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Decision Support

A group decision-making approach based on evidential reasoning for multiple criteria sorting problem with uncertainty

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ABSTRACT

A new group decision-making approach is developed to address a multiple criteria sorting problem with uncertainty. The uncertainty in this paper refers to imprecise evaluations of alternatives with respect to the considered criteria. The belief structure and the evidential reasoning approach are employed to represent and aggregate the uncertain evaluations. In our approach, the preference information elicited from a group of decision makers is composed of the assignment examples of some reference alternatives. The disaggregation–aggregation paradigm is utilized to infer compatible preference models from these assignment examples. To help the group reach an agreement on the assignment of alternatives, we propose a consensus-reaching process. In this process, a consensus degree is defined to measure the agreement among the decision makers' opinions. When the decision makers are not satisfied with the consensus degree, possible solutions are explored to help them adjust assignment examples in order to improve the consensus level. If the consensus degree arrives at a satisfactory level, a linear program is built to determine the collective assignment of alternatives. The application of the proposed approach to a customer satisfaction analysis is presented at the end of the paper.

1. Introduction

The multiple criteria sorting (MCS) problem is concerned with the assignment of a set of decision alternatives evaluated on a family of criteria into predefined ordered categories. In practice, MCS is used in many fields, such as supplier management (Araz & Ozkarahan, 2007), water resource planning (Chen, Hipel, & Kilgour, 2007), ABC inventory classification (Chen, Li, Kilgour, & Hipel 2008), energy management (Neves, Martins, Antunes, & Dias, 2008), credit rating (Doumpos & Zopounidis, 2011), assisted reproduction (Figueira et al., 2011), construction project management (Mota & Almeida, 2012), regional competitiveness analysis (Fernandez, Navarro, Duarte, & Ibarra, 2013), water contamination risk assessment (Macary, Almeida-Dias, Uny, & Probst, 2013), recommender system (Marin, Isern, Moreno, & Valls, 2013), climate classification (Mailly, Abi-Zeid, & Pepin, 2014), and research unit evaluation (Kadziński & Słowiński, 2015).

Various approaches for addressing the MCS problem have been proposed in the literature. These approaches can be classified into the following four categories: (1) the methods inspired by the outranking relations (e.g., Almeida-Dias, Figueira, & Roy, 2010, 2012; Janssen & Nemery, 2013; Köksalan, Mousseau, Özpeynirci, & Özpeynirci, 2009; Kadziński, Tervonen, & Figueira, 2014; Nemery & Lamboray, 2008; Rocha & Dias, 2008); (2) the methods motivated by the value functions (e.g., Doumpos, Zanakis, & Zopounidis, 2001; Doumpos & Zopounidis, 2004; Greco, Kadziński, & Słowiński, 2011; Greco, Mousseau, & Słowiński, 2010; Kadziński, Ciomek, & Słowiński 2015; Köksalan & Özpeynirci, 2009; Kadziński & Tervonen, 2013; Köksalan & Ulu, 2003); (3) the methods based on the weighted Euclidean distance (e.g., Chen et al., 2007, 2008; Vetschera, Chen, Hipel, & Kilgour, 2010); and (4) the rule induction-oriented methods (e.g., Dembczyński, Greco, & Słowiński, 2009; Greco, Matarazzo, & Słowiński, 2010; Kadziński, Greco, & Słowiński, 2014). These methods were originally designed to address the MCS problem for which the preferences are expressed by a single decision maker (DM). However, group decision-making is the most important and frequently encountered process within companies and organizations (Greco, Kadziński, Mousseau, & Słowiński, 2012). Therefore, it is important to study the MCS problem in the context of group decision-making. Moving from a single-DM setting to a multiple-DM setting introduces a great deal of complexity into the MCS problem. In the extant literature, Dias and Clímaco (2000) addressed the sorting problem of a group of DMs with imprecise information on the parameters. Damart, Dias, and Mousseau (2007) provided a methodology that could help a group

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iteratively reach an agreement on how to sort exemplary alternatives. Jabeur and Martel (2007) proposed a sorting method that determines at least one collective assignment from the individual preference systems. Bregar, Györkös, and Jurič (2008) implemented an active iterative mechanism for group consensus seeking that could automatically unify the opinions of the DMs. Greco et al. (2012) and Kadziński, Greco, and Słowiński (2013) extended the robust ordinal regression to the sorting problem in the context of group decision-making. Cai, Liao, and Wang (2012) proposed an interactive sorting approach based on the assignment examples of multiple DMs with different priorities. Bezerra, Melo, and Costa (2014) applied the visual, interactive and comparative analysis methodology to the ELECTRE Tri method (Yu, 1992) for the consensus-building in cooperative groups.

Most of the MCS approaches proposed in the literature assume that the evaluation of alternatives on criteria is accurately defined. However, in real life, the evaluation of certain alternatives on particular criteria may involve various types of uncertainties such as ignorance, fuzziness, interval data, and interval belief degrees (Fu & Yang, 2010). It is therefore necessary to use a scheme to represent and process such uncertain information. Within the field of MCS, few methods suit scenarios in which the evaluation of alternatives is not precisely defined. One of the exceptions is the study developed by Janssen and Nemery (2013) in which the authors extended the FlowSort method (Nemery & Lamboray, 2008) to the case of imprecision in the input data which are defined by intervals. Dembczyński et al. (2009) considered an extension of DRSA (the Dominance-based Rough Set Approach) to the context of the imprecise evaluation of alternatives on criteria and the imprecise assignment of alternatives to categories. However, the two methods are not suitable for handling situations in which the data for measuring the performance of an alternative on a criterion is absent or incomplete or in the form of a probability distribution.

Within the field of multiple criteria decision analysis (MCDA), the evidential reasoning (ER) approach (Yang & Xu, 2002a, 2002b) is a well-known method for addressing decision situations characterized by uncertainty. Unlike the majority of conventional MCDA methods, the ER approach describes the performance of an alternative on a criterion with a distributed assessment using a belief structure. In addition, the ER approach uses a new procedure for aggregating multiple criteria based on the distributed assessment framework. The ER approach provides a new avenue for handling various types of uncertainties in a unified format that includes precise data, the absence of data, incomplete data and probability uncertainty. It has been applied to decision problems in many areas, such as engineering design (Chin, Yang, Guo, & Lam, 2009; Yang, Xu, Xie, & Maddulapalli, 2011), environmental impact assessment (Wang, Yang, & Xu, 2006), business management (Hilhorst, Ribbers, van Heck, & Smits, 2008), and group decision-making (Fu & Yang, 2010).

In this paper, we propose a group decision-making approach based on ER to address the MCS problem with uncertainty. We employ the belief structure and the ER algorithm to represent and aggregate uncertain assessments, respectively. In the ER approach, it is necessary for the DMs to specify the utilities of grades in the assessment framework to obtain the global evaluation of alternatives. This elicitation of the preference information is often referred to as the direct elicitation technique, in which the DMs are guided to directly specify the values of the parameters used in the preference model (Greco, Mousseau, & Słowiński, 2008). However, such direct elicitation is considered to be cognitively cumbersome and difficult for the DMs. Alternatively, there is a preference elicitation technique known as indirect elicitation, which is of interest because it requires relatively less cognitive efforts from the DM (Corrente, Greco, & Słowiński, 2012; Kadziński & Słowiński, 2013; Köksalan et al., 2009). For the indirect elicitation technique, the DMs specify certain examples of holistic judgments, from which compatible values of the preference model parameters are induced using the disaggregation–aggregation (or regression) paradigm. The approach proposed in this paper utilizes the indirect preference elicitation technique. Specifically, the DMs participating in the sorting process are required to provide assignment examples for some reference alternatives. A methodology using the disaggregation–aggregation paradigm is then adopted to infer a set of compatible utility function models integrating the utilities of grades from these assignment examples, which is applied to sort other alternatives.

In the context of group decision-making, one major issue to consider is how to help a group cooperatively develop a common MCS model to sort alternatives into categories. However, due to the DMs' different knowledge bases and levels of experience, there may be a diversity of opinions among the group members. What makes the process more difficult is that an agreement between the DMs may have to be reached in spite of the diversity of judgments and the subjective perception of reality (Damart et al., 2007). Thus, prior to the actual sorting of alternatives it is necessary to reach some type of consensus among the DMs in the decision process. In the proposed approach, a consensus-reaching process is developed to help the DMs to interact for the purpose of iteratively reaching an agreement on how to sort alternatives, with consistency preserved at both the individual level and the collective level. A consensus degree is defined to analyze, control and monitor the consensus-reaching process. The consensus degree measures the agreement among the DMs' opinions. The iterative process is composed of several rounds of consensus-reaching. When the consensus degree is at a low level and the DMs are not satisfied with it, we explore possible solutions to help the group adjust assignment examples in order to improve the consensus level. If the consensus degree arrives at a satisfactory level, we build a linear program to determine the collective assignment of alternatives.

The approach proposed in this paper is distinguished from the previous methods by the following new features. First, the ER approach is extended to the case of the sorting problem for the first time. Despite its successful application to ranking and choice problems with uncertainty, no previous method has introduced the ER approach to the sorting problem. The approach proposed in this paper reveals the applicability and flexibility of the ER approach in modelling the uncertainty in the MCS problem. Second, the proposed approach employs the disaggregation–aggregation paradigm to infer a compatible utility function model from the assignment examples provided by the DMs. This technique reduces the cognitive efforts of the DMs to specify the utilities of assessment grades in the ER approach. Third, this paper develops a consensus-reaching process to support interaction among the DMs, helping them to reach a consensus on how to sort the alternatives. The process contributes to the search for a transparent, justifiable and collectively constructed consensus. An additional appeal of this paper stems from the fact that it provides a definition of the consensus measure for the MCS problem in the context of group decision-making. The definition of this consensus measure could be easily integrated with the other existing MCS approaches to the case of group decision-making and could help accelerate the consensus-reaching process.

The remainder of this paper is organized as follows. In Section 2, we provide a brief introduction to the ER approach. In Section 3, we present a group decision-making approach based on ER for the MCS problem with uncertainty. Section 4 demonstrates the approach using an example. The paper ends with conclusions and discussion regarding future research.

2. Brief introduction to the ER approach

The ER approach is a general approach for analyzing MCDA problems characterized by various types of uncertainty using a unified framework – belief structures (Xu, 2012). With belief structures, the performance of an alternative on a criterion is described as a distribution assessment using belief degrees.

We consider a decision problem in which a finite set of alternatives, denoted by $A = \{a_1, a_2, \ldots, a_i, \ldots, a_K\}$, can have uncertain evaluations with respect to L evaluation criteria $G = \{g_1, g_2, \ldots, g_l, \ldots, g_L\}$. Suppose the weights of criteria are given as $\omega = (\omega_1, \omega_2, \ldots, \omega_l, \ldots, \omega_L)$, where ω_l is the relative importance of the lth criterion g_l with $\omega_l > 0$, $l = 1, 2, \ldots, L$, and $\sum_{l=1}^{L} \omega_l = 1$. Suppose N distinctive evaluation grades are defined to collectively provide a complete set of standards for assessing a criterion, as represented by

$$H = \{H_1,\ H_2, \ldots, H_n, \ldots, H_N\},$$

where H_n is the nth evaluation grade. Without the loss of generality, it is assumed that H_{n+1} is preferred to H_n, $n = 1, 2, \ldots, N-1$. The assessment of alternative a_i on criterion g_l can be represented as the following distribution:

$$s_l(a_i) = \{(H_1, \beta_{1,l}(a_i)), (H_2, \beta_{2,l}(a_i)), \ldots, (H_N, \beta_{N,l}(a_i))\},$$

where $\beta_{n,l}(a_i) \geq 0$, $n = 1, 2, \ldots, N$, and $\sum_{n=1}^{N} \beta_{n,l}(a_i) \leq 1$ and $\beta_{n,l}(a_i)$ denotes the belief degree of alternative a_i being assessed to grade H_n on criterion g_l. An assessment $s_l(a_i)$ is said to be complete if $\sum_{n=1}^{N} \beta_{n,l}(a_i) = 1$ and incomplete if $\sum_{n=1}^{N} \beta_{n,l}(a_i) < 1$. The assessments of the K alternatives on the L criteria can be represented by the following performance matrix:

$$S = (s_l(a_i))_{K\times L}.$$

We should mention that for the assessment of different qualitative criteria, however, different sets of evaluation grades may need to be defined to facilitate data collection. Moreover, some criteria are quantitative and may be assessed using precise or random numbers. This increases the complexity in criteria aggregation. Yang (2001) proposed the rule and utility based information transformation techniques to deal with various types of assessment information under a unified belief structure defined by the same set of grades.

Based on the unified belief structure, Yang and Xu (2002a, 2002b) developed the ER algorithm for the aggregation of multiple criteria. The algorithm in detail can be found in Appendix A. After all the L criteria have been aggregated, we obtain the combined belief degrees for the grade H_n, denoted by $\beta_n(a_i)$, $n = 1, 2, \ldots, N$, and the combined belief degree that is not assigned to any individual grade, denoted by $\beta_H(a_i)$.

If it is required to rank alternatives or choose the best one or assign them to predefined categories, their performances represented by belief structures may not be directly comparable. Hence the concept of expected utility is introduced as a measure to evaluate the performances of alternatives, which requires that the utility of one assessment grade is disjoint from the utility of any other assessment grade in the belief structure. The expected utility of alternative a_i is given by:

$$\begin{aligned} U(a_i) &= \sum_{n=1}^{N} \left(\beta_n(a_i) + \frac{\beta_n(a_i)}{\sum_{n'=1}^{N} \beta_{n'}(a_i)} \beta_H(a_i)\right) u(H_n) \\ &= \sum_{n=1}^{N} \left(\beta_n(a_i) + \frac{\beta_n(a_i)}{1 - \beta_H(a_i)} \beta_H(a_i)\right) u(H_n) \\ &= \sum_{n=1}^{N} \frac{\beta_n(a_i)}{1 - \beta_H(a_i)} u(H_n), \end{aligned}$$

where $u(H_n)$ is the utility of grade H_n satisfying that $0 = u(H_1) < u(H_2) < \cdots < u(H_N) = 1$. Note that $\beta_H(a_i)$ is proportionally assigned to each grade H_n, $n = 1, 2, \ldots, N$.

