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Decision Support

Group decision making with expertons and uncertain generalized probabilistic weighted aggregation operators

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ABSTRACT

Expertons and uncertain aggregation operators are tools for dealing with imprecise information that can be assessed with interval numbers. This paper introduces the uncertain generalized probabilistic weighted averaging (UGPWA) operator. It is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. Moreover, it is able to assess uncertain environments that cannot be assessed with exact numbers but it is possible to use interval numbers. Thus, we can analyze imprecise information considering the minimum and the maximum result that may occur. Further extensions to this approach are presented including the quasi-arithmetic uncertain probabilistic weighted averaging operator and the uncertain generalized probabilistic weighted moving average. We analyze the applicability of this new approach in a group decision making problem by using the theory of expertons in strategic management.
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1. Introduction

Aggregation operators are very common in the literature (Beliakov, Pradera, & Calvo, 2007; Xu & Da, 2003; Yager, Kacprzyk, & Beliakov, 2011). They are very useful for assessing the available information in a more efficient way. One of the most common aggregation operators is the weighted average. It aggregates the information by giving different degrees of importance to the arguments considered in the problem. It has been used in an astonishingly wide range of applications (Beliakov et al., 2007; Torra, 1997). Another very common aggregation operator is the probabilistic aggregation. It aggregates the information by using probabilities in the analysis. It has also been applied in a lot of applications (Merigó, 2010; Yager, Engemann, & Filev, 1995; Yang, Yang, Liu, & Li, 2013; Yang, 2001).

Another type of aggregation operators that are also becoming very popular in the literature are the generalized aggregation operators (Beliakov et al., 2007). Their main characteristic is that they generalize a wide range of aggregation operators by using generalized and quasi-arithmetic means. Thus, we can include in the same formulation arithmetic aggregations, geometric aggregations or quadratic aggregations. For example, Yager (2004) introduced the generalized ordered weighted averaging (GOWA) operator and Fodor, Marichal, and Roubens (1995) the quasi-arithmetic OWA (Quasi-OWA) operator. Merigó and Gil-Lafuente (2009) extended these approaches by using induced aggregation operators. Merigó and Casanovas (2010, 2011a) developed several extensions by using different types of interval and fuzzy numbers. Zhou and Chen (2010) suggested an extension by using logarithmic aggregation operators and Zhou, Chen, and Liu (2012) by using power aggregation operators. Xu and Wang (2012), Xu and Xia (2011) and Zhao, Xu, Ni, and Liu (2010) introduced several extensions when dealing with intuitionistic information.

Recently, Merigó (2012a) has suggested a new approach that unifies the probability and the weighted average in the same formulation and considering the degree of importance of each concept in the aggregation. He has called it the probabilistic weighted averaging (PWA) operator. Its main advantage is that it can consider subjective and objective information in the same formulation. Thus, it is able to assess the information in a more complete way.

Usually, when dealing with these aggregation operators we assume that the information is clearly known and can be assessed with exact numbers. However, in real world problems, it is not so easy to assess the information because usually it is very complex and affected by different types of uncertainties. Thus, the use of exact numbers is not enough because it provides incomplete information and sometimes this may lead to wrong decisions. Therefore, in order to properly assess the information we need to use other techniques for representing the uncertainty in a more complete way such as the use of interval numbers (Moore, 1966). Its main advantage is that it can consider a wide range of scenarios

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from the minimum to the maximum. Note that by using interval numbers, the previous aggregation operators are known as the uncertain weighted average (UWA) and the uncertain probabilistic aggregation (UPA).

The aim of this paper is to present the uncertain generalized probabilistic weighted averaging (UGPWA) operator. It is a new aggregation operator that unifies the UWA and the UPA operator in the same formulation taking into account the degree of importance that each concept has in the analysis. Thus, it can represent the information considering subjective and objective perspectives and uncertain environments assessed with interval numbers. Moreover, it also uses generalized means that provide a general framework that includes a wide range of particular cases including the UWA, the UPA, the uncertain generalized weighted average (UGWA), the uncertain generalized probabilistic aggregation (UGPA), the uncertain average (UA), the PWA operator, the uncertain PWA (UPWA) and many others. Note that the main advantage of using uncertain information is that the information can be represented in a more complete way considering the most pessimistic and optimistic scenarios and the most possible ones.

We further generalize this approach by using quasi-arithmetic means, obtaining the uncertain quasi-arithmetic PWA (Quasi-UPWA) operator. It provides a more robust generalization that includes the UGPWA operator as a particular case and many other situations. We also extend this framework by using moving averages (Evans, 2002; Merigó, 2012a; Yager, 2008) in order to represent the information in a dynamic way. Thus, we form the uncertain generalized probabilistic weighted moving average (UGPWMA). Its main advantage is that it can deal with problems to be solved in more than one period of time (or equivalent).

We study the applicability of the new approach in a multi-person decision making problem regarding the selection of strategies by using the theory of expertons (Kaufmann, 1988; Kaufmann & Gil-Aluja, 1993). The theory of expertons extends the concept of probabilistic set (Hirota, 1981) for uncertain environments that can be assessed with interval numbers. The main advantage of this approach is that we can analyze the information of the group in a more complete way considering all the individual opinions and producing a final single result. Thus, by using expertons in group decision making problems, the information becomes more robust because it is assessed by several experts and usually the use of several experts in the analysis leads to better decisions.

This paper is organized as follows. In Section 2 we briefly describe some basic concepts. Section 3 introduces the UGPWA operator and Section 4 analyzes several families. Section 5 presents a generalization by using moving averages. Section 6 develops an application in group decision making with the theory of expertons and Section 7 summarizes the main results of the paper.

2. Preliminaries

In this Section we briefly review some basic concepts regarding the interval numbers, the uncertain generalized weighted average and the PWA operator.

2.1. The interval numbers

The interval numbers (Moore, 1966) are a very useful and simple technique for representing the uncertainty that has been used in a wide range of applications.

The interval numbers can be expressed in different forms. For example, if we assume a 4-tuple (a_1, a_2, a_3, a_4), that is, a quadruplet; we could consider that a_1 and a_4 represents the minimum and the maximum of the interval number, and a_2 and a_3, the interval with the highest probability or possibility, depending on the use we want to give to the interval numbers. Note that $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. If $a_1 = a_2 = a_3 = a_4$, then, the interval number is an exact number; if $a_2 = a_3$, it is a 3-tuple known as triplet; and if $a_1 = a_2$ and $a_3 = a_4$, it is a simple 2-tuple interval number.

