ELSEVIER

European Journal of Operational Research 126 (2000) 541–556

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

Theory and Methodology

Minimax reference point approach and its application for multiobjective optimisation

Jian-Bo Yang *

Manchester School of Management, University of Manchester, Institute of Science and Technology (UMIST) P.O. Box 88, Manchester M60 1QD, UK

Received 1 September 1998; accepted 27 April 1999

Abstract

In multiobjective optimisation, one of the most common ways of describing the decision maker's preferences is to assign targeted values (goals) to conflicting objectives as well as relative weights and priority levels for attaining the goals. In linear and convex decision situations, traditional goal programming provides a pragmatic and flexible manner to cater for the above preferences. In certain real world decision situations, however, multiobjective optimisation problems are non-convex. In this paper, a minimax reference point approach is developed which is capable of handling the above preferences in non-convex cases. The approach is based on ∞-norm formulation and can accommodate both preemptive and non-preemptive goal programming. A strongly non-linear multiobjective ship design model is presented and fully examined using the new approach. This simulation study is aimed to illustrate the implementation procedures of the approach and to demonstrate its potential application to general multiobjective optimisation problems. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Multiobjective optimisation; Minimax formulation; Reference point approach; Goal programming; Computer aided design

1. Introduction

Most real world decision problems in management and engineering involve multiple, potentially conflicting requirements reflecting technical and economical performance. The ultimate goal of multiobjective optimisation is to reach a decision maker's most preferred solution.

The solution of a multiobjective optimisation problem is dependent upon the decision maker's preferences, which could be represented by a utility function that aggregates all objective functions into a scalar criterion. In most decision situations, a global utility function is not known explicitly and only local information about the utility function could be elicited. This leads to interactive procedures facilitating tradeoff analysis. For instance, implicit tradeoff analysis allows the decision maker to search the efficient frontier in natural and progressive manners (Chankong and

* Tel.: +44-161-236-3311; fax: +44-161-200-3505.
E-mail address: jian-bo.yang@umist.ac.uk (J.-B. Yang).

PII: S0377-2217(99)00309-4

Haimes, 1983; Vanderpooten and Vinck, 1989; Yang et al., 1990). Such search procedures could be enhanced by estimating local utility functions using the pairwise comparisons of efficient solutions generated in the interactive processes (Jacquet-Lagreze and Sisko, 1982; Jacquet-Lagreze et al., 1987; Yang and Sen, 1996a). Explicit tradeoff analysis may also be conducted by identifying the normal vectors of an efficient frontier and estimating utility gradients in terms of marginal rates of substitution (Li and Yang, 1996; Yang, 1999).

On the other hand, preferences may also be described by assigning targeted values (goals) to objectives as well as relative weights and priority levels for attaining these goals (Hillier and Lieberman, 1990; Yang, 1996). This way of communicating preferences is particularly useful if the decision maker does not directly participate in a computerised interactive decision analysis process or a group of decision makers are involved in deciding the most preferred solutions.

As one of the most popular multiobjective optimisation techniques, goal programming provides a pragmatic and flexible way to cater for the above preferences (Charnes and Cooper, 1977; Hillier and Lieberman, 1990). Traditional goal programming techniques are well suited to dealing with linear and convex problems, though they may work like a weighting method in a local region defined by the designated goal (Yang, 1996). In certain decision situations, however, multiobjective optimisation problems are strongly non-linear and non-convex, for example in system reliability analysis (Li, 1996; Li and Yang, 1996) and in optimal design of complex engineering products (Yang and Sen, 1996b; Sen and Yang, 1998).

Traditional minimax approaches are capable of identifying any efficient (non-dominated, non-inferior, or Pareto-optimal) solution of a convex or non-convex problem by regulating weights (Lightner and Director, 1981; Steuer and Choo, 1983; Li and Yang, 1996). An efficient solution closest to a designated reference (ideal) point could be generated using a traditional minimax formulation by assigning so-called canonical weights (Lightner and Director, 1981). In some literature, such a minimax scheme is referred to as goal attachment minimax approach. In comparison with traditional goal programming, the goal attachment minimax approach is incapable of accommodating general preferences such as those addressed in non-preemptive and preemptive goal programming. This is due to the fact that traditional minimax approaches are based on a ∞-norm distance measure between an ideal point and feasible solutions.

This paper is intended to develop a minimax reference point approach capable of handling general preference information in non-convex cases just as traditional goal programming does in convex cases. Reference point based approaches have been investigated for dealing with linear, non-linear and discrete problems (Wierzbicki, 1979,1982; Vanderpooten and Vincke, 1989). Such methods are normally based on p-norm $(1 \leqslant p < \infty)$ formulations. For a sufficiently large p, p-norm based approaches are capable of identifying all efficient solutions of a problem whether it is convex, non-convex, continuous or discrete (White, 1988; Li, 1996).

The new approach is based on ∞-norm formulation and the assignment of a reference solution. If the reference solution is chosen to be an ideal solution taking the best values of all objectives, then the new approach degenerates to a traditional minimax (ideal point) approach. By means of equivalent transformation, minimax reference point formulations are developed, which can be solved using existing linear and non-linear programming algorithms. The formulations are then tailored to develop minimax goal programming schemes facilitating both preemptive and non-preemptive goal programming in non-convex cases. In multiobjective linear programming, the new approach is capable of identifying the exact efficient solution achieving the tradeoffs among objectives as required by the decision maker while traditional goal programming may unexpectedly end up with an extreme efficient solution.

In the paper, different minimax reference point formulations are explored first. Non-preemptive and preemptive minimax goal programming schemes are then developed. A strongly non-linear multiobjective ship design model is summarised (Yang and Sen, 1996b) and reformulated to test the proposed approach. The new approach is

incorporated into a large multiobjective decision support system (DSS) (Sen and Yang, 1995; Yang and Sen, 1996b) where several multiple criteria decision analysis methods are employed in an integrated manner, including the traditional goal programming method (Charnes and Cooper, 1977), the minimax method (Lightner and Director, 1981; Steuer and Choo, 1983), Geoffrion's method (Geoffrion et al., 1972) and the ISTM method (Yang et al., 1990). The first two methods are based on distance measures and the last two methods are of an interactive nature based on implicit trade-off analysis. The new approach developed in this paper is a kin to the first two methods. It takes advantages of the two traditional methods but is equipped with its own features. A ship design model is examined to demonstrate these features using the enhanced DSS. This simulation study is also intended to demonstrate the potential application and implementation procedures of the new approach.

2. Minimax reference point formulations

2.1. Direct minimax reference point formulation

A multiobjective optimisation problem may be represented as follows:

$$\max \quad f(x) = (f_1(x) \cdots f_k(x)) \tag{1a}$$

s.t.

$$x \in \Omega, \tag{1b}$$

$$x = [x_1\ x_2 \cdots x_n]^{\mathrm{T}}, \tag{1c}$$

$$\Omega = \{x | g_j(x) \leqslant 0,\ h_l(x) = 0;\ j = 1, \ldots, m_1,\ l = 1, \ldots, m_2\}, \tag{1d}$$

where $f_i(x)$ is the ith objective function, x_j the jth decision variable and Ω the feasible decision space. $g_j(x)$ and $h_l(x)$ are inequality and equality constraint functions. The feasible objective space is then given by $f(\Omega)$.

Suppose an aspiration level (targeted value or goal) for an objective $f_i(x)$ is provided, denoted by $\hat{f}_i$. A reference point is then represented by

$$\hat{f} = [\hat{f}_1\ \hat{f}_2 \cdots \hat{f}_k]^{\mathrm{T}}. \tag{2}$$

In this section, it is assumed that the best compromise solution is the one that is the closest to the reference point. This assumption will be relaxed to handle more general preferences in next sections.