3. The proposed approach

3.1. Problem description

Assume that several DMs, denoted by $E = \{e_1, e_2, \ldots, e_t \ldots, e_T\}$, collaborate to sort a set of alternatives A into predefined ordered categories $C = \{C_1, C_2, \ldots, C_q, \ldots, C_Q\}$, such that C_{q+1} is preferred to C_q, $q = 1, 2, \ldots, Q-1$. The evaluation of alternative a_i on criterion g_l is expressed by belief structure $s_l(a_i)$. The DMs share the same "description" of the sorting problem, i.e., the set of alternatives A, the family of criteria G, the performance matrix S, and the set of categories C (Greco et al., 2012). We consider the different role of each DM in the decision process and hence differentiate their priorities by a weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_T)$ such that $\sum_{t=1}^{T} \lambda_t = 1$ and $\lambda_t > 0$, $t = 1, 2, \ldots, T$.

In this study, we focus on introducing the ER approach to address the MCS problem and developing a consensus-reaching process. Note that the raw data of alternatives on criteria could be either qualitative or quantitative in different formats and we should transform them into the unified belief structure in advance. Moreover, similarly to some multiple criteria group decision-making methods (Chin et al., 2009; Fu & Yang, 2010; Huang, Chang, Li, & Lin, 2013; Tavana, Sodenkamp, & Suhl, 2010), we assume that all members of the group accept the common weights of criteria, i.e., $\omega = (\omega_1, \omega_2, \ldots, \omega_l, \ldots, \omega_L)$. The collective weights of criteria can be obtained by using some methods such as direct rating (Doyle, Green, & Bottomley, 1997), eigenvector method (Takeda, Cogger, & Yu, 1987), and Delphi method (Hwang & Yoon, 1981). Concerning the DMs' priorities, the leader of the group could specify the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_T)$, or we could ask each DM to evaluate other DMs' importance and then determine the weight vector (Figueira & Roy, 2002).

We assume that the preference information of each DM e_t, $t = 1, 2, \ldots, T$, is a set of assignment examples for the reference alternatives $a^* \in A_t^R \subset A$. The assignment example provided by the DM e_t specifies a desired assignment $a^* \to [C_{L_t^{DM}(a^*)}, C_{R_t^{DM}(a^*)}]$ on a corresponding reference alternative $a^* \in A_t^R$, where $[C_{L_t^{DM}(a^*)}, C_{R_t^{DM}(a^*)}]$ is an interval of contiguous categories $C_{L_t^{DM}(a^*)}, C_{L_t^{DM}(a^*)+1}, \ldots, C_{R_t^{DM}(a^*)}$ with $L_t^{DM}(a^*) \leq R_t^{DM}(a^*)$. An assignment example can be precise if $L_t^{DM}(a^*) = R_t^{DM}(a^*)$ or imprecise, otherwise.

In the context of group decision-making, the DMs may have different utility values on $u(H_n)$, $n = 1, 2, \ldots, N$. Thus, we use $u_t(H_n)$ to denote the utility of the DM e_t on grade H_n and $U_t(a)$ to denote the expected utility of the DM e_t for alternative α. In the proposed method, we take the threshold-based sorting procedure (Greco, Mousseau et al., 2010), and the sorting is performed through the comparison between the utility of each alternative with category thresholds $b_0^t, b_1^t, \ldots, b_Q^t$, i.e.,

$$b_{q-1}^t \leq U_t(a) < b_q^t, \quad t = 1, 2, \ldots, T \Leftrightarrow a \to C_q, \tag{1.1}$$

where b_h^t, $h = 0, 1, \ldots, Q$, such that $b_0^t < b_1^t < b_2^t < \cdots < b_Q^t$, are the thresholds of the DM e_t. b_{q-1}^t and b_q^t are the lower and upper thresholds of category C_q, $q = 1, 2, \ldots, Q$, respectively.

3.2. Outline of the process

The iterative process of the approach proposed in this paper is depicted in Fig. 1, and its phases are described in detail below.

Step 1. Define the sorting problem and collect related information, including the set of alternatives A, the set of criteria G, the general scale of evaluation grades H, the performance matrix S, and the set of categories C. Transform the raw data into the unified belief structure. Organize a group

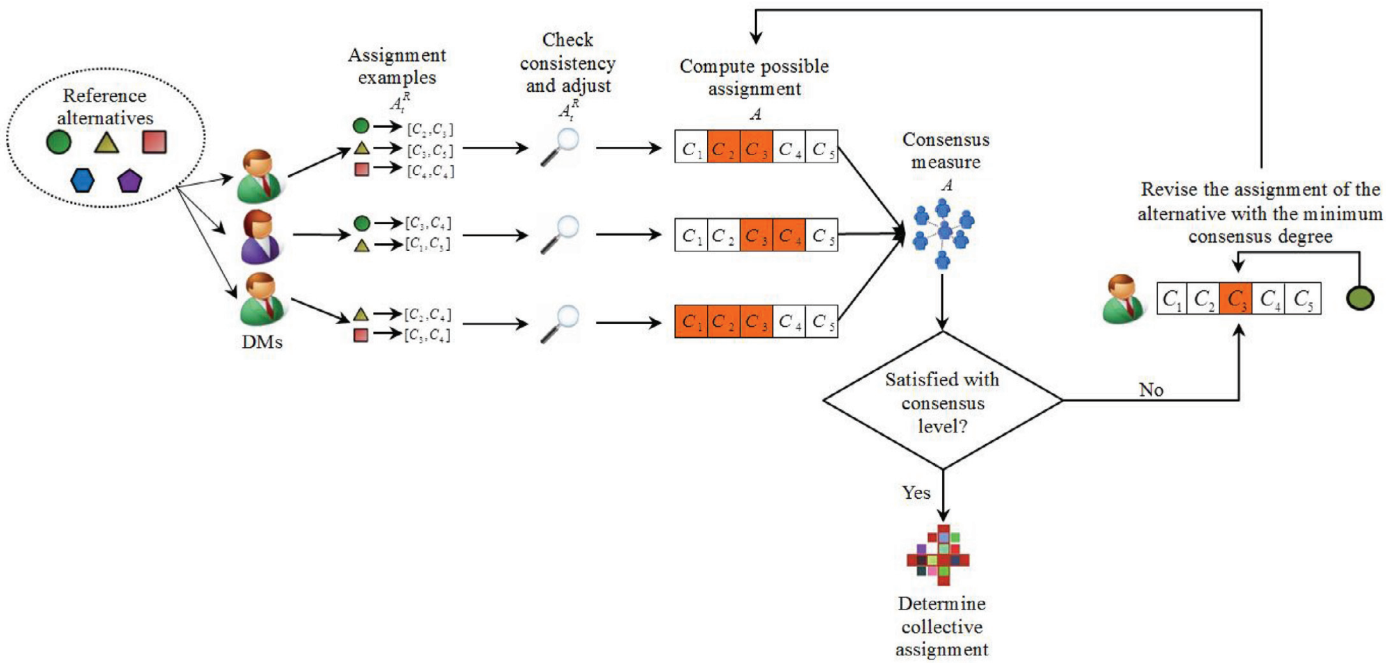


Fig. 1. General framework of the proposed approach.

discussion and help the DMs to determine the weight vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_l, \ldots, \omega_L)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_T)$.

Step 2. Apply the ER algorithm to aggregate the belief structures on multiple criteria into the combined belief degrees and then present the outcomes to the group. Each DM is asked to provide his/her assignment examples based on the combined belief degrees.

Step 3. Detect and solve the possible inconsistency of assignment examples provided by each DM $e_t, t = 1, 2, \ldots, T$, at the individual level (see Section 3.3 for more details).

Step 4. For each alternative $a \in A$, compute its possible assignment $[C_{L_t(a)}, C_{R_t(a)}]$ for each DM $e_t, t = 1, 2, \ldots, T$, at the individual level (see Section 3.4 for more details).

Step 5. Calculate the consensus degree $cog(a)$ for each alternative $a \in A$ and the average consensus degree $\overline{cog}$ at the group level (see Section 3.5.1 for more details).

Step 6. The members in the group verify the consensus level $\overline{cog}$. If they are not satisfied with it, choose the alternative $a^* \in \bigcup_{t=1,2,\ldots,T} A_t^R$ with the minimum consensus degree $cog(a^*)$ and ask the DMs to revise its assignment example (see Section 3.5.2 for more details) and then proceed to Step 4. If the members are satisfied with $\overline{cog}$ or they want to terminate the consensus-reaching process, proceed to Step 7.

Step 7. Determine the collective assignment of alternatives (see Section 3.6 for more details) and present the sorting results to the group.

The MCS approach proposed in this paper extends the disaggregation–aggregation paradigm to the case of group decision-making. In this case, a group of DMs collaborate to sort a set of alternatives into predefined ordered categories. The approach aims to support interaction among the DMs and to help them to reach a consensus on the assignment of alternatives. In this context, we identify two types of inconsistency that may occur at the individual level and the group level. The former stems from the possible inconsistency in the assignment examples of an individual DM. Indeed, the preferences of a DM may be incoherent and unstable, so it is difficult for him/her to provide consistent assignment examples. The second type of inconsistency arises from the group's pursuance of an agreement on the assignment of alternatives. Due to the difference of the DMs' preferences, there is usually a divergence about the assignment among the DMs. Thus, it is required to make the entire group move toward a collective assignment while keeping the individual assignment consistent.

To meet this requirement, we develop a process to help the group reach a consensus on the assignment of alternatives progressively. On the one hand, to maintain consistency at the individual level, any possible inconsistency in the assignment examples is detected and solved by constructing linear programs based on the principle of the minimum sum of shifts. On the other hand, the possible assignment of each non-reference alternative is computed based on the consistent assignment examples, and an index is defined to measure the consensus level of the group. If the DMs are not satisfied with the consensus level, possible adjustment solutions are explored and presented to the DMs and they could choose among these solutions to improve the consensus level. When the consensus level is satisfactory, we build a linear program to determine the collective assignment of alternatives. Note that we assume that all members in the group are sufficiently cooperative to follow this procedure.

3.3. Analysis of the inconsistency at the individual level

The preference information provided by the DM e_t, $t = 1, 2, \ldots, T$, can be represented by a pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$, where $\boldsymbol{u}_t = (u_t(H_1), u_t(H_2), \ldots, u_t(H_N))$ is a vector of utilities of grades and $\boldsymbol{b}_t = (b_0^t, b_1^t, \ldots, b_Q^t)$ is a vector of thresholds (Greco, Mousseau et al., 2010). Let us define the following set of constraints $E_{A_t^R}$ which incorporates the preference information provided by the DM e_t.

$E_{A_t^R}$:

$$U_t(a^*) = \sum_{n=1}^{N} \frac{\beta_n(a^*)}{1 - \beta_H(a^*)} u_t(H_n), \quad a^* \in A_t^R, \tag{3.1}$$

$$U_t(a^*) \geq b^t_{L_t^{DM}(a^*)-1}, \quad a^* \in A_t^R, \tag{3.2}$$

$$U_t(a^*) + \varepsilon \leq b^t_{R_t^{DM}(a^*)}, \quad a^* \in A_t^R, \tag{3.3}$$

$$u_t(H_{n+1}) - u_t(H_n) \geq \varepsilon, \quad n = 1, 2, \ldots, N-1, \tag{3.4}$$

$$u_t(H_1) = 0, \quad u_t(H_N) = 1, \tag{3.5}$$

$$b^t_{q+1} - b^t_q \geq \varepsilon, \quad q = 0, 1, 2, \ldots, Q-1, \tag{3.6}$$

$$b^t_0 = 0, \quad b^t_Q = 1 + \varepsilon. \tag{3.7}$$

For constraints (3.2) and (3.3), the expected utility $U_t(a^*)$ of the reference alternative a^* should be larger than or equal to the lower threshold $b^t_{L^{\rm DM}_t(a^*)-1}$ and less than the upper threshold $b^t_{R^{\rm DM}_t(a^*)}$, if the DM e_t desires the reference alternative a^* to be assigned to the interval of contiguous categories $[C_{L^{\rm DM}_t(a^*)}, C_{R^{\rm DM}_t(a^*)}]$. Constraints (3.4) and (3.7) ensure the monotonicity and normalization of utility values $u_t(H_n)$, $n = 1, 2, \ldots, N$, and thresholds b_q, $q = 0, 1, 2, \ldots, Q$, for the DM e_t.

Let $\boldsymbol{\partial}^R_t$ be the set of pairs $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ satisfying the constraint set $E_{A^R_t}$. To check that $\boldsymbol{\partial}^R_t$ is not empty, it is sufficient to solve the following linear program (Greco, Mousseau et al., 2010):

$$LP1: \quad \max f_1 = \varepsilon$$
$$\text{s.t.} \quad E_{A^R_t}.$$

If $f^*_1 > 0$, where $f^*_1 = \max f_1$, s.t., $E_{A^R_t}$, then there exists at least a pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ compatible with the assignment examples provided by the DM e_t; otherwise, no compatible pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ exists. This incompatibility may result from the following reasons: (1) the assignment examples of the DM do not match the expected utility model; (2) the statements provided by the DM are contradictory because, for instance, his/her preferences are unstable, and certain hidden criteria are taken into account (Greco et al., 2008). In such a case, we could help the DM to find the assignment examples which cause the inconsistency and then to adjust them to make the constraint set $E_{A^R_t}$ feasible.