In the following, we are going to review some basic interval number operations as follows. Let A and B be two triplets, where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then:

(1) $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
(2) $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.
(3) $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$.
(4) $A \times B = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3)$; for R^+.
(5) $A \div B = (a_1 \div b_3, a_2 \div b_2, a_3 \div b_1)$; for R^+.

Note that other operations could be studied (Merigó & Casanovas, 2011a; Moore, 1966; Wei, 2009; Xu & Da, 2002) but in this paper we focus on these ones. Note also that we call uncertain aggregation operators, to those operators that use interval numbers in the analysis.

2.2. The uncertain generalized weighted average

The uncertain generalized weighted average (UGWA) is an aggregation operator that generalizes the uncertain weighted average (UWA) operator by using generalized means. Thus, it can be implemented in the same problems where the UWA operator has been used because we can always reduce the UGWA to the UWA. Moreover, it can be used in a wide range of other situations because it also includes geometric, quadratic and harmonic aggregations as particular cases. It is defined as follows.

Definition 1. Let Ω be the set of interval numbers. An UGWA operator of dimension n is a mapping $UGWA$: $\Omega^n \to \Omega$ that has an associated weighting vector W of dimension n with $\sum_{i=1}^{n} \tilde{w}_i = 1$ and $\tilde{w}_i \in [0, 1]$ such that:

$$UGWA\ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\sum_{i=1}^{n} \tilde{w}_i \tilde{a}_i^{\lambda} \right)^{1/\lambda}, \tag{1}$$

where each $\tilde{a}_i$ is an interval number, and λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$.

It includes a wide range of particular cases, such as the UWA (when $\lambda = 1$), the uncertain weighted geometric average (UWGA) (when $\lambda \to 0$), the uncertain weighted quadratic average (UWQA) (when $\lambda = 2$) and the uncertain weighted harmonic average (UWHA) (when $\lambda = -1$). Note that we will use the λ value that better fits our needs in the specific problem considered. It is very common to use $\lambda = 1$ because it provides the classical weighted average adapted for situations that requires the use of interval numbers. However, we may find a lot of other studies that use other values including the analysis of multiplicative preference relations that is assessed with geometric means (Xu, 2007).

2.3. The probabilistic weighted average

The probabilistic weighted averaging (PWA) operator (Merigó, 2012a) is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. It is defined as follows.

Definition 2. A PWA operator of dimension n is a mapping PWA: $R^n \to R$ such that:

$$PWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_i a_i, \tag{2}$$

where the a_i are the argument variables, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic weight p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_i = \beta p_i + (1-\beta) v_i$ with $\beta \in [0, 1]$ and $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation.

As we can see, with the PWA operator we can combine objective probabilities with subjective probabilities. Note that this may be very practical in a wide range of situations where information is assessed from different perspectives. For example, when we analyze the probability of rain tomorrow, we may have a study from previous years that establish an objective probability of rain. However, depending on the weather we have today, our subjective opinion may be used to modify the initial objective probability because we have the subjective feeling that tomorrow the weather is going to be similar to today's weather. Thus, with the PWA operator we can update much better probabilistic information by taking into account the different sources of information that affect the analysis of the specific problem we are considering.

Note that if $\beta = 1$, we get the classical probabilistic aggregation (expected value) and if $\beta = 0$, the weighted average that often represents some kind of subjective probability. The closer β to 1, the more importance we give to probabilities and vice versa.

3. The uncertain generalized probabilistic weighted average

The uncertain generalized probabilistic weighted averaging (UGPWA) operator is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. Its main advantage is that it can assess uncertain environments that cannot be assessed with exact numbers but it is possible to use interval numbers. Thus, we can analyze imprecise information considering the minimum and the maximum result that can occur in each situation. This is important because when analyzing uncertain information with exact numbers we may lose information and sometimes this can be critical.

For example, let us assume two different analyses. The first one assesses the expected benefit of a situation x with an exact result of 20 million €. Thus, the assumption is that on average the company can get benefit. In the second analysis, interval numbers are used to predict an expected benefit of 20 million € while in a pessimistic scenario it could be – 30 million € and in an optimistic scenario, it could be 50 million €. The interval numbers are more realistic and informative because we know all the potential results that may occur. Furthermore, in this example the decision maker is aware that there is the risk of a potential loss. So he knows that he should prepare some strategies that could cover mitigate this high risk. While not every situation includes such high risk from a general perspective, it is important to remark that the use of interval numbers and other tools for dealing with uncertain information is useful for correctly representing the problem without losing information.

Moreover, the UGPWA operator uses generalized means in order to provide a more general formulation that includes a wide range of particular cases such as the uncertain PWA (UPWA), the geometric uncertain probabilistic weighted average (GUPWA), the harmonic uncertain probabilistic weighted average (HUPWA), the quadratic uncertain probabilistic weighted average (QUPWA) and a lot of other cases. Note that in this aggregation, we unify the UGWA with the uncertain generalized probabilistic aggregation (UGPA) and we are able to include other unifications such as the uncertain weighted geometric average (UWGA) with the uncertain probabilistic geometric average (UPGA) and the uncertain weighted quadratic average (UWQA) with the uncertain probabilistic quadratic average (UPQA). It is defined as follows.

Definition 3. Let Ω be the set of interval numbers. An UGPWA operator is a mapping $UGPWA$: $\Omega^n \to \Omega$ of dimension n that has an associated probabilistic vector P, with $\sum_{i=1}^{n} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0, 1]$ and a weighting vector V that affects the weighted average, with $\sum_{i=1}^{n} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0, 1]$, such that:

$$UGPWA\ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{\beta} \left(\sum_{j=1}^{n} \tilde{p}_i \tilde{a}_i^{\lambda} \right)^{1/\lambda} + (1-\tilde{\beta}) \left(\sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i^{\sigma} \right)^{1/\sigma}, \tag{3}$$

where the $\tilde{a}_i$ are the argument variables represented in the form of interval numbers, $\tilde{\beta} \in [0, 1]$ and λ and σ are parameters such that $\lambda, \sigma \in (-\infty, \infty) - \{0\}$.

