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Distance-based intuitionistic multiplicative multiple criteria decision-making methods for healthcare management in West China Hospital

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Abstract

Intuitionistic multiplicative sets use an asymmetric, unbalanced scale to express information from positive, negative, and indeterminate information. They have been found capable of comprehensively and objectively representing a person's intuitive understanding and hence have attracted much attention. Distance techniques are widely used to measure the degree to which arguments deviate from one another. Several fuzzy set extensions have been developed, but little research has been conducted on measures of distance between intuitionistic multiplicative sets. In this paper, we start by presenting a variety of measures of the distance between intuitionistic multiplicative sets, including Hausdorff distance measures, weighted distance measures, ordered weighted distance measures, and continuous weighted distance measures. We then develop a distance-based intuitionistic multiplicative-technique for order preference by similarity to ideal solution method and a distance-based intuitionistic multiplicative-Vlsekriterijumska Optimizacija I Kompromisno Resenje method for handling multiple criteria decision-making problems with intuitionistic multiplicative evaluation information. To demonstrate the practical application of these distance measures and the proposed methods, we provide a case study of hospital management of inpatient admission. The paper ends with comparative analyses of the two methods and some concluding remarks.

KEYWORDS
distance measures, hospital management, inpatient admission, intuitionistic multiplicative sets, multiple criteria decision-making

1 | INTRODUCTION

Hospitals are complex systems, always involving various complex information and processes, and the issue of finding ways to manage them effectively is highly important. The imbalance between supply and demand for scarce medical resources is increasing in China with a huge population pressure. Because the service provided by primary medical and health institutions tends to be poor, patients in China prefer to go to large public hospitals than small ones, regardless how serious their illness is (Zhang et al., 2016). In the case of patients with severe diseases or acute medical problems, a delay of treatment may lead to a high risk of disease progression or death, but patients with minor ailments should be directed to community hospitals or community clinics for treatment. China has many more primary health care institutions than large public hospitals, so the

Chinese government is trying to establish a hierarchical system of healthcare triage, which not only helps patients select the most appropriate institution but also helps hospital administrators determine which patients should be treated first. The patient admission process is very important in improving the efficiency with which medical resources are used, especially in large public hospitals such as the West China Hospital (WCH).

Unfortunately, the task of ranking patients is a difficult and subjective one, and an efficient, scientific decision support techniques are needed to aid the process. More importantly, for the patient admission evaluation, the intricacies of objective matters mean it is difficult for people to describe their opinions accurately, especially in the decision-makers' (DMs) opinions on qualitative criteria. Fuzzy set (FS) theory provides a tool for investigating such problems. It mainly uses a certain value for the interval (0, 1), called the membership degree, to represent the degree to which an element belongs to a set. Through deeper research, scholars have found that it is difficult to determine the exact membership degree for the traditional FS. Atanassov (1986) proposed the use of the intuitionistic FS (IFS) to solve this problem; the IFS is characterized by a membership degree, a non-membership degree, and a hesitancy degree to represent people's cognition from three different aspects. The use of the IFS has led to great achievements in the past three decades since 1986 (Liu & Liao, 2017; Yu & Liao, 2016), but IFSs can only represent people's cognition symmetrically by using a balanced scale, that is, 0.1–0.9 scale, whereas in many practical cases, the preferences of experts or DMs are actually uniformly and symmetrically distributed. One simple example comes from the law of diminishing marginal utility in economics (Xia, Xu, & Liao, 2013): if inefficient and efficient companies are given the same resources, the growth rate of the latter should be much higher. It is noted that Saaty (1990) proposed the asymmetric 1/9–9 scale, which can be used to describe asymmetric problems. Xia et al. (2013) combined the asymmetric 1/9–9 scale with the concept of the IFS and introduced the intuitionistic multiplicative preference relation and the concept of intuitionistic multiplicative sets (IMSs). IMSs use an asymmetric, unbalanced scale to express three different aspects of information and thus can provide a comprehensive, objective representation of intuitive information. A lot of work has been done with IMSs (Jiang, Xu, & Gao, 2015; Jiang, Xu, & Shu, 2016; Jiang, Xu, & Yu, 2015; Luo, Zhang, & Liao, 2019; Ren, Xu, & Liao, 2016; Xu, 2013; Zhang, Liao, & Luo, 2019; Zhang & Pedrycz, 2017, 2018). As the fundamental theory for investigating the IMS, distance measure for IMS plays a significant role in measuring the difference between numbers (points or sets). Meanwhile, it is also the core part for most multiple criteria decision-making (MCDM) methods, such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and Vlsekriterijumska Optimizacija I Kompromisno Resenje (VIKOR) in Serbian.