Definition 1. Let $a^* \in A^R_t$. The DM e_t originally assigned a^* to the interval $[C_{L^{\rm DM}_t(a^*)}, C_{R^{\rm DM}_t(a^*)}]$. Suppose e_t adjusts it to the interval $[C_{L^{\rm DM'}_t(a*)}, C_{R^{\rm DM'}_t(a*)}]$ with $L^{\rm DM'}_t(a^*) \leq R^{\rm DM'}_t(a^*)$. The adjustment range $\zeta_t(a^*)$ of $a*$ by e_t is defined as the sum of the "shifts" (Damart et al., 2007) of the minimum and maximum assignment, i.e.,

$$\zeta_t(a^*) = \left|L^{\rm DM}_t(a) - L^{\rm DM'}_t(a)\right| + \left|R^{\rm DM}_t(a) - R^{\rm DM'}_t(a)\right|. \tag{3.8}$$

When pursuing to make the constraint set $E_{A^R_t}$ consistent, we wish to maintain original assignment examples as much as possible, that is, to minimize the sum of shifts of all reference alternatives. The following linear program can then be performed to provide an adjustment solution for the DM e_t.

$$LP2: \min f_2 = \sum_{a^* \in A^R_t} (\zeta^+_t(a^*) + \zeta^-_t(a^*)) \tag{3.9}$$

$$\text{s.t.} \quad U_t(a^*) - b^t_{q-1} \geq y^t_q(a^*) - 1, \quad q = 1, 2, \ldots, Q, \quad a^* \in A^R_t, \tag{3.10}$$

$$U_t(a^*) - b^t_q + \varepsilon \leq 1 - y^t_q(a^*), \quad q = 1, 2, \ldots, Q, \quad a^* \in A^R_t, \tag{3.11}$$

$$\sum_{q=1}^{Q} y^t_q(a^*) = 1, \quad a^* \in A^R_t, \tag{3.12}$$

$$y^t_q(a^*) \in \{0, 1\}, \quad q = 1, 2, \ldots, Q, \quad a^* \in A^R_t, \tag{3.13}$$

$$h_t(a^*) = \sum_{q=1}^{Q} q y^t_q(a^*), \quad a^* \in A^R_t, \tag{3.14}$$

$$L^{\rm DM'}_t(a^*) \leq h_t(a^*), \quad a^* \in A^R_t, \tag{3.15}$$

$$h_t(a^*) \leq R^{\rm DM'}_t(a^*), \quad a^* \in A^R_t, \tag{3.16}$$

$$L^{\rm DM}_t(a^*) - L^{\rm DM'}_t(a^*) \leq \zeta^-_t(a^*), \quad a^* \in A^R_t, \tag{3.17}$$

$$L^{\rm DM'}_t(a^*) - L^{\rm DM}_t(a^*) \leq \zeta^-_t(a^*), \quad a^* \in A^R_t, \tag{3.18}$$

$$R^{\rm DM}_t(a^*) - R^{\rm DM'}_t(a^*) \leq \zeta^+_t(a^*), \quad a^* \in A^R_t, \tag{3.19}$$

$$R^{\rm DM'}_t(a^*) - R^{\rm DM}_t(a^*) \leq \zeta^+_t(a^*), \quad a^* \in A^R_t, \tag{3.20}$$

$$L^{\rm DM'}_t(a^*),\ R^{\rm DM'}_t(a^*) \in \{1, 2, \ldots, Q\}, \quad a^* \in A^R_t, \quad \text{Constraints (3.1), (3.4)–(3.7)}, \tag{3.21}$$

where ε is a small positive value representing the smallest discernable difference. The objective function (3.9) is to minimize the sum of shifts of all reference alternatives $a^* \in A^R_t$. $y^t_q(a^*)$ is a binary variable such that $y^t_q(a^*) = 1$ if a^* is assigned to C_q; otherwise, $y^t_q(a^*) = 0$. For constraints (3.10) and (3.11), the expected utility $U_t(a^*)$ should be larger than or equal to the lower threshold b^t_{q-1} and less than the upper threshold b^t_q, if a^* is assigned to C_q. Constraint (3.12) ensures that each alternative is assigned to a single category. Constraints (3.14)–(3.16) indicate the assignment interval of a^* after adjustment. Constraints (3.17) and (3.18) specify the shift of the minimum assignment, while constraints (3.19) and (3.20) measure the shift of the maximum assignment.

The optimal solution of *LP*2 shows one possible solution with the minimum sum of shifts. In some cases, there are multiple possible solutions corresponding to the minimum sum of shifts. We can search for more solutions by solving *LP*2 with additional constraints and then present these solutions to the DM and ask him/her to choose one. See Appendix B for more details about this issue.

3.4. Computation of the possible assignment of alternative

If $\boldsymbol{\partial}^R_t$ is not empty, there is usually more than one compatible pair $(\boldsymbol{u}_t, \boldsymbol{b}_t)$. Because different instances of $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ may produce different assignment for the same alternative, we consider a sorting procedure based on a set of pairs $(\boldsymbol{u}_t, \boldsymbol{b}_t)$.

Definition 2. Given a set A^R_t of assignment examples provided by the DM e_t and a corresponding set $\boldsymbol{\partial}^R_t$ of pairs $(\boldsymbol{u}_t, \boldsymbol{b}_t)$, for any alternative $a \in A$, the possible assignment $C^P_t(a)$ (Greco, Mousseau et al., 2010; Köksalan et al., 2009) is defined as a set of categories C_q for which there exists at least one pair of $(\boldsymbol{u}_t, \boldsymbol{b}_t) \in \boldsymbol{\partial}^R_t$ assigning α to C_q, that is:

$$C^P_t(a) = \{C_q : \exists (\boldsymbol{u}_t, \boldsymbol{b}_t) \in \boldsymbol{\partial}^R_t \text{ for which } b^t_{q-1} \leq U_t(a) < b^t_q, \quad q = 1, 2, \ldots, Q\}. \tag{4.1}$$

Proposition 1. *If $C_q, C_{q'} \in C^P_t(a)$, with $q < q'$, then $C_{q''} \in C^P_t(a)$, $\forall q'' \in [q, q']$.*

Proof. See Appendix C. □

Proposition 1 states that any category between two possible assignments is also a possible assignment.

Definition 3. Given a set A^R_t of assignment examples provided by the DM e_t and a corresponding set $\boldsymbol{\partial}^R_t$ of pairs $(\boldsymbol{u}_t, \boldsymbol{b}_t)$, for any alternative $a \in A$, we define the following categories (Damart et al., 2007):

- minimum possible assignment $C_{L_t(a)}$, where $L_t(a)$ is defined by:

$$L_t(a) = \min\{q : C_q \in C^P_t(a)\}, \tag{4.2}$$

- maximum possible assignment $C_{R_t(a)}$, where $R_t(a)$ is defined by:

$$R_t(a) = \max\{q : C_q \in C^P_t(a)\}. \tag{4.3}$$

According to Proposition 1, for any alternative $a \in A$, if its minimum possible assignment $C_{L_t(a)}$ and maximum possible assignment

$C_{R_t(a)}$ have been obtained, its possible assignment $C_t^P(a)$ can be obtained as $[C_{L_t(a)}, C_{R_t(a)}]$. The category index of the minimum and the maximum possible assignment of α can be calculated as follows:

$$\begin{aligned} LP3: \ & L_t(a) = \min h_t(a)\ (R_t(a) = \max h_t(a)) \\ \text{s.t.} \ & \text{Constraints}(3.1), (3.10) - (3.14) \text{ for } \alpha, \\ & E_{A_t^R}, \end{aligned}$$

where ε is a small positive value representing the smallest discernable difference.

Note that the possible assignment $[C_{L_t(a^*)}, C_{R_t(a^*)}]$ for a reference alternative $a^* \in A_t^R$ may differ from the range $[C_{L_t^{DM}(a^*)}, C_{R_t^{DM}(a^*)}]$ desired by the DM e_t, and $[C_{L_t(a^*)}, C_{R_t(a^*)}]$ could be more precise. This means that if the DM specifies an imprecise assignment example, some categories from this range may not be feasible.

3.5. Group consensus and improvement

3.5.1. Consensus degree of alternative

The group consensus is an important issue for finding the collective solution of a group decision-making problem. In this section, we will propose a new method to measure the consensus degree for the MCS problem in the context of group decision-making.

The consensus measure of a single alternative is constructed at three levels: the pairwise comparison level, the individual level and the group level. With the possible assignment of alternative, the measures at the above three levels can be defined as follows.

Definition 4. Let e_t, $e_{t'} \in E$, $t \neq t'$, $a \in A$. The consensus degree of alternative α at the pairwise comparison level is defined as the consistency between $[C_{L_t(a)}, C_{R_t(a)}]$ and $[C_{L_{t'}(a)}, C_{R_{t'}(a)}]$, which is

$$\begin{aligned} & cop_{t,t'}(a) \\ & = \frac{|\{L_t(a), L_t(a)+1, \ldots, R_t(a)\} \cap \{L_{t'}(a), L_{t'}(a)+1, \ldots, R_{t'}(a)\}|}{|\{L_t(a), L_t(a)+1, \ldots, R_t(a)\} \cup \{L_{t'}(a), L_{t'}(a)+1, \ldots, R_{t'}(a)\}|}, \\ & \qquad a \in A, \quad t, t' = 1, 2, \ldots, T. \end{aligned} \tag{5.1}$$

Definition 5. Let $e_t \in E$, $a \in A$. The consensus degree of alternative α for the DM e_t at the individual level is defined as the consistency between the DM e_t and other DMs, which is

$$cod_t(a) = \frac{\sum_{t'=1;\, t' \neq t}^{T} cop_{t,t'}(a)}{T-1}, \quad a \in A, \quad t = 1, 2, \ldots, T. \tag{5.2}$$

Definition 6. Let $a \in A$. The consensus degree of alternative α at the group level is defined as the consistency among the group, which is

$$cog(a) = \sum_{t=1}^{T} \lambda_t cod_t(a), \quad a \in A. \tag{5.3}$$

$cog(a)$ is bound within the interval [0,1]. For any alternative $a \in A$, if the consensus degree $cog(a)$ is close to 1, the group has a high consensus level on the assignment of α; on the other hand, if the consensus degree $cog(a)$ is close to 0, the group has a large divergence on the assignment of α.

Example 1. Suppose that the assignment of alternative α provided by the DMs e_1, e_2, e_3, e_4 is depicted with gray cells in Fig. 2. The normalized priority vector of the DMs is $\lambda = (0.4, 0.2, 0.3, 0.1)$.

According to Definitions 4, 5 and 6, we obtain the following consensus degrees of alternative α based on the assignment of each DM:

(a) *The consensus degree at the pairwise comparison level.* The consensus degrees $cop_{t,t'}(a)$ of alternative α at the pairwise comparison level are $cop_{1,2}(a) = 1/3$, $cop_{1,3}(a) = 0$, $cop_{1,4}(a) = 2/3$, $cop_{2,3}(a) = 1/3$, $cop_{2,4}(a) = 2/3$ and $cop_{3,4}(a) = 1/4$.

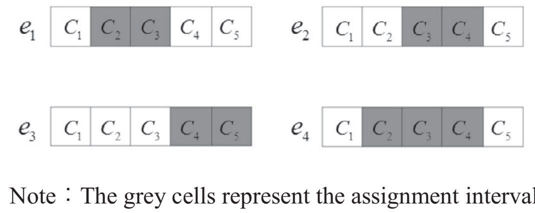


Fig. 2. The assignment of alternative a in Example 1.

(b) *The consensus degree at the individual level.* The consensus degrees $cod_t(a)$ of alternative α for each DM at the individual level are $cod_1(a) = 1/3$, $cod_2(a) = 4/9$, $cod_3(a) = 7/36$ and $cod_4(a) = 19/36$.

(c) *The consensus degree at the group level.* The consensus degree $cog(a)$ of alternative α at the group level is $cog(a) = 1/3$.

Definition 7. Based on the consensus measure of a single alternative, the average consensus degree of all alternatives is defined as follows:

$$\overline{cog} = \frac{\sum_{a \in A} cog(a)}{K}. \tag{5.4}$$

The average consensus degree of all alternatives can serve as an index to measure the consensus level of the group's opinions and be used to monitor the process of consensus reaching. $\overline{cog}$ is bound within the interval [0,1].

3.5.2. Improvement of the consensus level

According to the process of consensus reaching, if $\overline{cog}$ is small and the DMs are not satisfied with the consensus level, the way to improve consensus level should be explored to help the DMs to revise their preference information. In this paper we propose a simple method which is implemented as follows.

Definition 8. Let $a^* \in \bigcup_{t=1,2,\ldots,T} A_t^R$. Suppose the DM e_t adjusts his/her assignment example of a^* as $a^* \to [C_{L_t^{DM'}(a^*)}, C_{R_t^{DM'}(a^*)}]$. Let $\overline{cog}$ and $\overline{cog'}$ be the average consensus degrees before and after this adjustment, respectively. The improvement of consensus level can be defined as follows:

$$\Delta\overline{cog} = \overline{cog'} - \overline{cog}. \tag{5.5}$$

If $\Delta\overline{cog} > 0$, it means that the adjustment of $a^* \to [C_{L_t^{DM'}(a^*)}, C_{R_t^{DM'}(a^*)}]$ by the DM e_t will improve the consensus level and it is called an efficient adjustment.

Note that if $a^* \in A_t^R$, this adjustment means that the DM e_t assigns a^* to a different interval of categories, and the number of possible assignments for e_t to adjust his/her assignment example of a^* is $(1+Q)Q/2 - 2$. For example, suppose $Q = 5$ and the DM e_2 originally assigned the alternative a_5 to the interval $[C_2, C_4]$. Thus there are 13 possible assignments for e_2 to make an adjustment of a_5, i.e., C_1, $C_2, C_3, C_4, C_5, [C_1, C_2], [C_2, C_3], [C_3, C_4], [C_4, C_5], [C_1, C_3], [C_3, C_5], [C_1, C_4]$, and $[C_2, C_5]$. In contrast, if $a^* \notin A_t^R$, this adjustment means that e_t adds a new assignment example of a^*, and the number of possible assignments is $(1+Q)Q/2 - 1$. For example, suppose $Q = 5$ and the DM e_3 did not provide any assignment example for the alternative a_2 initially. Thus there are 14 possible assignments for e_3 to provide a new assignment example for a_2, i.e., $C_1, C_2, C_3, C_4, C_5, [C_1, C_2], [C_2, C_3]$, $[C_3, C_4], [C_4, C_5], [C_1, C_3], [C_2, C_4], [C_3, C_5], [C_1, C_4]$, and $[C_2, C_5]$. We do not consider the whole range $[C_1, C_5]$ of all categories as it does not incorporate any preference information.

The following procedure is used to find the efficient adjustments to improve consensus level:

Step 1. Choose $a^* \in \bigcup_{t=1,2,\ldots,T} A_t^R$ with the minimum consensus degree $cog(a^*)$.