Similar to the PWA operator, we can see that the UGPWA operator also permits to combine different types of probabilities in the same formulation in order to assess complex environments in a more informative and flexible way. Note that if the weighting vector of objective probabilities or subjective information is not normalized, then, the UGPWA operator is expressed as follows:

$$UGPWA\ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\tilde{\beta}}{\tilde{P}} \left(\sum_{j=1}^{n} \tilde{p}_i \tilde{a}_i^{\lambda} \right)^{1/\lambda} + \frac{(1-\tilde{\beta})}{\tilde{V}} \left(\sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i^{\sigma} \right)^{1/\sigma}. \tag{4}$$

where $\tilde{P}$ represents the sum of the objective weighting vector $\tilde{P} = \sum_{i=1}^{n} p_i \neq 1$ and $\tilde{V}$ the sum of the subjective weighting vector $\tilde{V} = \sum_{i=1}^{n} v_i \neq 1$.

Note that it is possible to use a wide range of interval numbers in the aggregation process such as the 2-tuples, the triplets and the quadruplets. Moreover, note that sometimes it is not clear how to reorder the interval numbers. Therefore, it is necessary to establish a criterion for ranking interval numbers (Gil-Lafuente, 2005; Merigó & Casanovas, 2011a; Merigó & Wei, 2011). For simplicity, we recommend the following method.

- For 2-tuples, calculate the arithmetic mean of the interval, with $(a_1 + a_2)/2$.
- For 3-tuples and above, calculate a weighted average that yields more importance to the central values. That is, for 3-tuples, $(a_1 + 3a_2 + a_3)/5$. Note that in this case, we assume that we give a degree of importance of 60% to the central value and 20% for the minimum and another 20% to the maximum.
- For 4-tuples, we calculate: $(a_1 + 3a_2 + 3a_3 + a_4)/8$.
- And so on.

In the case of a tie between the intervals, we select the interval with the lowest difference, i.e., $(a_2 - a_1)$. For 3-tuples and above odd-tuples, we select the interval with the highest central value. Note that for 4-tuples and above even-tuples, we must calculate the average of the central values following the initial criteria.

The UGPWA is bounded, idempotent and monotonic. It is bounded because the UGPWA aggregation is delimitated by the minimum and the maximum. That is, $\text{Min}\{\tilde{a}_i\} \leqslant UGPWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leqslant \text{Max}\{\tilde{a}_i\}$. It is idempotent because if $\tilde{a}_i = a$, for all i, then, $UGPWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = a$. It is monotonic because if $\tilde{a}_i \geqslant u_i$, for all$i$, then, $UGPWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \geqslant UGPWA(u_1, u_2, \ldots, u_n)$.

The UGPWA operator can be further generalized by using quasi-arithmetic means (Fodor et al., 1995; Merigó & Casanovas, 2011a) obtaining the uncertain quasi-arithmetic PWA (Quasi-UPWA) operator. It is defined as follows.

Definition 4. Let Ω be the set of interval numbers. A Quasi-UPWA operator is a mapping *QUPWA*: $\Omega^n \to \Omega$ of dimension n, if it has an associated probabilistic vector P, with $\sum_{i=1}^n \tilde{p}_i = 1$ and $\tilde{p}_i \in [0,1]$ and a weighting vector V that affects the weighted average, with $\sum_{i=1}^n \tilde{v}_i = 1$ and $\tilde{v}_i \in [0,1]$, such that:

$$Quasi-UPWA\ (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{\beta} g^{-1}\left(\sum_{j=1}^n \tilde{p}_i g(\tilde{a}_i)\right) + (1-\tilde{\beta}) h^{-1}\left(\sum_{i=1}^n \tilde{v}_i h(\tilde{a}_i)\right), \quad (5)$$

where the $\tilde{a}_i$ are the argument variables represented in the form of interval numbers, $\tilde{\beta} \in [0,1]$ and g and h are strictly continuous monotonic functions.

As we can see, if $g(a) = a^\lambda$ and $h(a) = a^\sigma$, the Quasi-UPWA operator becomes the UGPWA operator. Thus, it is straightforward to develop similar families of Quasi-UPWA operators as it is presented in Section 4 for the UGPWA operator.

Another issue to analyze is the measures for characterizing the weighting vector $\widehat{V}$. For example, the entropy of dispersion (Merigó, 2012b; Shannon, 1948; Yager, 1988) measures the amount of information used in the aggregation. For the UGPWA operator, it is defined as follows.

$$H(\widehat{V}) = -\left(\tilde{\beta}\sum_{i=1}^n \tilde{v}_i \ln(\tilde{v}_i) + (1-\tilde{\beta})\sum_{i=1}^n \tilde{p}_i \ln(\tilde{p}_i)\right). \quad (6)$$

Note that $\tilde{p}_i$ is the ith probability of the UGPA aggregation and $\tilde{v}_i$ the ith weight of the UGWA aggregation. As we can see, if $\tilde{\beta} = 1$ or $\tilde{\beta} = 0$, we get the classical Shannon entropy of dispersion by using interval numbers.

Observe that in the literature there are other aggregation operators closely related to the UGPWA operator including the uncertain induced generalized probabilistic ordered weighted averaging weighted average (UIGPOWAWA) (Merigó & Gil-Lafuente, 2012). This operator provides a deeper generalization by using OWA aggregations in the UGPWA operator. However, note that in this paper the objective is to analyze the unification between probabilities (objective information) and weighted averages (subjective information). Therefore, the OWA operator is not considered. Note that the use of OWA operators (Yager, 1988, 1993) provides a parameterized family of aggregation operators between the minimum and the maximum that could be incorporated in the UGPWA by using the UIGPOWAWA operator.

4. Families of UGPWA operators

The UGPWA operator includes a wide range of particular cases that are useful in some specific situations.

Remark 1. If $\beta = 0$, we obtain the uncertain generalized weighted average (UGWA) and if $\tilde{\beta} = 1$, the uncertain generalized probabilistic aggregation (UGPA). Observe that it is possible to consider partial cases such as $\tilde{\beta} = (0.8, 1)$ where for some part of the interval we get the UGPA (or the UGWA) but not in the whole interval. Moreover, note that the UGWA (or the UGPA) is useful when dealing with one weighting vector (subjective information). However, the UGPWA operator is needed when more sources of information are available in the problem. Especially, when dealing with subjective and objective information or probabilities. UGPWA includes UGWA as a particular case when only one weighting vector is used or both weighting vectors are equal.

Remark 2. If $\tilde{p}_i = 1/n$ and $\tilde{v}_i = 1/n$, for all i, then, we get the uncertain generalized average (UGA). Note that the UGM is also found if $\tilde{\beta} = 1$ and $\tilde{p}_i = 1/n$, for all i, and if $\tilde{\beta} = 0$ and $\tilde{v}_i = 1/n$, for all i.