So far, distance measures can be classified into traditional distance measures (such as the Hamming distance, the Euclidean distance, and the Hausdorff distance) and weighted distance measures (including ordered weighted distance [Xu & Chen, 2008], continued weighted distance [Xu, 2012], and hybrid weighted distance [Xu, 2008]). Many scholars have investigated distance measures for different kinds of fuzzy information, such as IFSs (Burillo & Bustince, 1996; Du & Hu, 2015; Grzegorzewski, 2004; Narukawa & Torra, 2006; Szmidt & Kacprzyk, 2000; Wang & Xin, 2005; Xu & Chen, 2011), hesitant FSs (HFSs; Xu & Xia, 2011), and hesitant fuzzy linguistic term sets (HFLTSs; Liao & Xu, 2015; Liao, Xu, & Zeng, 2014). As we can see, there have been many advances in approaches to measuring the distance between IFSs but little research on measuring the distance between IMSs (Garg, 2017; Jiang et al., 2016; Jiang, Xu, & Gao, 2015). Thus, in this paper, we focus on measures of the distance between IMSs, which are basic and highly important, especially in solving MCDM problems with IMSs.

In addition, many scholars have investigated this issue regarding the patient admission evaluation, but there are drawbacks to all the approaches proposed so far. First, the approaches proposed for handing inpatient admission management in other countries (see Ashour & Kremer, 2016; Mariotti, Siciliani, Rebba, et al., 2014; Rahimi, Jamshidi, Ait-Kadi, & Bartolome, 2015; Rahimi, Jamshidi, Ruiz, & Ait-kadi, 2016; Solans-Domènech, Adam, Tebé, & Espallargues, 2013; Valente et al., 2009) are not appropriate to the China context. Second, although some Chinese scholars (Zhang et al., 2016) have considered the problem, they have only considered evaluation criteria such as "urgency," "need for hospitalization," "value of clinical pathology," and not other factors such as the limitations on the person's activity or time spent on the waiting list. Here, we present a case study of inpatient admission management by the admission centre of the WCH to illustrate how distance-based MCDM methods with IMSs can be used to circumvent these drawbacks by combining with the specific situations of Chinese hospital.

Inspired by these motivations, in this paper, we investigate the Hausdorff distance measure for IMSs associated with its hybrid form, generalized weighted form, ordered weighted form, and continuous weighted form. Comparing with the existing distance measures, including the Minkowski distance, Hamming distance, and Euclidean distance (Garg, 2017; Jiang et al., 2016; Jiang, Xu, & Gao, 2015), the Hausdorff distance has good noise immunity, especially in pattern recognition, image matching, and distinct improved algorithms. Moreover, both the ordered weighted form and the continuous form are analysed in this paper, by which the DMs can select the optimal one according to their demands. Subsequently, based on these novel distance measures between IMSs, we investigate a distance-based intuitionistic multiplicative (IM)-TOPSIS method and a distance-based IM-VIKOR method. Both of these two methods can be used to solve MCDM problems with IMS information. Finally, we apply the proposed methods to tackle the problem related to inpatients admission evaluation, which can provide some insights for hospital management in China.

In summary, the contributions of this paper are summarized as follows:

1 We propose a variety of distance measures for IMSs, based on traditional distance measures such as the Hausdorff distance measures, hybrid distance measures, weighted distance measures, ordered weighted distance measures, and weighted distance measures in continuous form. We also provide some numerical illustrations.

2 We present a distance-based IM-TOPSIS method and a distance-based IM-VIKOR method. These two methods have wide applicability in solving MCDM problems with IMSs. It should be noted that any type of distance measure could be applied in these two methods, so they provide DMs a wider choice for solving MCDM problems.
3 We apply the proposed distance-based IM-TOPSIS method and distance-based IM-VIKOR method to inpatient admission assessment to aid the WCH admission centre. These methods can be used to select patients scientifically and are useful for allocating medical resources in a rational way.

The remainder of this paper is organized as follows: In Section 2, we review some concepts relevant to IMSs as well as the existing measures of distance between IMSs. Section 3 presents various measures of the distance between IMSs. Section 4 investigates the IM-TOPSIS method and IM-VIKOR method based on measures of the distance between IMSs. A case study of application of the IM-TOPSIS and IM-VIKOR methods to inpatient admission assessment is presented in Section 5. Section 6 provides some concluding remarks.

2 | IMSS AND THEIR EXISTING DISTANCE MEASURES

To start our study, we first present the existing distance measures of IMSs.

For a reference set X, an IMS on X can be defined as (Xia et al., 2013)

$$D = \{ < x, (\rho_D(x), \sigma_D(x)) > | x \in X\}, \tag{1}$$

where $\rho_D(x)$ and $\sigma_D(x)$ are the membership degree the non-membership degree of $x \in X$ to the set D, respectively, and they satisfy $1/9 \le \rho_D(x)$, $\sigma_D(x) \le 9$, $\rho_D(x)\sigma_D(x) \le 1$, and $\forall x \in X$. $\tau_D(x) = 1/\rho_D(x)\sigma_D(x)$ is called the hesitancy degree, and $(\rho_D(x), \sigma_D(x))$ is called an intuitionistic multiplicative number (IMN). For any two IMNs, they can be compared by the score function $s(\alpha) = \rho_\alpha/\sigma_\alpha$ and the accuracy function $h(\alpha) = \rho_\alpha\sigma_\alpha$.