Under this assumption and using ∞-norm to measure distance, it is easy to show that the best compromise solution can be generated by solving the following minimax problem:

$$\begin{aligned} \min \quad & r \\ \text{s.t.} \quad & \omega_i \left| \hat{f}_i - f_i(x) \right| \leqslant r, \quad i = 1, \ldots, k, \\ & x \in \Omega, \end{aligned} \tag{3}$$

where $\omega_i (\geqslant 0)$ is defined as $\hat{\omega}_i / \bar{\omega}_i$ with $\bar{\omega}_i$ being a normalising factor and $\hat{\omega}_i$ a relative weight for an objective $f_i(x)$. If the reference point is given as the ideal point taking the best values of all objectives, then the absolute value sign in the objective constraints of problem (3) is not needed and the problem degenerates to a traditional minimax problem.

Formulation (3) is a non-smooth optimisation problem. To facilitate the solution of the above problem using existing mathematical programming algorithms, we transform the non-smooth problem to two new formulations, each of which has its own features and drawbacks.

In formulation (3), a non-smooth objective constraint $\omega_i |\hat{f}_i - f_i(x)| \leqslant r$ can be simply replaced by the following two equivalent smooth constraints:

$$\omega_i (\hat{f}_i - f_i(x)) \leqslant r, \tag{4a}$$

$$-\omega_i (\hat{f}_i - f_i(x)) \leqslant r. \tag{4b}$$

A new formulation equivalent to formulation (3) can then be constructed directly using Eq. (4a) and (4b) as follows:

$$\min \quad r$$

s.t.

$$\omega_i (\hat{f}_i - f_i(x)) \leqslant r, \tag{5a}$$

$$-\omega_i(\hat{f}_i - f_i(x)) \leqslant r, \quad i = 1, \ldots, k, \tag{5b}$$

$$x \in \Omega, \tag{5c}$$

which is a smooth mathematical programming problem and can be readily solved using existing optimisation software packages.

2.2. Deviation variable based minimax reference point formulation

It should be noted that the underlying preference assumption we made in formulation (5a)–(5c) is that the best compromise solution is the one in the feasible decision space that is closest to the designated reference point in the sense of ∞-norm. In many decision situations, however, the decision maker may wish to express his preferences in more flexible ways as will be discussed in the section "Minimax goal programming". Based on this understanding, we introduce deviation variables to transform formulation (3) to another formulation different from formulation (5a)–(5c). First, let us introduce the following deviation variables:

$$d_i^+ = \frac{1}{2}\left\{\left|\hat{f}_i - f_i(x)\right| - \left(\hat{f}_i - f_i(x)\right)\right\}, \tag{6}$$

$$d_i^- = \frac{1}{2}\left\{\left|\hat{f}_i - f_i(x)\right| + \left(\hat{f}_i - f_i(x)\right)\right\}, \tag{7}$$

where d_i^+ is a deviation variable, representing the degree to which $f_i(x)$ over achieves $\hat{f}_i$, and d_i^- denotes the degree that $f_i(x)$ under achieves $\hat{f}_i$. Obviously the following conclusions are true:

$$d_i^+ > 0 \quad \text{and} \quad d_i^- = 0 \quad \text{if } f_i(x) > \hat{f}_i, \tag{8}$$

$$d_i^+ = 0 \quad \text{and} \quad d_i^- > 0 \quad \text{if } f_i(x) < \hat{f}_i, \tag{9}$$

$$d_i^+ = 0 \quad \text{and} \quad d_i^- = 0 \quad \text{if } f_i(x) = \hat{f}_i. \tag{10}$$

Eqs. (6)–(10) can be transformed to the following equivalent:

$$d_i^+ + d_i^- = \left|\hat{f}_i - f_i(x)\right|, \tag{11}$$

$$d_i^+ - d_i^- = f_i(x) - \hat{f}_i, \tag{12}$$

$$d_i^+ \times d_i^- = 0, \tag{13}$$

$$d_i^+,\ d_i^- \geqslant 0. \tag{14}$$

Combining problem (3) with Eqs. (11)–(14), we obtain the following smooth *minimax reference point formulation* based on the introduction of deviation variables:

min $\quad r$

s.t.

$$\omega_i(d_i^+ + d_i^-) \leqslant r, \quad i = 1, \ldots, k, \tag{15a}$$

$$d_i^+ - d_i^- = f_i(x) - \hat{f}_i, \quad i = 1, \ldots, k, \tag{15b}$$

$$d_i^+ \times d_i^- = 0, \quad i = 1, \ldots, k, \tag{15c}$$

$$d_i^+,\ d_i^- \geqslant 0, \quad i = 1, \ldots, k, \tag{15d}$$

$$x \in \Omega. \tag{15e}$$

In problem (15a)–(15e), there are $3k$ additional constraints added to the original constraint set Ω. Formulation (15a)–(15e) is more flexible in representing preferences than formulation (5a)–(5c) as will be made clear in Section 2.3. However, the former requires the non-linear complementarity condition (15c). If the original multiobjective problem (problem (1a)–(1d)) is linear, then problem (5a)–(5c) is linear, and so is problem (15a)–(15e) except that Eq. (15c) is non-linear. Fortunately, in this case problem (15a)–(15e) can be solved using a modified simplex method with d_i^+ and d_i^- not selected as basic variables simultaneously. If problem (1a)–(1d), is non-linear, then both formulations (5a)–(5c) and (15a)–(15e) are ordinary non-linear programming problems and may be solved using for example sequential linear or quadratic programming techniques.

It is easy to show that the traditional minimax (ideal point) formulation is a special case of formulation (5a)–(5c) or (15a)–(15e). In fact, if $\hat{f}_1\ \hat{f}_2 \cdots \hat{f}_k$ are assigned to the best values of the corresponding objectives, we always have $f_i(x) \leqslant \hat{f}_i$ for any $x \in \Omega$. Therefore, constraint (5b) becomes redundant, $d_i^+ \equiv 0$ and $d_i^- = \hat{f}_i - f_i(x)$ in formu-

lation (15a)–(15e). Formulations (5a)–(5c) and (15a)–(15e) are then equivalent to the following ideal point formulation:

$$\begin{aligned} \min \quad & r \\ \text{s.t.} \quad & \omega_i(\hat{f}_i - f_i(x)) \leqslant r, \quad i = 1, \ldots, k, \\ & x \in \Omega. \end{aligned} \tag{16}$$

Note that an optimal solution of problem (16) is an efficient solution of problem (1a)–(1d) if all ω_i are positive or if the optimal solution is unique (Steuer and Choo, 1983). However, an optimal solution x^* of problem (5a)–(5c) or (15a)–(15e) is not guaranteed to be always efficient. For example, the optimal solution will be an inferior solution if the reference point $\hat{f} = [\hat{f}_1\ \hat{f}_2 \cdots \hat{f}_k]^{\mathrm{T}}$ happens to be an interior point of problem (1a)–(1d). In general, x^* may be an inferior solution if all deviation variables d_i^+ and d_i^- $(i = 1, \ldots, k)$ are zero at x^*. If a reference point is assigned northeast of the efficient frontier of problem (1a)–(1d), it can be shown that an optimal solution of problem (15a)–(15e) is an efficient solution of problem (1a)–(1d).

Instead of proving the above conclusion and developing rules for guiding the assignment of reference points, an auxiliary problem as defined below is designed to check whether an optimal solution of problem (5a)–(5c) or (15a)–(15e) is efficient. If not, an efficient solution will result from solving the following problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{k} \omega_i y_i \\ \text{s.t.} \quad & f_i(x) - f_i(x^*) \geqslant y_i, \quad i = 1, \ldots, k, \\ & x \in \Omega, \quad y_i \geqslant 0, \quad i = 1, \ldots, k, \end{aligned} \tag{17}$$

where x^* is an optimal solution of problem (15a)–(15e) and y_i an auxiliary variable to be maximised. Let $[\bar{x}^{\mathrm{T}}\ \bar{y}_1 \cdots \bar{y}_k]^{\mathrm{T}}$ be the optimal solution of problem (17). If $\omega_i > 0$ and all $\bar{y}_i$ are zero, then x^* is efficient. Otherwise, x^* is an inferior solution. However, the resultant optimal solution $\bar{x}$ of problem (17) is an efficient solution dominating x^* (Yang et al., 1990; Yang, 1999).