Remark 3. If $\tilde{v}_i = 1/n$, for all i, we get the uncertain generalized arithmetic probabilistic aggregation (UGA-PA).

$$UGA-PA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\left(\sum_{i=1}^n \tilde{p}_i \tilde{a}_i^\lambda\right)^{1/\lambda} + (1-\tilde{\beta})\left(\frac{1}{n}\sum_{i=1}^n \tilde{a}_i^\sigma\right)^{1/\sigma}. \quad (7)$$

Remark 4. If $\tilde{p}_i = 1/n$, for all i, we get the uncertain generalized arithmetic WA (UGA-WA).

$$UGA-WA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\left(\frac{1}{n}\sum_{i=1}^n \tilde{a}_i^\lambda\right)^{1/\lambda} + (1-\tilde{\beta})\left(\sum_{i=1}^n \tilde{v}_i \tilde{a}_i^\sigma\right)^{1/\sigma}. \quad (8)$$

Theorem 1. *If the interval numbers are reduced to the usual exact numbers, then, the UGPWA operator becomes the generalized PWA (GPWA) operator.*

Proof. Assume a quadruplet = (a_1, a_2, a_3, a_4). If $a_1 = a_2 = a_3 = a_4$, then (a_1, a_2, a_3, a_4) = a. Thus, we get the GPWA operator.

Remark 5. Note that if the available information is given in different types of interval numbers, then, we have to adapt them to the same structure. Thus, we have to construct an interval that includes all the other ones. For example, if we have one interval with 2-tuples and another one with triplets, then, we have to convert both of them to quadruplets. The 2-tuple is constructed as follows: [a_1, a_2] = [a_1, a_1, a_2, a_2] and the triplet in the following way: [a_1, a_2, a_3] = [a_1, a_2, a_2, a_3].

If we analyze different values of the parameter λ, we obtain another group of particular cases such as the usual UPWA operator, the geometric UPWA (GUPWA) operator, the quadratic UPWA (QUPWA) operator and the harmonic UPWA (HUPWA) operator.

Remark 6. When $\lambda = \sigma = 1$, the UGPWA operator becomes the UPWA operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\sum_{i=1}^n \tilde{p}_i \tilde{a}_i + (1-\tilde{\beta})\sum_{i=1}^n \tilde{v}_i \tilde{a}_i. \quad (9)$$

Note that if $p_i = 1/n$, for all $\tilde{a}_i$, we get the uncertain arithmetic weighted average (UAWA) and if $v_i = 1/n$, for all $\tilde{a}_i$, the uncertain arithmetic probabilistic aggregation (UAPA) operator.

Remark 7. When $\lambda \to 0$ and $\sigma \to 0$, the UGPWA operator becomes the geometric uncertain probabilistic weighted averaging (GUPWA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\prod_{i=1}^n \tilde{a}_i^{\tilde{p}_i} + (1-\tilde{\beta})\prod_{i=1}^n \tilde{a}_i^{\tilde{v}_i}. \quad (10)$$

If $p_i = 1/n$, for all $\tilde{a}_i$, we obtain the uncertain geometric averaging weighted geometric average (UGAWGA) and if $v_i = 1/n$, for all $\tilde{a}_i$, we get the uncertain probabilistic geometric averaging geometric average (UPGAGA) operator. Note that if $\beta = 1$, we get the uncertain probabilistic geometric aggregation (UPGA).

Remark 8. When $\lambda = \sigma = -1$, we get the harmonic uncertain probabilistic weighted averaging (HUPWA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\frac{1}{\sum_{i=1}^n \frac{\tilde{p}_i}{\tilde{a}_i}} + (1-\tilde{\beta})\frac{1}{\sum_{i=1}^n \frac{\tilde{v}_i}{\tilde{a}_i}}. \quad (11)$$

If $p_i = 1/n$, for all $\tilde{a}_i$, we form the uncertain harmonic averaging weighted harmonic average (UHAWHA) and if $v_i = 1/n$, for all $\tilde{a}_i$, the uncertain probabilistic harmonic averaging harmonic average (UPHAHA) operator. Note that if $\beta = 1$, we get the uncertain probabilistic harmonic aggregation (UPHA).

Remark 9. When $\lambda = \sigma = 2$, we get the quadratic uncertain probabilistic weighted averaging (QUPWA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\left(\sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i^2\right)^{1/2} + (1-\tilde{\beta})\left(\sum_{i=1}^{n} \hat{v}_i \tilde{a}_i^2\right)^{1/2}. \quad (12)$$

If $p_i = 1/n$, for all $\tilde{a}_i$, we get the uncertain quadratic averaging weighted quadratic average (UQAWQA) and if $v_i = 1/n$, for all $\tilde{a}_i$, we get the uncertain probabilistic quadratic averaging quadratic average (UPQAQA) operator. Note that if $\beta = 1$, we get the uncertain probabilistic quadratic aggregation (UPQA).

Moreover, we can also consider situations where $\lambda \neq \sigma$. For example, we can form the following aggregation operators. If $\lambda = 1$ and $\sigma = 2$, we get the uncertain probabilistic weighted quadratic averaging (UPWQA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i + (1-\tilde{\beta})\left(\sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i^2\right)^{1/2}. \quad (13)$$

If $\lambda = 1$ and $\sigma = 3$, we form the uncertain probabilistic weighted cubic averaging (UPWCA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i + (1-\tilde{\beta})\left(\sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i^3\right)^{1/3}. \quad (14)$$

If $\lambda = 2$ and $\sigma = 1$, we get the uncertain probabilistic quadratic weighted averaging (UPQWA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\left(\sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i^2\right)^{1/2} + (1-\tilde{\beta})\sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i. \quad (15)$$

If $\lambda = 2$ and $\sigma = 3$, we obtain the uncertain probabilistic quadratic weighted cubic averaging (UPQWCA) operator.

$$UGPWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\left(\sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i^2\right)^{1/2} + (1-\tilde{\beta})\left(\sum_{i=1}^{n} \hat{v}_i \tilde{a}_i^3\right)^{1/3}. \quad (16)$$

Note that people usually tend to use the UPWA operator (arithmetic aggregation) because it is the most simple one and very easy to use. However, we may find more complex environments where we need to use more complex aggregations. For example, when dealing with multiplicative preference relations, we may prefer to use the GUPWA operator. Furthermore, in extremely complex situations where we analyze a lot of dimensions and complex forms of understanding, it becomes very useful to use more complex aggregations where the value of λ and σ increases to one hundred, one thousand or even more. Note that further situations could be considered by using mixture operators and norm aggregations (Merigó & Casanovas, 2011b; Yager, 2010).