Assume that a DM evaluates two alternatives x_i and x_j on a criterion and gives his/her opinions as IMS $(\rho_D(x), \sigma_D(x))$. For an IMS $(\rho_D(x), \sigma_D(x))$, the term $\rho_D(x) \in \left[\frac{1}{9}, 1\right)$ denotes that the alternative x_i is inferior to x_j, that is, the alternative x_j is prior to x_i. In contrast, the term $\rho_D(x) \in (1,9]$ means the alternative x_i is preferred to x_j. In addition, several special cases can be described: (a) $\rho_D(x) = \sigma_D(x) = 1$ represents that there is no difference between x_i and x_j; (b) $\rho_D(x) = 9$ or $\sigma_D(x) = \frac{1}{9}$ indicates that x_i is absolutely preferred to x_j; (c) $\rho_D(x) = \frac{1}{9}$ or $\sigma_D(x) = 9$ denotes that x_i is absolutely inferior to x_j. For details, please refer to Xia et al. (2013).

For two IMSs, $A = \{(\rho_A(x_i), \sigma_A(x_i), \tau_A(x_i)) \ | x_i \in X\}$ and $B = \{(\rho_B(x_i), \sigma_B(x_i), \tau_B(x_i)) \ | x_i \in X\}$, Jiang, Xu, and Gao (2015) proposed that the distance measure $d(A,B)$ should satisfy the following basic axioms: (a) $0 \le d(A,B) \le 1$; (b) $d(A,B) = 0$, if and only if $A = B$; (c) $d(A,B) = d(B,A)$; (d) For $A,B,C \in M$, if $A \subseteq B \subseteq C$, then $d(A,C) \ge d(A,B)$, and $d(A,C) \ge d(B,C)$. Then, they defined the Minkowski distance measure and the normalized Minkowski distance measure between IMSs A and B as

$$d_{md}(A,B) = \left(\frac{1}{2^{\lambda+1}}\sum_{i=1}^{n}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right)\right)^{1/\lambda}, \lambda \ge 1 \tag{2}$$

$$d_{nmd}(A,B) = \left(\frac{1}{2^{\lambda+1}n}\sum_{i=1}^{n}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right)\right)^{1/\lambda}, \lambda \ge 1. \tag{3}$$

If $\lambda = 1$, the Minkowski distance becomes the Hamming distance, and the normalized Minkowski distance becomes the normalized Hamming distance; If $\lambda = 2$, the Minkowski distance becomes the Euclidean distance, and the normalized Minkowski distance becomes the normalized Euclidean distance (Jiang et al., 2016).

Theorem 1. *For two IMSs,* A *and* B, *if* $\to +\infty$, *the normalized Minkowski distance measure reduces to the following:*

$$\lim_{\lambda\to+\infty} d_{nmd}(A,B) = \frac{1}{2}\max_i\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|, \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|, \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}. \tag{4}$$

Proof. Let $\alpha = \frac{1}{2}\max_i\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|, \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|, \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}$. Then by Equation (3), we have

$$\lim_{\lambda\to+\infty} d_{nmd}(A,B) = \lim_{\lambda\to+\infty}\left(\frac{1}{2^{\lambda+1}n}\sum_{i=1}^{n}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda} + \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda} + \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right)\right)^{1/\lambda}$$

$$= \lim_{\lambda\to+\infty}\alpha\left[\frac{1}{2^{\lambda+1}n}\sum_{i=1}^{n}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|/\alpha\right)^{\lambda} + \left(\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|/\alpha\right)^{\lambda} + \left(\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|/\alpha\right)^{\lambda}\right]^{1/\lambda}$$

$$= \alpha\lim_{\lambda\to+\infty}\left(\frac{1}{2n}\right)^{\frac{1}{\lambda}}\cdot\lim_{\lambda\to+\infty}\left[\sum_{i=1}^{n}\left(\left(\left|\frac{1}{2}\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|/\alpha\right)^{\lambda} + \left(\left|\frac{1}{2}\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|/\alpha\right)^{\lambda} + \left(\left|\frac{1}{2}\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|/\alpha\right)^{\lambda}\right)\right]^{1/\lambda} = \alpha.$$

This completes the proof of Theorem 1.

3 | HAUSDORFF METRIC-BASED DISTANCE MEASURES FOR IMSS

This section introduces new distance measures between IMSs.