2.3. Geographic interpretation of the above formulations

In Fig. 1, a two-objective optimisation problem is illustrated and the shaded area is the feasible objective space as denoted by $f(\Omega)$. The curve $\widetilde{CD}$ (the northeast boundary of $f(\Omega)$) constitute the efficient frontier which is non-convex. The efficient solutions on the curve $\widetilde{AB}$ form the non-convex part of the frontier. Such solutions can never be identified by using the simple weighting method. A reference point is defined by $\hat{f} = [\hat{f}_1\ \hat{f}_2]$.

In Fig. 1, the contour of a traditional goal programming problem is as shown by a diamond with $\hat{f}$ as its centre. Solving the problem is equivalent to expanding the diamond around the reference point $\hat{f}$ until it just touches the feasible objective space $f(\Omega)$. It is clear from Fig. 1 that if the reference point is located northeast of the efficient frontier and $\hat{f}_1 \leqslant \bar{f}_1$, $\hat{f}_2 \leqslant \bar{f}_2$, then solving the traditional goal programming problem can only identify either solution A' or B' whatever weights $\hat{\omega}_i \geqslant 0$ $(i = 1, 2)$ may be assigned.

Similarly, the contour of ∞-norm is as shown by a rectangle with $\hat{f}$ as its centre. Solving problem (3), ((5a)–(5c) or (15a)–(15e)) is equivalent to expanding the rectangle around the reference point $\hat{f}$ until it just touches the feasible objective space. It is clear from Fig. 1 that any efficient solution on the curve $\widetilde{A'B'}$ can be identified by solving problem

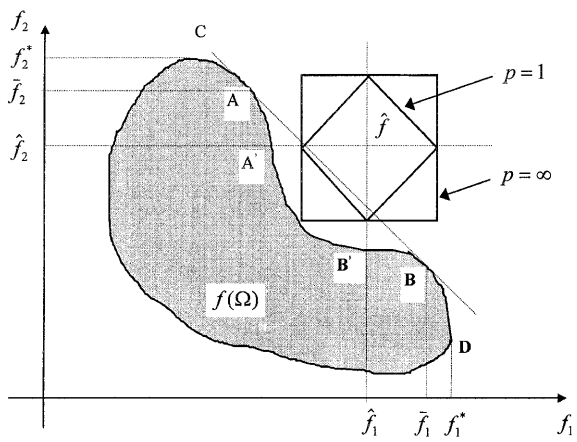

Fig. 1. Interpretation of weighted 1-norm and ∞-norm methods.

(3) with regulating ω_i $(i = 1, 2)$. From Fig. 1, it is also clear that if the reference point is assigned to satisfy $\hat{f}_1 \geqslant f_1^*$, $\hat{f}_2 \geqslant f_2^*$, then formulation (3) will be equivalent to the ideal point formulation.

3. Minimax goal programming

In formulations (5a)–(5c) and (15a)–(15e), it is assumed that the best compromised solution is the one that is closest to the reference point. This assumption is now dropped to accommodate different types of preferences for developing minimax goal programming models. In this section formulation (15a)–(15e) is modified to construct non-preemptive and preemptive minimax goal programming models.

3.1. Non-preemptive minimax goal programming

In non-preemptive goal programming, all goals are of roughly comparable importance. Some objectives may be required to be as close to their goals as possible, referred to as two-sided goals. However, other objectives may be required to be above their respective goals, called upper one-sided goals, or below their respective goals, called lower one-sided goals (Hillier and Lieberman, 1990). In this section, we develop non-preemptive models based on the minimax reference point formulation.

If objective l is required to be maintained at its targeted level $\hat{f}_l$, then both deviation variables d_l^+ and d_l^- need be minimised and Eq. (15a) becomes

$$\omega_l^+ d_l^+ + \omega_l^- d_l^- \leqslant r, \tag{18}$$

where ω_l^+ is relative importance (penalty) for the over-achievement of the goal $\hat{f}_l$ and ω_l^- for the under-achievement. The index set of all such objectives is given by

$$I^= \{l | f_l(x) = \hat{f}_l \text{ is required}, \quad l = 1, \ldots, k\}. \tag{19}$$

If objective i is required to be achieved above its goal level $\hat{f}_i$ as far as possible, then only d_i^- needs to be minimised and Eq. (15a) becomes

$$\omega_i d_i^- \leqslant r. \tag{20}$$

The index set of all such objectives is given by

$$I^{\geqslant} = \{i | f_i(x) \text{ is required above } \hat{f}_i \text{ as far as possible}, \quad i = 1, \ldots, k\}. \tag{21}$$

If objective j is required to be achieved below its goal level $\hat{f}_j$ as far as possible, then only d_j^+ needs to be minimised and Eq. (15a) becomes

$$\omega_j d_j^+ \leqslant r. \tag{22}$$

The index set of all such objectives is given by

$$I^{\leqslant} = \{j | f_j(x) \text{ is required below } \hat{f}_j \text{ as far as possible}, \quad j = 1, \ldots, k\}. \tag{23}$$

To summarise the above discussions, we formulate a *non-preemptive minimax goal programming* model as follows:

$$\min \quad r$$

s.t.

$$\omega_l^+ d_l^+ + \omega_l^- d_l^- \leqslant r \quad \text{for } l \in I^=, \tag{24a}$$

$$\omega_i d_i^- \leqslant r \quad \text{for } i \in I^{\geqslant}, \tag{24b}$$

$$\omega_j d_j^+ \leqslant r \quad \text{for } j \in I^{\leqslant}, \tag{24c}$$

$$d_i^+ - d_i^- = f_i(x) - \hat{f}_i, \quad i = 1, \ldots, k, \tag{24d}$$

$$d_i^+ \times d_i^- = 0, \quad i = 1, \ldots, k, \tag{24e}$$

$$d_i^+, \ d_i^- \geqslant 0, \quad i = 1, \ldots, k, \tag{24f}$$

$$x \in \Omega. \tag{24g}$$

If all objective functions and constraint functions are linear, the above problem is linear except constraint (24e). Fortunately, problem (24a)–(24g) can still be solved using a modified simplex method with d_i^+ and d_i^- not selected as basic variables simultaneously.

If all the goals are fully achievable, at an optimal solution of problem (24a)–(24g) we will have

$d_l^+ = d_l^- = d_j^+ = d_i^- = 0$ for all $l \in I^=$, $i \in I^{\geqslant}$ and $j \in I^{\leqslant}$. If in this case the optimal solution of problem (24a)–(24g) is not unique, we could attain the decision maker's most preferred solution by solving the following problem:

$$\max \quad \sum_{i \in I^{\geqslant}} \omega_i d_i^+ + \sum_{i \in I^{\leqslant}} \omega_j d_j^-$$

s.t.

$$d_i^+ = f_i(x) - \hat{f}_i \quad \text{for } i \in I^{\geqslant}, \tag{25a}$$

$$-d_j^- = f_j(x) - \hat{f}_j \quad \text{for } j \in I^{\leqslant}, \tag{25b}$$

$$d_i^+,\ d_j^- \geqslant 0 \quad \text{for } i \in I^{\geqslant},\ j \in I^{\leqslant}, \tag{25c}$$

$$f_l(x) = \hat{f}_l \quad \text{for } l \in I^=, \tag{25d}$$

$$x \in \Omega. \tag{25e}$$

Combining problem (24a)–(24g) with problem (25a)–(25e) the non-preemptive minimax goal programming model can be summarised as follows:

$$\begin{aligned} \min \quad & P_1 \times r + P_2 \left(\sum_{i \in I^{\geqslant}} - \omega_i d_i^+ + \sum_{i \in I^{\leqslant}} - \omega_j d_j^- \right) \\ \text{s.t.} \quad & \text{constraints (24a) to (24g)}, \end{aligned} \tag{26}$$

where P_1 and P_2 are priority weights and $P_1 \gg P_2$. Obviously, if the first priority problem has a unique optimal solution, then the second priority problem does not need to be solved.