5. Moving averages in the UGPWA operator

The use of moving averages in the UGPWA operator permits to represent the information in a dynamic way. By using moving averages (Merigó & Yager, 2013), we form the uncertain generalized probabilistic weighted moving average (UGPWMA). Its main advantage is that it unifies probabilities (objective information) and weighted averages (subjective information) in the same formulation and considering the degree of importance that each concept has in the moving average. Thus, we can analyze the information in a more complete way because we can analyze dynamic systems with subjective and objective information that adapts to the changes found throughout the time. It can be defined as follows.

Definition 5. Let Ω be the set of interval numbers. An UGPWMA operator of dimension m is a mapping $UGPWMA$: $\Omega^m \to \Omega$ such that:

$$UGPWMA\ (\tilde{a}_{1+t}, \tilde{a}_{2+t}, \ldots, \tilde{a}_{m+t}) = \tilde{\beta}\left(\sum_{i=1+t}^{m+t} \tilde{p}_i \tilde{a}_i^{\lambda}\right)^{1/\lambda} + (1-\tilde{\beta})\left(\sum_{i=1+s}^{l+s} \tilde{v}_i \tilde{a}_i^{\sigma}\right)^{1/\sigma}, \quad (17)$$

where the $\tilde{a}_i$ are the argument variables represented in the form of interval numbers, each argument $\tilde{a}_i$ has an associated weight (WA) $\tilde{v}_i$ with $\sum_{i=1+t}^{m+t} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0,1]$, and a probabilistic weight $\tilde{p}_i$ with $\sum_{i=1+t}^{m+t} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0,1]$, with $\tilde{\beta} \in [0,1]$, m and l are the total number of arguments considered from the whole sample, t and s indicate the movement done in the aggregation from the initial analysis and λ and σ are parameters such that $\lambda, \sigma \in (-\infty, \infty) - \{0\}$.

An interesting issue when analyzing the dynamic process is that the size of the probabilities and the weighted averages used may be equal or different. If it is equal, we assume that both concepts are assessing the information in a similar way. If they are different, it means that they are not dealing exactly with the same information and therefore the dynamic process is not the same.

Observe that β may also change during the dynamic process because the degree of importance of each concept in the aggregation may change over time. Since for each period (sequence) the probabilities and the weights of the PWMA may change we have used a different notation for both concepts, m and t for the probability and l and s for the weighted average.

The UGPWMA operator includes a wide range of particular cases. For example, if $\tilde{\beta} = 1$, the UGPWMA operator becomes the uncertain generalized probabilistic moving average (UGPMA) and if $\tilde{\beta} = 0$, the uncertain generalized weighted moving average (UGWMA). In Table 1 we present some of the main families of UGPWMA operators.

Note that a lot of other families could be studied by giving different values to λ and σ, and analyzing different situations with m, t, l, s and $\tilde{\beta}$. Furthermore, we could generalize the UGPWMA operator by using quasi-arithmetic means forming the quasi-arithmetic uncertain probabilistic weighted moving average (Quasi-UPWMA) in a similar way as it has been done with the UGPWA in Definition 4.

6. Group decision making with the theory of expertons

6.1. Decision making approach

In the literature, there are an extremely wide range of methodologies for decision making (Canós & Liern, 2008; Figueira, Greco, & Ehrgott, 2005; Gil-Aluja, 1999; Kacprzyk & Zadrozny, 2009; Xu, 2009; Xu, Yang, & Wang, 2006; Yang, Wang, Xu, & Chin, 2006, 2011; Zavadskas & Turskis, 2011). In this section, we consider an uncertain multi-person decision making problem by using the theory of expertons (Kaufmann, 1988; Kaufmann & Gil-Aluja, 1993). Note that an experton is an extension of the concept of probabilistic set (Hirota, 1981) for uncertain environments that cannot be assessed with exact numbers but it is possible to use interval numbers. It is very useful in group decision making problems because we can deal with the opinion of several experts in the analysis in a more efficient way since we can assess the information

Table 1
Families of UGPWMA operators.

Particular type	UGPWMA
$p_i = 1/n, \forall i$	Uncertain generalized arithmetic weighted moving average (UGAWMA)
$v_i = 1/n, \forall i$	Uncertain generalized arithmetic probabilistic moving average (UGAPMA)
$p_i = 1/n,\ v_i = 1/n, \forall i$	Uncertain generalized moving average (UGMA)
$\lambda = \sigma = 1$	Uncertain probabilistic weighted moving average (UPWMA)
$\lambda = \sigma = 2$	Quadratic UPWMA (QUPWMA)
$\lambda = \sigma \to 0$	Geometric UPWMA (GUPWMA)
$\lambda = \sigma = -1$	Harmonic UPWMA (HUPWMA)
$\lambda = \sigma = 3$	Cubic UPWMA (CUPWMA)
$\lambda = 2, \sigma = 1$	Uncertain quadratic probabilistic weighted moving average (UQPWMA)
$\lambda = 1,\ \sigma = 2$	Uncertain probabilistic weighted quadratic moving average (UPWQMA)
$\lambda = 1,\ \sigma = 3$	Uncertain probabilistic weighted cubic moving average (UPWCMA)
$\lambda = 1,\ \sigma \to 0$	Uncertain probabilistic weighted geometric moving average (UPWGMA)
$\lambda = 1,\ \sigma = -1$	Uncertain probabilistic weighted harmonic moving average (UPWHMA)
$\lambda = 2,\ \sigma = 3$	Uncertain quadratic probabilistic weighted cubic moving average (UQPWCMA)
λ and $\sigma \to \infty$	Maximum moving average
λ and $\sigma \to -\infty$	Minimum moving average

showing various details on their information and the general tendency of the opinion of the group. The process to follow with expertons and UGPWA operators in group decision making problems, can be summarized as follows.

Step 1. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of finite alternatives, $S = \{S_1, S_2, \ldots, S_n\}$, a set of finite states of nature, forming the payoff matrix $(\tilde{a}_{hi})_{m\times n}$. Let $E = \{e_1, e_2, \ldots, e_q\}$ be a finite set of decision-makers. We assume that all the experts have the same degree of importance. Each decision-maker provides his own payoff matrix $\left(\tilde{a}_{hi}^{(k)}\right)_{m\times n}$.