Step 2. Sort $\lambda_t cod_t(a^*)$, $t = 1, 2, \ldots, T$, in the ascending order. For the DM e_t with the minimum $\lambda_t cod_t(a^*)$, enumerate all possible adjustments of a^*.

Step 3. For any possible adjustment of $a^* \to [C_{L_t^{DM'}(a^*)}, C_{R_t^{DM'}(a^*)}]$, employ the linear program $LP1$ to check the compatibility between this adjustment and other assignment examples provided by e_t. If this adjustment is compatible with other assignment examples, the adjustment is feasible; otherwise, it is not feasible.

Step 4. For any feasible adjustment of $a^* \to [C_{L_t^{DM'}(a^*)}, C_{R_t^{DM'}(a^*)}]$, calculate the possible assignment of any alternative $a \in A$ after adjustment, consensus degree $cog'(a)$ and average consensus degree $\overline{cog}'$.

Step 5. Verify the improvement of consensus level for all feasible adjustments. If there is no efficient adjustment, find the DM $e_{t'}$ whose $\lambda_{t'} cod_{t'}(a^*)$ is greater than and adjacent to $\lambda_t cod_t(a^*)$ and then enumerate all possible adjustments of a^* for $e_{t'}$ and proceed to Step 3; otherwise, present all efficient adjustments to e_t in the descending order of $\overline{cog}'$ and e_t is allowed to choose one to revise his/her preference information and then terminate the procedure.

Note that in this method we need to enumerate all possible adjustment solutions and then verify the improvement of consensus level and it seems time-consuming. Nevertheless, we should remember that decision analysis problems often deal with relatively small data sets (Corrente, Greco, Kadziński, & Słowiński, 2013) and the enumeration can be implemented with a modern computer in acceptable time.

3.6. Determination of the collective assignment of alternative

Through several rounds of consensus-reaching, the consensus level will improve and the average consensus degree $\overline{cog}$ will get closer to one. The group could continue such rounds until $\overline{cog} = 1$. However, in many cases the improvement of consensus level will be not significant when $\overline{cog}$ is close to one. Therefore, it will take many more rounds to arrive at $\overline{cog} = 1$ and the group may not wish to continue. In this section we provide a method to determine the collective assignment of alternatives when the DMs are satisfied with the consensus level and decide to terminate the consensus-reaching process.

3.6.1. Supporting degree of assigning alternative to each category

In order to determine the collective assignment of alternatives, we need to consider the assignment of all the DMs comprehensively. Due to the difference of the DMs' preference information, the assignment of the DMs may be inconsistent. Thus we follow the idea of the proportion of DMs accepting each possible assignment in Damart et al. (2007) and propose a measure to indicate the supporting degree for each possible assignment when the DMs have different priorities.

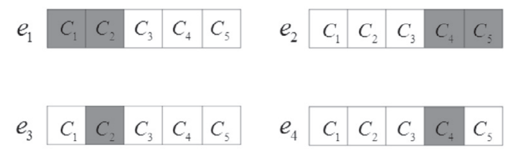
Definition 9. The measure of support for assigning $a \in A$ to C_q is defined as follows:

$$sup_q(a) = \begin{cases} E_q(a), & q = 1, Q \\ \max\left\{E_q(a), \min\left\{\max_{q'<q}\{E_{q'}(a)\}, \max_{q'>q}\{E_{q'}(a)\}\right\}\right\}, & 1 < q < Q \end{cases}$$

$$\text{with } E_q(a) = \sum_{\substack{t \mid a \in A \text{ and} \\ L_t(a) \le q \le R_t(a)}} \frac{\lambda_t}{|\{L_t(a), L_t(a)+1, \ldots, R_t(a)\}|}. \tag{6.1}$$

In this definition, $E_q(a)$ is the sum of the equal allocation of DMs' weights to their assignment intervals. The definition of $sup_q(a)$ means that $E_q(a)$ of category C_q is compared with $E_{q'}(a)$ of lower categories $C_{q'}$ (i.e., $q' < q$) and $E_{q''}(a)$ of upper categories $C_{q''}$ (i.e., $q'' > q$). If $E_q(a)$ is greater than the minimum of $E_{q'}(a)$ and $E_{q''}(a)$, $sup_q(a)$ will be equal to $E_q(a)$; otherwise, $sup_q(a)$ will be equal to the minimum of $E_{q'}(a)$ and $E_{q''}(a)$. This reinforces the supporting degree of assigning α to a middle category for the purpose of consensus-reaching.

Remark that for $q, q', q'' \in \{1, 2, ..., Q\}$, if $q < q'' < q'$, $sup_q(a) > 0$ and $sup_{q'}(a) > 0$, we have $sup_{q''}(a) > 0$. Definition 9 ensures the continuity of supporting degrees. The property is illustrated by the following example.



Note：The grey cells represent the assignment interval.

Fig. 3. The assignment of alternative a in Example 2.

Example 2. Suppose that the assignment of alternative α is depicted with gray cells in Fig. 3. The normalized priority vector of the DMs is $\lambda = (0.4, 0.2, 0.1, 0.3)$.

Based on the definition of the supporting degree of each assignment, we would have $E_1(a) = 0.2$, $E_2(a) = 0.3$, $E_3(a) = 0$, $E_4(a) = 0.4$, and $E_5(a) = 0.1$. Thus, the supporting degrees of assigning α to each category are computed and listed as follows: $sup_1(a) = 0.2$, $sup_2(a) = 0.3$, $sup_3(a) = 0.3$, $sup_4(a) = 0.4$, and $sup_5(a) = 0.1$. In this case, we observe that the opinions of the DMs are divided into two sub-groups, with e_1 and e_3 intending to assign α to lower categories C_1 and C_2 but e_2 and e_4 deciding to assign α to upper categories C_4 and C_5. When we seek for consensus assignment, it is highlighted that assigning α to C_3 should be considered. Although no DM assigns α to C_3, we can consider C_3 as a trade-off for consensus reaching.

3.6.2. Linear program to determine the collective assignment of alternatives

We propose a linear program to determine the collective assignment of alternatives. The linear program has two objectives, with the first one being to maximize the sum of supporting degrees of all alternatives and the second one to maximize the sum of margins between the utilities of alternatives and the category thresholds.

$$LP4: \quad \max f_3 = \rho \sum_{a \in A} \sum_{q=1}^{Q} sup_q(a) y_q(a) + \sum_{a \in A} \sum_{q=1}^{Q} (d_q^+(a) + d_q^-(a)) \tag{6.2}$$

$$\text{s.t.} \quad U(a) = \sum_{n=1}^{N} \frac{\beta_n(a)}{1 - \beta_H(a)} u(H_n), \quad a \in A, \tag{6.3}$$

$$U(a) - b_{q-1} \ge y_q(a) - 1, \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.4}$$

$$U(a) - b_q + \varepsilon \le 1 - y_q(a), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.5}$$

$$\sum_{q=1}^{Q} y_q(a) = 1, \quad a \in A, \tag{6.6}$$

$$y_q(a) \in \{0, 1\}, \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.7}$$

$$d_q^+(a) \ge 0, \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.8}$$

$$d_q^+(a) \le y_q(a), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.9}$$

$$d_q^+(a) \geq b_q - U(a) - (1 - y_q(a)), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.10}$$

$$d_q^+(a) \leq b_q - U(a) + (1 - y_q(a)), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.11}$$

$$d_q^-(a) \geq 0, \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.12}$$

$$d_q^-(a) \leq y_q(a), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.13}$$

$$d_q^-(a) \geq U(a) - b_{q-1} - (1 - y_q(a)), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.14}$$

$$d_q^-(a) \leq U(a) - b_{q-1} + (1 - y_q(a)), \quad q = 1, 2, \ldots, Q, \quad a \in A, \tag{6.15}$$

$$u(H_{n+1}) - u(H_n) \geq \varepsilon, \quad n = 1, 2, \ldots, N - 1, \tag{6.16}$$

$$u(H_1) = 0, \quad u(H_N) = 1, \tag{6.17}$$

$$b_{q+1} - b_q \geq \varepsilon, \quad q = 0, 1, 2, \ldots, Q - 1, \tag{6.18}$$

$$b_0 = 0, \quad b_Q = 1 + \varepsilon, \tag{6.19}$$

where ε is a small positive value representing the smallest discernable difference and ρ is a sufficiently large value satisfying $\rho > \frac{K}{\varepsilon}$ (recall that $|A| = K$). As *LP*4 is to determine the collective assignment, we omit the subscript t and use the collective preference model. In the objective function (6.2), $\sum_{a\in A}\sum_{q=1}^{Q} sup_q(a)y_q(a)$ is the sum of supporting degrees of all the alternatives and $\sum_{a\in A}\sum_{q=1}^{Q}(d_q^+(a) + d_q^-(a))$ is the sum of margins between the utilities of alternatives and the category thresholds. The variables $d_q^+(a)$ and $d_q^-(a)$ are the margins between the utility $U(a)$ and the upper threshold b_q and the lower threshold b_{q-1} of the category C_q, respectively. Constraints (6.8)–(6.15) ensure that if alternative α is assigned to C_q (i.e. $y_q(a) = 1$), $d_q^+(a)$ and $d_q^-(a)$ will be equal to $b_q - U(a)$ and $U(a) - b_{q-1}$, respectively; otherwise, $d_q^+(a)$ and $d_q^-(a)$ will be equal to zero. With the optimal solution of *LP*4, the collective assignment of alternative α can be obtained as $C_{h^*(a)}$, where $h^*(a)$ is the optimum of $h(a)$.

Proposition 2. *The optimal solution of LP4 is also the optimal solution of the linear program max* $f_3' = \sum_{a\in A}\sum_{q=1}^{Q} sup_q(a)y_q(a)$ *subject to the same constraints of LP4.*

Proof. See Appendix D. □

Proposition 2 states that $\max\sum_{a\in A}\sum_{q=1}^{Q} sup_q(a)y_q(a)$ is the primary objective of *LP*4 and $\max\sum_{a\in A}\sum_{q=1}^{Q}(d_q^+(a) + d_q^-(a))$ is also incorporated into the formulation of *LP*4 as a secondary goal to avoid possible arbitrariness in determining the collective assignment.

4. A case study to illustrate the new approach

4.1. Problem description

In this section, the proposed approach is applied to a customer satisfaction analysis for the demonstration of its detailed implementation process and potential applications in the MCS problem with uncertainty.

The MCS problem in question is related to a software corporation in China that provides 20 software services for enterprise applications, such as OA, ERP, SCM, and CRM. The corporation has a large customer population in the nation. The corporation is planning to measure and analyze customer satisfaction with its software services based on seven evaluation criteria identified by the corporation: g_1 = "Core functions (CF)", g_2 = "Extended functions (EF)", g_3 = "Validity of solution (VS)", g_4 = "Implementation experience (IE)", g_5 = "Technical support level (TSL)", g_6 = "Value for money regarding software expense (VMSE)", and g_7 = "Value for money regarding consultation expense (VMCE)". The following general grade set is defined to evaluate the services:

$$\begin{aligned} H &= \{H_n, \ n = 1, 2, \ldots, 5\} \\ &= \{\text{poor}, \ \text{average}, \ \text{good}, \ \text{very good}, \ \text{excellent}\} \\ &= \{P, \ A, \ G, \ V, \ E\}. \end{aligned}$$

Table 1 lists the belief structures of the services referred to as $a_1, a_2, \ldots, a_{20}$. The belief structures are collected from the returned 100 questionnaires emailed to the customers who have adopted the 20 services. The basic probability mass $\beta_{n,l}$ in the belief structure is represented by the proportion of the customers who gave the grade rating H_n. Note that the proportion of the customers who did not give any grade rating is denoted by the remaining probability mass $\beta_{H,l}$. For example, to evaluate the service a_1 on the criterion g_2 = "Extended functions (EF)", six customers rated "Poor", nine rated "Average", seven rated "Good", four rated "Very good", sixty-six rated "Excellent", and the remaining eight did not give any rating.

The manager of the corporation, denoted by e_1, an expert from the R&D department, denoted by e_2, an expert from the financial department, denoted by e_3, and an expert from the project department, denoted by e_4, are invited to participate in the process, and they act as the decision group. They analyze the belief structures and decide to sort the 20 services into five categories: C_1 = "Poor", C_2 = "Average", C_3 = "Good", C_4 = "Very good", and C_5 = "Excellent". Suppose that the weight vector of the seven criteria is $\omega = \{1/7, 1/7, \ldots, 1/7\}$ and that the weight vector of the DMs is $\lambda = \{0.2, 0.3, 0.15, 0.35\}$ as specified by the manager based on their different positions and roles in the corporation.

4.2. The decision process

4.2.1. The assignment examples provided by the DMs

The ER algorithm is used to aggregate the belief structures on multiple criteria into the combined belief degrees and the outcomes are presented in Table 2. All the DMs analyze the combined belief degrees and then provide assignment examples based on their knowledge and experiences. Their assignment examples are listed in Table 3(a). The reference sets of the DMs are $A_1^R = \{a_1, a_3, a_4, a_6, a_8\}$, $A_2^R = \{a_2, a_4, a_5, a_7, a_{10}\}$, $A_3^R = \{a_3, a_5, a_6, a_8, a_9\}$, $A_4^R = \{a_2, a_4, a_6, a_8, a_9\}$.

4.2.2. The analysis of the inconsistency at the individual level

The linear program *LP*1 is built to check the consistency of the assignment examples at the individual level. IBM ILOG CPLEX® is used to solve *LP*1. The assignment examples provided by the DM e_1 are found to be inconsistent, while those provided by the DMs e_2, e_3 and e_4 are consistent. To make the assignment examples consistent, the linear program *LP*2 is solved to help the DM e_1 to adjust his assignment examples. In the *LP*2 model, ε and M are set to be 10^{-5} and 6, respectively. We obtain three possible solutions as follows: (1) $\{a_3 \to [C_3, C_5], a_4 \to [C_1, C_3], \ a_6 \to [C_2, C_3]\}$; (2) $\{a_3 \to [C_2, C_5], a_4 \to [C_1, C_3]\}$; (3) $\{a_3 \to [C_2, C_5], a_8 \to [C_2, C_3]\}$. All these solutions correspond to the minimum sum of shifts of four. Suppose the DM e_1 chooses the solution $\{a_3 \to [C_2, C_5], a_4 \to [C_1, C_3]\}$ to adjust his assignment examples. Thus, the adjusted assignment examples of the DM e_1 are $a_1 \to [C_2, C_3]$, $a_3 \to [C_2, C_5]$, $a_4 \to [C_1, C_3]$, $a_6 \to C_2$, and

Table 1
The belief structures of services.