3.2. Preemptive minimax goal programming

In preemptive goal programming, there is a hierarchy of priority levels for the goals, so that the goals of primary importance receive first-priority attention, those of secondary importance receive second-priority attention, and so forth. Such a problem arises when one or more of the goals is clearly far more important than the others.

One way of solving an preemptive minimax goal programming problem is to solve a sequence of problem (15a)–(15e) or (24a)–(24g) which may be referred to as sequential minimax goal programming. At the first stage of the approach, the only goals included in problem (15a)–(15e) or (24a)–(24g) are the first-priority goals. That is, we first attempt to solve the following problem:

$$\min \quad r$$

s.t.

$$\omega_i^+ d_i^+ + \omega_i^- d_i^- \leqslant r, \quad i \in I^{\mathrm{f}}, \tag{27a}$$

$$d_i^+ - d_i^- = f_i(x) - \hat{f}_i, \quad i \in I^{\mathrm{f}}, \tag{27b}$$

$$d_i^+ \times d_i^- = 0, \quad i \in I^{\mathrm{f}}, \tag{27c}$$

$$d_i^+,\ d_i^- \geqslant 0, \quad i \in I^{\mathrm{f}}, \tag{27d}$$

$$x \in \Omega, \tag{27e}$$

where I^{f} denotes the index set of all objectives with the first-priority goals. Note that in (27a)–(27e) only two-sided goals are listed.

Suppose $\tilde{x}$, $\tilde{d}_i^+$, $\tilde{d}_i^-$ for $i \in I^{\mathrm{f}}$ are the optimal solution of problem (27a)–(27e). If the optimal solution is unique, it will be adopted as the best compromise solution without considering any additional goals.

However, if there are multiple optimal solutions, we move to the second stage by adding the second-priority goals to problem (15a)–(15e) or (24a)–(24g) and keeping the first-priority goals at the levels they have achieved at the first stage. Let I^{s} denote the index set of all objectives with the second-priority goals. We then attempt to solve the following problem:

$$\min \quad r$$

s.t.

$$\omega_i^+ d_i^+ + \omega_i^- d_i^- \leqslant r, \quad i \in I^{\mathrm{s}}, \tag{28a}$$

$$d_i^+ - d_i^- = f_i(x) - \hat{f}_i, \quad i \in I^{\mathrm{s}}, \tag{28b}$$

$$d_i^+ \times d_i^- = 0, \quad i \in I^{\mathrm{s}}, \tag{28c}$$

$$d_i^+,\ d_i^- \geqslant 0, \quad i \in I^{\mathrm{s}}, \tag{28d}$$

$$f_l(x) = f_l(\tilde{x}) + \tilde{d}_l^+ - \tilde{d}_l^-, \quad l \in I^{\mathrm{f}}, \tag{28e}$$

$$x \in \Omega. \tag{28f}$$

Note that in problem (28a)–(28f) $\tilde{d}_l^+$ and $\tilde{d}_l^-$ are constant. If the optimal solution of the above problem is not unique, we repeat the same process for any lower-priority goals. If the optimal solution of the lowest priority goal is not unique, we formulate problem (25a)–(25e) for this goal.

When we deal with goals on the same priority level, any of the three types of goals (lower one-sided, two-sided, upper one-sided) can arise. The approach for non-preemtive goal programming can then be applied. In problem (27a)–(27e) or (28a)–(28f), constraint (27a) or (28a) could be replaced by constraints (24a)–(24c) as appropriate, depending upon the types of goals. In formulating (28a)–(28f), however, constraint (28e) needs to be replaced by

$$f_l(x) = f_l(\tilde{x}) + \tilde{d}_l^+ - \tilde{d}_l^-, \quad l \in I^{=} \subseteq I^{\mathrm{f}},$$

$$f_l(x) \geqslant f_l(\tilde{x}) + \tilde{d}_l^+ - \tilde{d}_l^-, \quad l \in I^{\geqslant} \subseteq I^{\mathrm{f}},$$

$$f_l(x) \leqslant f_l(\tilde{x}) + \tilde{d}_l^+ - \tilde{d}_l^-, \quad l \in I^{\leqslant} \subseteq I^{\mathrm{f}}.$$

4. Multiobjective preliminary ship design

4.1. A brief description of a preliminary ship design model

The detailed design of an engineering product such as a large ship is a complex process. At early design stages, however, a mathematical design model is useful for preliminary design analysis and synthesis. By examining such a model, useful guidelines can be generated for determining a preferred design space where detailed design could be carried out at later design stages. This subsection is aimed to present a verbal preliminary ship design model. This model will be transformed to a conventional optimisation problem and then analysed using the minimax reference point approach explored in the previous sections.

Developing a mathematical ship design model is not a trivial task. It requires the domain specific knowledge and the in-depth understanding of the design problem in hand. On the other hand, substantial knowledge in optimisation and decision analysis is needed to guide the modelling process. In addition, it is often the case that curve fitting techniques need to be employed to smooth the relationships among certain design variables, which are otherwise represented by tables and diagrams.

A ship design model was built to investigate a family of bulk carriers, ranging from 3000 to 500,000 tonnes deadweight, with speeds ranging from about 14 to 18 knots. In summary, the model includes three performance objectives: minimisation of annual transportation cost, minimisation of light ship mass, and maximisation of annual cargo. Six independent design variables are taken into account, which are ship length (L), draft (T), depth (D), block coefficient (C_{B}), breadth (B) and speed (V). To help the readers understand the relationships among the design variables and how the objectives and the constraints are constructed, a verbal preliminary ship design model is presented in Appendix A. More details about the modelling process can be found in the references (Yang and Sen, 1996b; Sen and Yang, 1998).

4.2. A transformed multiobjective non-linear ship design model

Mathematically, the ship design model as presented in Appendix A can be transformed to a conventional multiobjective non-linear optimisation problem, consisting of three objectives $f_i(x)$ $(i = 1, \ldots, 3)$ and six variables x_i $(i = 1, \ldots, 6)$. Table 1 lists the definitions of the objective functions and the design variables.

To represent the model in a concise way, intermediate variables are introduced in Table 2. From Appendix A, one can see that these intermediate variables provide insight into the ship design problem. In fact, designers may use some of

Table 1
Definition of objective functions and design variables

Objective		Variable	
Name	Definition	Name	Definition
$f_1(x)$	Transportation cost (pounds/tonne)	x_1	Length (m)
		x_2	Draft (m)
$f_2(x)$	Light ship mass (10,000 tonnes)	x_3	Depth (m)
		x_4	Block coefficient
$f_3(x)$	Annual cargo (million tonnes)	x_5	Breadth (m)
		x_6	Speed (knot)

Table 2
Definition of intermediate variables

Intermediate variables	
Name	Definition
γ_1	Displacement
γ_2	Power
γ_3	Machine mass
γ_4	Outfit mass
γ_5	Steel mass
γ_6	Ship cost
γ_7	Deadweight
γ_8	Daily consumption
γ_9	Voyage cost
γ_{10}	Cargo deadweight
γ_{11}	Round trip per annum

these intermediate variables to judge whether or not a generated design is preferred.