Step 2. Construct the experton by using the expert's opinions (Kaufmann, 1988). Note that the construction process of an experton is as follows. First, calculate the absolute frequencies (the number of experts that suggest each result). Second, calculate the relative frequencies (divide the absolute frequencies by the total number of experts). Third, calculate the accumulated relative frequency of the results (sum from $\alpha = 1$, the relative frequencies in an accumulated way until $\alpha = 0$).

Step 3. Transform the expertons into simple interval numbers by using the expected value of the experton. That is, $EV(Exp.) = (1/10)\sum_{\alpha=0.1}^{1} \tilde{u}_{\exp}$, where $\tilde{u}_{\exp}$ represents the argument variables considered in the analysis. Since we use a decimal scale when constructing the experton, we consider all the decimal levels α excepting $\alpha = 0$. Thus, we consider 10 levels from 0.1 to 1. However, note that it is possible to consider other scales such as a vigesimal or a centesimal system.

Step 4. Calculate the weighting vector $P = (p_1, p_2, \ldots, p_n)$ such that $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, and $V = (v_1, v_2, \ldots, v_n)$ such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, to be used in the UGPWA aggregation. Define also the parameters λ, σ and $\tilde{\beta}$ to be used in the aggregation. Note that we define them according to the available information and in a subjective way that adapts to the specific needs of the problem. Usually, due to its simplicity it becomes very common to assume that $\lambda = \sigma = 1$. For $\tilde{\beta}$, the higher the value we use, the higher importance we assume that the probability has in the aggregation. Regarding the weighting vectors, we recall any classical method used to obtain probabilistic information or weighted averages.

Step 5. Calculate the aggregated results using the UGPWA operator explained in Eq. (3).

Step 6. Adopt decisions according to the results found in the previous steps. Select the alternative(s) that provides the best result(s). Moreover, establish a ranking of the alternatives.

Table 2
Expert 1.

	S_1	S_2	S_3	S_4
A_1	[0.2, 0.3, 0.4]	[0.6, 0.7, 0.8]	[0.2, 0.3, 0.4]	[0.5, 0.6, 0.7]
A_2	0.5	[0.3, 0.4, 0.5]	[0.4, 0.5, 0.7]	[0.4, 0.5, 0.6]
A_3	[0.1, 0.2, 0.4]	[0.6, 0.8, 0.9]	[0.8, 0.9, 1]	[0.7, 0.8, 0.9]
A_4	[0.3, 0.4, 0.5]	[0.3, 0.4, 0.6]	0.5	[0.3, 0.4, 0.5]

Table 3
Expert 2.

	S_1	S_2	S_3	S_4
A_1	[0.2, 0.3, 0.4]	[0.2, 0.3, 0.5]	[0.3, 0.4, 0.6]	[0.3, 0.5, 0.7]
A_2	[0.7, 0.8, 1]	[0.3, 0.4, 0.6]	[0.1, 0.2, 0.3]	[0.4, 0.6, 0.7]
A_3	[0.6, 0.7, 0.8]	[0.5, 0.6, 0.7]	[0.4, 0.5, 0.6]	[0.2, 0.3, 0.5]
A_4	0.6	[0.3, 0.4, 0.5]	[0.4, 0.5, 0.6]	0.2

Table 4
Expert 3.

	S_1	S_2	S_3	S_4
A_1	[0.2, 0.3, 0.4]	[0.5, 0.6, 0.7]	[0.3, 0.5, 0.6]	[0.4, 0.6, 0.7]
A_2	[0.1, 0.2, 0.4]	0.7	[0.6, 0.7, 0.8]	[0.7, 0.8, 0.9]
A_3	[0.6, 0.8, 0.9]	[0.2, 0.4, 0.6]	[0.4, 0.5, 0.6]	[0.5, 0.6, 0.7]
A_4	0.4	0.3	[0.1, 0.2, 0.4]	[0.2, 0.4, 0.7]

Table 5
Expert 4.

	S_1	S_2	S_3	S_4
A_1	[0.2, 0.3, 0.4]	[0.5, 0.6, 0.7]	[0.2, 0.3, 0.4]	[0.4, 0.6, 0.7]
A_2	[0.2, 0.4, 0.5]	[0.2, 0.3, 0.4]	0.4	[0.7, 0.8, 0.9]
A_3	[0.3, 0.4, 0.6]	0.6	[0.3, 0.4, 0.7]	[0.2, 0.3, 0.6]
A_4	[0.2, 0.3, 0.4]	[0.4, 0.5, 0.6]	[0.3, 0.4, 0.5]	0.6

Table 6
Expert 5.

	S_1	S_2	S_3	S_4
A_1	[0.1, 0.4, 0.5]	[0.6, 0.7, 0.8]	[0.1, 0.3, 0.4]	[0.5, 0.6, 0.8]
A_2	[0.5, 0.6, 0.8]	[0.5, 0.6, 0.8]	[0.2, 0.3, 0.4]	[0.3, 0.4, 0.5]
A_3	[0.3, 0.5, 0.6]	[0.1, 0.2, 0.3]	[0.4, 0.6, 0.7]	[0.5, 0.6, 0.7]
A_4	[0.1, 0.3, 0.5]	0.4	[0.4, 0.6, 0.7]	[0.5, 0.6, 0.7]

Table 7
Expertons for each strategy and state of nature.