3.1 | Hausdorff distance measures and hybrid distance measures between IMSs

Jiang, Xu, and Gao (2015), Jiang et al. (2016), and Garg (2017) introduced a series of distance measures for IMSs, including the Manhattan distance, the Euclidean distance, and the Minkowski distance, together with the corresponding normalized and weighted forms. However, the Euclidean distance, one of the most widely used distance measures, fails to measure the difference between different types of attribute. We know that different types of attribute have different dimensions, and the Euclidean distance treats them equally, such that the results are inconsistent with reality. The Hausdorff distance, a new definition of distance, not only overcomes the defect of Euclidean distance but also has good noise immunity, and so it has been widely applied to pattern recognition, image matching, and distinct improved algorithms (Ashour & Kremer, 2016; Du & Hu, 2015). The application of the Hausdorff distance to various information forms, such as IFSs (Grzegorzewski, 2004), HFSs (Xu & Xia, 2011), and HFLTSs (Liao et al., 2014) has also been investigated. However, there has been little research into use of the Hausdorff distance measure with IMSs. We therefore introduce some measures of the distance between IMSs based on the Hausdorff metric.

Inspired by the generalized Hausdorff measures of the distance between IFSs (Grzegorzewski, 2004) and taking into account the relationships between IMSs and IFSs, we can define a generalized Hausdorff measure of the distance between two IMSs A and B as

$$d_{ghaud}(A,B) = \left(\frac{1}{2^{\lambda}}\sum_{i=1}^{n}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda}, \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda}, \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right\}\right)^{1/\lambda}, \lambda > 0. \tag{5}$$

Property. The generalized Hausdorff distance measure shown as Equation (5) satisfies the following properties:

1 $d_{ghaud}(A,B) = 0$, if and only if $A = B$;
2 $d_{ghaud}(A,B) = d_{ghaud}(B,A)$;
3 If $A \subseteq B \subseteq C$, then $d_{ghaud}(A,B)+d_{ghaud}(B,C) \geq d_{ghaud}(A,C)$.

The proof of Property is given in the Appendix.

If $\lambda = 1$, then the generalized Hausdorff distance becomes the Hamming–Hausdorff distance:

$$d_{hhaud}(A,B) = \frac{1}{2}\sum_{i=1}^{n}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|, \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|, \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}. \tag{6}$$

If $\lambda = 2$, then the generalized Hausdorff distance becomes the Euclidean–Hausdorff distance:

$$d_{ehaud}(A,B) = \sqrt{\frac{1}{4}\sum_{i=1}^{n}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2, \left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2, \left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right\}}. \tag{7}$$

In addition, if we combine the Hausdorff distance with the Hamming distance and the Euclidean distance, we can obtain some hybrid measures of the distance between A and B, such as the hybrid Hamming distance, the hybrid Euclidean distance, and the generalized hybrid distance, which are shown below in that order:

$$d_{hhd}(A,B)=\sum_{i=1}^{n}\left(\frac{1}{8}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right)+\frac{1}{4}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}\right), \tag{8}$$

$$d_{hed}(A,B)=\left(\sum_{i=1}^{n}\left(\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right\}\right)\right)^{1/2}, \tag{9}$$

$$d_{ghd}(A,B)=\left(\sum_{i=1}^{n}\left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right)+\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right\}\right)\right)^{1/\lambda},\lambda>0. \tag{10}$$

Property. The generalized hybrid distance measure shown as Equation (10) satisfies the following properties:

1 $d_{ghd}(A,B)=0$, if and only if $A=B$;
2 $d_{ghd}(A,B)=d_{ghd}(B,A)$;
3 If $A\subseteq B\subseteq C$, then $d_{ghd}(A,B)+d_{ghd}(B,C)\geq d_{ghd}(A,C)$.

The proof of Property is given in the Appendix.

3.2 | Distinct weighted distance measures between IMSs

In this subsection, we mainly introduce the distance measures between IMSs from three aspects, which are the weighted distance measures, the ordered weighted distance measures, and the continuously weighted distance measures.

3.2.1 | Weighted distance measures between IMSs

In many real-world situations, the weights of the elements $x_i\in X(i=1,2,\cdots,n)$ should be considered. Jiang et al. (2016) proposed some weighted measures of the distance between IMSs, such as the weighted Minkowski distance, the weighted Hamming distance, and the weighted Euclidean distance. In addition, Garg (2017) presented the normalized weighted Hamming distance and the normalized weighted Euclidean distance. Below, we introduce more weighted distance measures for IMSs.

