for generation of X^l.

To generate X^l, the DM is expected to indicate which objectives need to improved from their current values at X^{l-1}, which must be kept at least at their current values and which may be sacrificed for the improvement of the designated objectives. This first step is actually used to determine the trade-off direction, along which X^l may lie. As a result, the set of the objectives can be classified into three subsets. Suppose W denotes the index subset of objective functions $f_i(X)$ which need to be improved from $f_i(X^{l-1})$, R the index subset of objective functions $f_j(X)$ which must be kept at $f_j(X^{l-1})$, and Z the index subset of objective functions $f_k(X)$ which may be sacrificed from $f_k(X^{l-1})$. Let

$$W = \{i \mid i = i_1, \dots, i_w\}, \quad R_s = \{j \mid j = j_1, \dots, j_r\}, \quad Z = \{k \mid k = k_1, \dots, k_z\}, \tag{5}$$

where $W \cup R \cup Z = \{1, \dots, K\}$ and $W \cap R = W \cap Z = R \cap Z = \emptyset$.

Then, the DM is queried about the extent to which he would like to sacrifice $f_k(X)$ $(k \in Z)$ so that a new (perhaps better) solution X^l can be generated at which $f_i(X)$ $(i \in W)$ will be improved as greatly as possible. This second step is actually used to choose the trade-off step sizes. Suppose $\mathrm{d}f_k(X^{l-1})$ is the step size assigned for $f_k(X)$ $(k \in Z)$ while $(f_k(X^{l-1}) - \mathrm{d}f_k(X^{l-1}))$ must not be smaller than f_k^0. ISTM provides a few trade-off questions to help assign the step sizes by predicting the increment of $f_i(X)$ given a small decrement $\mathrm{d}f_k(X^{l-1})$ of $f_k(X)$ [24].

Question Q_k^i: Suppose all $f_l(X)$ $(l = 1, \dots, K;\ l \neq i, k)$ are kept at their current levels, $f_l(X^{l-1})$. If $f_k(X)$ is sacrificed from $f_k(X^{l-1})$ by a small amount of $\mathrm{d}f_k(X^{l-1})$, $f_i(X)$ will be improved from $f_i(X^{l-1})$ to $f_i(X^l)$ by an amount of about $h_i \lambda_k^{l-1}\ \mathrm{d}f_k(X^{l-1})$. Then, assign $\mathrm{d}f_k(X^{(l-1})$ for $f_k(X)$ as the amount the DM is willing to sacrifice for the improvement of $f_i(X)$.

In Q_k^i, X^l is the optimal solution of an auxiliary problem AP^l to be defined in the following, λ_k^{l-1} the optimal Kuhn–Tucker or simplex multiplier of AP^{l-1} with respect to an objective constraint $f_k(X) \geq f_k(X^{l-2}) - \mathrm{d}f_k(X^{l-2})$, and h_i a constant. To assign $\mathrm{d}f_k X^{(l-1})$, at most the DM may need to answer w trade-off questions (i.e., $Q_k^i, \dots, Q_k^{i_w}$).

In the third step, an auxiliary problem is constructed, which embodies the above the trade-off analysis. The optimal solution of this problem must be consistent with the trade-off analysis. Suppose d_i and c_i express the

Table 1
Trade-off table for preference learning

Efficient solutions Γ	Objectives				
	$f_1(X^t)$	$\cdots$	$f_i(X^t)$	$\cdots$	$f_K(X^t)$
X^0	$f_1(X^0)$	$\cdots$	$f_i(X^0)$	$\cdots$	$f_K(X^0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X^h	$f_1(X^h)$	$\cdots$	$f_i(X^h)$	$\cdots$	$f_K(X^h)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X^l	$f_1(X^l)$	$\cdots$	$f_i(X^l)$	$\cdots$	$f_K(X^l)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X^T	$f_1(X^T)$	$\cdots$	$f_i(X^T)$	$\cdots$	$f_K(X^T)$

best value and the lower bound of the i-th objective function, respectively, so that $d_i > c_i$. Let

$$h_i = d_i - c_i, \quad i = 1, \dots, K. \tag{6}$$

Suppose $X_w = [X^T, y_{i_1}, \dots, y_{i_w}]^T$, where y_i ($i \in W$) are auxiliary variables. Then, an auxiliary problem AP^t can be defined as follows [24]:

(AP^t)

$$\begin{aligned} \max \quad & y = \sum_{i \in W} \tau_i y_i \\ \text{s.t.} \quad & X_w \in \Omega_w \end{aligned} \tag{7}$$

where

$$\Omega_w = \left\{ X_w \;\middle|\; \begin{array}{l} f_i(X) - h_i y_i \geq f_i(X^{t-1}), \quad y_i \geq 0, \quad i \in W \\ f_j(X) \geq f_j(X^{t-1}), \quad j \in R \\ f_k(X) \geq f_k(X^{t-1}) - \mathrm{d}f_k(X^{t-1}), \quad k \in Z \\ X \in \Omega \end{array} \right\},$$

and where y_i is maximized to improve $f_i(X)$ as greatly as possible and τ_i is a positive weighting factor representing the relative importance of the i-th objective among the w objectives for improvement. Normally, let $\tau_i = 1$ for $i \in W$. The optimal solution of AP^t is denoted by X^t. It has been proven that X^t is an efficient solution of the original problem (1) [24]. It may be noted that the optimal solution X^{t-1} of AP^{t-1} is a feasible solution of AP^t. X^{t-1} may therefore be used as a feasible initial point to search for the optimum of AP^t.

ISTM provides a strategy to construct the first auxiliary problem AP^0 for generating the first acceptable X^0, as discussed in Section 5. Then, the second efficient solution denoted by X^1 can be generated from X^0 while X^1 should be comparable with X^0, either X^0PX^1 or X^1PX^0 or X^1IX^0. From X^1, the third efficient solution X^2 may be generated, which should be comparable with either X^0 or X^1 or both. As the process proceeds, the DM may gain an increasingly better insight into the acceptable decision space of the problem.

As a result, more and maybe better preferences may be gathered in terms of the pairwise comparisons of the generated acceptable efficient solutions. This learning process of preferences is terminated either if the DM is able to judge that one of the generated solutions is precisely the best compromise solution he expects, or if he recognizes that he has gained a sufficiently comprehensive insight into the acceptable decision space and that the preferences he has provided reflect the insight. In the latter case, a search process can be initiated to help the DM search for the best compromise solution.

As a result of the learning process, a trade-off table, as shown in Table 1, can be built as a basis for constructing a search process. In Table 1, T acceptable efficient solutions are generated, each of which is pairwisely comparable with at least one of the others. Let Γ be the set of the generated acceptable solution, or $\Gamma = \{X^0, \dots, X^h, \dots, X^l, \dots, X^T\}$. A pairwise comparison on two comparable solutions X^l and X^h, denoted by (X^l, X^h), may be X^lPX^h or X^hPX^l or X^lIX^h. Let Ω_P denote the set of all pairs of solutions with the preference relation P and Ω_I the set of all pairs of solutions with the preference relation I, that is,

$$\Omega_P = \{(X^l, X^h) \mid X^lPX^h \text{ for } all\; X^l, X^h \in \Gamma\}, \tag{8}$$

$$\Omega_I = \{(X^l, X^h) \mid X^lIX^h \text{ for } all\; X^l, X^h \in \Gamma\}. \tag{9}$$

3. Estimation of local utility functions

3.1. Piecewise linear local utility functions

In the additive local utility function defined by (4), a marginal utility function, say $u_i(f_i(X))$, may be constructed as a piecewise linear function valid in the acceptable interval of $f_i(X)$. Using the same idea as in the UTA method [8], we can cut the acceptable interval $[f_i^0, f_i^*]$ into N_i equal sub-intervals $[f_i^{j-1}, f_i^j]$. The end points of these intervals are given by

$$f_i^j = f_i^0 + (j/N_i)(f_i^* - f_i^0), \quad j = 0, 1, \ldots, N_i, \tag{10}$$

where $f_i^{N_i} = f_i^*$. The heuristics for selection of N_i can be found in [8]. For instance, we may assign N_i so that $1 \le N_i \le 10$. Note that some of the equations and constraints for the estimation model explored in this section originate from the UTA method and citation is therefore made. It is for the integrity of discussion that they are listed here. Details about these equations and constraints can be found in the references.

The marginal utilities at the end points, i.e. $u_i(f_i^j)$ $(j = 0, 1, N_i;\ i = 1, \ldots, K)$, are the variables to be estimated. When the optimal estimation of $u_i(f_i^j)$ is obtained, the marginal utility of an acceptable solution X^l on an objective $f_i(X)$, i.e. $u_i(f_i(X^l))$, can then be approximated by a linear interpolation $\bar{u}_i(f_i(X^l))$, that is,

$$u_i(f_i(X^l)) = \bar{u}_i(f_i(X^l)) + \sigma_{il} \quad \text{for } f_i(X^l) \in [f_i^0, f_i^*], \tag{11}$$

$$\bar{u}_i(f_i(X^l)) = u_i(f_i^{j-1}) + \frac{f_i(X^l) - f_i^{j-1}}{f_i^j - f_i^{j-1}}\left[u_i(f_i^j) - u_i(f_i^{j-1})\right] \quad \text{if } f_i(X^l) \in [f_i^{j-1}, f_i^j], \tag{12}$$

where σ_{il} is a real number, representing an approximation error caused by using the linear interpolation $\bar{u}_i(f_i(X^l))$ to approximate the marginal utility $u_i(f_i(X^l))$. If $\sigma_{il} = 0$, $u_i(f_i(X^l))$ is equivalent to $\bar{u}_i(f_i(X^l))$. The utility of X^l is given by

$$u(F(X^l)) = \sum_{i=1}^{K} u_i(f_i(X^l)) = \sum_{i=1}^{K} \bar{u}_i(f_i(X^l)) + \sigma_l \quad \text{for } X^l \in \Omega_a, \tag{13}$$

where $\sigma_l = \sum_{i=l}^{K} \sigma_{il}$. σ_l is a real number, indicating the total approximation error. It is desirable that the variables $u_i(f_i^j)$ $(j = 0, 1, \ldots, N_i;\ i = 1, \ldots, K)$ are so estimated that $|\sigma_l|$ is minimized. Note that in UTA a similar error variable was defined but assumed to be non-negative. Such an assumption essentially means that true utility of a solution is never lower than its approximate utility obtained by linear interpolation. This assumption may hold in some cases for example if u is concave, but may not always be justified. Allowing σ_l to take any real number necessarily leads to a nonlinear estimation model as shown later.