		S_1			S_2			S_3			S_4		
A_1	0	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1
	0.2	0.8	1	1	1	1	1	0.8	1	1	1	1	1
	0.3	0	1	1	0.8	1	1	0.4	1	1	1	1	1
	0.4	0	0.2	1	0.8	0.8	1	0	0.4	1	0.8	1	1
	0.5	0	0	0.2	0.8	0.8	1	0	0.2	0.4	0.4	1	1
	0.6	0	0	0	0.4	0.8	0.8	0	0	0.4	0	0.8	1
	0.7	0	0	0	0	0.4	0.8	0	0	0	0	0	1
	0.8	0	0	0	0	0	0.4	0	0	0	0	0	0.2
	0.9	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0
A_2	0	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1
	0.2	0.8	1	1	1	1	1	0.8	1	1	1	1	1
	0.3	0.6	0.8	1	0.8	1	1	0.6	0.8	1	1	1	1
	0.4	0.6	0.8	1	0.4	0.8	1	0.6	0.6	0.8	0.8	1	1
	0.5	0.6	0.6	0.8	0.4	0.4	0.8	0.2	0.4	0.4	0.4	0.8	1
	0.6	0.2	0.4	0.4	0.2	0.4	0.6	0.2	0.2	0.4	0.4	0.6	0.8
	0.7	0.2	0.2	0.4	0.2	0.2	0.4	0	0.2	0.4	0.4	0.4	0.6
	0.8	0	0.2	0.4	0	0	0.2	0	0	0.2	0	0.4	0.4
	0.9	0	0	0.2	0	0	0	0	0	0	0	0	0.4
	1	0	0	0.2	0	0	0	0	0	0	0	0	0
A_3	0	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1
	0.2	0.8	1	1	0.8	1	1	1	1	1	1	1	1
	0.3	0.8	0.8	1	0.6	0.8	1	1	1	1	0.6	1	1
	0.4	0.4	0.8	1	0.6	0.8	0.8	0.8	1	1	0.6	0.6	1
	0.5	0.4	0.6	0.8	0.6	0.6	0.8	0.2	0.8	1	0.6	0.6	1
	0.6	0.4	0.4	0.8	0.4	0.6	0.8	0.2	0.4	1	0.2	0.6	0.8
	0.7	0	0.4	0.4	0	0.2	0.4	0.2	0.2	0.6	0.2	0.2	0.6
	0.8	0	0.2	0.4	0	0.2	0.2	0.2	0.2	0.2	0	0.2	0.2
	0.9	0	0	0.2	0	0	0.2	0	0.2	0.2	0	0	0.2
	1	0	0	0	0	0	0	0	0	0.2	0	0	0
A_4	0	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1
	0.2	0.8	1	1	1	1	1	0.8	1	1	1	1	1
	0.3	0.6	1	1	1	1	1	0.8	0.8	1	0.6	0.8	0.8
	0.4	0.4	0.6	1	0.4	0.8	0.8	0.6	0.8	1	0.4	0.8	0.8
	0.5	0.2	0.2	0.6	0	0.2	0.6	0.2	0.6	0.8	0.4	0.4	0.8
	0.6	0.2	0.2	0.2	0	0	0.4	0	0.2	0.4	0.2	0.4	0.6
	0.7	0	0	0	0	0	0	0	0	0.2	0	0	0.4
	0.8	0	0	0	0	0	0	0	0	0	0	0	0
	0.9	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0

6.2. Illustrative example

In the following, we develop an example in an uncertain multi-person decision making problem in strategic management by using the theory of expertons and the UGPWA operator. Although the example does not use real data, it represents a common real-world situation.

Step 1. Assume a company that operates in the United Kingdom is analyzing its general strategy for the next year and is considering an expansion policy to a new market. After careful review with the board of directors, it considers four possible alternatives.

- A_1: Expand to the French market.
- A_2: Expand to the German market.
- A_3: Expand to the Italian market.
- A_4: Expand to the Spanish market.

In order to evaluate these strategies, the company is assessed with the opinion of five experts. They consider that the key factor for the determination of the expected benefits with each strategy is the European economic situation for the next year. They have summarized the possible scenarios that could occur in the future:

- S_1: Bad economic situation.
- S_2: Regular economic situation.
- S_3: Good economic situation.
- S_4: Very good economic situation.

Each expert evaluates the expected benefits of the company according to the economic situation for the next year. They give their opinion in the interval [0, 1] being 0 the lowest expected benefits (or highest loses) and 1 the highest ones. As the available information is very uncertain, the experts provide their information with interval numbers represented in the form of triplets. The results are shown in Tables 2–6.

Table 8
Experton for A_1 and S_2.

		Absolute frequencies				*Relative frequencies*				*Experton*		
	0	0	0	0		0	0	0		0	1	1	1
	0.1	0	0	0		0.1	0	0		0.1	1	1	1
	0.2	1	0	0		0.2	0.2	0		0.2	1	1	1
	0.3	0	1	0		0.3	0	0.2	0	0.3	0.8	1	1
	0.4	0	0	0		0.4	0	0	0	0.4	0.8	0.8	1
A_1	0.5	2	0	1	⇒	0.5	0.4	0	0.2	⇒ 0.5	0.8	0.8	1
	0.6	2	2	0		0.6	0.4	0.4	0	0.6	0.4	0.8	0.8
	0.7	0	2	2		0.7	0	0.4	0.4	0.7	0	0.4	0.8
	0.8	0	0	2		0.8	0	0	0.4	0.8	0	0	0.4
	0.9	0	0	0		0.9	0	0	0	0.9	0	0	0
	1	0	0	0		1	0	0	0	1	0	0	0

Step 2. With this information, we construct the expertons. Note that when using triplets, we refer to them as m-expertons (Kaufmann & Gil-Aluja, 1993). The results are shown in Table 7.

In order to calculate the results shown in Table 7, we use the following methodology presented in Table 8 for the experton A_1 with S_2.

That is, first we calculate the absolute frequencies (the number of experts that gives each result). Next, we calculate the relative frequencies (we divide the absolute frequencies by the total number of experts) and finally, the accumulated relative frequency of the results (we sum from $\alpha = 1$, the relative frequencies in an accumulated way until $\alpha = 0$) (Kaufmann, 1988; Kaufmann & Gil-Aluja, 1993). The results obtained in the experton indicate the following. The result 0.4 in level 0.6 indicates that 40% of the experts believe that the results are 0.6. The result 0.8 in levels 0.3, 0.4 and 0.5, states that 80% of the experts believe that the results are equal or higher than 0.3, 0.4, and 0.5, respectively. The result 1 in levels 0, 0.1 and 0.2 indicates that all the experts (100%) believe that this level will be reached. The same should be done for all the other expertons.

Note that in the construction of the expertons we could consider different methodologies such as the case when the experts have different degrees of importance or when the scale itself is heterogeneous for different levels of α. For further information, see Kaufmann and Gil-Aluja (1993).

Step 3. Next, we calculate the expected value of the expertons. For doing so, we sum all the levels of membership α except the 0 and divide the result by 10. Note that in this example it is assumed that the aggregation is done with the arithmetic mean. However, in future research we will consider other aggregation operators that can assess the information in this step. The results are shown in Table 9.