Service	CF	EF	VS	IE	TSL	VMSE	VMCE
a_1	{(P,0.02), (A,0.03), (G,0), (V,0.1), (E,0.85), (H,0)}	{(P,0.06), (A,0.09), (G,0.07), (V,0.04), (E,0.66), (H,0.08)}	{(P,0.65), (A,0.32), (G,0), (V,0.02), (E,0), (H,0.01)}	{(P,0.57), (A,0.29), (G,0.06), (V,0), (E,0.06), (H,0.02)}	{(P,0), (A,0.02), (G,0), (V,0.52), (E,0.45), (H,0.01)}	{(P,0.05), (A,0.35), (G,0.58), (V,0.01), (E,0), (H,0.01)}	{(P,0.03), (A,0.03), (G,0.63), (V,0.29), (E,0.01), (H,0.01)}
a_2	{(P,0.01), (A,0.87), (G,0.03), (V,0), (E,0.07), (H,0.02)}	{(P,0.02), (A,0.29), (G,0.54), (V,0.01), (E,0.07), (H,0.07)}	{(P,0), (A,0.01), (G,0), (V,0.69), (E,0.3), (H,0)}	{(P,0.01), (A,0.02), (G,0.01), (V,0.69), (E,0.26), (H,0.01)}	{(P,0.04), (A,0.09), (G,0.01), (V,0.54), (E,0.22), (H,0.1)}	{(P,0.04), (A,0.03), (G,0.69), (V,0.22), (E,0), (H,0.02)}	{(P,0.02), (A,0.89), (G,0.02), (V,0.04), (E,0.01), (H,0.02)}
a_3	{(P,0.7), (A,0.26), (G,0.03), (V,0), (E,0), (H,0.01)}	{(P,0.04), (A,0.01), (G,0.02), (V,0.57), (E,0.33), (H,0.03)}	{(P,0.02), (A,0.68), (G,0.15), (V,0.04), (E,0.05), (H,0.06)}	{(P,0.47), (A,0.35), (G,0.02), (V,0.04), (E,0.06), (H,0.06)}	{(P,0.06), (A,0.3), (G,0.44), (V,0.06), (E,0.09), (H,0.05)}	{(P,0.05), (A,0.08), (G,0.07), (V,0.48), (E,0.24), (H,0.08)}	{(P,0.07), (A,0.03), (G,0.05), (V,0.65), (E,0.1), (H,0.1)}
a_4	{(P,0.03), (A,0.21), (G,0.6), (V,0.08), (E,0.08), (H,0)}	{(P,0.03), (A,0.06), (G,0.04), (V,0.3), (E,0.55), (H,0.02)}	{(P,0), (A,0.01), (G,0.01), (V,0.13), (E,0.84), (H,0.01)}	{(P,0.04), (A,0.05), (G,0.06), (V,0.48), (E,0.28), (H,0.09)}	{(P,0), (A,0.05), (G,0.72), (V,0.23), (E,0), (H,0)}	{(P,0.07), (A,0.03), (G,0.01), (V,0.22), (E,0.56), (H,0.11)}	{(P,0.06), (A,0.07), (G,0.01), (V,0.11), (E,0.66), (H,0.09)}
a_5	{(P,0), (A,0), (G,0.01), (V,0.55), (E,0.43), (H,0.01)}	{(P,0.06), (A,0.01), (G,0.03), (V,0.64), (E,0.21), (H,0.05)}	{(P,0.02), (A,0.18), (G,0.58), (V,0.09), (E,0.05), (H,0.08)}	{(P,0.05), (A,0.25), (G,0.55), (V,0.02), (E,0.08), (H,0.05)}	{(P,0.03), (A,0), (G,0.06), (V,0.8), (E,0.1), (H,0.01)}	{(P,0.04), (A,0.07), (G,0.01), (V,0.42), (E,0.37), (H,0.09)}	{(P,0.76), (A,0.14), (G,0.01), (V,0.02), (E,0.02), (H,0.05)}
a_6	{(P,0.09), (A,0.12), (G,0.7), (V,0.03), (E,0.04), (H,0.02)}	{(P,0.03), (A,0.01), (G,0.36), (V,0.54), (E,0.02), (H,0.04)}	{(P,0.06), (A,0.05), (G,0.04), (V,0.43), (E,0.3), (H,0.12)}	{(P,0), (A,0.03), (G,0.02), (V,0.28), (E,0.65), (H,0.02)}	{(P,0.06), (A,0.06), (G,0), (V,0.33), (E,0.45), (H,0.1)}	{(P,0.04), (A,0.03), (G,0.64), (V,0.27), (E,0.01), (H,0.01)}	{(P,0), (A,0.01), (G,0.22), (V,0.77), (E,0), (H,0)}
a_7	{(P,0.02), (A,0.32), (G,0.64), (V,0.01), (E,0), (H,0.01)}	{(P,0.82), (A,0.1), (G,0.01), (V,0.07), (E,0), (H,0)}	{(P,0.06), (A,0.04), (G,0.04), (V,0.66), (E,0.17), (H,0.03)}	{(P,0), (A,0.05), (G,0), (V,0.21), (E,0.74), (H,0)}	{(P,0.03), (A,0.03), (G,0.04), (V,0.58), (E,0.3), (H,0.02)}	{(P,0.02), (A,0.09), (G,0.09), (V,0.09), (E,0.65), (H,0.06)}	{(P,0.77), (A,0.08), (G,0.04), (V,0), (E,0.06), (H,0.05)}
a_8	{(P,0.26), (A,0.59), (G,0.05), (V,0), (E,0.09), (H,0.01)}	{(P,0.64), (A,0.11), (G,0.01), (V,0.08), (E,0.06), (H,0.1)}	{(P,0.02), (A,0.56), (G,0.29), (V,0.05), (E,0.01), (H,0.07)}	{(P,0.57), (A,0.38), (G,0), (V,0.04), (E,0), (H,0.01)}	{(P,0.02), (A,0), (G,0), (V,0.06), (E,0.91), (H,0.01)}	{(P,0.08), (A,0.05), (G,0.04), (V,0.64), (E,0.15), (H,0.04)}	{(P,0.04), (A,0.01), (G,0), (V,0.48), (E,0.45), (H,0.02)}
a_9	{(P,0), (A,0.04), (G,0.02), (V,0.76), (E,0.17), (H,0.01)}	{(P,0.01), (A,0), (G,0), (V,0.45), (E,0.54), (H,0)}	{(P,0), (A,0), (G,0.72), (V,0.07), (E,0.09), (H,0.12)}	{(P,0.02), (A,0.5), (G,0.44), (V,0), (E,0.02), (H,0.02)}	{(P,0.08), (A,0.44), (G,0.4), (V,0), (E,0.08), (H,0)}	{(P,0.05), (A,0.08), (G,0.06), (V,0.64), (E,0.06), (H,0.11)}	{(P,0.59), (A,0.15), (G,0.08), (V,0.02), (E,0.03), (H,0.13)}
a_{10}	{(P,0.06), (A,0.04), (G,0.02), (V,0.31), (E,0.55), (H,0.02)}	{(P,0.09), (A,0.02), (G,0.07), (V,0.07), (E,0.7), (H,0.05)}	{(P,0), (A,0.08), (G,0), (V,0.24), (E,0.67), (H,0.01)}	{(P,0.03), (A,0.02), (G,0.06), (V,0.7), (E,0.19), (H,0)}	{(P,0.04), (A,0.02), (G,0.07), (V,0.72), (E,0.14), (H,0.01)}	{(P,0.02), (A,0.04), (G,0.71), (V,0.08), (E,0.04), (H,0.11)}	{(P,0.08), (A,0.02), (G,0.07), (V,0.37), (E,0.34), (H,0.12)}
a_{11}	{(P,0.54), (A,0.36), (G,0.04), (V,0.03), (E,0.02), (H,0.01)}	{(P,0.04), (A,0.33), (G,0.54), (V,0.05), (E,0.03), (H,0.01)}	{(P,0.01), (A,0.08), (G,0.15), (V,0.72), (E,0.03), (H,0.01)}	{(P,0.31), (A,0.43), (G,0.03), (V,0.09), (E,0.09), (H,0.05)}	{(P,0.02), (A,0.04), (G,0.01), (V,0.9), (E,0), (H,0.03)}	{(P,0.05), (A,0.07), (G,0.67), (V,0.04), (E,0.03), (H,0.14)}	{(P,0.12), (A,0.79), (G,0.02), (V,0.03), (E,0.02), (H,0.02)}
a_{12}	{(P,0.27), (A,0.54), (G,0.08),	{(P,0), (A,0), (G,0.17),	{(P,0.08), (A,0.01), (G,0.45),	{(P,0.67), (A,0.24), (G,0.01),	{(P,0.01), (A,0.04), (G,0.18),	{(P,0.55), (A,0.31), (G,0.04),	{(P,0.27), (A,0.66), (G,0.02),

(*continued on next page*)

Table 1 (*continued*)

Service	CF	EF	VS	IE	TSL	VMSE	VMCE
	(V,0.02), (E,0.08), (H,0.01)}	(V,0.78), (E,0.03), (H,0.02)}	(V,0.27), (E,0.09), (H,0.1)}	(V,0), (E,0.04), (H,0.04)}	(V,0.76), (E,0), (H,0.01)}	(V,0.03), (E,0.04), (H,0.03)}	(V,0.04), (E,0), (H,0.01)}
a_{13}	{(P,0), (A,0.85), (G,0.14), (V,0.01), (E,0), (H,0)}	{(P,0), (A,0.01), (G,0.38), (V,0.61), (E,0), (H,0)}	{(P,0.04), (A,0.03), (G,0.28), (V,0.61), (E,0.03), (H,0.01)}	{(P,0), (A,0), (G,0.73), (V,0.26), (E,0.01), (H,0)}	{(P,0), (A,0.57), (G,0.33), (V,0.06), (E,0), (H,0.04)}	{(P,0.01), (A,0.07), (G,0.6), (V,0.28), (E,0.03), (H,0.01)}	{(P,0), (A,0.02), (G,0.33), (V,0.55), (E,0.06), (H,0.04)}
a_{14}	{(P,0.06), (A,0.06), (G,0.07), (V,0.26), (E,0.5), (H,0.05)}	{(P,0.02), (A,0.71), (G,0.11), (V,0.07), (E,0.06), (H,0.03)}	{(P,0.05), (A,0.06), (G,0.05), (V,0.67), (E,0.03), (H,0.14)}	{(P,0.01), (A,0.06), (G,0.83), (V,0), (E,0.09), (H,0.01)}	{(P,0.04), (A,0.05), (G,0.19), (V,0.65), (E,0.03), (H,0.04)}	{(P,0.42), (A,0.53), (G,0), (V,0.03), (E,0.01), (H,0.01)}	{(P,0.01), (A,0.95), (G,0.01), (V,0.03), (E,0), (H,0)}
a_{15}	{(P,0.21), (A,0.78), (G,0.01), (V,0), (E,0), (H,0)}	{(P,0.06), (A,0.06), (G,0.81), (V,0.01), (E,0.03), (H,0.03)}	{(P,0.53), (A,0.34), (G,0.03), (V,0.02), (E,0.07), (H,0.01)}	{(P,0.85), (A,0.03), (G,0.01), (V,0.05), (E,0.05), (H,0.01)}	{(P,0.51), (A,0.46), (G,0.02), (V,0), (E,0), (H,0.01)}	{(P,0.06), (A,0.07), (G,0.08), (V,0.55), (E,0.19), (H,0.05)}	{(P,0.01), (A,0.08), (G,0.43), (V,0.36), (E,0.09), (H,0.03)}
a_{16}	{(P,0), (A,0), (G,0.01), (V,0.19), (E,0.8), (H,0)}	{(P,0.14), (A,0.71), (G,0.01), (V,0.05), (E,0.02), (H,0.07)}	{(P,0.06), (A,0.02), (G,0.5), (V,0.32), (E,0.03), (H,0.07)}	{(P,0.56), (A,0.26), (G,0.05), (V,0), (E,0.04), (H,0.09)}	{(P,0.07), (A,0.05), (G,0.25), (V,0.48), (E,0.03), (H,0.12)}	{(P,0), (A,0.03), (G,0.08), (V,0.88), (E,0.01), (H,0)}	{(P,0), (A,0), (G,0.94), (V,0.03), (E,0.02), (H,0.01)}
a_{17}	{(P,0.06), (A,0.09), (G,0.55), (V,0.15), (E,0.08), (H,0.07)}	{(P,0), (A,0.03), (G,0.59), (V,0.37), (E,0.01), (H,0)}	{(P,0.4), (A,0.39), (G,0.01), (V,0.06), (E,0.05), (H,0.09)}	{(P,0.02), (A,0), (G,0.04), (V,0.74), (E,0.14), (H,0.06)}	{(P,0.01), (A,0.08), (G,0.28), (V,0.49), (E,0.05), (H,0.09)}	{(P,0.03), (A,0.85), (G,0.02), (V,0.06), (E,0.04), (H,0)}	{(P,0.01), (A,0.02), (G,0), (V,0.87), (E,0.06), (H,0.04)}
a_{18}	{(P,0), (A,0.01), (G,0), (V,0.45), (E,0.54), (H,0)}	{(P,0.05), (A,0.18), (G,0.53), (V,0.05), (E,0.05), (H,0.14)}	{(P,0), (A,0.07), (G,0.89), (V,0.02), (E,0), (H,0.02)}	{(P,0.07), (A,0.25), (G,0.65), (V,0.02), (E,0), (H,0.01)}	{(P,0.57), (A,0.37), (G,0), (V,0.04), (E,0.01), (H,0.01)}	{(P,0.01), (A,0.06), (G,0.06), (V,0.35), (E,0.51), (H,0.01)}	{(P,0.08), (A,0.64), (G,0.09), (V,0.08), (E,0.05), (H,0.06)}
a_{19}	{(P,0.17), (A,0.76), (G,0.02), (V,0.04), (E,0), (H,0.01)}	{(P,0.33), (A,0.43), (G,0.06), (V,0.02), (E,0.05), (H,0.11)}	{(P,0.01), (A,0.26), (G,0.63), (V,0.05), (E,0.02), (H,0.03)}	{(P,0.03), (A,0.01), (G,0.34), (V,0.56), (E,0.02), (H,0.04)}	{(P,0.07), (A,0.75), (G,0.05), (V,0.05), (E,0.04), (H,0.04)}	{(P,0.02), (A,0.35), (G,0.51), (V,0.04), (E,0.02), (H,0.06)}	{(P,0.05), (A,0.02), (G,0.02), (V,0.79), (E,0.04), (H,0.08)}
a_{20}	{(P,0.28), (A,0.57), (G,0.05), (V,0.01), (E,0.05), (H,0.04)}	{(P,0.01), (A,0), (G,0.16), (V,0.81), (E,0.01), (H,0.01)}	{(P,0), (A,0.05), (G,0.57), (V,0.34), (E,0), (H,0.04)}	{(P,0.44), (A,0.45), (G,0.01), (V,0.06), (E,0), (H,0.04)}	{(P,0.04), (A,0.03), (G,0.01), (V,0.37), (E,0.53), (H,0.02)}	{(P,0.03), (A,0.47), (G,0.24), (V,0.08), (E,0.07), (H,0.11)}	{(P,0.66), (A,0.17), (G,0.03), (V,0.03), (E,0.04), (H,0.07)}

Note: The grade H is used to represent unknown in the assessment.