Suppose f_i^- is the least preferred value (lower bound) of the i-th objective with $f_i^- \le f_i^0$. The utility function defined by (4) may be normalized by the following constraints [8,11]:

$$\sum_{i=1}^{K} u_i(f_i^*) = 1, \qquad u_i(f_i^-) = 0 \quad \text{for all } i = 1, \ldots, K. \tag{14}$$

This normalization transforms $u_i(f_i^*)$ to be equivalent to the relative weight of the objective function $f_i(X)$ [8]

3.2. Estimation of optimal utility functions

The marginal utilities of the end points, i.e., $u_i(f_i^j)$ ($j=0, 1, \ldots, N_i$; $i=1, \ldots, K$), may be estimated using a regression model, such as a linear goal programming model to be investigated in this subsection. In addition to the boundary conditions defined by (14) and (15), the consistency and monotonicity conditions for a utility function may also need to be taken into consideration.

It is essential that a utility function be consistent with the DM's preferences [3,8,11], that is,

$$u(F(X^l)) > u(F(X^h)) \quad \Leftrightarrow \quad (X^l, X^h) \in \Omega_P. \tag{16}$$

$$u(F(X^l)) = u(F(X^h)) \quad \Leftrightarrow \quad (X^l, X^h) \in \Omega_I. \tag{17}$$

The pairwise comparisons of the generated solutions listed in Table 1 may then be transformed into the following consistency constraints:

$$u(F(X^l)) - u(F(X^h)) + d_{lh}^+ \geq \delta \quad \text{if } (X^l, X^h) \in \Omega_P, \tag{18}$$

$$u(F(X^l)) - u(F(X^h)) + d_{lh} = 0 \quad \text{if } (X^l, X^h) \in \Omega_I, \tag{19}$$

where $\delta(>0)$ is a small constant, d_{lh}^+ a non-negative real number and d_{lh} a real number. δ may be assigned as in UTA.

d_{lh}^+ (or d_{lh}) is introduced as a new inconsistency error variable. d_{lh}^+ (or $|d_{lh}|$) indicates the extent to which the preference relation X^lPX^h (or X^lIX^h) is inconsistent with the other comparisons in Ω_P and Ω_I with respect to the multiple criteria evaluations of X^l for $X^l \in \Gamma$. In other words, d_{lh}^+ (or $|d_{lh}|$) denotes the degree of inconsistency which needs to be minimized. This is because d_{lh}^+ (or d_{lh}) is defined only for the comparison (X^l, X^h), independently of any other comparisons or the assumption about the structure of u. The magnitude of d_{lh}^+ (or $|d_{lh}|$) can thus be used as a means for the identification of inconsistency errors.

Considering (13), constraints (18) and (19) can be re-written as the following equivalent forms:

$$\sum_{i=1}^{K} \{\bar{u}_i(f_i(X^l)) - \bar{u}_i(f_i(X^h))\} + \sigma_l - \sigma_h + d_{lh}^+ \geq \delta \quad \text{if } (X^l, X^h) \in \Omega_P, \tag{20}$$

$$\sum_{i=1}^{K} \{\bar{u}_i(f_i(X^l)) - \bar{u}_i(f_i(X^h))\} + \sigma_l - \sigma_h + d_{lh} = 0 \quad \text{if } (X^l, X^h) \in \Omega_I, \tag{21}$$

where $\bar{u}_i(f_i(X^l))$ is calculated using the formula (12). Note that σ_l will appear in more than one constraints if X^l is compared with more than one solutions, that is, σ_l is generally not peculiar to a single constraint. This means that σ_l is generally incapable of clearly indicating which comparison may cause the most serious inconsistency error.

Besides, the marginal unitities need to satisfy the following monotonicity conditions [8]:

$$u_i(f_i^j) \geq u_i(f_i^{j-1}) \quad \text{for } j=1, \ldots, N_i, \quad i=1, \ldots, K. \tag{22}$$

In the preferences as shown by (8) and (9), the information about the relative importance of objectives is already implied to some extent. To capture more accurate weight information about objectives, the DM could be asked to precisely assign the weights or provide a minimal amount of weight information [17]. In the former case, the weights of objectives are explicitly given. As mentioned before, the marginal utility $u_i(f_i^{N_i})$ at the best value $f_i^{N_i}$ of $f_i(X)$ is equivalent to the weight of $f_i(X)$ [8,11]. In this case, $u_i(f_i^{N_i})$ is a constant and no longer needs to be estimated in the following estimation model.

Generally, pairwise comparisons about the relative importance of objectives may be captured [17]. If objective i is judged to be at least c_{ij} times as important as objective j, for example, we could formulate the following weight constraint:

$$u_i\left(f_i^{N_i}\right) - c_{ij}u_j\left(f_j^{N_j}\right) + s_{ij}^{+} \geq 0, \tag{23}$$

where s_{ij}^{+} is a non-negative inconsistency error variable. If $f_i(X)$ is judged to be exactly c_{ij} times as important as $f_j(X)$, the following weight constraint could be formulated:

$$u_i\left(f_i^{N_i}\right) - c_{ij}u_j\left(f_j^{N_j}\right) - s_{ij}^{-} + s_{ij}^{+} = 0, \tag{24}$$

$$s_{ij}^{-} \times s_{ij}^{+} = 0, \tag{25}$$

where s_{ij}^{+} and s_{ij}^{+} are non-negative inconsistency error variables. Such judgments could be made for any pair of objectives. For the convenience of further discussion, let us define Ω_f as a set of all such judgments:

$$\Omega_f = \left\{(i, j) \mid f_i(X) \text{ is comparable with } f_j(X),\ i,\ j = 1, \ldots, K;\ i \neq j\right\}. \tag{26}$$

It needs to be pointed out that the above weight information is complementary to that given in (8) and (9). Such information, if available, should be acquired to improve the quality of the estimation of a set of utility functions.

It is always expected that an estimated utility function can model the DM's preferences as consistently and precisely as possible. The objective of the estimation model may therefore be to minimize the total absolute inconsistency errors and the total absolute approximation errors. As the approximation errors only indicate the limitation of a piecewise linear utility function, it may be sensible to assume that the DM's first priority is to minimize the inconsistency errors so that the utility function can be estimated to be as consistent with his preferences as possible. In other words, it would be inappropriate to allow the approximation errors to compensate for the inconsistency errors. The following objective function is therefore suggested:

$$E = P_1\left\{\sum_{(X^l, X^h)\in\Omega_P} d_{lh}^{+} + \sum_{(X^l, X^h)\in\Omega_I} |d_{lh}| + \sum_{(i,j)\in\Omega_f}\left(s_{ij}^{+} + s_{ij}^{-}\right)\right\} + P_2\sum_{X^l\in\Gamma}|\sigma_l|, \tag{27}$$

where P_1 and P_2 represent the priorities for the minimization of the two types of errors and $P_1 \gg P_2$.

Define deviation variables d_{lh}^{+}, d_{lh}^{-}, σ_l^{+} and σ_l^{-} as follows:

$$d_{lh}^{+} = \tfrac{1}{2}(|d_{lh}| + d_{lh}),\quad d_{lh}^{-} = \tfrac{1}{2}(|d_{lh}| - d_{lh}),\qquad \sigma_l^{+} = \tfrac{1}{2}(|\sigma_l| + \sigma_l),\quad \sigma_l^{-} = \tfrac{1}{2}(|\sigma_l| - \sigma_l). \tag{28}$$

Then we have

$$d_{lh}^{+} + d_{lh}^{-} = |d_{lh}|,\quad d_{lh}^{+} - d_{lh}^{-} = d_{lh},\quad d_{lh}^{+} \times d_{lh}^{-} = 0,\quad d_{lh}^{+},\ d_{lh}^{-} \geq 0, \tag{29}$$

$$\sigma_l^{+} + \sigma_l^{-} = |\sigma_l|,\quad \sigma_l^{+} - \sigma_l^{-} = \sigma_l,\quad \sigma_l^{+} \times \sigma_l^{-} = 0,\quad \sigma_l^{+},\ \sigma_l^{-} \geq 0. \tag{30}$$

The marginal utilities, $u_i(f_i^j)$ ($j = 0, 1, \ldots, N_i$; $i = 1, \ldots, K$) may then be estimated by the following linear goal programming (LGP):