Let $X=\{x_1,x_2,\cdots,x_n\}$be a finite set, $A=\{\rho_A(x_i),\sigma_A(x_i),\tau(x_i))\ \ |x_i\in X\}$ and $B=\{(\rho_B(x_i),\sigma_B(x_i),\tau_B(x_i))\ |x_i\in X\}$ be two IMSs in X associated with the weight vector $w=(w_1,w_2,\cdots,w_n)^T$, where $0\leq w_i\leq 1$ and $\sum_{i=1}^{n}w_i=1$. The generalized weighted Hausdorff distance between A and B is defined as

$$d_{gwhaud}(A,B)=\left(\frac{1}{2^{\lambda}}\sum\nolimits_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right\}\right)^{1/\lambda},\lambda>0. \tag{11}$$

If $\lambda=1$, we can obtain the weighted Hamming–Hausdorff distance between A and B:

$$d_{whhaud}(A,B)=\frac{1}{2}\sum_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}. \tag{12}$$

If $\lambda=2$, we can obtain the weighted Euclidean–Hausdorff distance between A and B:

$$d_{wehaud}(A,B)=\sqrt{\frac{1}{4}\sum_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right\}}. \tag{13}$$

In particular, if $w=(1/n,1/n,\cdots,1/n)^T$, then the generalized weighted distance, the weighted Hamming distance, and the weighted Euclidean distance are equal to the normalized Minkowski distance, the normalized Hamming distance, and the normalized Euclidean distance, respectively.

Obviously, we can also obtain some hybrid weighted distance measures on the basis of the above distance measures, for example,

1 The hybrid weighted Hamming distance between A and B:

$$d_{hwhd}(A,B)=\sum_{i=1}^{n}w_i\left[\frac{1}{8}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right)+\frac{1}{4}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}\right]. \tag{14}$$

2 The hybrid weighted Euclidean distance between A and B:

$$d_{hwed}(A,B)=\left[\sum_{i=1}^{n}w_i\left(\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right\}\right)\right]^{1/2}. \tag{15}$$

3 The generalized hybrid weighted distance between A and B:

$$d_{ghwd}(A,B)=\left[\sum_{i=1}^{n}w_i\left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right)+\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)\right]^{1/\lambda},\lambda>0. \tag{16}$$

3.2.2 | Ordered weighted measures of the distance between IMSs

The concept of ordered weighted distance was introduced by Xu and Chen (2008) and has been studied by many scholars. So far, the concept of ordered weighted distance has been extended to IFSs (Xu & Chen, 2011), HFSs (Xu & Xia, 2011), and HFLTSs (Liao et al., 2014; Liao & Xu, 2015). A prominent characteristic of ordered weighted distance is that it can diminish or enhance the influence of unduly large or small deviations on the aggregation results by assigning them low or high weights (Xu & Chen, 2008). This desirable property makes ordered weighted distance measures extremely useful in decision-making processes; however, our review of the existing literature revealed that there has been no research on ordered weighted measures of the distance between IMSs, so in this subsection, we present some ordered weighted measures of the distance between IMSs.

Let A and B be two IMSs, and their associated weight vector be given as above. The generalized ordered weighted distance between A and B is defined as

$$d_{gowd}(A,B)=\left[\frac{1}{2^{\lambda+1}}\sum_{i=1}^{n}w_i\left(\left|\log_9\frac{\rho_A(x_{\eta(i)})}{\rho_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_{\eta(i)})}{\sigma_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\tau_A(x_{\eta(i)})}{\tau_B(x_{\eta(i)})}\right|^\lambda\right)\right]^{1/\lambda},\lambda>0, \tag{17}$$

where $\eta(1),\eta(2),\cdots,\eta(n)$ is a permutation of $(1,2,\cdots,n)$ such that

$$\left(\left|\log_9\frac{\rho_A(x_{\eta(i-1)})}{\rho_B(x_{\eta(i-1)})}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_{\eta(i-1)})}{\sigma_B(x_{\eta(i-1)})}\right|^\lambda+\left|\log_9\frac{\tau_A(x_{\eta(i-1)})}{\tau_B(x_{\eta(i-1)})}\right|^\lambda\right)\geq\left(\left|\log_9\frac{\rho_A(x_{\eta(i)})}{\rho_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_{\eta(i)})}{\sigma_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\tau_A(x_{\eta(i)})}{\tau_B(x_{\eta(i)})}\right|^\lambda\right).$$

If $\lambda=1$, Equation (17) reduces to an ordered weighted Hamming distance between A and B:

$$d_{owhd}(A,B)=\frac{1}{4}\sum_{i=1}^{n}w_i\left(\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|+\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|+\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|\right). \tag{18}$$

If $\lambda = 2$, Equation (17) reduces to an ordered weighted Euclidean distance between A and B:

$$d_{owed}(A,B)=\sqrt{\frac{1}{8}\sum_{i=1}^{n}w_i\left(\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^2+\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^2+\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^2\right)}. \tag{19}$$