$a_8 \to C_3$. The assignment examples after adjustment are presented in Table 3(b).

4.2.3. The group consensus and improvement (Round 1)

Based on the consistent assignment examples, we can compute possible assignment of services $a \in A$ for each DM e_t, $t = 1, 2, 3, 4$, and the results are presented in Table 4. We observe that the possible assignment of some reference alternatives differs from the range desired by some DMs. For example, the possible assignment of service a_1 for the DM e_1 is C_3, which is more precise than the assignment example $[C_2, C_3]$ specified by him.

Table 4 shows that the services a_{12} and a_{15} have the maximum consensus degree of 0.78333 and the service a_9 has the minimum consensus degree of 0.22916. The average consensus degree $\overline{cog}$ of all the services is 0.51279.

As a_9 has the minimum consensus degree and e_4 has the minimum $\lambda_4 cod_4(a_9) = 0.02916$, we explore the efficient adjustment solutions of a_9 for e_4 to improve the consensus level. Table 5 lists all efficient adjustment solutions in the descending order of $\overline{cog'}$. It is presented to e_4 and he is asked to choose one solution to revise his preference information. Suppose e_4 chooses the first solution and adjusts a_9 to the interval $[C_2, C_3]$ and then we proceed to the next round.

4.2.4. The group consensus and improvement (Round 2)

Since only e_4 adjusts his assignment examples, the possible assignment of the other three DMs remain the same. We compute

Table 2
The combined belief degrees of belief structures.

Service	Combined belief degree	Service	Combined belief degree
a_1	{(P,0.19563), (A,0.16048), (G,0.18931), (V,0.13654), (E,0.29973), (H,0.01830)}	a_{11}	{(P,0.15121), (A,0.31347), (G,0.20468), (V,0.26709), (E,0.02909), (H,0.03445)}
a_2	{(P,0.01827), (A,0.32130), (G,0.17828), (V,0.32393), (E,0.12773), (H,0.03050)}	a_{12}	{(P,0.26943), (A,0.26160), (G,0.13010), (V,0.27374), (E,0.03700), (H,0.02813)}
a_3	{(P,0.20101), (A,0.25088), (G,0.10724), (V,0.26965), (E,0.12117), (H,0.05004)}	a_{13}	{(P,0.00618), (A,0.20515), (G,0.41730), (V,0.34303), (E,0.01613), (H,0.01222)}
a_4	{(P,0.02933), (A,0.06258), (G,0.19743), (V,0.21687), (E,0.45444), (H,0.03936)}	a_{14}	{(P,0.08206), (A,0.36612), (G,0.17473), (V,0.24437), (E,0.09693), (H,0.03579)}
a_5	{(P,0.12884), (A,0.08791), (G,0.17274), (V,0.38935), (E,0.17792), (H,0.04324)}	a_{15}	{(P,0.33314), (A,0.26439), (G,0.19265), (V,0.13446), (E,0.05733), (H,0.01804)}
a_6	{(P,0.03602), (A,0.04001), (G,0.28342), (V,0.40121), (E,0.20105), (H,0.03829)}	a_{16}	{(P,0.11372), (A,0.14825), (G,0.27035), (V,0.29084), (E,0.13026), (H,0.04660)}
a_7	{(P,0.24574), (A,0.09791), (G,0.11709), (V,0.23422), (E,0.28292), (H,0.02212)}	a_{17}	{(P,0.06912), (A,0.20201), (G,0.20831), (V,0.41982), (E,0.05687), (H,0.04386)}
a_8	{(P,0.23539), (A,0.24769), (G,0.05191), (V,0.19111), (E,0.24034), (H,0.03357)}	a_{18}	{(P,0.10580), (A,0.22855), (G,0.33057), (V,0.14097), (E,0.16196), (H,0.03216)}
a_9	{(P,0.10073), (A,0.17213), (G,0.25118), (V,0.28682), (E,0.13910), (H,0.05004)}	a_{19}	{(P,0.09073), (A,0.39202), (G,0.23099), (V,0.21520), (E,0.02472), (H,0.04635)}
a_{10}	{(P,0.04086), (A,0.03057), (G,0.12930), (V,0.36799), (E,0.39224), (H,0.03903)}	a_{20}	{(P,0.20897), (A,0.25537), (G,0.14967), (V,0.24836), (E,0.09505), (H,0.04259)}

Table 3
The assignment examples provided by each DM.

DM	Assignment examples
(a)	
e_1	$a_1 \to [C_2, C_3]$,$a_3 \to C_5$,$a_4 \to [C_1, C_2]$,$a_6 \to C_2$,$a_8 \to C_3$
e_2	$a_2 \to C_4$,$a_4 \to C_5$,$a_5 \to [C_3, C_4]$,$a_7 \to C_2$,$a_{10} \to C_4$
e_3	$a_3 \to [C_1, C_3]$,$a_5 \to C_3$,$a_6 \to C_5$,$a_8 \to C_1$,$a_9 \to [C_2, C_3]$
e_4	$a_2 \to C_4$,$a_4 \to [C_4, C_5]$,$a_6 \to C_5$,$a_8 \to [C_1, C_2]$,$a_9 \to C_4$
(b)	
e_1	$a_1 \to [C_2, C_3]$,$a_3 \to [C_2, C_5]$,$a_4 \to [C_1, C_3]$,$a_6 \to C_2$,$a_8 \to C_3$
e_2	$a_2 \to C_4$,$a_4 \to C_5$,$a_5 \to [C_3, C_4]$,$a_7 \to C_2$,$a_{10} \to C_4$
e_3	$a_3 \to [C_1, C_3]$,$a_5 \to C_3$,$a_6 \to C_5$,$a_8 \to C_1$,$a_9 \to [C_2, C_3]$
e_4	$a_2 \to C_4$,$a_4 \to [C_4, C_5]$,$a_6 \to C_5$,$a_8 \to [C_1, C_2]$,$a_9 \to C_4$

possible assignment of services $a \in A$ for e_4 and the results are presented in Table 6. We observe that the services a_{12} and a_{15} still have the maximum consensus degrees of 0.78333 and the service a_5 has the minimum consensus degree of 0.23166. The average consensus degree $\overline{cog}$ of all the services is 0.52779.

As a_5 has the minimum consensus degree and e_1 has the minimum $\lambda_1 cod_1(a_5) = 0.01333$, we explore the efficient adjustment solutions of a_5 for e_1 to improve the consensus level. However, there is no efficient adjustment solution and thus e_1 cannot provide a new assignment example for a_5 to improve the consensus level. This is due to the fact that a_5 can only be assigned to C_2 by e_1 based on his preference information.

For e_3 has the second minimum $\lambda_3 cod_3(a_5) = 0.03499$, we explore the efficient adjustment solutions of a_5 for e_3 to improve the consensus level. Table 7 lists all efficient adjustment solutions in the descending order of $\overline{cog'}$. It is presented to e_3 and he is asked to choose one solution to revise his preference information. Suppose e_3 chooses the first solution and adjusts a_5 to the interval $[C_3, C_4]$ and then we proceed to the next round.

4.2.5. The group consensus and improvement (Round 3–Round 11)

For the compactness of presentation, we shall not present more details of the following rounds of consensus-reaching. Table 8 presents the selected adjustment solutions from Round 3 to Round 11. Fig. 4 depicts the improvement of average consensus degree $\overline{cog}$ during the whole process.

4.2.6. The determination of the collective assignment of services

After the adjustment in Round 11, the group finds that the $\overline{cog}$ is greater than 0.8 and they are satisfied with the consensus level. Moreover, they consider that it may need more rounds to arrive at $\overline{cog} = 1$. Hence they decide to terminate the process and determine the collective assignment of services.

The possible assignment at the end of Round 11 is presented in Table 9. Based on the possible assignment, we obtain the supporting degrees of assigning services to each category which is listed in Table 10.

Then we employ the linear program *LP*4 to determine the collective assignment of services. In the *LP*4 model, ε and ρ are set to be 10^{-5} and 3×10^6, respectively. With the optimal solution of *LP*4, the collective assignment of each service is obtained as: a_8, a_{12}, a_{15} and a_{20} are assigned to the category C_1, a_1, a_3, a_7 and a_{11} are assigned to the category C_2, a_5, a_9, a_{16}, a_{18} and a_{19} are assigned to the category C_3, a_2, a_6, a_{13}, a_{14} and a_{17} are assigned to the category C_4, a_4 and a_{10} are assigned to the category C_5. The classification results are also presented in Table 11.

We observe that for any service $a \in A$, it is collectively assigned to the corresponding category with the maximum supporting degree. When there are multiple categories corresponding to the maximum supporting degree, the second goal of *LP*4 is used to determine the collective assignment of α by maximizing the margins between its utility and the category thresholds. For example, for the service a_5, we have the maximum supporting degrees of assigning a_5 to the categories a_3 and a_4, i.e., $sup_3(a_5) = sup_4(a_5) = 0.36666$ and its collective assignment of $a_5 \to C_3$ is determined through the second goal of *LP*4.

Table 4
The possible assignment and consensus degrees of services (Round 1).

Service	e_1	e_2	e_3	e_4	$cog(a)$	Service	e_1	e_2	e_3	e_4	$cog(a)$
a_1	C_3	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.56166	a_{11}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_4]$	0.70694
a_2	C_2	C_4	$[C_1, C_5]$	C_4	0.30333	a_{12}	$[C_1, C_2]$	$[C_1, C_2]$	C_1	$[C_1, C_2]$	0.78333
a_3	C_2	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_4]$	0.58055	a_{13}	$[C_1, C_2]$	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_5]$	0.72000
a_4	C_3	C_5	$[C_3, C_5]$	$[C_4, C_5]$	0.30833	a_{14}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_4]$	0.69166
a_5	C_2	$[C_3, C_4]$	C_3	$[C_1, C_5]$	0.23166	a_{15}	$[C_1, C_2]$	$[C_1, C_2]$	C_1	$[C_1, C_2]$	0.78333
a_6	C_2	$[C_3, C_5]$	C_5	C_5	0.28888	a_{16}	C_2	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_5]$	0.56416
a_7	C_3	C_2	$[C_1, C_5]$	$[C_1, C_5]$	0.29999	a_{17}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.66333
a_8	C_3	$[C_1, C_4]$	C_1	$[C_1, C_2]$	0.27083	a_{18}	C_2	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.56166
a_9	C_2	$[C_1, C_4]$	$[C_2, C_3]$	C_4	0.22916	a_{19}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_4]$	0.69166
a_{10}	C_3	C_4	$[C_3, C_5]$	$[C_4, C_5]$	0.30833	a_{20}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_4]$	0.70694

Table 5
Efficient adjustment solutions of service a_9 (Round 1).

No.	Service	DM	Interval	$\overline{cog'}$	No.	Service	DM	Interval	$\overline{cog'}$
1	a_9	e_4	$[C_2, C_3]$	0.52779	6	a_9	e_4	$[C_3, C_4]$	0.52230
2	a_9	e_4	$[C_2, C_4]$	0.52709	7	a_9	e_4	$[C_3, C_5]$	0.52052
3	a_9	e_4	$[C_1, C_4]$	0.52540	8	a_9	e_4	C_3	0.52008
4	a_9	e_4	$[C_1, C_3]$	0.52394	9	a_9	e_4	C_2	0.51536
5	a_9	e_4	$[C_2, C_5]$	0.52331					

Table 6
The possible assignment and consensus degrees of services (Round 2).

Service	e_1	e_2	e_3	e_4	$cog(a)$	Service	e_1	e_2	e_3	e_4	$cog(a)$
a_1	C_3	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.56166	a_{11}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.72499
a_2	C_2	C_4	$[C_1, C_5]$	C_4	0.30333	a_{12}	$[C_1, C_2]$	$[C_1, C_2]$	C_1	$[C_1, C_2]$	0.78333
a_3	C_2	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.58333	a_{13}	$[C_1, C_2]$	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_5]$	0.72000
a_4	C_3	C_5	$[C_3, C_5]$	$[C_4, C_5]$	0.30833	a_{14}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_4]$	0.69166
a_5	C_2	$[C_3, C_4]$	C_3	$[C_1, C_5]$	0.23166	a_{15}	$[C_1, C_2]$	$[C_1, C_2]$	C_1	$[C_1, C_2]$	0.78333
a_6	C_2	$[C_3, C_5]$	C_5	C_5	0.28888	a_{16}	C_2	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_5]$	0.56416
a_7	C_3	C_2	$[C_1, C_5]$	$[C_1, C_5]$	0.29999	a_{17}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.66333
a_8	C_3	$[C_1, C_4]$	C_1	$[C_1, C_2]$	0.27083	a_{18}	C_2	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	0.56166
a_9	C_2	$[C_1, C_4]$	$[C_2, C_3]$	$[C_2, C_3]$	0.54166	a_{19}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_3]$	0.63472
a_{10}	C_3	C_4	$[C_3, C_5]$	$[C_2, C_5]$	0.31388	a_{20}	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.72499

Table 7
Efficient adjustment solutions of service a_5 (Round 2).