(LGP)

$$\min\ P_1\left\{\sum_{(X^l, X^h)\in\Omega_P} d_{lh}^{+} + \sum_{(X^l, X^h)\in\Omega_I}\left(d_{lh}^{+} + d_{lh}^{-}\right) + \sum_{(i,j)\in\Omega_f}\left(s_{ij}^{+} + s_{ij}^{-}\right)\right\} + P_2\sum_{X^l\in\Gamma}\left(\sigma_l^{+} + \sigma_l^{-}\right) \tag{31}$$

$$\text{s.t.}\quad \sum_{i=1}^{K}\left\{\bar{u}_i\left(f_i(X^l)\right) - \bar{u}_i\left(f_i(X^h)\right)\right\} + \sigma_l^{+} - \sigma_l^{-} - \sigma_h^{+} + \sigma_h^{-} + d_{lh}^{+} \geq \delta \quad \text{if } (X^l, X^h) \in \Omega_P,$$

$$\sum_{i=1}^{K}\left\{\bar{u}_i(f_i(X^l))-\bar{u}_i(f_i(X^h))\right\}+\sigma_l^+-\sigma_l^--\sigma_h^++\sigma_h^-+d_{lh}^+-d_{lh}^-=0 \quad \text{if } (X^l, X^h)\in\Omega_I,$$

$$u_i(f_i^j)\geq u_i(f_i^{j-1})\geq 0 \quad \text{for } j=1,\ldots,N_i, \quad i=1,\ldots,K,$$

$$\sum_{i=1}^{K}u_i(f_i^{N_i})=1, \qquad u_i(f_i^0)-\frac{f_i^0-f_i^-}{f_i^1-f_i^0}\left[u_i(f_i^1)-u_i(f_i^0)\right]=0 \quad \text{for } i=1,\ldots,K,$$

$$u_i(f_i^{N_i})-c_{ij}u_j(f_j^{N_j})+s_{ij}^+\geq 0, \text{ or}$$

$$u_i(f_i^{N_i})-c_{ij}u_j(f_j^{N_j})-s_{ij}^-+s_{ij}^+=0 \quad \text{for } all\ (i, j)\in\Omega_f,$$

$$\sigma_l^+\times\sigma_l^-=0, \quad d_{lh}^+\times d_{lh}^-=0, \quad s_{ij}^+\times s_{ij}^-=0,$$

$$\sigma_l^+,\ \sigma_l^-\geq 0, \quad d_{lh}^+,\ d_{ij}^-\geq 0, \quad s_{ij}^+,\ s_{ij}^-\geq 0$$

for *all* $X^l\in\Gamma$; $(X^l, X^h)\in\Omega_P\cup\Omega_I$; and $(i, j)\in\Omega_f$.

The solution of LGP is denoted by $\hat{u}_i(f_i^j)$ $(j=0, 1,\ldots,N_i; 1,\ldots,K)$, $\hat{\sigma}_l^+$, $\hat{\sigma}_l^-$ for all $X^l\in\Gamma$, $\hat{d}_{lh}^+$, $\hat{d}_{lh}^-$ for all $(X^l, X^h)\in\Omega_P\cup\Omega_I$ and $\hat{s}_{ij}^+$ and $\hat{s}_{ij}^-$ for all $(i, j)\in\Omega_f$. $\hat{u}_i(f_i^j)$ is referred to as the optimal marginal utility of f_i^j. The utility of an acceptable solution X may then be calculated by the following optimal local utility function:

$$u(F(X))=\sum_{i=1}^{K}\left\{\hat{u}_i(f_i^{j-1})+\frac{f_i(X)-f_i^{j-1}}{f_i^j-f_i^{j-1}}\left[\hat{u}_i(f_i^j)-\hat{u}_i(f_i^{j-1})\right] \quad \text{if } f_i(X)\in\left[f_i^{j-1}\ f_i^j\right]\right\}$$

$$\text{for } X\in\Omega_a. \tag{32}$$

If $\hat{d}_{lh}^+=\hat{d}_{lh}^-=0$ for all $(X^l, X^h)\in\Omega_P\cup\Omega_I$ and $\hat{s}_{ij}^+=\hat{s}_{ij}^-=0$ for all $(i, j)\in\Omega_f$, the obtained optimal marginal utilities are consistent with the DM's preferences. If any $\hat{d}_{lh}^+$, $\hat{d}_{lh}^-$, $\hat{s}_{ij}^+$ or $\hat{s}_{ij}^-\neq 0$, either the DM has provided inconsistent preferences, or his preferences may not be appropriately represented using the local additive utility function.

If $\hat{\sigma}_l^+=\hat{\sigma}_l^-=0$ for all $X^l\in\Gamma$, the assessed optimal piecewise linear utility function (32) can precisely represent the DM's preferences for all of the generated solutions. If any $\hat{\sigma}_l^+$ or $\hat{\sigma}_l^-\neq 0$, (32) can only approximately represent the preferences for the solution X^l.

If $\hat{d}_{lh}^+=\hat{d}_{lh}^-=0$ for all $(X^l, X^h)\in\Omega_P\cup\Omega_I$, $\hat{\sigma}_l^+=\hat{\sigma}_l^-=0$ for all $X^l\in\Gamma$ and $\hat{s}_{ij}^+=\hat{s}_{ij}^-=0$ for all $(i, j)\in\Omega_f$, the assessed optimal utility function (32) can precisely and consistently model the DM's preferences. In this case, there often exist other optimal utility functions which all lead to a perfect representation of the preferences. It is therefore desirable to investigate the set of the possible optimal utility functions so as to select one or a sub-set of favourable optimal utility functions.

3.3. Estimation of mean utility function

An optimal utility function may be more favorable if it is smoother. Although it is generally difficult to find the smoothest optimal utility function, it may always be expected that the mean function of a representative set of optimal utility functions may be a good smooth utility function.

If the upper and lower bounds of the permissible optimal values for each marginal utility $u_i(f_i^j)$, i.e. $\bar{u}_i^j$ and $\underline{u}_i^j$, can be found and if a sufficient number of the optimal values for $u_i(f_i^j)$ between the two bounds can be generated, we can obtain a mean value for $u_i(f_i^j)$. A mean utility function may then be fitted using such mean marginal utilities. In LGP, the set of the optimal marginal utilities with all $\hat{d}_{lh}^+$, $\hat{d}_{lh}^-$, $\hat{\sigma}_l^+$, $\hat{\sigma}_l^-$, $\hat{s}_{ij}^+$ and $\hat{s}_{ij}^- = 0$ can be defined by U_{opt} as follows:

$$U_{\mathrm{opt}} = \left\{ U \;\middle|\; \begin{array}{l} \sum_{i=1}^{K} \{\bar{u}_i(f_i(X^l)) - \bar{u}_i(f_i(X^h))\} \geq \delta \quad \text{if } (X^l, X^h) \in \Omega_P \\ \sum_{i=1}^{K} \{\bar{u}_i(f_i(X^l)) - \bar{u}_i(f_i(X^h))\} = 0 \quad \text{if } (X^l, X^h) \in \Omega_I \\ u_i(f_i^j) \geq u_i(f_i^{j-1}) \geq 0 \quad \text{for } j = 1, \ldots, N_i, \quad i = 1, \ldots, K \\ \sum_{i=1}^{K} u_i(f_i^{N_i}) = 1;\; u_i(f_i^0) - \frac{f_i^0 - f_i^-}{f_i^1 - f_i^0}\left[u_i(f_i^1) - u_i(f_i^0)\right] = 0 \quad \text{for } i = 1, \ldots, K \\ u_i(f_i^{N_i}) - c_{ij}u_j(f_j^{N_j}) \geq 0 \text{ or } u_i(f_i^{N_i}) - c_{ij}u_j(f_i^{N_j}) = 0 \quad \text{for all } (i, j) \in \Omega_f \end{array} \right\} \tag{33}$$

where $U = [u_1(f_1^0), \ldots, u_1(f_1^N), \ldots, u_i(f_i^j), \ldots, u_K(f_K^0), \ldots, u_K(f_K^{N_K})]^{\mathrm{T}}$.

To sample U_{opt}, let us define the following problems for $j = 0, 1, \ldots, N_i;\ 1, \ldots, K$:

$$\begin{array}{ll} (\mathrm{LP}(\bar{u}_i^j)) & (\mathrm{LP}(\underline{u}_i^j)) \\ \max \quad u_i(f_i^j) & \min \quad u_i(f_i^j) \\ \text{s.t.} \quad U \in U_{\mathrm{opt}} & \text{s.t.} \quad U \in U_{\mathrm{opt}} \end{array} \tag{34}$$

the solutions of which are expressed by $\bar{U}^{ij}$ and $\underline{U}^{ij}$ respectively:

$$\bar{U}^{ij} = \left[\bar{u}_1^{ij}(f_1^0), \ldots, \bar{u}_i^{ij}(f_i^j), \ldots, \bar{u}_K^{ij}(f_K^{N_K})\right]^{\mathrm{T}}, \qquad \underline{U}^{ij} = \left[\underline{u}_1^{ij}(f_1^0), \ldots, \underline{u}_i^{ij}(f_i^j), \ldots, \underline{u}_K^{ij}(f_K^{N_K})\right]^{\mathrm{T}}.$$

The upper and lower bounds of each marginal utility $u_i(f_i^j)$ are denoted by $\bar{u}_i^j = \bar{u}_i^{ij}(f_i^j)$ and $\underline{u}_i^j = \underline{u}_i^{ij}(f_i^j)$, respectively.

Let $N = \sum_{i=1}^{K}(N_i + 1)$. By solving (34), we can thus obtain a sub-set of $2N$ optimal solutions $\bar{U}^{ij}$ and $\underline{U}^{ij}$ ($j = 0, 1, \ldots, N_i;\ j = 1, \ldots, K$). The mean of these solutions is given by

$$\hat{U} = \frac{1}{2N} \sum_{i=1}^{K} \sum_{j=0}^{N_i} \left(\bar{U}^{ij} + \underline{U}^{ij}\right) = \left[\hat{u}_1(f_1^0), \ldots, \hat{u}_1(f_1^{N_1}), \ldots, \hat{u}_i(f_i^j), \ldots, \hat{u}_K(f_K^0), \ldots, \hat{u}_K(f_K^{N_K})\right]^{\mathrm{T}}, \tag{35}$$

where $\hat{u}_i(f_i^j)$ is a mean optimal value of $u_i(f_i^j)$.