Based on the Hausdorff metric, we can define a generalized ordered weighted Hausdorff distance between A and B:

$$d_{gowhaud}(A,B)=\left[\frac{1}{2^{\lambda}}\sum_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^{\lambda}\right\}\right]^{1/\lambda},\ \lambda>0, \tag{20}$$

where $\eta(1),\eta(2),\cdots,\eta(n)$ is a permutation of $(1,2,\cdots,n)$, such that

$$\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^{\lambda}\right\}\geq\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i+1)}\right)}{\rho_B\left(x_{\eta(i+1)}\right)}\right|^{\lambda},\left|\log_9\frac{\sigma_A\left(x_{\eta(i+1)}\right)}{\sigma_B\left(x_{\eta(i+1)}\right)}\right|^{\lambda},\left|\log_9\frac{\tau_A\left(x_{\eta(i+1)}\right)}{\tau_B\left(x_{\eta(i+1)}\right)}\right|^{\lambda}\right\}.$$

If $\lambda = 1$, then Equation (20) reduces to an ordered weighted Hamming–Hausdorff distance between A and B:

$$d_{owhhaud}(A,B)=\frac{1}{2}\sum_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|,\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|,\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|\right\}. \tag{21}$$

If $\lambda = 2$, then Equation (20) reduces to an ordered weighted Euclidean–Hausdorff distance between A and B:

$$d_{owehaud}(A,B)=\sqrt{\frac{1}{4}\sum_{i=1}^{n}w_i\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^2,\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^2,\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^2\right\}}. \tag{22}$$

Obviously, we can also develop some hybrid ordered weighted distance measures, such as the hybrid generalized ordered weighted distance, the hybrid ordered weighted Hamming distance, and the hybrid ordered weighted Euclidean distance between A and B, which are shown below in that order:

$$d_{ghowd}(A,B)=\left[\sum_{i=1}^{n}w_i\left(\frac{1}{2^{i+2}}\left(\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^{\lambda}+\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^{\lambda}+\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^{\lambda}\right)+\frac{1}{2^{i+1}}\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^{\lambda},\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^{\lambda}\right\}\right)\right]^{1/\lambda},\ \lambda>0, \tag{23}$$

$$d_{howhd}(A,B)=\sum_{i=1}^{n}w_i\left(\frac{1}{8}\left(\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|+\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|+\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|\right)+\frac{1}{4}\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|,\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|,\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|\right\}\right), \tag{24}$$

$$d_{howed}(A,B)=\left[\sum_{i=1}^{n}w_i\left(\frac{1}{16}\left(\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^2+\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^2+\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^2\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A\left(x_{\eta(i)}\right)}{\rho_B\left(x_{\eta(i)}\right)}\right|^2,\left|\log_9\frac{\sigma_A\left(x_{\eta(i)}\right)}{\sigma_B\left(x_{\eta(i)}\right)}\right|^2,\left|\log_9\frac{\tau_A\left(x_{\eta(i)}\right)}{\tau_B\left(x_{\eta(i)}\right)}\right|^2\right\}\right]^{1/2}, \tag{25}$$

where $\eta(1),\eta(2),\cdots,\eta(n)$ is any permutation of $(1,2,\cdots,n)$, such that

$$\left[\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x_{\eta(i)})}{\rho_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_{\eta(i)})}{\sigma_B(x_{\eta(i)})}\right|^\lambda+\left|\log_9\frac{\tau_A(x_{\eta(i)})}{\tau_B(x_{\eta(i)})}\right|^\lambda\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x_{\eta(i)})}{\rho_B(x_{\eta(i)})}\right|^\lambda,\left|\log_9\frac{\sigma_A(x_{\eta(i)})}{\sigma_B(x_{\eta(i)})}\right|^\lambda,\left|\log_9\frac{\tau_A(x_{\eta(i)})}{\tau_B(x_{\eta(i)})}\right|^\lambda\right\}\right]\geq,$$
$$\left[\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x_{\eta(i+1)})}{\rho_B(x_{\eta(i+1)})}\right|^\lambda+\left|\log_9\frac{\sigma_A(x_{\eta(i+1)})}{\sigma_B(x_{\eta(i+1)})}\right|^\lambda+\left|\log_9\frac{\tau_A(x_{\eta(i+1)})}{\tau_B(x_{\eta(i+1)})}\right|^\lambda\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x_{\eta(i+1)})}{\rho_B(x_{\eta(i+1)})}\right|^\lambda,\left|\log_9\frac{\sigma_A(x_{\eta(i+1)})}{\sigma_B(x_{\eta(i+1)})}\right|^\lambda,\left|\log_9\frac{\tau_A(x_{\eta(i+1)})}{\tau_B(x_{\eta(i+1)})}\right|^\lambda\right\}\right].$$

3.2.3 | Weighted distances between IMSs in the continuous case

We find that all the above distance measures are based on discrete values. However, sometimes the universe of discourse and the weights of elements may be continuous. This subsection mainly discusses that case.