Step 4. In this paper, we consider the UA, the UWA, the UPA and the UGPWA operator when $\lambda = \sigma = 1$. The main advantage of using these operators is that they can deal with subjective and objective information in the same formulation. Thus, they can analyze situations where there is some previous objective information and at the same time make adjustments with subjective opinions that may appear in the specific problem considered. In this example, it is assumed that UWA and UPA have the degrees of importance of 70% and 30% respectively. For the UWA and UPA weights, each expert gives their own opinions. These opinions are aggregated with a simple arithmetic mean in order to form the collective weights to be used in the aggregation process. The results are presented in Table 10. As shown in the above table, the opinions of the experts are integrated into a collective result, forming the following weighting vectors to be used in the UGPWA operator for the weighted average V and the probabilities P: $V = (0.2, 0.2, 0.3, 0.3)$ and $P = (0.1, 0.3, 0.3, 0.3)$.

Table 10
UWA and UPA weights.

	UWA				*UPA*			
	S_1	S_2	S_3	S_4	S_1	S_2	S_3	S_4
Expert 1	0.1	0.2	0.4	0.3	0.1	0.2	0.3	0.4
Expert 2	0.2	0.3	0.2	0.3	0.1	0.2	0.3	0.4
Expert 3	0.3	0.2	0.3	0.2	0.1	0.4	0.3	0.2
Expert 4	0.2	0.2	0.3	0.3	0.1	0.3	0.4	0.2
Expert 5	0.2	0.1	0.3	0.4	0.1	0.4	0.2	0.3
Collective	0.2	0.2	0.3	0.3	0.1	0.3	0.3	0.3

Step 5. The results obtained in Table 9 can be used in an aggregation process in order to obtain a single result that permits us to see the expected benefits by using each alternative. Note that in this aggregation process we use several particular cases of the UGPWA operator. The reason for doing so is because the assessment provides forecast but no one knows what is going to happen in the future. Therefore, it is good to see different scenarios according to different key interpretations of the UGPWA operator. However, the results provided by the UGPWA operator should be those that the decision maker follows in order to make a decision. The results are shown in Table 11.

Step 6. As we can see, each particular type of UGPWA operator may lead to different results and decisions. In this example, it seems that A_3 is the optimal choice because for any aggregation operator considered provides the highest expected results. Moreover, we can also establish a ranking of the alternatives for each method. Thus, we get the results shown in Table 12.

Note that in this example all the methods provide the same ranking but sometimes we may find different rankings, especially when the alternatives provide similar results. The main advantage of this approach is that it is possible to combine subjective and objective information in the same formulation so it is possible to represent complex information in a more flexible way because the UGPWA operator can be adapted to the specific needs of the problem considered. This is relevant because real world problems often have a lot of heterogeneities and flexible models that adapt to different situations are needed.

Table 9
Expected value of the expertons for each strategy and state of nature.

	S_1	S_2	S_3	S_4
A_1	[0.18, 0.32, 0.42]	[0.48, 0.58, 0.7]	[0.22, 0.36, 0.48]	[0.42, 0.58, 0.72]
A_2	[0.4, 0.5, 0.6]	[0.4, 0.48, 0.6]	[0.34, 0.42, 0.52]	[0.5, 0.62, 0.72]
A_3	[0.38, 0.52, 0.66]	[0.4, 0.52, 0.62]	[0.46, 0.58, 0.72]	[0.42, 0.52, 0.68]
A_4	[0.32, 0.4, 0.48]	[0.34, 0.4, 0.48]	[0.34, 0.44, 0.54]	[0.36, 0.44, 0.54]

Table 11
Aggregated results.

	UA	*UPA*	*UWA*	*UGPWA*
A_1	[0.325, 0.46, 0.58]	[0.354, 0.488, 0.612]	[0.324, 0.462, 0.584]	[0.333, 0.469, 0.592]
A_2	[0.41, 0.505, 0.61]	[0.412, 0.506, 0.612]	[0.412, 0.508, 0.612]	[0.412, 0.507, 0.612]
A_3	[0.415, 0.535, 0.67]	[0.422, 0.538, 0.672]	[0.42, 0.538, 0.676]	[0.420, 0.538, 0.674]
A_4	[0.34, 0.42, 0.51]	[0.344, 0.424, 0.516]	[0.342, 0.424, 0.516]	[0.343, 0.424, 0.516]

Table 12
Ranking of the strategies.

	Ranking		Ranking
UA	$A_3 \succ A_2 \succ A_1 \succ A_4$	*UWA*	$A_3 \succ A_2 \succ A_1 \succ A_4$
UPA	$A_3 \succ A_2 \succ A_1 \succ A_4$	*UGPWA*	$A_3 \succ A_2 \succ A_1 \succ A_4$

7. Conclusions

The UGPWA operator has been introduced. It is an aggregation operator that unifies the uncertain probabilistic aggregation and the uncertain weighted average in the same formulation and considering the degree of importance that each concept has in the aggregation. Thus, it is very useful for uncertain environments where the available information is imprecise and cannot be assessed with exact numbers but it is possible to use interval numbers. We have seen that the use of interval numbers permits to represent the information in a more complete way because we can assess the imprecise information by using the minimum and the maximum that guarantee that the information is contained inside these bounds.

Moreover, by using generalized aggregation operators, the UGPWA operator includes a wide range of particular cases including the UPA, the UWA, the UGPA, the UGWA, the PWA, the GPWA, the UGA-PA and the UGA-WA operator. Further generalizations to the UGPWA operator have been presented by using quasi-arithmetic means obtaining the Quasi-UPWA operator and by using moving averages forming the UGPWMA operator.

An application of the UGPWA operator in a multi-person decision making problem regarding the selection of business strategies has been proposed. We have used the theory of expertons in order to assess the multi-person analysis. Thus, we have represented the information in a more general way that considers the individual opinions inside the group and produces a final representative result. The main advantage of the UGPWA operator is that it can consider a wide range of scenarios by using different particular cases and select the one in closest accordance with the specific interests of the decision maker.

In future research further developments to this approach will be developed by adding more characteristics in the analysis such as the use of the ordered weighted averaging (OWA) operator and the unified aggregation operators (UAO). Moreover, we will also use other types of uncertain information such as fuzzy numbers or linguistic variables and other potential applications giving special attention to decision theory (Durbach & Stewart, 2012) and statistics. In this context, it is worth noting the importance of testing this new approach in real-world problems. The main concern here is that many difficulties appear in the analysis and sometimes the model solves some issues but not all the complexities of the real world. Therefore, more difficulties may appear when dealing with real data which implies a deeper analysis of external factors that may affect the new approach.

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