It is easy to show that $\hat{U} \in U_{\mathrm{opt}}$. In fact, $\bar{U}^{ij}$, $\underline{U}^{ij} \in U_{\mathrm{opt}}$ for all $j = 0, 1, \ldots, N_i$; $i = 1, \ldots, K$, and U_{opt} is a convex set. As a convex combination of all $\bar{U}^{ij}$ and $\underline{U}^{ij}$, $\hat{U}$ is hence still in U_{opt}. The utility of an acceptable solution X may then be calculated by the following mean local utility function:

$$u(F(X)) = \sum_{i=1}^{K} \left\{ \hat{u}_i(f_i^{j-1}) + \frac{f_i(X) - f_i^{j-1}}{f_i^j - f_i^{j-1}}\left[\hat{u}_i(f_i^j) - \hat{u}_i(f_i^{j-1})\right] \quad \text{if } f_i(X) \in \left[f_i^{j-1}, f_i^j\right] \right\}$$

$$\text{for } X \in \Omega_a. \tag{36}$$

4. Evolution of best compromise solutions

4.1. Representation and optimization of local utility functions

The estimated additive piecewise linear utility function, denoted by (32) or (36), can be transformed into the following equivalent form [22]:

$$u(F(X)) = \sum_{i=1}^{K} \left\{ \sum_{j=1}^{N_i-1} \alpha_{ij} \,|\, f_i(X) - f_i^j \,|\, + \beta_i f_i(X) + \gamma_i \right\} \text{ for } X \in \Omega_a \tag{37}$$

where

$$\alpha_{ij} = \tfrac{1}{2}(t_{i,j+1} - t_{i,j}), \quad \beta_i = \tfrac{1}{2}(t_{i,1} + t_{i,N_i}) \text{ and } \gamma_i = \tfrac{1}{2}(s_{i,1} + s_{i,N_i}), \tag{38}$$

and $t_{i,j}$ is the slope and $s_{i,j}$ the y-intercept for the j-th section of the marginal utility function $u_i(f_i)$, starting from f_i^{j-1} and being terminated at f_i^j, that is,

$$t_{i,j} = \left(\hat{u}_i(f_i^j) - \hat{u}_i(f_i^{j-1})\right) / \left(f_i^j - f_i^{j-1}\right), \tag{39}$$

$$s_{i,1} = \hat{u}_i(f_i^0) - t_{i,1} f_i^0, \qquad s_{i,N_i} = \hat{u}_i(f_i^{N_i}) - t_{i,N_i} f_i^{N_i}. \tag{40}$$

Let us introduce the following auxiliary variables, a_{ij}^+ and a_{ij}^-:

$$a_{ij}^+ = \tfrac{1}{2}\left\{ |\, f_i(X) - f_i^j \,| + \left(f_i(X) - f_i^j\right)\right\}, \qquad a_{ij}^- = \tfrac{1}{2}\left\{ |\, f_i(X) - f_i^j \,| - \left(f_i(X) - f_i^j\right)\right\}. \tag{41}$$

Then the utility function denoted by (37) can be represented by

$$u(F(X) = \sum_{i=1}^{K} \left\{ \sum_{j=1}^{N_i-1} \alpha_{ij}\left(a_{ij}^+ + a_{ij}^-\right) + \beta_i f_i(X) + \gamma_i \right\} \quad \text{for } X \in \Omega_a, \tag{42}$$

under the restrictions

$$a_{ij}^+ - a_{ij}^- = f_i(X) - f_i^j, \qquad a_{ij}^+ \times a_{ij}^- = 0, \qquad a_{ij}^+, a_{ij}^- \geq 0, \quad j = 1, \ldots, N_i - 1, \quad i = 1, \ldots, K. \tag{43}$$

The best compromise solution may then be obtained by solving the following auxiliary goal programming (GP) problem:

(GP)

$$\begin{aligned} \max \quad & u(F(X)) = \sum_{i=1}^{K} \left\{ \sum_{j=1}^{N_i-1} \alpha_{ij}\left(a_{ij}^+ + a_{ij}^-\right) + \beta_i f_i(X) + \gamma_i \right\} \\ \text{s.t.} \quad & X \in \Omega_a, \\ & f_i(X) - a_{ij}^+ + a_{ij}^- = f_i^j, \quad j = 1, \ldots, N_i - 1, \quad i = 1, \ldots, K, \\ & a_{ij}^+ \times a_{ij}^- = 0, \qquad a_{ij}^+, a_{ij}^- \geq 0, \quad j = 1, \ldots, N_i - 1, \quad i = 1, \ldots, K. \end{aligned} \tag{44}$$

If the original problem MOP denoted by (1) is linear, the problem GP defined by (44) can be treated as a linear goal programming problem. Otherwise, GP is a general non-linear (goal) programming problem, which could be solved using existing software for nonlinear programming. It is easy to prove that the optimal solution of (44) is an efficient solution of (1) if each estimated marginal utility function, say $u_i(f_i(X))$, is a strictly monotonically increasing function of $f_i(X)$ and $\Omega_a \neq \emptyset$.

4.2. Post-optimality analysis

The best compromise solution obtained by solving (44) may be only preferred in a local sense that on the basis of the ordinal pairwise comparisons of the efficient solutions already generated the utility function is assessed which is only valid within the pre-assigned acceptable decision space. If the DM is not confident in or satisfied with this solution, he could conduct post-optimality analysis as follows, so that the confidence in the solution may be built or a more favourable solution may be evolved.

Suppose the pairwise comparisons of the generated efficient solutions are not consistent with one another because of the assumption of the objectives being preferentially independent in the acceptable decision space. Then, the DM can either modify those comparisons with non-zero values of d_{lh}^+ or d_{lh}^- as obtained by solving LGP, or assign a new acceptable decision space Ω_a' by for example increasing the standard levels of some objectives so that all the objectives may become preferentially independent in Ω_a'. In the latter case, a generated efficient solution with an objective value being below the new standard level of the objective will no longer be taken into account for utility assessment and new acceptable solutions may therefore need to be generated for eliciting more preference information.

Suppose the assumed piecewise linear utility function can not precisely represent the DM's preferences when some σ_l^+ or σ_l^- in the solution of LGP is not zero. Then, the acceptable interval of an objective function needs to be divided into smaller equal sub-intervals. However, the sizes of the problems LGP and GP increase linearly with the increase of the number of the equal sub-intervals for an objective function, such as N_i for $f_i(X)$. It is therefore necessary to make a compromise between the number of the equal sub-intervals and the sizes of LGP and GP. It has also been realized, as will be shown by the example of next section, that an estimated utility function is usually non-convex with respect to objective functions. A large number of the equal sub-intervals for objective functions may worsen such non-convexity and may therefore result in a high possibility that an estimated utility function has multiple local optimums. This will make it more difficult to search for the best compromise solution maximizing the utility function.

Suppose the DM is not satisfied with the assignment of the relative weights of the objectives, equivalent to the estimated marginal utilities, i.e. $\omega_i = \hat{u}_i(f_i^*) = \hat{u}_i(f_i^{N_i})$ $(i = 1, \ldots, K)$. He can assign a new set of weights using other methods for weight assignment [7]. Let ω_i be the newly assigned weights for the objective $f_i(X)$. The assessed marginal utilities may then be modified to $\hat{v}_i(f_i^j)$:

$$\hat{v}_i(f_i^j) = \omega_i \hat{u}_i(f_i^j) / \hat{u}_i(f_i^{N_i}), \quad j = 0, 1, \ldots, N_i, \quad i = 1, \ldots, K. \tag{45}$$

It should be noted that this modification does not change the normalized marginal utilities. In fact, suppose $\tilde{u}_i(f_i^j)$ and $\tilde{v}_i(f_i^j)$ are the normalized marginal utilities obtained by dividing $\hat{u}_i(f_i^j)$ and $\hat{v}_i(f_i^j)$ by $\hat{u}_i(f_i^{N_i})$ and $\hat{v}_i(f_i^{N_i})$, respectively. Then, $\tilde{v}(f_i^j) = \tilde{u}_i(f_i^j)$ for all $j = 0, 1, \ldots, N_i$; $i = 1, \ldots, K$. If the modified utilities obtained using (45) become inconsistent with the preference relations defined by (8) and (9), the marginal utilities may need to be re-estimated using the model LGP while the marginal utilities $u_i(f_i^{N_i})$ are fixed to be ω_i for $i = 1, \ldots, K$.

5. Example of application

5.1. Problem description and interactive trade-off analysis

The modified Bow River Valley water quality management problem [7] is taken as example to demonstrate the new technique developed above. This management problem is represented as a nonlinear multiple objective optimization problem, defined as follows, where $f_1(X)$ expresses DO level at Robin State Park, $f_2(X)$ the percentage return on equity at Pierce-Cannery and $f_3(X)$ the addition to the tax rate at Bowville. The treatment

Table 2
Pay-off table

	$f_1(\hat{X}^i)$	$f_2(\hat{X}^i)$	$f_3(\hat{X}^i)$
$\hat{X}^1$	6.79	0.34	9.68
$\hat{X}^2$	6.35	6.28	9.68
$\hat{X}^3$	4.86	0.34	1.04

levels of waste discharges at the Pierce-Cannery, Bowville, and Plympto, denoted by x_1, x_2 and x_3, respectively, are the decision variables to be determined [7].

$$\begin{cases} \max & f_1(X) = 2.0 + 0.524(x_1 - 0.3) + 2.79(x_2 - 0.3) + 0.882(w_1 - 0.3) + 2.65(w_2 - 0.3) \\ \max & f_2(X) = 7.5 - 0.012\left(59/\left(1.09 - x_1^2\right) - 59\right) \\ \min & f_3(X) = 1.8 \times 10^{-3}\left(532/\left(1.09 - x_2^2\right) - 532\right) \end{cases} \tag{46}$$

$$\text{s.t.} \quad X \in \Omega, \quad X = [x_1, x_2, x_3]^{\mathrm{T}},$$

where

$$\Omega = \left\{ X \left| \begin{array}{ll} g_1(X): & 4.75 + 2.27(x_1 - 0.3) \geq 6.0 \\ g_2(X): & 5.1 + 0.177(x_1 - 0.3) + 0.978(x_2 - 0.3) + 0.216(w_1 - 0.3) \\ & +0.768(w_2 - 0.3) \geq 6.0 \\ g_3(X): & 2.50 \times 10^{-3}\left(450/\left(1.09 - x_3^2\right) - 450\right) \leq 1.5 \\ g_4(X): & 1.0 + 0.0332(x_1 - 0.3) + 0.0186(x_2 - 0.3) + 3.34(x_3 - 0.3) \\ & + 0.0204(w_1 - 0.3) + 0.78(w_2 - 0.3) + 2.62(w_3 - 0.3) \geq 3.5 \\ \multicolumn{2}{l}{0.3 \geq x_i \leq 1.0, \quad w_i = 0.39/\left(1.39 - x_i^2\right), \quad i = 1, 2, 3} \end{array} \right. \right\}. \tag{47}$$

By optimizing each of the three objective functions, we can obtain the three single-objective optimal solutions, denoted by $\hat{X}^1$, $\hat{X}^2$ and $\hat{X}^3$ maximizing $f_1(X)$ and $f_2(X)$ and minimizing $f_3(X)$, respectively. The pay-off table and the values of the three decision variables at $\hat{X}^1$, $\hat{X}^2$ and $\hat{X}^3$ are obtained as listed in Table 2 and Table 3.