Let A and B be two IMSs in X with the weight of $x \in X = [a,b]$ being $w(x)$, where and $\int_b^a w(x)dx = 1$. We define a generalized continuous weighted distance, a continuous weighted Hamming distance, and a continuous weighted Euclidean distance between the IMSs A and B, respectively, as follows:

$$d_{gcwd}(A,B)=\left(\frac{1}{2^{\lambda+1}}\int_a^b w(x)\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^\lambda+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^\lambda+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^\lambda\right)dx\right)^{1/\lambda},\lambda>0, \tag{26}$$

$$d_{cwhd}(A,B)=\frac{1}{4}\int_a^b w(x)\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right)dx, \tag{27}$$

$$d_{cwed}(A,B)=\left(\frac{1}{8}\int_a^b w(x)\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right)dx\right)^{1/2}. \tag{28}$$

If $w(x) = 1/(b-a)$ for any $x \in [a,b]$, then the above distances respectively reduce to a generalized continuous normalized weighted distance, a continuous normalized Hamming distance, and a continuous normalized Euclidean distance, as follows:

$$d_{gcnd}(A,B)=\left(\frac{1}{2^{\lambda+1}(b-a)}\int_a^b\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^\lambda+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^\lambda dx\right)\right)^{1/\lambda},\lambda>0 \tag{29}$$

$$d_{cnhd}(A,B)=\frac{1}{4(b-a)}\int_a^b\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right)dx \tag{30}$$

$$d_{cned}(A,B)=\left(\frac{1}{8(b-a)}\int_a^b\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right)dx\right)^{1/2} \tag{31}$$

On the basis of the Hausdorff metric, we can introduce a generalized continuous weighted Hausdorff distance, a continuous weighted Hamming-Hausdorff distance and a continuous weighted Euclidean-Hausdorff distance between A and B as follows:

$$d_{gcwhaud}(A,B)=\frac{1}{2}\left(\int_a^b w(x)\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^\lambda,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^\lambda,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^\lambda\right\}dx\right)^{1/\lambda},\lambda>0, \tag{32}$$

$$d_{cwhhaud}(A,B)=\frac{1}{2}\int_a^b w(x)\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right\}dx, \tag{33}$$

$$d_{cwehaud}(A,B)=\frac{1}{2}\left(\int_a^b w(x)\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right\}dx\right)^{1/2}. \tag{34}$$

If $w(x) = 1/(b-a)$ for any $x \in [a,b]$, then Equations (32)–(34) reduce respectively to a generalized continuous normalized Hausdorff distance, a continuous normalized Hamming–Hausdorff distance, and a continuous normalized Euclidean–Hausdorff distance between A and B as follows:

$$d_{gcnwhaud}(A,B)=\frac{1}{2}\left(\frac{1}{b-a}\int_a^b \max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^{\lambda}\right\}dx\right)^{1/\lambda},\lambda>0, \tag{35}$$

$$d_{cnhhaud}(A,B)=\frac{1}{2(b-a)}\int_a^b \max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right\}dx, \tag{36}$$

$$d_{cnehaud}(A,B)=\frac{1}{2}\left(\frac{1}{b-a}\int_a^b \max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right\}dx\right)^{1/2}. \tag{37}$$

We can, of course, derive some hybrid continuous weighted distance measures, such as a generalized hybrid continuous weighted distance, a hybrid continuous weighted Hamming distance, and a hybrid continuous weighted Euclidean distance between A and B as follows:

$$d_{ghcwd}(A,B)=\left(\int_a^b w(x)\left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right)+\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right\}\right)dx\right)^{1/\lambda},\lambda>0, \tag{38}$$

$$d_{hcwhd}(A,B)=\int_a^b w(x)\left(\frac{1}{8}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right)+\frac{1}{4}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|\right\}\right)dx, \tag{39}$$

$$d_{hcwed}(A,B)=\left(\int_a^b w(x)\left(\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^2,\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^2,\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^2\right\}\right)dx\right)^{1/2}. \tag{40}$$

If $w(x) = 1/(b-a)$ for any $x \in [a,b]$, then Equations (38)–(40) reduce respectively to a generalized hybrid continuous normalized distance, a hybrid continuous normalized Hamming distance, and a hybrid continuous normalized Euclidean distance between A and B as follows:

$$d_{ghcnd}(A,B)=\left(\frac{1}{(b-a)}\int_a^b\left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^{\lambda}+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^{\lambda}+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^{\lambda}\right)+\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^{\lambda}\right\}\right)dx\right)^{1/\lambda},\lambda>0, \tag{41}$$

$$d_{hcwhd}(A,B)=\frac{1}{(b-a)}\int_a^b\left(\frac{1}{8}\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right)+\frac{1}{4}\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|\right\}\right)dx, \tag{42}$$

$$d_{hcwed}(A,B)=\left(\frac{1}{(b-a)}\int_a^b\left(\frac{1}{16}\left(\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2+\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2+\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right)+\frac{1}{8}\max\left\{\left|\log_9\frac{\rho_A(x)}{\rho_B(x)}\right|^2,\left|\log_9\frac{\sigma_A(x)}{\sigma_B(x)}\right|^2,\left|\log_9\frac{\tau_A(x)}{\tau_B(x)}\right|^2\right\}\right)dx\right)^{1/2}. \tag{43}$$