Let $x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^{\mathrm{T}}$. With reference to the verbal model presented in Appendix A, it is straightforward to show that the relationships among the six design variables and the eleven intermediate variables can be summarised as follows:

$$\gamma_1(x) = 1.025x_1x_2x_4x_5, \tag{29}$$

$$\gamma_2(x) = \gamma_1^{2/3}(x)x_6^3\frac{(9.8065x_1)^{0.5}}{b(x_4)x_6 + a(x_4)(9.8065x_1)^{0.5}}, \tag{30}$$

where $b(x_4)$ and $a(x_4)$ are auxiliary functions fitted for identifying the relations between two parameters, Admiralty Coefficient and Froude Number (Yang and Sen, 1996b). $b(x_4)$ and $a(x_4)$ are quadratic functions of the design variable x_4 as follows:

$$b(x_4) = -10{,}847.2x_4^2 + 12{,}817x_4 - 6960.32, \tag{31a}$$

$$a(x_4) = 4977.06x_4^2 - 8105.61x_4 + 4456.51. \tag{31b}$$

Other intermediate variables are given as follows:

$$\gamma_3(x) = 0.17\gamma_2^{0.9}(x), \tag{32}$$

$$\gamma_4(x) = 1.0x_1^{0.8}x_3^{0.3}x_4^{0.1}x_5^{0.6}, \tag{33}$$

$$\gamma_5(x) = 0.034x_1^{1.7}x_3^{0.4}x_4^{0.5}x_5^{0.7}, \tag{34}$$

$$\gamma_6(x) = 1.3(2000\gamma_5^{0.85}(x) + 3500\gamma_4(x) + 2400\gamma_2^{0.8}(x)), \tag{35}$$

$$\gamma_7(x) = \gamma_1(x) - \gamma_3(x) - \gamma_4(x) - \gamma_5(x), \tag{36}$$

$$\gamma_8(x) = 0.00456\gamma_2(x) + 0.2, \tag{37}$$

$$\gamma_9(x) = 21{,}875\frac{\gamma_8(x)}{x_6} + 6.3\gamma_7^{0.8}(x), \tag{38}$$

$$\gamma_{10}(x) = \gamma_7(x) - 2\gamma_7^{0.5}(x) - \gamma_8(x)\left(\frac{208.33}{x_6} + 5\right), \tag{39}$$

$$\gamma_{11}(x) = \frac{1{,}400{,}000x_6}{833{,}333.33 + (\gamma_{10}(x) + 4000)x_6}. \tag{40}$$

Note that the above intermediate variables are all dependent variables and can be expressed as direct functions of the six independent design variables. The ship design model can now be expressed using the above variables as the following strongly non-linear, multiobjective optimisation problem:

$$\min \quad f_1(x) = \frac{0.2\gamma_6(x) + 40{,}000\gamma_7^{0.3}(x) + \gamma_9(x)\gamma_{11}(x)}{\gamma_{10}(x)\gamma_{11}(x)} \tag{41a}$$

$$\min \quad f_2(x) = (\gamma_3(x) + \gamma_4(x) + \gamma_5(x))/10{,}000 \tag{41b}$$

$$\max \quad f_3(x) = \gamma_{10}(x)\gamma_{11}(x)/1,000,000, \tag{41c}$$

s.t.

$$-x_1 + 6x_5 \leqslant 0, \tag{41d}$$

$$x_1 - 15x_3 \leqslant 0, \tag{41e}$$

$$x_1 - 19x_2 \leqslant 0, \tag{41f}$$

$$x_2 - 0.45\gamma_7^{0.31}(x) \leqslant 0, \tag{41g}$$

$$x_2 - 0.7x_3 - 0.7 \leqslant 0, \tag{41h}$$

$$3000 \leqslant \gamma_7^{0.31}(x) \leqslant 500{,}000, \tag{41i}$$

$$0.63 \leqslant x_4 \leqslant 0.75, \tag{41j}$$

$$14 \leqslant x_6 \leqslant 18, \tag{41k}$$

$$x_6 - 0.32(9.8065x_1)^{0.5} \leqslant 0, \tag{41l}$$

$$-0.53x_2 + 0.52x_3 + 0.07x_5 - \frac{(0.085x_4 - 0.002)x_5^2}{x_2x_4} + 1 \leqslant 0, \tag{41m}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geqslant 0. \tag{41n}$$

4.3. Extreme and efficient ideal designs

The proposed minimax reference point approach is well suited to dealing with the above problem. The approach has been incorporated into a multiobjective decision support system (Sen and Yang, 1995; Yang, 1999). In the system, a scalar optimisation problem is solved using a modified sequential linear programming solver capable of identifying oscillations around an optimal solution. Several multiobjective optimisation methods have been integrated into the system, including the traditional minimax (ideal point) method and goal programming.

By optimising each of the three objectives individually, three extreme efficient designs are obtained, denoted by $\hat{x}^1$, $\hat{x}^2$ and $\hat{x}^3$, respectively. Table 3 shows the values of the six design variables at each of the three extreme efficient designs. From these extreme designs, a payoff table is constructed as in Table 4.

From Table 4, we can determine the best feasible and the least preferred values of the objectives as follows:

$$\begin{aligned} f_1^* &= 9.4584, & f_1^- &= 17.3413, \\ f_2^* &= 0.7163, & f_2^- &= 9.6145, \\ f_3^* &= 1.2702, & f_3^- &= 0.3719. \end{aligned} \tag{42}$$

Note from Table 4 that the three objectives are in conflict with one another. Minimising annual cost does not lead to the minimisation of light ship mass or the maximisation of annual cargo. On the other side, maximising annual cargo results in a heavy ship with high annual cost. If a light ship is preferred, then only low annual cargo could be achieved. It is therefore necessary to conduct tradeoff analysis to find a compromise design.

The traditional minimax (ideal point) method has been widely used for conducting tradeoff analysis. In reference point formulation (15a)–(15e) let $\hat{f}_i = f_i^*$ $(i = 1, 2, 3)$. We then have the ideal point formulation (16). The weighting and normalising factor ω_i is defined by $\omega_i = \hat{\omega}_i/\bar{\omega}_i$.

Table 3
Extreme efficient designs for the ship design problem

Extreme design	Variable					
	x_1	x_2	x_3	x_4	x_5	x_6
$\hat{x}^1$	271.09	16.08	21.97	0.63	45.18	14.00
$\hat{x}^2$	195.15	10.27	13.67	0.63	24.14	14.00
$\hat{x}^3$	499.63	26.30	36.57	0.63	70.27	18.00

Table 4
Payoff table for the ship design problem

Extreme design	Objective		
	$f_1(x)$	$f_2(x)$	$f_3(x)$
$\hat{x}^1$	9.4584	2.2355	0.8886
$\hat{x}^2$	12.8140	0.7163	0.3719
$\hat{x}^3$	17.3413	9.6145	1.2702

The normalising factor $\bar{\omega}_i$ is calculated by $\bar{\omega}_i = |f_i^* - f_i^-|$. The ideal point formulation for the problem is then constructed as follows:

$$\begin{aligned} \min \quad & r \\ \text{s.t.} \quad & 0.1269\,\hat{\omega}_1(f_1(x) - 9.4584) \leqslant r, \\ & 0.1124\,\hat{\omega}_2(f_2(x) - 0.7163) \leqslant r, \\ & 1.1132\,\hat{\omega}_3(1.2702 - f_3(x)) \leqslant r, \\ & x \in \Omega, \end{aligned} \tag{43}$$

where Ω is the feasible region of problem (41a)–(41n). Ω is defined as follows:

$$\Omega = \{x | \text{any } x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^{\mathrm{T}} \text{ that satisfies constraints (41d) to (41n)}\}. \tag{44}$$

Note that in problem (43) the two objectives $f_1(x)$ and $f_2(x)$ are for minimisation and $f_3(x)$ is for maximisation. It should also be noted that minimising $f_1(x)$ is equivalent to maximising $(-f_1(x))$. Solving problem (43) with all $\hat{\omega}_i = 1$ $(i = 1, 2, 3)$ results in the following efficient ideal design $\hat{x}^4$:

$$\hat{x}^4 = [292.9\ 19.15\ 26.36\ 0.75\ 48.82\ 14.26]^{\mathrm{T}},$$

$$f_1(\hat{x}^4) = 11.55, \quad f_2(\hat{x}^4) = 30{,}716.95/10{,}000,$$

$$f_3(\hat{x}^4) = 1{,}032{,}415.81/1{,}000{,}000.$$

4.4. Compromise design via changing reference points and weights

The above ideal design does not fully achieve any of the targeted values of the three objectives and may not be preferred by the decision maker. It is possible to generate other efficient designs using formulation (43) by regulating the weights $\hat{\omega}_i$ $(i = 1, 2, 3)$. However, it is difficult to estimate a set of weights that precisely represent the decision maker's preferences. On the other hand, it is relatively easy to assign targeted values for the objectives. The rest of the paper is aimed to illustrate how this can be done for the ship design problem using the new minimax reference point approach by moving reference points and setting weights around the points.