From Table 2, the best and the worst values of the objectives can be listed by

$$F^* = [f_1^*, f_2^*, f_3^*]^{\mathrm{T}} = [6.79, 6.28, 1.04]^{\mathrm{T}}, \qquad F^- = [f_1^-, f_2^-, f_3^-]^{\mathrm{T}} = [4.86, 0.34, 9.68]^{\mathrm{T}}. \tag{48}$$

It should be noted that f_i^- is not necessarily the worst feasible value of $f_i(X)$.

Table 3
Values of decision variables

	$\hat{x}_1^i$	$\hat{x}_2^i$	$\hat{x}_3^i$
$\hat{X}^1$	1.0	1.0	0.741
$\hat{X}^2$	0.851	1.0	0.744
$\hat{X}^3$	1.0	0.782	0.813

It is assumed that a decision X^i is acceptable if the objective values at X^i satisfy the following conditions:

$$f_1(X^i) \geq 5.0, \quad f_2(X^i) \geq 3.0, \quad f_3(X^i) \leq 6.0, \tag{49}$$

where 5.0, 3.0 and 6.0 express the acceptable (standard) levels for $f_1(X)$, $f_2(X)$ and $f_3(X)$, respectively, and they are assigned and may be modified by the DM. The acceptable intervals of the three objectives and the acceptable decision space of problem (46) may thus be defined by

$$(f_1^0, f_1^*) = (5.0, 6.79), \quad (f_2^0, f_2^*) = (3.0, 6.28), \quad (f_3^0, f_3^*) = (6.0, 1.04),$$
$$\Omega_a = \{X \mid f_i(X) \in (f_i^0, f_i^*),\ i = 1, 2, 3;\ X \in \Omega\}. \tag{50}$$

It is further assumed that the evaluations of any of the three objectives are independent of others if the objective values all fall into the acceptable decision space Ω_a. Obviously, the three optimal solutions generated above are all unacceptable although they are efficient solutions of (46). It is therefore necessary to examine other efficient solutions.

The first acceptable solution can be generated from any one of the three optimal solutions using the ISTM process as described in Section 2. However, we introduce an alternative way as follows. The ideal (infeasible) point F^* is defined by (48). In ISTM [23,24], an ideal feasible solution is generated as the solution of the following problem, which is a variant of formulation (7) obtained by defining $y_i = (f_i^* - f_i^{l-1})/h_i - y_i'$:

$$\min \quad y' = \sum_{i=1}^{K} \tau_i y_i' \tag{51}$$

$$\text{s.t.} \quad X_w \in \Omega_w, \quad X_w = [X^T, y_1', \dots, y_K']^T,$$

where

$$\Omega_w = \{X_w \mid f_i(X) + h_i y_i' \geq f_i^*, \quad y_i' \geq 0,\ i = 1, \dots, K; \quad X \in \Omega\}$$

and where $\tau_i > 0$ $(i = 1, \dots, K)$. If $\tau_1 y_1' = \tau_2 y_2' = \dots = \tau_K y_K' = \lambda$, then (51) is equivalent to the minimax formulation [19].

For this example, suppose $\tau_i = 1$ and let $h_i = |f_i^* - f_i^-| = (i = 1, 2, 3)$. Then, the initial solution may be generated by constructing the following problem:

(AP^1)

$$\min \quad y' = \sum_{i=1}^{3} y_i' \tag{52}$$

$$\text{s.t.} \quad X_w \in \Omega_w \qquad X_w = [X^T\ y_1'\ y_2'\ y_3']^T,\ y_1', y_2', y_3' \geq 0,$$

where

$$\Omega_w = \left\{ X_w \left| \begin{array}{l} f_1(X) + 1.93 y_1' \geq 6.79,\ f_2(X) + 5.94 y_2' \geq 6.28 \\ f_3(X) - 8.64 y_3' \leq 1.04 \\ X \in \Omega, \quad y_1' = y_2' = y_3' \end{array} \right. \right\}.$$

Note that in (52), $f_3(X)$ is for minimization while $-f_3(X)$ is for maximization.

Solving (52). We can obtain the following efficient solution, denoted by $X^1 = [x_1^1, x_2^1, x_3^1]^T$ and $F^1 = [f_1^1, f_2^1, f_3^1]^T$:

$$X^1 = [0.962, 0.956, 0.767]^T, \qquad F^1 = [6.025, 3.922, 4.469]^T. \tag{53}$$

It is clear that $X^1 \in \Omega_a$. If it happens that $X^1 \notin \Omega_a$ for $\tau_i = 1$ $(i = 1, 2, 3)$, we can always assign to τ_i some values other than 1 so that $X^1 \in \Omega_a$ if $\Omega_a \neq \emptyset$.

Suppose from X^1 the DM appreciates to improve $f_2(X)$ and $f_3(X)$ at the expense of $f_1(X)$ while the maximum decrement of $f_1(X)$ is assigned by $\Delta f_1(X^1) = f_1^1 - f_1^0 = 1.025$. The auxiliary problem AP^2, defined by (7) with $t = 2$, can then be constructed as follows:

(AP^2)

$$\begin{aligned} \max \quad & y = y_2 + y_3 && (54)\\ \text{s.t.} \quad & X \in \Omega, \qquad y_2,\ y_3 \geq 0,\\ & f_1(X) \geq 5.0, \quad f_2(X) - 5.94 y_2 \geq 3.922, \quad f_3(X) + 8.64 y_3 \leq 4.469. \end{aligned}$$

The auxiliary problem AP^t, such as AP^2, can be automatically constructed and solved using the IMC-DSS [15,25] once the trade-off analysis is conducted. Solving (54), we can obtain a new efficient solution X^2:

$$X^2 = [0.851, 0.874, 0.795]^{\mathrm{T}}, \quad F^2 = [5.0, 6.276, 1.975]^{\mathrm{T}}, \qquad (55)$$

with $X^2 \in \Omega_a$. By comparing the achievement levels of the objectives F^1 and F^2, the DM confirms that he prefers X^2 to X^1, or $X^2 P X^1$.

From X^2, the DM is in favour of improving $f_1(X)$ at the expense of $f_2(X)$ by keeping $f_3(X)$ at $f_3(X^2)$ while the maximum decrement of $f_2(X)$ is given by $\Delta f_2(X^2) = f_2^2 - f_2^0 = 3.276$. Then, a new auxiliary problem AP^3 can be formulated by

(AP^3)

$$\begin{aligned} \max \quad & y = y_1 && (56)\\ \text{s.t.} \quad & X \in \Omega, \qquad y_1 \geq 0,\\ & f_1(X) - 1.93 y_1 \geq 5.0, \quad f_2(X) \geq 3.0, \quad f_3(X) \leq 1.975. \end{aligned}$$

The optimal solution of (56) is given by

$$X^3 = [0.977, 0.874, 0.793]^{\mathrm{T}}, \qquad F^3 = [5.335, 3.0, 1.975]^{\mathrm{T}}, \qquad (57)$$

With $X^3 \in \Omega_a$. By comparing F^3 with F^2, the DM also confirms that $X^3 P X^2$.

By conducting similar trade-off analysis interactively, the following efficient solutions are obtained, denoted by X^4, X^5 and X^6, respectively:

$$\begin{aligned} X^4 &= [0.851, 0.976, 0.759]^{\mathrm{T}}, & F^4 &= [5.993, 6.276, 6.0]^{\mathrm{T}},\\ X^5 &= [0.944, 0.958, 0.767]^{\mathrm{T}}, & F^5 &= [5.993, 4.635, 4.603]^{\mathrm{T}},\\ X^6 &= [0.944, 0.967, 0.762]^{\mathrm{T}}, & F^6 &= [6.103, 4.635, 5.233]^{\mathrm{T}}, && (58) \end{aligned}$$

where X^4, X^5 and $X^6 \in \Omega_a$. The DM has provided his evaluations of these solutions as $X^4 P X^3$, $X^5 P X^4$ and $X^6 P X^5$. These generated acceptable efficient solutions consist of a sub-set of the efficient frontier of problem (46), which is denoted by

$$\Gamma = \{X^1, X^2, X^3, X^4, X^5, X^6\}. \qquad (59)$$

In the above trade-off analysis, the DM tried to learn what efficient solutions could possibly be obtained within the acceptable decision space that he has defined, so that he could be in a better position to respond to provide his preferences. In this example, the DM attempted to generate a sub-set of acceptable solutions with X^{l+1} being more favourable than X^l. The preference relations among the generated solutions could be summarized by Ω_P:

$$\Omega_P = \{(X^l, X^h) \mid X^l P X^h,\ h = i,\ l = i + 1,\ i = 1, \ldots, 5\}. \qquad (60)$$

Table 4
Trade-off table for problem (46)

	$f_1(X^i)$	$f_2(X^i)$	$f_3(X^i)$
X^1	6.025	3.922	4.469
X^2	4.995	6.276	1.975
X^3	5.335	2.996	1.975
X^4	5.993	6.276	6.005
X^5	5.993	4.635	4.603
X^6	6.103	4.635	5.233

Eq. (60) implies that $X^6PX^5PX^4PX^3PX^2PX^1$ or the relations in Ω_P are transitive. In general cases, the DM may be less ambitious and may only try to generate a subset of acceptable solutions with any of these solutions being comparable with at least one of others. Anyway, it is desirable that the DM has gained a significant insight into the acceptable decision space through the analysis and that the preference relations summarized by Ω_P reflect this insight. As a result of the above interactive trade-off analysis, a trade-off table can be constructed, as shown by Table 4.