No.	Service	DM	Interval	$\overline{cog'}$	No.	Service	DM	Interval	$\overline{cog'}$
1	a_5	e_3	$[C_3, C_4]$	0.53320	6	a_5	e_3	$[C_4, C_5]$	0.53190
2	a_5	e_3	$[C_2, C_4]$	0.53279	7	a_5	e_3	C_4	0.53161
3	a_5	e_3	$[C_1, C_4]$	0.53272	8	a_5	e_3	$[C_1, C_3]$	0.52966
4	a_5	e_3	$[C_2, C_5]$	0.53259	9	a_5	e_3	$[C_2, C_3]$	0.52959
5	a_5	e_3	$[C_3, C_5]$	0.53225					

Table 8
The selected adjustment solutions (Round 3–Round 11).

Round	Service	DM	Interval	$\overline{cog'}$	Round	Service	DM	Interval	$\overline{cog'}$
3	a_8	e_1	$[C_1, C_2]$	0.59909	8	a_{10}	e_2	$[C_3, C_5]$	0.78194
4	a_6	e_1	$[C_3, C_5]$	0.67720	9	a_6	e_3	$[C_4, C_5]$	0.78318
5	a_4	e_1	$[C_4, C_5]$	0.71184	10	a_6	e_1	$[C_3, C_4]$	0.78909
6	a_2	e_3	C_4	0.72856	11	a_6	e_4	$[C_3, C_4]$	0.81677
7	a_8	e_3	$[C_1, C_2]$	0.75919					

Table 9
The possible assignment and consensus degrees of services (at the end of Round 11).

Service	e_1	e_2	e_3	e_4	$cog(a)$	Service	e_1	e_2	e_3	e_4	$cog(a)$
a_1	$[C_2, C_3]$	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_5]$	0.72000	a_{11}	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.83333
a_2	$[C_2, C_5]$	C_4	C_4	C_4	0.64999	a_{12}	$[C_1, C_2]$	$[C_1, C_2]$	$[C_1, C_2]$	$[C_1, C_2]$	1.00000
a_3	$[C_2, C_3]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.72499	a_{13}	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_5]$	1.00000
a_4	$[C_4, C_5]$	C_5	$[C_4, C_5]$	$[C_4, C_5]$	0.73333	a_{14}	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	1.00000
a_5	$[C_2, C_4]$	$[C_3, C_4]$	$[C_3, C_4]$	$[C_1, C_4]$	0.66805	a_{15}	$[C_1, C_2]$	$[C_1, C_2]$	$[C_1, C_2]$	$[C_1, C_2]$	1.00000
a_6	$[C_3, C_4]$	$[C_3, C_5]$	$[C_4, C_5]$	$[C_3, C_4]$	0.63333	a_{16}	$[C_2, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	0.88333
a_7	$[C_1, C_5]$	C_2	$[C_1, C_5]$	$[C_1, C_4]$	0.51416	a_{17}	$[C_1, C_4]$	$[C_1, C_5]$	$[C_1, C_5]$	$[C_1, C_4]$	0.86666
a_8	$[C_1, C_2]$	$[C_1, C_4]$	$[C_1, C_2]$	$[C_1, C_2]$	0.73333	a_{18}	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	1.00000
a_9	$[C_2, C_4]$	$[C_1, C_4]$	$[C_2, C_3]$	$[C_2, C_3]$	0.67500	a_{19}	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	$[C_1, C_4]$	1.00000
a_{10}	$[C_2, C_5]$	$[C_3, C_5]$	$[C_3, C_5]$	$[C_2, C_5]$	0.83333	a_{20}	$[C_1, C_3]$	$[C_1, C_4]$	$[C_1, C_3]$	$[C_1, C_3]$	0.86666

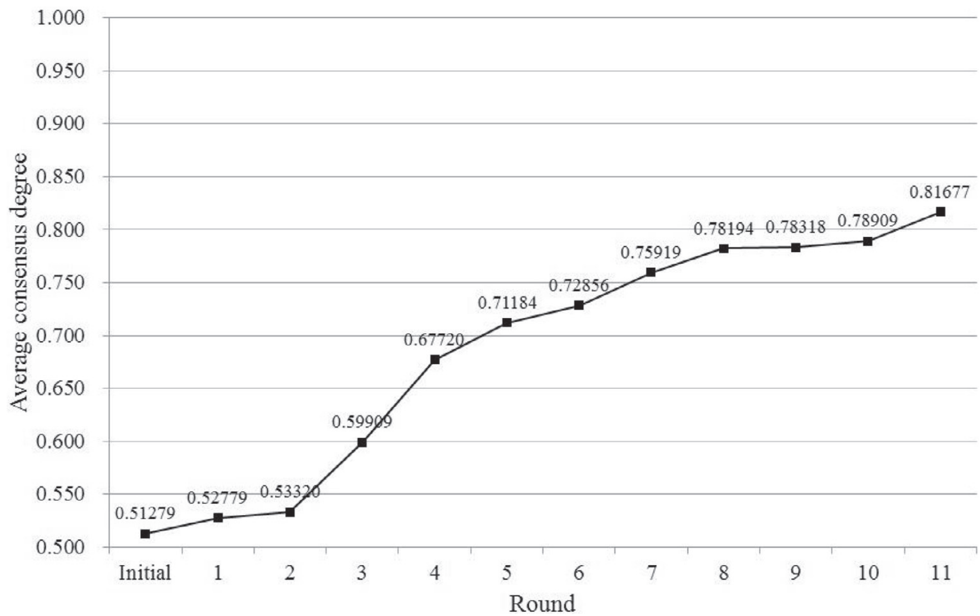


Fig. 4. The improvement of average consensus degree $\overline{cog}$.

Table 10
The supporting degrees of services.

Service	C_1	C_2	C_3	C_4	C_5	Service	C_1	C_2	C_3	C_4	C_5
a_1	0.16000	0.26000	0.26000	0.16000	0.16000	a_{11}	0.29166	0.29166	0.29166	0.12500	0.00000
a_2	0.00000	0.05000	0.05000	0.85000	0.05000	a_{12}	0.50000	0.50000	0.00000	0.00000	0.00000
a_3	0.24166	0.29166	0.29166	0.12500	0.05000	a_{13}	0.20000	0.20000	0.20000	0.20000	0.20000
a_4	0.00000	0.00000	0.00000	0.35000	0.65000	a_{14}	0.25000	0.25000	0.25000	0.25000	0.00000
a_5	0.10000	0.16666	0.36666	0.36666	0.00000	a_{15}	0.50000	0.50000	0.00000	0.00000	0.00000
a_6	0.00000	0.00000	0.40000	0.45000	0.15000	a_{16}	0.20000	0.26666	0.26666	0.26666	0.00000
a_7	0.16000	0.45999	0.16000	0.16000	0.06000	a_{17}	0.23000	0.23000	0.23000	0.23000	0.08000
a_8	0.42500	0.42500	0.07500	0.07500	0.00000	a_{18}	0.25000	0.25000	0.25000	0.25000	0.00000
a_9	0.07500	0.39166	0.39166	0.14166	0.00000	a_{19}	0.25000	0.25000	0.25000	0.25000	0.00000
a_{10}	0.00000	0.15000	0.28333	0.28333	0.28333	a_{20}	0.30833	0.30833	0.30833	0.07500	0.00000

Table 11
The final sorting results of services.

Service	C_1	C_2	C_3	C_4	C_5	Service	C_1	C_2	C_3	C_4	C_5
a_1		⋆				a_{11}		⋆			
a_2				⋆		a_{12}	⋆				
a_3		⋆				a_{13}				⋆	
a_4					⋆	a_{14}				⋆	
a_5			⋆			a_{15}	⋆				
a_6				⋆		a_{16}			⋆		
a_7		⋆				a_{17}				⋆	
a_8	⋆					a_{18}			⋆		
a_9			⋆			a_{19}			⋆		
a_{10}					⋆	a_{20}	⋆				

Note: ⋆ represents the collective assignment of each service.

5. Conclusions

In this paper, we presented a group decision support approach for the MCS problem with uncertainty. The belief structure and the ER approach are employed to represent and aggregate uncertain information. To specify the sorting model, each DM is asked to provide individual assignment examples (the desired assignment for specific reference alternatives) that yield a set of compatible utility functions. Taking into account the set of assignment examples as preference information, the approach provides a possible assignment interval for each alternative at the individual level. A consensus degree is defined to measure the agreement among the DMs' opinions. When the DMs are not satisfied with the consensus degree, possible solutions are explored to help them adjust assignment examples in order to improve the consensus level. If the consensus degree arrives at a satisfactory level, a linear program is built to determine the collective assignment of alternatives.

Note that the consensus-reaching process developed in this paper can be applied to other sorting models for dealing with the case of group decision-making in a straightforward way. In this study, we focus on the design of the consensus-reaching process for MCS and thus assume that all the DMs share the same belief structure and the same criteria weights. When the DMs use different belief structures to express their uncertain judgments or the weights of criteria could be different for the DMs, it will be challenging but interesting to solve such problems, which requires further research.

However, we should acknowledge that the consensus-reaching process may require many rounds to reach a satisfactory consensus level, especially when there is a large divergence among the opinions of the DMs. In such a case, the DMs may feel that the tedious process is boring and may not have confidence in participating in the consensus-reaching process. This is due to the fact that only one DM is asked to revise a single assignment example at each round and thus the improvement of consensus level is not significant. In future research, more efficient adjustment solutions should be explored to accelerate the consensus-reaching process.

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Appendix A. The ER algorithm

The ER algorithm operates on probability masses which take into account the weights of criteria and are defined as follows. Let $m_{n,l}(a_i)$ be a basic probability mass representing the degree to which the lth basic criterion supports the hypothesis that the general criterion is assessed to the nth grade H_n. Let $m_{H,l}(a_i)$ be a remaining probability mass unassigned to any individual grade after all N probability masses $m_{n,l}(a_i)$ $(n = 1, 2, ..., N)$ have been assessed. Then, $m_{n,l}(a_i)$ and $m_{H,l}(a_i)$ are given by

$$m_{n,l}(a_i) = \omega_i \beta_{n,l}(a_i),\ \ n = 1, 2, \ldots, N,$$

$$m_{H,l}(a_i) = 1 - \sum_{n=1}^{N} m_{n,l}(a_i) = 1 - \omega_i \sum_{n=1}^{N} \beta_{n,l}(a_i).$$

$m_{H,l}(a_i)$ can be decomposed into two parts: $\bar{m}_{H,l}(a_i)$ and $\tilde{m}_{H,l}(a_i)$ with $m_{H,l}(a_i) = \bar{m}_{H,l}(a_i) + \tilde{m}_{H,l}(a_i)$, where

$$\bar{m}_{H,l}(a_i) = 1 - \omega_i \quad \text{and} \quad \tilde{m}_{H,l}(a_i) = \omega_i \Big(1 - \sum_{n=1}^{N} \beta_{n,l}(a_i)\Big).$$

Let $I_{n,l}(a_i)$ $(n = 1, 2, \ldots, N)$, $\tilde{I}_{H,l}(a_i)$ and $\bar{I}_{H,l}(a_i)$ denote the combined probability masses generated by aggregating the first l criteria. The $(l+1)$th criterion is then combined with the first l criteria in a recursive manner as follows.

$$
\begin{aligned}
I_{n,1}(a_i) &= m_{n,1}(a_i), \quad n = 1, 2, \ldots, N,\\
I_{H,1}(a_i) &= m_{H,1}(a_i),\\
\tilde{I}_{H,1}(a_i) &= \tilde{m}_{H,1}(a_i),\\
\bar{I}_{H,1}(a_i) &= \bar{m}_{H,1}(a_i),\\
K_{l+1}(a_i) &= \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1\\ j\neq t}}^{N} I_{t,l}(a_i) m_{j,l+1}(a_i)\right]^{-1},\\
I_{n,l+1}(a_i) &= K_{l+1}(a_i)[I_{n,l}(a_i) m_{n,l+1}(a_i) + I_{H,l}(a_i) m_{n,l+1}(a_i)\\
&\quad + I_{n,l}(a_i) m_{H,l+1}(a_i)], \quad n = 1, 2, \ldots, N,\\
\tilde{I}_{H,l+1}(a_i) &= K_{l+1}(a_i)[\tilde{I}_{H,l}(a_i)\tilde{m}_{H,l+1}(a_i) + \bar{I}_{H,l}(a_i)\tilde{m}_{H,l+1}(a_i)\\
&\quad + \tilde{I}_{H,l}(a_i)\bar{m}_{H,l+1}(a_i)],\\
\bar{I}_{H,l+1}(a_i) &= K_{l+1}(a_i)[\bar{I}_{H,l}(a_i)\bar{m}_{H,l+1}(a_i)],\\
I_{H,l+1}(a_i) &= \tilde{I}_{H,l+1}(a_i) + \bar{I}_{H,l+1}(a_i),\\
&\quad l = 1, 2, \ldots, L-1.
\end{aligned}
$$

The process continues until $l + 1 = L$, and $I_{n,L}(a_i)$ $(n = 1, 2, \ldots, N)$, $\bar{I}_{H,L}(a_i)$ and $\tilde{I}_{H,L}(a_i)$ are obtained, where $I_{n,L}(a_i)$ is the combined probability mass assigned to the nth grade, $\bar{I}_{H,L}(a_i)$ is the combined probability mass that needs to be redistributed to the N grades, and $\tilde{I}_{H,L}(a_i)$ is the remaining combined probability mass that is unable to be redistributed to any specific grade due to insufficient information and $I_{H,L}(a_i) = \tilde{I}_{H,L}(a_i) + \bar{I}_{H,L}(a_i)$.