Suppose a reference point in objective space is provided at which the transportation cost is 10 pounds per tonne, the light ship mass 3×10000 tonnes and the annual cargo 1 million tonnes, or

$$\hat{f}_1 = 10, \qquad \hat{f}_2 = 3, \qquad \hat{f}_3 = 1. \tag{45}$$

Then, problem (15a)–(15e) can be constructed as follows to search for a feasible design achieving the targeted objective values as closely as possible:

$$\min \quad r \tag{46a}$$

s.t.

$$0.1269\,\hat{\omega}_1(d_1^+ + d_1^-) \leqslant r, \tag{46b}$$

$$0.1124\,\hat{\omega}_2(d_2^+ + d_2^-) \leqslant r, \tag{46c}$$

$$1.1132\,\hat{\omega}_3(d_3^+ + d_3^-) \leqslant r, \tag{46d}$$

$$d_1^+ - d_1^- = -f_1(x) + 10, \tag{46e}$$

$$d_2^+ - d_2^- = -f_2(x) + 3, \tag{46f}$$

$$d_3^+ - d_3^- = f_3(x) - 1, \tag{46g}$$

$$d_1^+ \times d_1^- = 0, \quad d_2^+ \times d_2^- = 0, \quad d_3^+ \times d_3^- = 0, \tag{46h}$$

$$d_1^+,\ d_1^-,\ d_2^+,\ d_2^-,\ d_3^+,\ d_3^- \geqslant 0, \tag{46i}$$

$$x \in \Omega. \tag{46j}$$

In problem (46a)–(46j), Ω is defined in Eq. (44), $f_1(x)$ and $f_2(x)$ are for minimisation and $f_3(x)$ is for maximisation. Suppose the three objectives are of equal importance, i.e. $\hat{\omega}_i = 1$ $(i = 1, 2, 3)$. Solving problem (46a)–(46j) then yields the following feasible design $\hat{x}^5$:

$$\hat{x}^5 = [304.43\ 18.35\ 25.22\ 0.66\ 50.74\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^4) = 10, \qquad f_2(\hat{x}^4) = 3, \qquad f_3(\hat{x}^4) = 1.$$

It is clear that at $\hat{x}^5$ all the three objectives precisely achieve their targeted values. Although this may please the decision maker, the design $\hat{x}^5$ is not efficient and there exist other designs dominating $\hat{x}^5$. For instance, problem (17) can be constructed as follows with $f_i(x^*)$ replaced by $f_i(\hat{x}^4)$:

$$\begin{aligned} &\max \quad 0.1269\,\hat{\omega}_1 y_1 + 0.1124\hat{\omega}_2 y_2 + 1.1132\hat{\omega}_3 y_3 \\ &\text{s.t.} \\ &10 - f_1(x) \geqslant y_1, \\ &3.0 - f_2(x) \geqslant y_2, \\ &f_3(x) - 1.0 \geqslant y_3, \\ &x \in \Omega. \end{aligned} \tag{47}$$

Solving problem (47) with regulating the weights $\hat{\omega}_i$ can generate the efficient designs dominating $\hat{x}^5$. Let $\hat{\omega}_i = 1$ $(i = 1, 2, 3)$ for example. We can then obtain the following efficient design $\hat{x}^6$:

$$\hat{x}^6 = [304.92\ 18.52\ 25.46\ 0.65\ 50.82\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^6) = 9.93, \qquad f_2(\hat{x}^6) = 3, \qquad f_3(\hat{x}^6) = 1.$$

Clearly $\hat{x}^6$ dominates $\hat{x}^5$, though there is not much difference between the two designs. This indicates that the given reference point is an internal point of the feasible design space but very close to the efficient frontier.

If the decision maker is interested in low light ship mass, he may wish to provide another reference point different from that given by Eq. (45), for example,

$$\hat{f}_1 = 10, \qquad \hat{f}_2 = 2, \qquad \hat{f}_3 = 1. \tag{48}$$

Similarly, a new minimax reference point problem can be constructed using the above preferences. The resultant new problem is the same as problem (46a)–(46j) except that Eq. (46f) is replaced by $d_2^+ - d_2^- = -f_2(x) + 2$. Solving this new problem with $\hat{\omega}_i = 1$ $(i = 1, 2, 3)$ leads to the following efficient design $\hat{x}^7$:

$$\hat{x}^7 = [275.43\ 17.63\ 24.19\ 0.71\ 45.91\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^7) = 10.43, \qquad f_2(\hat{x}^7) = 24{,}850.94/10{,}000,$$

$$f_3(\hat{x}^7) = 951{,}028.25/1{,}000{,}000.$$

At $\hat{x}^7$ all the three objectives have been catered for to some extent but none of them fully achieves its targeted value. It is therefore of interest to investigate how the targeted objective values could be attained if the objectives are given different relative weights or priorities.

Suppose the targeted value of the first objective receives first-priority attention and the other two objectives are of equal importance. Then the following two minimax problems can be constructed sequentially:

$$\min \quad r \tag{49a}$$

s.t.

$$0.1269(d_1^+ + d_1^-) \leqslant r, \tag{49b}$$

$$d_1^+ - d_1^- = -f_1(x) + 10, \tag{49c}$$

$$d_1^+ \times d_1^- = 0, \tag{49d}$$

$$d_1^+,\ d_1^- \geqslant 0, \tag{49e}$$

$$x \in \Omega; \tag{49f}$$

$$\min \quad r \tag{50a}$$

s.t.

$$0.1124(d_2^+ + d_2^-) \leqslant r, \tag{50b}$$

$$1.1132(d_3^+ + d_3^-) \leqslant r, \tag{50c}$$

$$d_2^+ - d_2^- = -f_2(x) + 2, \tag{50d}$$

$$d_3^+ - d_3^- = f_3(x) - 1, \tag{50e}$$

$$d_2^+ \times d_2^- = 0, \quad d_3^+ \times d_3^- = 0, \tag{50f}$$

$$d_2^+,\ d_2^-,\ d_3^+,\ d_3^- \geqslant 0, \tag{50g}$$

$$x \in \Omega, \tag{50h}$$

$$f_1(x) = 10 - \tilde{d}_1^+ + \tilde{d}_1^-. \tag{50i}$$

In Eq. (50i) $\tilde{d}_1^+$ and $\tilde{d}_1^-$ are the optimal solution of problem (49a)–(49f).