It is possible that the best compromise solution could be evolved by this implicit trade-off analysis [24]. This non-directed search technique, however, may suffer from depending on too much guesswork on the part of the DM, with no real sense of which direction to search for the best compromise solution [2]. A directed search can relieve the DM from undertaking too much guesswork and it may be implemented as follows through the estimation and optimization of local utility functions based on the DM's preferences provided by (60) and Table 4.

5.2. Estimation of local utility functions

First of all, suppose the acceptable intervals of the three objectives, or (f_i^0, f_i^*) $(i = 1, 2, 3)$ are divided into N_i equal sub-intervals $(i = 1, 2, 3)$ as follows:

$$N_1 = 3, \quad N_2 = 3, \quad N_3 = 4. \tag{61}$$

The objective values of the end points of these equal sub-intervals are calculated using formulas (10), as shown in Table 5.

In Table 5, u_i^j is a marginal utility to estimate with respect to f_i^j.

Taking into account the information given by (48), (49), (59), (60), (61), Table 4 and Table 5, we can construct an LGP for this MODM problem as defined in Section 3.2 with $\delta = 0.02$ and $\Omega_I = \emptyset$. In the LGP, there are all together 30 variables, including 13 marginal utility variables u_i^j $(j = 0, 1, \ldots, N_i;\ i = 1, 2, 3)$, 12 approximation error variables σ_l^+ and σ_l^- $(l = 1, \ldots, 6)$, and 5 inconsistency error variables d_{lh}^+ for all

Table 5
Objective values of end points

j	$f_1(X)$		$f_2(X)$		$f_3(X)$	
	f_1^j	u_1^j	f_2^j	u_2^j	f_3^j	u_3^j
0	5.0	u_1^0	3.0	u_2^0	6.0	u_3^0
1	5.597	u_1^1	4.093	u_2^1	4.76	u_3^1
2	6.193	u_1^2	5.187	u_2^2	3.52	u_3^2
3	6.79	u_1^3	6.28	u_2^3	2.28	u_3^3
4					1.04	u_3^4

$(X^l, X^h) \in \Omega_P$. The LGP has 25 constraints, including 15 linear inequality constraints, 4 linear equality constraints and 6 nonlinear equality constraints.

This linear goal programming is readily solved using the IMC-DSS in which the goal programming (linear or nonlinear) is also adopted as one of the MODM methods. The optimal solution of the LGP shows that the optimal value of one inconsistency error variable is nonzero, that is, $d_{54} = 0.0365$. This indicates that the preference relation X^5PX^4 is inconsistent with the other four relations contained in Ω_P in terms of the multiple criteria evaluations of $X^l (X^l \in \Gamma)$ given by Table 4 with assuming that the preferential independence condition is satisfied by the three objectives for $X \in \Omega_a$.

Recognizing the inconsistency, the DM still assumes that the three objectives can be evaluated independently in Ω_a defined by (50). He then appreciates to change the preference relation between X^5 and X^4 from X^5PX^4 to X^4PX^5 instead of modifying any of the acceptable levels defined by (49). A new set of preference relations can thus be summarized by Ω'_P:

$$\Omega'_P = \{(X^l, X^h) \mid X^2PX^1,\ X^3PX^2,\ X^4PX^3,\ X^4PX^5,\ X^6PX^5\}. \tag{62}$$

A new LGP for the MODM problem can then be constructed with Ω_P being replaced by Ω'_P. Solving the new problem, we can obtain the optimal estimation of the marginal utilities. This time, all error variables are zero, which indicates that the preference relations contained in (62) are consistent with the multiple criteria evaluations and that the preferences can be precisely fitted using the piecewise linear marginal utility functions defined by (32) and Table 5. The obtained optimal marginal utilities are listed in Table 6. The piecewise linear optimal marginal utility functions can then be fitted, as shown in Fig. 1.

The optimal utilities of the six generated efficient solutions are calculated using (32) as follows:

$$[\hat{u}(X^1), \hat{u}(X^2), \hat{u}(X^3), \hat{u}(X^4), \hat{u}(X^5), \hat{u}(X^6)]^{\mathrm{T}} = [0.468, 0.488, 0.508, 0.529, 0.506, 0.527]^{\mathrm{T}}, \tag{63}$$

which is consistent with the preference relations contained in Ω'_P.

However, the optimal estimation of the marginal utilities may generally not be unique when all optimal error variables are zero. In this example, a set of optimal marginal utilities do exist, which is defined by $\mathbf{U}_{\mathrm{opt}}$ as shown in (33). To sample $\mathbf{U}_{\mathrm{opt}}$, 26 linear programming problems, with 13 variables (marginal utilities) and 19 constraints, including 4 linear equality constraints and 15 linear inequality constraints, can be formulated as defined by (34). 26 different optimal estimates of the marginal utilities are therefore obtained. The mean marginal utilities of these 26 estimates are calculating using (35). The maximum, minimum and mean marginal utilities with respect to each end point are listed in Table 7. The piecewise linear mean marginal utility functions and their upper and lower bounds at each end point can thus be drawn, as shown in Fig. 2 along with the initially obtained optimal marginal utility functions.

It can be seen from Fig. 2 that for this example the mean marginal utility functions look smoother than the initial optimal marginal utility functions. The mean utilities of the six generated efficient solutions can be

Table 6
Optimal marginal utilities of end points

j	$f_1(X)$		$f_2(X)$		$f_3(X)$	
	f_1^j	u_1^j	f_2^j	u_2^j	f_3^j	u_3^j
0	5.0	0.0476	3.0	0.0	6.0	0.0748
1	5.597	0.2502	4.093	0.0	4.76	0.1
2	6.193	0.4144	5.187	0.0945	3.52	0.1
3	6.79	0.4144	6.28	0.0945	2.28	0.3
4					1.04	0.4912

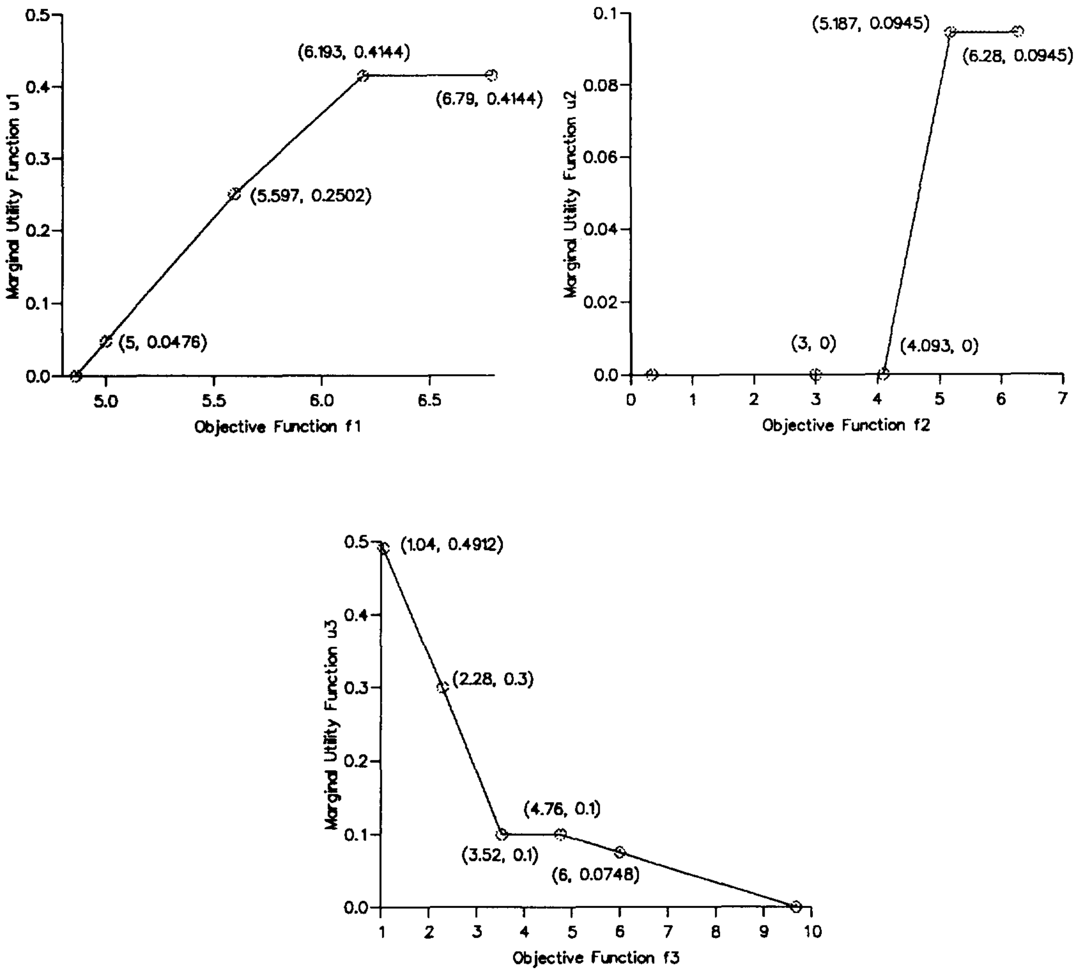

Fig. 1. Optimal piecewise linear marginal utility functions.