After all L criteria have been aggregated and $I_{n,L}(a_i)$ $(n = 1, 2, \ldots, N)$, $\bar{I}_{H,L}(a_i)$ and $\tilde{I}_{H,L}(a_i)$ are obtained, the remaining probability mass that needs to be assigned to individual grades, $\bar{I}_{H,L}(a_i)$, is assigned to all individual grades proportionally using the following normalization process so as to generate the combined degrees of belief to the grade H_n,

$$\beta_n(a_i) = \frac{I_{n,L}(a_i)}{1 - \bar{I}_{H,L}(a_i)},\ \ n = 1, 2, \ldots, N.$$

The degree of belief that is not assigned to any individual grades is assigned to the whole set H by

$$\beta_H(a_i) = \frac{\tilde{I}_{H,L}(a_i)}{1 - \bar{I}_{H,L}(a_i)}.$$

It has been proved that the combined degrees of belief, $\beta_n(a_i)$ $(n = 1, 2, \ldots, N)$, satisfy the axioms of independence, consensus, completeness and incompleteness (Xu, 2012).

Appendix B. Searching for more adjustment solutions

Let f_2^* be the optimum of the objective function (3–9) and $\zeta_t^{+*}(a^*)$, $\zeta_t^{-*}(a^*)$, $L_t^{\mathrm{DM}'*}(a^*)$ and $R_t^{\mathrm{DM}'*}(a^*)$, $a^* \in A_t^R$, be the optimal solution of $LP2$. Let also $S_L = \{a^* \in A_t^R | L_t^{\mathrm{DM}'*}(a^*) \neq L_t^{\mathrm{DM}}(a^*)\}$ and $S_R = \{a^* \in A_t^R | R_t^{\mathrm{DM}'*}(a^*) \neq R_t^{\mathrm{DM}}(a^*)\}$. The additional constraints are as follows

$$L_t^{\mathrm{DM}'}(a^*) - M(1 - \nu_{L1}(a^*)) \leq L_t^{\mathrm{DM}'} * (a^*) - 1, \quad a^* \in S_L, \tag{B.1}$$

$$L_t^{\mathrm{DM}'}(a^*) + M(1 - \nu_{L2}(a^*)) \geq L_t^{\mathrm{DM}'} * (a^*) + 1, \quad a^* \in S_L, \tag{B.2}$$

$$R_t^{\mathrm{DM}'}(a^*) - M(1 - \nu_{R1}(a^*)) \leq R_t^{\mathrm{DM}'} * (a^*) - 1, \quad a^* \in S_R, \tag{B.3}$$

$$R_t^{\mathrm{DM}'}(a^*) + M(1 - \nu_{R2}(a^*)) \geq R_t^{\mathrm{DM}'} * (a^*) + 1, \quad a^* \in S_R, \tag{B.4}$$

$$\sum_{a^* \in S_L} (\nu_{L1}(a^*) + \nu_{L2}(a^*)) + \sum_{a^* \in S_R} (\nu_{R1}(a^*) + \nu_{R2}(a^*)) \geq 1, \tag{B.5}$$

$$\nu_{L1}(a^*), \nu_{L2}(a^*) \in \{0, 1\}, \quad a^* \in S_L, \tag{B.6}$$

$$\nu_{R1}(a^*), \nu_{R2}(a^*) \in \{0, 1\}, \quad a^* \in S_R, \tag{B.7}$$

where M is a sufficiently large value satisfying $M > Q$. In constraints (B.1) and (B.2), if the binary variable $\nu_{L1}(a^*)$ or $\nu_{L2}(a^*)$ is equal to 1, the minimum assignment of the reference alternative a^* will change. In constraints (B.3) and (B.4), if the binary variable $\nu_{R1}(a^*)$ or $\nu_{R2}(a^*)$ is equal to 1, the maximum assignment of the reference alternative a^* will change. Constraint (B.5) ensures that at least one reference alternative will be adjusted to a different interval of categories and thus another adjustment solution will be generated. Such additional constraints are added to $LP2$ and they will forbid finding again the same solution. We continue solving $LP2$ with the extended constraint set until the optimum of the objective function is greater than f_2^*. The optimal solution at each iteration indicates different adjustment solutions.

Appendix C. Proof of Proposition 1

Proposition 1. *If $C_q, C_{q'} \in C_t^P(a)$, with $q < q'$, then $C_{q''} \in C_t^P(a)$, $\forall q'' \in [q, q']$.*

Proof. We follow the proof of the no-jump property in Greco, Mousseau et al. (2010). □

First we have to prove that ∂_t^R is convex. Observe that, for any $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ and $(\boldsymbol{u}'_t, \boldsymbol{b}'_t) \in \partial_t^R$, then we construct the pair $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ such that $u''_t(H_n) = \alpha u_t(H_n) + (1-\alpha)u'_t(H_n)$, $n = 1, 2, \ldots, N$, and $b_q^{t,''} = \alpha b_q^t + (1-\alpha) b_q^{t,'}$, $q = 0, 1, 2, \ldots, Q$, and $\alpha \in [0, 1]$. If $b_{q-1}^t \leq U_t(a) = \sum_{n=1}^N \frac{\beta_n(a)}{1-\beta_H(a)} u_t(H_n) < b_q^t$ and $b_{q-1}^{t,'} \leq U'_t(a) = \sum_{n=1}^N \frac{\beta_n(a)}{1-\beta_H(a)} u'_t(H_n) < b_q^{t,'}$, we have that $\alpha b_{q-1}^t + (1-\alpha) b_{q-1}^{t,'} \leq \sum_{n=1}^N \frac{\beta_n(a)}{1-\beta_H(a)} (\alpha u_t(H_n) + (1-\alpha) u'_t(H_n)) < \alpha b_q^t + (1-\alpha) b_q^{t,'}$, i.e.,

$b_{q-1}^{t,''} \le U_t''(a) < b_q^{t,''}$. Therefore, $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ is a compatible pair satisfying E_{AR}^t. Thus, we conclude that $\boldsymbol{\partial}_t^R$ is convex.

Next, we prove if $C_q, C_{q'} \in C_t^P(a)$, with $q < q'$, then $C_{q''} \in C_t^P(a)$, $\forall q'' \in [q, q']$. Because $C_q, C_{q'} \in C_t^P(a)$, we assume that there exists $(\boldsymbol{u}_t, \boldsymbol{b}_t)$ and $(\boldsymbol{u}'_t, \boldsymbol{b}'_t) \in \boldsymbol{\partial}_t^R$ such that $b_{q-1}^t \le U_t(a) = \sum_{n=1}^N \frac{\beta_n(a)}{1-\beta_H(a)} u_t(H_n) < b_q^t$ and $b_{q'-1}^{t,'} \le U_t'(a) = \sum_{n=1}^N \frac{\beta_n(a)}{1-\beta_H(a)} u'_t(H_n) < b_{q'}^{t,'}$.

(1) If $\frac{b_{q'}^{t,'} - b_{q''}^{t,'}}{b_{q''}^t - b_q^t + b_{q'}^{t,'} - b_{q''}^{t,'}} \le \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}$, we construct the pair $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ such that $u_t''(H_n) = \alpha u_t(H_n) + (1-\alpha) u_t'(H_n)$, $n = 1, 2, \ldots, N$, and $b_q^{t,''} = \alpha b_q^t + (1-\alpha) b_q^{t,'}$, $q = 0, 1, 2, \ldots, Q$, and

$$\alpha \in \left[\frac{b_{q'}^{t,'} - b_{q''}^{t,'}}{b_{q''}^t - b_q^t + b_{q'}^{t,'} - b_{q''}^{t,'}},\ \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}} \right].$$

For $\alpha \le \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}$, $b_{q-1}^t \le U(a)$, $b_{q'-1}^{t,'} \le U'(a)$, we have

$$\begin{aligned} U_t''(a) &= \alpha U_t(a) + (1-\alpha) U_t'(a) \ge \alpha b_{q-1}^t + (1-\alpha) b_{q'-1}^{t,'} \\ &\ge \alpha b_{q''-1}^t + (1-\alpha) b_{q''-1}^{t,'} = b_{q''-1}^{t,''}; \end{aligned}$$

For $\frac{b_{q'}^{t,'} - b_{q''}^{t,'}}{b_{q''}^t - b_q^t + b_{q'}^{t,'} - b_{q''}^{t,'}} \le \alpha$, $U(a) \le b_q^t$, $U'(a) \le b_{q'}^{t,'}$, we have

$$\begin{aligned} U_t''(a) &= \alpha U_t(a) + (1-\alpha) U_t'(a) < \alpha b_q^t + (1-\alpha) b_{q'}^{t,'} \\ &\le \alpha b_{q''}^t + (1-\alpha) b_{q''}^{t,'} = b_{q''}^{t,''}. \end{aligned}$$

Thus $b_{q''-1}^{t,''} \le U_t''(a) < b_{q''}^{t,''}$. Because $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ is a compatible pair satisfying E_{AR}^t, and we conclude that $C_{q''} \in C_t^P(a)$.

(2) If $\frac{b_{q'}^{t,'} - b_{q''}^{t,'}}{b_{q''}^t - b_q^t + b_{q'}^{t,'} - b_{q''}^{t,'}} > \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}$, we construct the pair $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ such that $u_t''(H_n) = \alpha u_t(H_n) + (1-\alpha) u_t'(H_n)$, $n = 1, 2, \ldots, N$, and $b_q^{t,''} = \alpha b_q^t + (1-\alpha) b_q^{t,'}$, $q = 0, 1, 2, \ldots, Q$, and

$$\alpha \in \left[0,\ \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}} \right].$$

For $\alpha \le \frac{b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}{b_{q''-1}^t - b_{q-1}^t + b_{q'-1}^{t,'} - b_{q''-1}^{t,'}}$, $b_{q-1}^t \le U(a)$, $b_{q'-1}^{t,'} \le U'(a)$, we have

$$\begin{aligned} U_t''(a) &= \alpha U_t(a) + (1-\alpha) U_t'(a) \ge \alpha b_{q-1}^t + (1-\alpha) b_{q'-1}^{t,'} \\ &\ge \alpha b_{q''-1}^t + (1-\alpha) b_{q''-1}^{t,'} = b_{q''-1}^{t,''}; \end{aligned}$$

For $\alpha < \frac{b_{q'}^{t,'} - b_{q''}^{t,'}}{b_{q''}^t - b_q^t + b_{q'}^{t,'} - b_{q''}^{t,'}}$, $U(a) \le b_q^t$, $U'(a) \le b_{q'}^{t,'}$, we have

$$\begin{aligned} U_t''(a) = \alpha U_t(a) + (1-\alpha) U_t'(a) &< \alpha b_q^t + (1-\alpha) b_{q'}^{t,'} < \alpha b_{q'}^t \\ &+ (1-\alpha) b_{q'}^{t,'} = b_{q'}^{t,''}, \quad \text{and} \quad \alpha b_q^t + (1-\alpha) b_{q'}^{t,'} > \alpha b_{q''}^t \\ &+ (1-\alpha) b_{q''}^{t,'} = b_{q''}^{t,''}. \end{aligned}$$

Thus, if $b_{q''-1}^{t,''} \le U_t''(a) < b_{q'}^{t,''}$, we would have $\exists q''' \in (q'', q']$, $C_{q'''} \in C_t^P(a)$. Thus, we can repeat the same reasoning until $C_{q''} \in C_t^P(a)$.

This means that there exists a compatible pair $(\boldsymbol{u}''_t, \boldsymbol{b}''_t)$ assigning a to $C_{q''}$, i.e., $C_{q''} \in C_t^P(a)$. □

Appendix D. Proof of Proposition 2

Proposition 2. *The optimal solution of LP4 is also the optimal solution of the linear program max* $f_3' = \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q(a)$ *subject to the same constraints of LP4.*

Proof. Let $y_q^*(a)$, $d_q^{+,*}(a)$ and $d_q^{-,*}(a)$ $(a \in A)$ be the optimal solution of *LP*4. Suppose $y_q^{\#}(a)$, $d_q^{+,\#}(a)$ and $d_q^{-,\#}(a)$ $(a \in A)$ are the optimal solution of the linear program max $f_3{}' = \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q(a)$ subject to the same constraints of *LP*4 while $y_q^*(a)$, $d_q^{+,*}(a)$ and $d_q^{-,*}(a)$ $(a \in A)$ are not. □

For $0 \le sup_q(a) \le 1$ and $y_q(a) \in \{0, 1\}$, we have

$$\sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^{\#}(a) - \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^*(a) \ge \varepsilon.$$

For $0 \le U(a) \le 1$, $0 \le b_q \le 1 + \varepsilon (q = 0, 1, \ldots, Q)$, we have $0 \le d_q^+(a) + d_q^-(a) \le 1$. Thus

$$\sum_{a \in A} \sum_{q=1}^Q (d_q^{+,\#}(a) + d_q^{-,\#}(a)) - \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,*}(a) + d_q^{-,*}(a)) \ge (-1) \times K = -K.$$

Therefore, it can be concluded that

$$\begin{aligned} &\left(\rho \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^{\#}(a) + \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,\#}(a) + d_q^{-,\#}(a)) \right) \\ &\quad - \left(\rho \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^*(a) + \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,*}(a) + d_q^{-,*}(a)) \right) \\ &= \rho \left(\sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^{\#}(a) - \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^*(a) \right) \\ &\quad + \left(\sum_{a \in A} \sum_{q=1}^Q (d_q^{+,\#}(a) + d_q^{-,\#}(a)) - \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,*}(a) + d_q^{-,*}(a)) \right) \\ &> \frac{K}{\varepsilon} \cdot \varepsilon + (-K) = 0. \end{aligned}$$

Thus we have

$$\begin{aligned} &\rho \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^{\#}(a) + \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,\#}(a) + d_q^{-,\#}(a)) \\ &\quad > \rho \sum_{a \in A} \sum_{q=1}^Q sup_q(a) y_q^*(a) + \sum_{a \in A} \sum_{q=1}^Q (d_q^{+,*}(a) + d_q^{-,*}(a)). \end{aligned}$$

It contradicts the hypothesis that $y_q^*(a)$, $d_q^{+,*}(a)$ and $d_q^{-,*}(a)$ $(a \in A)$ are the optimal solution of *LP*4 and thus concludes the proof.□

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