Solving problem (49a)–(49f) and then problem (50a)–(50i) results in the following efficient design $\hat{x}^8$:

$$\hat{x}^8 = [280.85\ 17.56\ 24.09\ 0.68\ 46.81\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^8) = 10.0, \qquad f_2(\hat{x}^8) = 25{,}139.74/10{,}000,$$

$$f_3(\hat{x}^8) = 948{,}109.4/1{,}000{,}000.$$

At $\hat{x}^8$, the first objective is fully achieved but the other two objectives are both under-achieved. Let us further assume that the goal of objective 1 is given the first priority, that of objective 2 the second priority and that of objective 3 the third priority. We can then construct three minimax problems sequentially with the first one the same as problem (49a)–(49f) and the other two given as follows:

$$\min \quad r \tag{51a}$$

s.t.

$$0.1124(d_2^+ + d_2^-) \leqslant r, \tag{51b}$$

$$d_2^+ - d_2^- = -f_2(x) + 2, \tag{51c}$$

$$d_2^+ \times d_2^- = 0, \tag{51d}$$

$$d_2^+,\ d_2^- \geqslant 0, \tag{51e}$$

$$x \in \Omega, \tag{51f}$$

$$f_1(x) = 10 - \tilde{d}_1^+ + \tilde{d}_1^-; \tag{51g}$$

$$\min \quad r \tag{52a}$$

s.t.

$$1.1132(d_3^+ + d_3^-) \leqslant r, \tag{52b}$$

$$d_3^+ - d_3^- = f_3(x) - 1, \tag{52c}$$

$$d_3^+ \times d_3^- = 0, \tag{52d}$$

$$d_3^+,\ d_3^- \geqslant 0, \tag{52e}$$

$$x \in \Omega, \tag{52f}$$

$$f_1(x) = 10 - \tilde{d}_1^+ + \tilde{d}_1^-, \tag{52g}$$

$$f_2(x) = 2 - \overset{\approx}{d}{}_2^+ + \overset{\approx}{d}{}_2^-. \tag{52h}$$

In Eqs. (51g) and (52g) $\tilde{d}_1^+$ and $\tilde{d}_1^-$ is the optimal solution of problem (49a)–(49f) and in Eq. (52h) $\overset{\approx}{d}{}_2^+$ and $\overset{\approx}{d}{}_2^-$ the optimal solution of problem (51a)–(51f).

Solving problems (49a)–(49f), (51a)–(51f) and (52a)–(52h) sequentially results in a new efficient design $\hat{x}^9$:

$$\hat{x}^9 = [256.96\ 16.16\ 22.08\ 0.68\ 42.83\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^9) = 10.0, \quad f_2(\hat{x}^9) = 20{,}000.0/10{,}000,$$

$$f_3(\hat{x}^9) = 860{,}809.59/1{,}000{,}000.$$

Interestingly, at $\hat{x}^9$ the goals of objectives 1 and 2 are fully achieved simultaneously.

Similar sensitivity analysis can be conducted by assuming that the goal of objective 1 is given the first priority, that of objective 3 the second priority and that of objective 2 the third priority. This assumption leads to the following efficient design $\hat{x}^{10}$:

$$\hat{x}^{10} = [301.77\ 18.52\ 25.46\ 0.66\ 50.3\ 14]^{\mathrm{T}},$$

$$f_1(\hat{x}^{10}) = 10.0, \quad f_2(\hat{x}^{10}) = 29{,}642.4/10{,}000,$$

$$f_3(\hat{x}^{10}) = 1{,}000{,}000.0/1{,}000{,}000.$$

Thus, objectives 1 and 2 can also be achieved simultaneously. In the same way it can be shown that the goals of objectives 2 and 3 cannot be achieved simultaneously.

The reference point given by Eq. (45) is inside the feasible design space. In this case the most preferred solution is entirely determined by the reference point and the relative weights $\hat{\omega}_i$ play no role in this process whatever values they may take. The reference point given by Eq. (48) is outside the feasible space but fairly close to the efficient frontier. In such a case relative weights do play a part in determining the most preferred solution as demonstrated in the above analyses. Since the reference point limits the influence of relative

weights, however, changing the weights in this case does not lead to the significant changes of the solutions. In other words, given a reference point very close to the efficient frontier, the most preferred solution is insensitive to the changes of relative weights.

With a reference point moving away from the efficient frontier, relative weights become increasingly more important and the most preferred solution is more sensitive to the changes of weights. For instance, suppose the following reference point is provided:

$$\hat{f}_1 = 9.6, \qquad \hat{f}_2 = 1, \qquad \hat{f}_3 = 1.2. \tag{53}$$

Note that each of the targeted values is achievable as it is not as good as the best feasible value as given in Table 2. By assuming that the three objectives are of equal importance, we can generate the following efficient solution.

$$\hat{x}^{11} = [286.15\ 18.75\ 25.78\ 0.75\ 47.69\ 14.07]^{\mathrm{T}},$$
$$f_1(\hat{x}^{11}) = 11.27, \quad f_2(\hat{x}^{11}) = 28{,}814.16/10{,}000,$$
$$f_3(\hat{x}^{11}) = 1{,}010{,}062/1{,}000{,}000.$$

At $\hat{x}^{11}$, none of the three objectives achieves its targeted value.

If objective 2 is regarded to be much more important than the other two objectives with $\hat{\omega}_1 = \hat{\omega}_3 = 1$ and $\hat{\omega}_2 = 10$, then we can obtain a new solution $\hat{x}^{12}$:

$$\hat{x}^{12} = [224.97\ 14.4\ 19.58\ 0.69\ 37.5\ 14]^{\mathrm{T}},$$
$$f_1(\hat{x}^{12}) = 11.01, \qquad f_2(\hat{x}^{12}) = 14{,}655.29/10{,}000,$$
$$f_3(\hat{x}^{12}) = 730{,}119.37/1{,}000{,}000.$$

At $\hat{x}^{12}$ the goal of $f_2(x)$ is much better achieved than at $\hat{x}^{11}$ mainly at the expense of objective 3.

If the goal of objective 2 is given the first priority attention and those of objectives 1 and 3 the second with equal weights, then the following solution will be generated:

$$\hat{x}^{13} = [197.72\ 12.35\ 16.64\ 0.65\ 32.95\ 14]^{\mathrm{T}},$$
$$f_1(\hat{x}^{13}) = 11.37, \qquad f_2(\hat{x}^{13}) = 1,$$
$$f_3(\hat{x}^{13}) = 556{,}526.92/1{,}000{,}000.$$

At $\hat{x}^{13}$, the goal of objective 2 is fully achieved but that of objective 3 is poorly achieved. The above analyses show that the preferred solution becomes increasingly more sensitive to the changes of weights when a reference point moves away from the efficient frontier.

From the above analyses, 13 different designs are generated. The most preferred solution could be one of them, depending on the decision maker's preferences. New designs can also be generated if new preference information is provided.

5. Concluding remarks

In multiobjective optimisation, the most preferred solution is dependent upon the decision maker's preferences. Assigning targeted values (goals) to conflicting objectives as well as relative weights and priority levels for attaining these goals is one of the most common ways of communicating preferences. This explains why goal programming is among the most popular multiobjective optimisation techniques. The minimax reference point approach developed in this paper provides an alternative way for multiobjective optimisation and facilitates both non-preemptive and preemptive goal programming in non-convex cases. On the other hand, like traditional minimax methods it is capable of generating any efficient solution by setting appropriate reference points and regulating weights, though additional number (up to three times the number of objectives) of constraints is incurred.

The examined ship design problem is formulated as a three objective optimisation problem. It provides a fairly complex but manageable non-linear model that could be used to test other multiobjective optimisation techniques. The numerical analyses reported in this paper illustrated the computational procedures of the minimax reference point approach, which may help the user to implement the approach. It is evident from this investigation that the approach is suited to dealing with general multiobjective optimisation problems, whether they are linear or non-linear, convex or non-convex, continuous or discrete.

Acknowledgements

The original preliminary ship design model was developed by the author when he was working at the Engineering Design Centre of the University of Newcastle upon Tyne, UK. The author is in debit to the two anonymous referees for their constructive comments on this paper.