Table 7
Maximum, minimum and mean marginal utilities of end points

j	$f_1(X)$			$f_2(X)$			$f_3(X)$		
	$\bar{u}_1^j$	$\underline{u}_1^j$	$\hat{u}_1^j$	$\bar{u}_2^j$	$\underline{u}_2^j$	$\hat{u}_2^j$	$\bar{u}_3^j$	$\underline{u}_3^j$	$\hat{u}_3^j$
0	0.082	0.035	0.051	0.166	0.0	0.045	0.112	0.0	0.025
1	0.438	0.186	0.268	0.234	0.0	0.063	0.15	0.0	0.034
2	0.558	0.294	0.408	0.289	0.0	0.136	0.181	0.0	0.041
3	0.75	0.298	0.513	0.289	0.066	0.143	0.424	0.039	0.275
4							0.636	0.184	0.344

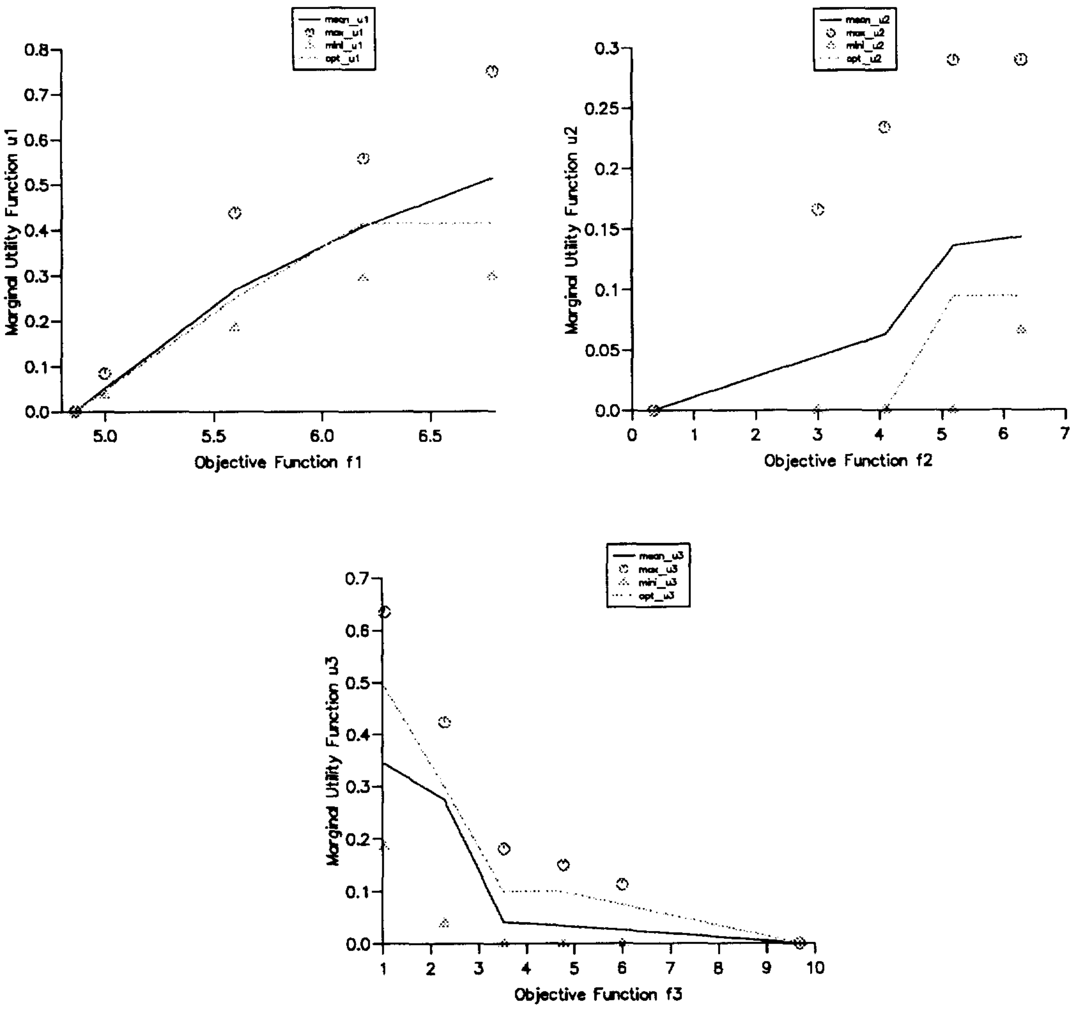

Fig. 2. Mean marginal utiliy functions and upper and lower bounds.

obtained using (36) as follows:

$$\left[\hat{u}(X^1),\ \hat{u}(X^2),\ \hat{u}(X^3),\ \hat{u}(X^4),\ \hat{u}(X^5),\ \hat{u}(X^6)\right]^{\mathrm{T}} = [0.464,\ 0.485,\ 0.509,\ 0.529,\ 0.495,\ 0.516]^{\mathrm{T}}, \tag{64}$$

which is also consistent with the preference relations given in Ω'_P.

5.3. Generation of the best compromise solution

The assessed mean utility function may be used to generate the best compromise solution. This function can be represented using (37), where f_i^j ($j = 1, \ldots, N_i - 1$; $i = 1, 2, 3$) are given in Table 5. The coefficients of the function, that is, α_{ij}, β_i and γ_i ($j = 1, \ldots, N_i - 1$; $i = 1, 2, 3$) defined by (38)–(40), can be obtained as follows using the data given in Tables 5 and 7:

$$\alpha_{11} = -0.064,\quad \alpha_{12} = -0.03,\quad \alpha_{21} = 0.025,\quad \alpha_{22} = -0.03,$$
$$\alpha_{31} = -0.001,\quad \alpha_{32} = 0.092,\quad \alpha_{33} = -0.067,$$
$$\beta_1 = 0.27,\quad \beta_2 = 0.012,\quad \beta_3 = 0.031,\quad \gamma_1 = -1.224,\quad \gamma_2 = 0.048,\quad \gamma_3 = 0.234. \tag{65}$$

Then, a nonlinear goal programming problem defined by (44) can be formulated, which is composed of 17 variables, including 3 decision variables x_i and 14 deviation variables a_{ij}^+ and a_{ij}^- ($j = 1, \ldots, N_i - 1$; $i = 1, 2, 3$), and 21 constraints, including 1 linear inequality constraint and 6 nonlinear inequality constraints for defining the acceptable decision space Ω_a determined by (50), or

$$\Omega_a = \{X \mid X \in \Omega;\ f_1(X) \geq 5.0,\ f_2(X) \geq 3.0,\ f_3(X) \leq 6.0\}, \tag{66}$$

and 14 nonlinear equality constraints for the objective values at the end points and for the deviation variables as defined in (44).

This nonlinear programming problem is readily solved using the IMC-DSS. The optimal solution, denoted by X^7, is given by

$$X^7 = [x_1^7,\ x_2^7,\ x_3^7]^T = [0.925, 0.976, 0.77]^T,$$
$$F^7 = [f_1^7,\ f_2^7,\ f_3^7]^T = [6.159, 5.187, 6.0]^T,\qquad \hat{\bar{u}}(X^7) = 0.561, \tag{67}$$

which may be regarded as the best compromise solution of (46) with $\hat{\bar{u}}(X^7) > \hat{\bar{u}}(X^i)$ ($i = 1, \ldots 6$).

X^7 is regarded as the best compromise solution in the sense that it is the optimal solution of the mean utility function in the acceptable decision space. However, both the acceptable space and the utility function may be changed if the DM wishes to modify his preferences. Such modification may be referred to as explicit sensitivity analysis and may result in the generation of new best compromise solutions which may or may not be the same as X^7. In Section 5.2, such modification was already made when the pairwise comparison between X^4 and X^5 was changed from X^5PX^4 to X^4PX^5 as shown in (60) and (62).

As an example of implementing such a post-optimality analysis, the initially assessed optimal utility function given in Table 6 is optimized. This optimization results in the generation of the same solution as X^7 although the optimal utility is given by $\hat{u}(X^7) = 0.575$, better than any utility given by (63).

The post-optimality analysis may also include the modification of the relative weights of the three objectives. These weights are already generated as a result of the assessment of the marginal utilities. From the assessed mean marginal utilities, for example, the relative weights of the three objective functions, denoted by $\hat{\omega}_1$, $\hat{\omega}_2$ and $\hat{\omega}_3$, can be obtained as follows:

$$\hat{\omega}_1 = \hat{\bar{u}}_1^3 = 0.513,\quad \hat{\omega}_2 = \hat{\bar{u}}_2^3 = 0.143,\quad \hat{\omega}_3 = \hat{\bar{u}}_3^4 = 0.344. \tag{68}$$

Table 8
Modified mean marginal utilities of end points

j	$f_1(X)$		$f_2(X)$		$f_3(X)$	
	f_1^j	$\hat{\nu}_1^j$	f_2^j	$\hat{\nu}_2^j$	f_3^j	$\hat{\nu}_3^j$
0	5.0	0.05	3.0	0.052	6.0	0.025
1	5.597	0.261	4.093	0.073	4.76	0.033
2	6.193	0.398	5.187	0.158	3.52	0.04
3	6.79	0.5	6.28	0.167	2.28	0.267
4					1.04	0.333

Table 9
Re-modified mean marginal utilities of end points

j	$f_1(X)$		$f_2(X)$		$f_3(X)$	
	f_1^j	$\hat{\hat{\nu}}_1^j$	f_2^j	$\hat{\hat{\nu}}_2^j$	f_3^j	$\hat{\hat{\nu}}_3^j$
0	5.0	0.018	3.0	0.086	6.0	0.04
1	5.597	0.095	4.093	0.12	4.76	0.054
2	6.193	0.145	5.187	0.259	3.52	0.065
3	6.79	0.182	6.28	0.273	2.28	0.436
4					1.04	0.546

Note that no explicit *a priori* knowledge about the weights was gathered for the utility estimation model in this example. Suppose the DM now recognizes that the relative importance of $f_2(X)$ is a little lower than he expected and he insists that $f_1(X)$ be only three times and $f_3(X)$ twice as important as $f_2(X)$. The new normalized weights may thus be given by

$$\omega_1 = 0.5, \quad \omega_1 = 0.1667, \quad \omega_1 = 0.3333. \tag{69}$$

Using (45), we can then obtain the modified mean marginal utilities, denoted by $\hat{\nu}_i^j$ and shown in Table 8.