Appendix A. A verbal preliminary ship design model

In the ship design model, the following three design objectives are taken into account:

- Minimisation of transportation cost (TC)

$$\text{transportation cost} = \frac{\text{annual costs}}{\text{annual cargo}}.$$

- Minimisation of light ship mass (LSM)

$$\text{light ship mass} = \text{steel mass} + \text{outfit mass} + \text{machinery mass}.$$

- Maximisation of annual cargo (AC)

$$\text{annual cargo} = \text{cargo deadweight} \times \text{RTPA}.$$

The terms in the objectives are defined with regard to the six design variables as follows:

$$\text{annual costs} = \text{capital charges} + \text{running costs} + \text{voyage costs} \times \text{RTPA},$$

$$\text{capital charges} = 0.2 \times \text{ship cost},$$

$$\text{ship cost} = 1.3 \times (2000 \times (\text{steel mass})^{0.85} + 3500 \times (\text{outfit mass}) + 2400 \times P^{0.8}),$$

$$\text{steel mass} = 0.034 \times L^{1.7} \times B^{0.7} \times D^{0.4} \times C_{\mathrm{B}}^{0.5},$$

$$\text{outfit mass} = 1.0 \times L^{0.8} \times B^{0.6} \times D^{0.3} \times C_{\mathrm{B}}^{0.1},$$

$$\text{machinery mass} = 0.17 \times P^{0.9},$$

$$P = \text{displacement}^{2/3} \times V^3 \times \frac{1}{b(C_{\mathrm{B}}) \times (V/(g \times L)^{0.5}) + a(C_{\mathrm{B}})},$$

$$\text{displacement} = 1.025 \times L \times B \times T \times C_{\mathrm{B}},$$

$$\text{running costs} = 40,000 \times \text{DW}^{0.3},$$

$$\text{DW} = \text{displacement} - \text{light ship mass},$$

$$\text{voyage costs} = \text{fuel cost} + \text{port cost},$$

$$\text{fuel cost} = 1.05 \times (\text{daily consumption}) \times (\text{sea days}) \times (\text{fuel price}),$$

$$\text{daily consumption} = P \times 0.19 \times 24/1000 + 0.2,$$

$$\text{sea days} = \frac{\text{round trip miles}}{24 \times V},$$

$$\text{round trip miles} = 5000 \text{ (nautical miles)},$$

$$\text{fuel price} = 100 \text{ (pounds/tonne)},$$

$$\text{port cost} = 6.3 \times \text{DW}^{0.8},$$

$$\text{RTPA} = \frac{350}{\text{sea days} + \text{port days}},$$

$$\text{port days} = 2 \times \left(\frac{\text{cargo deadwight}}{\text{cargo handling rate}} + 0.5\right),$$

$$\text{cargo deadweight} = \text{DW} - \text{fuel carried} - \text{crew, stores and water},$$

$$\text{fuel carried} = (\text{daily consumption}) \times (\text{sea days} + 5),$$

$$\text{crew, stores and water} = 2.0 \times \text{DW}^{0.5},$$

$$\text{cargo handling rate} = 8000 \text{ (tonnes/day)},$$

where RTPA is round trip per annual, DW deadweight, $g = 9.8065$ and $a(C_{\mathrm{B}})$ and $b(C_{\mathrm{B}})$ are fitted from test data and given as follows:

$$b(C_B) = -10,847.2C_B^2 + 12,817C_B - 6960.32,$$

$$a(C_B) = 4977.06C_B^2 - 8105.61C_B + 4456.51.$$

A feasible design needs to satisfy the following technical requirements that are interpreted as the constraints on ship dimensions and displacement, powering and stability.

A. *Dimensions and displacement*

- Length/breadth ratio $L/B \geqslant 6$.
- Length/depth ratio $L/D \leqslant 15$.
- Length/draft ratio $L/T \leqslant 19$.
- Draft constraints

$$T \leqslant 0.45 \times \mathrm{DW}^{0.31} \text{ and } T \leqslant 0.7 \times D + 0.7.$$

- Deadweight constraint $3000 \leqslant \mathrm{DW} \leqslant 500,000$.

B. *Powering*

- Block coefficient constraints $0.63 \leqslant C_B \leqslant 0.75$.
- Speed constraints $14 \leqslant V \leqslant 18$.
- Froude number constraint $V/(g \times L)^{0.5} \leqslant 0.32$.

C. *Stability*

- Metacentric height (GM)

$$\mathrm{GM} \geqslant 0.07 \times B,$$

$$\mathrm{GM} = \mathrm{KB} + \mathrm{BM} - \mathrm{KG},$$

$$\mathrm{KB} = 0.53 \times T,$$

$$\mathrm{BM} = \frac{(0.085 \times C_B - 0.002) \times B^2}{T \times C_B},$$

$$\mathrm{KG} = 1.0 + 0.52 \times D.$$

References

Chankong, V., Haimes, Y.Y., 1983. Multiobjective Decision Making: Theory and Methodology. North-Holland, New York.

Charnes, A., Cooper, W.W., 1977. Goal programming and multiple objective optimisation – Part I. European Journal of Operational Research 1, 39–54.

Geoffrion, A.M., Dyer, J.S., Feinberg, A., 1972. An interactive approach for multicriteria optimisation, with an application to the operation of an academic department. Management Science 19 (4), 357–368.

Hillier, F.S., Lieberman, G.J., 1990. Introduction to Operations Research, Fifth Edition. McGraw-Hill, New York.

Jacquet-Lagreze, E., Siskos, Y., 1982. Assessing a set of additive utility functions for multicriteria decision making, the UTA method. European Journal of Operational Research 10, 151–164.

Jacquet-Lagreze, E., Meziani, R., Slowinski, R., 1987. MOLP with an interactive assessment of a piecewise utility function. European Journal of Operational Research 31 (3), 350–357.

Li, D., 1996. Convexification of noninferior frontier. Journal of Optimisation Theory and Applications 88 (1), 177–196.

Li, D., Yang, J.B., 1996. Iterative parametric minimax method for a class of composite optimisation problems. Journal of Mathematical Analysis and Applications 198, 64–83.

Lightner, M.R., Director, S.W., 1981. Multiple criterion optimisation for the design of electronic circuits. IEEE Transactions on Circuits and Systems 28/3, 169–179.

Sen, P., Yang, J.B., 1995. Multiple criteria decision making in design selection and synthesis. Journal of Engineering Design 6 (3), 207–230.

Sen, P., Yang, J.B., 1998. Multiple Criteria Decision Support in Engineering Design. Springer, Berlin.

Steuer, R.E., Choo, E.U., 1983. An interactive weighted Tchebycheff procedure for multiple objective programming. Mathematical Programming 26, 326–344.

Vanderpooten, D., Vincke, P., 1989. Description and analysis of some representative interactive multicriteria procedures. Mathematical and Computer Modelling 12, 1221–1238.

White, D.J., 1988. Weighting factor extensions for finite multiple objective vector minimisation problems. European Journal of Operational Research 36, 256–265.

Wierzbicki, A.P., 1980. The use of reference objectives in multiobjective optimisation. In: Fandel, G., Gal, T. (Eds.), MCDM Theory and Application, Proceedings Hagen/Königswinter, F.R.G., 1979, LNEMS, vol. 177. Springer, Berlin, pp. 468–486.

Wierzbicki, A.P., 1982. A mathematical basis for satisficing decision making. Mathematical Modelling 3, 391–405.

Yang, J.B., Chen, C., Zhang, Z.J., 1990. The interactive step trade-off method (ISTM) for multiobjective optimisation. IEEE Transactions on Systems, Man, and Cybernetics 20 (3), 688–695.

Yang, J.B., 1996. Multiple Criteria Decision Making Methods and Applications. Hunan Publishing House, Changsha, P.R. China.

Yang, J.B., Sen, P., 1996a. Preference modelling by estimating local utility functions for multiobjective optimisation. European Journal of Operational Research 95, 115–138.

Yang, J.B., Sen, P., 1996b. Interactive tradeoff analysis and preference modelling for preliminary multiobjective ship design. Systems Analysis, Modelling, and Simulation 26, 25–55.

Yang, J.B., 1999. Gradient projection and local region search for multiobjective optimisation. European Journal of Operational Research 112, 432–459.