The modified piecewise linear mean utility function can be fitted from Table 8 and then optimized within Ω_a. Fortunately, this optimization also results in the same solution as X^7. The modified utilities of the seven generated efficient solutions are given by

$$\left[\hat{\nu}(X^1), \hat{\nu}(X^2), \hat{\nu}(X^3), \hat{\nu}(X^4), \hat{\nu}(X^5), \hat{\nu}(X^6), \hat{\nu}(X^7)\right]^{\mathrm{T}}$$
$$= [0.464, 0.478, 0.503, 0.543, 0.501, 0.522, 0.573]^{\mathrm{T}}, \tag{70}$$

which is also consistent with the preferences given by (62). In fact, the result given by (70) confirms that the modified mean marginal utilities still constitute an optimal estimation of the marginal utilities. The above analysis should help the DM establish confidence in the best solution generated.

To examine possible inconsistency which may occur in the sensitivity analysis, let us deliberately make the radical change of the weights given by (68) as follows:

$$\omega_1' = \tfrac{2}{11}, \quad \omega_2' = \tfrac{3}{11}, \quad \omega_3' = \tfrac{6}{11}. \tag{71}$$

Using (45), we can obtain the re-modified mean marginal utilities, denoted by $\hat{\hat{\nu}}_i^j$ and shown in Table 9.

The re-modified piecewise linear utility function can be fitted from Table 9 and then optimized. This optimization generates a different optimal (efficient) solution, as given by

$$X^8 = [0.925, 0.851, 0.799]^{\mathrm{T}}, \qquad F^8 = [5.0, 5.187, 1.657]^{\mathrm{T}}. \tag{72}$$

The re-modified utilities of the eight efficient solutions generated are given by

$$\left[\hat{\hat{\nu}}(X^1), \hat{\hat{\nu}}(X^2), \hat{\hat{\nu}}(X^3), \hat{\hat{\nu}}(X^4), \hat{\hat{\nu}}(X^5), \hat{\hat{\nu}}(X^6), \hat{\hat{\nu}}(X^7), \hat{\hat{\nu}}(X^8)\right]^{\mathrm{T}}$$
$$= [0.302, 0.753, 0.61, 0.44, 0.372, 0.375, 0.441, 0.769]^{\mathrm{T}}, \tag{73}$$

which is obviously inconsistent with the preferences given by (62). If the DM insists that the weights given by (71) more precisely represent his priorities on the three objectives than those given by (68) or (69), the marginal utilities will need to be re-estimated with $u_i(f_i^{N_i})$ being fixed at ω_i' $(i = 1, 2, 3)$.

6. Concluding remarks

The new MODM technique developed by incorporating into the ISTM method a search process explored in this paper can support the DM to search for his best compromise solution in a natural way without bearing of too much guesswork. The new technique allows the DM to define and modify his acceptable achievement levels for all objective functions so that any unacceptable solution can never be identified as the best compromise solution in the search process. This technique can also assist the DM to check and eliminate any inconsistent preference information. Furthermore, the DM can re-evaluate and modify the preferences he has already provided whenever necessary. This enables the DM to carry out sensitivity analysis in an explicit manner in order to examine the robustness of the obtained best compromise solution. Such an analysis may enhance the DM's confidence in the generated solutions. Besides, the explicit representation of the preferences for individual decision makers could provide a basis for conflict resolution in group decision making.

However, the assessment of additive utility functions relies on a fundamental restrictive hypothesis of the mutual preferential independence condition for multiple objectives. Although in the new method this condition only needs to be satisfied in a local decision space which can be regulated by the DM, such flexibility may still be of no use in some decision situations where utility functions inherently have non-separable features. In such circumstances, non-additive utility functions may need to be assessed [6], or more complex evaluation structures with hierarchy and uncertainty may need to be employed for preference modelling [26,27] or different search strategies should be adopted without explicit assessment of utility functions.

Acknowledgements

This work forms part of a project to develop a general purpose multiple criteria decision support environment supported by the UK Science and Engineering Research Council under Grant no. GR/F 95306.

References

[1] Buchanan, J., and Daellenbach, H.G., "A comparative evaluation of interactive solution methods for multiple objective decision models", *European Journal of Operational Research 24* (1987) 353–359.
[2] Buchanan, J., "A two-phase interactive solution method for multiple-objective programming problems", *IEEE Transactions on Systems, Man, and Cybernetics* 21 (1991) 743–749.
[3] Changkong, V., and Haimes, Y.Y., *Multiobjective Decision Making – Theory and Methodology*, Elsevier Science Publishers, Amsterdam, 1983.
[4] Dyer, J.S., Fishburn, P.C., Steuer, R.E., Wallenius, J., and Zionts, S., "Multiple criteria decision making, multiattribute utility theory: The next ten years", *Management Science* 38 (1992) 645–654.
[5] Fishburn, P.C., "Methods of estimating additive utilities", *Management Science* 13 (1967) 435–453.
[6] Fishburn, P.C., "Nonlinear preference and utility theory", AT & T Bell Laboratories, 1988.
[7] Huang, C.L., and Masnd, A.S., *Multiple Objective Decision Making Methods and Applications. A State-of-Art Survey*, Springer-Verlag, Berlin, 1979.
[8] Jacquet-Lagreze, E., and Siskos, J., "Assessing a set of additive utility functions for multicriteria decision making: The UTA method", *European Journal of Operational Research* 10 (1982) 151–164.
[9] Jacquet-Lagreze, E., and Shakun, M.F., "Decision support systems for semi-structured buying decisions", *European Journal of Operational Research* 16 (1984) 48–58.
[10] Jacquet-Lagreze, E., Meziani, R., and Slowinski, R., "MOLP with an interactive assessment of piecewise linear utility function", *European Journal of Operational Research* 31 (1987) 350–357.
[11] Keeney, R.L., and Raiffa, H., *Decisions with Multiple Objectives: Preferences and Value Trade-offs*, Wiley, New York, 1976.
[12] Krantz, D.H., Luce, R.D., Suppes, P., and Tversky, A., *Foundation of Measurement, Vol. 1*, Academic Press, New York, 1971.

[13] Moskowitz, H., Preckel, P.V., and Yang, A., "Multiple-criteria robust interactive decision analysis (MCRID) for optimizing public policies", *European Journal of Operational Research* 56 (1992) 219–236.
[14] Reeves, G.R., and Gonzaiez, J.J., "Comparison of two interactive MCDM procedures", *European Journal of Operational Research* 41 (1989) 203–209.
[15] Sen, P., and Yang, J.B., "A multiple criteria decision support environment for engineering design", in: *Proceedings of the 9th International Conference on Engineering Design*, The Hague, Netherlands, 1993.
[16] Sen, P., Yang, J.B., and Meldrum, P., "Interactive trade-off analysis in multiple criteria preliminary design of a semi-submersible", in: *Proceedings of 6th International Conference on Marine Engineering Systems*, University of Hamburg, Germany, 1993.
[17] Sen, P., and Yang, J.B., "Design decision making based upon multiple attribute evaluation and minimal preference information", *Mathematical and Computer Modelling* 20/3 (1994) 107–124.
[18] Shin, W.S., and Ravindran, A., "Interactive multiple objective optimization. Survey I. Continuous case", *Computers & Operations Research* 18 (1991) 97–114.
[19] Steuer, R.E., and Choo, E.U., "An interactive weighted Tchebycheff procedure for multiple objective programming", *Mathematical Programming* 26 (1983) 326–344.
[20] Vanderpooten, D., and Vincke, P., "Description and analysis of some representative interactive multicriteria procedures", *Mathematical and Computer Modelling* 12 (1989) 1221–1238.
[21] White, D.J., "A bibliography on the applications of mathematical programming multiple-objective methods", *Journal of the Operational Research Society* 41 (1990) 669–691.
[22] Yang, J.B., Chen, C., and Zhang, Z.J., "A multiobjective fuzzy decisionmaking method for planning management expert systems", in: *Proceedings of International Academic Conference on Economics and Management*, Shanghai, China, 1987.
[23] Yang, J.B., Chen, C., and Zhang, Z.J., "The interactive decomposition method for multiobjective linear programming and its applications", *Information and Decision Technologies* 14 (1988) 275–288.
[24] Yang, J.B., Chen, C., and Zhang, Z.J., "The interactive step trade-off method (ISTM) for multiobjective optimization", *IEEE Transactions on Systems, Man, and Cybernetics* 20 (1990) 688–695.
[25] Yang, J.B., "An integrated multicriteria decision support system for engineering design", Research Report, EDCN/MCDM/PAPERS/3/2, Engineering Design Centre, University of Newcastle upon Tyne, UK, 1992.
[26] Yang, J.B., and Singh, M.G., "An evidential reasoning approach for multiple attribute decision making with uncertainty", *IEEE Transactions on Systems, Man, and Cybernetics* 24/1 (1994) 1–18.
[27] Yang, J.B., and Sen, P., "A general multi-level evaluation process for hybrid MADM with uncertainty", *IEEE Transactions on Systems, Man, and Cybernetics* 24/10 (1994) 1458–1473.