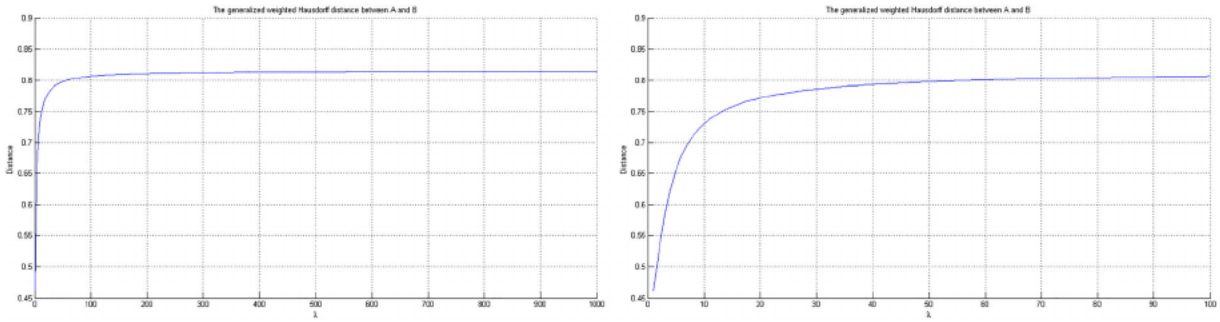
Note. In Section 3.2, various weighted distance measures for IMSs based on the generalized weighted Hausdorff distance and the generalized hybrid weighted distance were investigated. It is easy to demonstrate that all the proposed weighted distance measures also have the properties shown in Properties 1 and 2.

Example 1. Let A and B be two IMSs in $X = \{x_1, x_2\}$ and $A = \{(1/6,4),(2,1/6)\}$,$B = \{(6,1/9),(7,1/9)\}$. Their weight vector is $w = (1/3,2/3)^T$.

1. Using Equation (11), we can obtain the generalized weighted Hausdorff distance between A and B. The results are shown in Figure 1. The left side of Figure 1 shows the variations and distributions of the generalized weighted Hausdorff distance between A and B when $0 \le \lambda \le 1{,}000$, whereas the right side shows these data for $0 \le \lambda \le 100$. When $\lambda = 1$ and $\lambda = 2$, we can obtain the weighted Hamming–Hausdorff distance and the weighted Euclidean–Hausdorff distance between A and B: $d_{whhaud}(A,B) = 0.4619$ and $d_{wehaud}(A,B) = 0.5252$.

2. Using Equation (16), we can obtain the generalized hybrid weighted distance between A and B. The results are shown in Figure 2. The left side of Figure 2 shows the variations and distributions of the generalized hybrid weighted distance between A and B when $0 \le \lambda \le 1{,}000$, whereas the right side shows the data for $0 \le \lambda \le 100$. We can obtain the hybrid weighted Hamming distance and the hybrid weighted Euclidean distance between A and B for $\lambda = 1$ and $\lambda = 2$, respectively: $d_{hwhd}(A,B) = 0.4619$ and $d_{hwed}(A,B) = 0.5195$.

We can see from Figures 1 and 2 that the variations and distributions of these generalized measures of the distance between A and B are almost the same. Both Figures 1 and 2 exhibit a slow decrease in the rate of growth of the distance between A and B, whereas all the distances

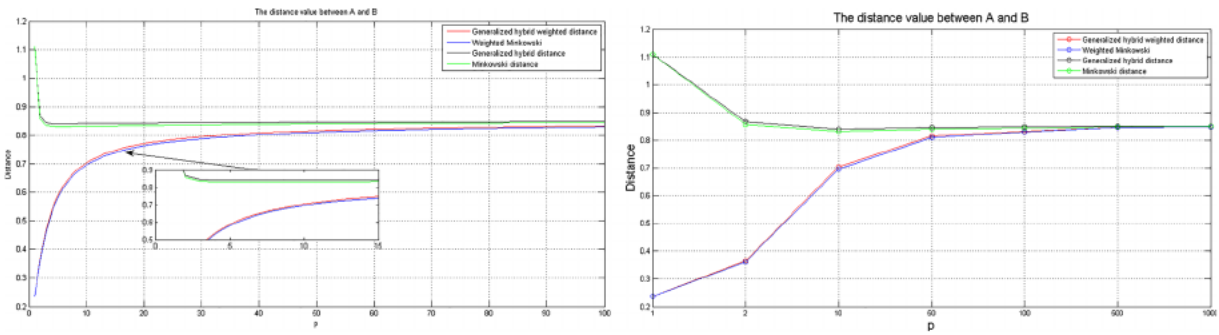


The generalized weighted Hausdorff distance between *A* and *B*

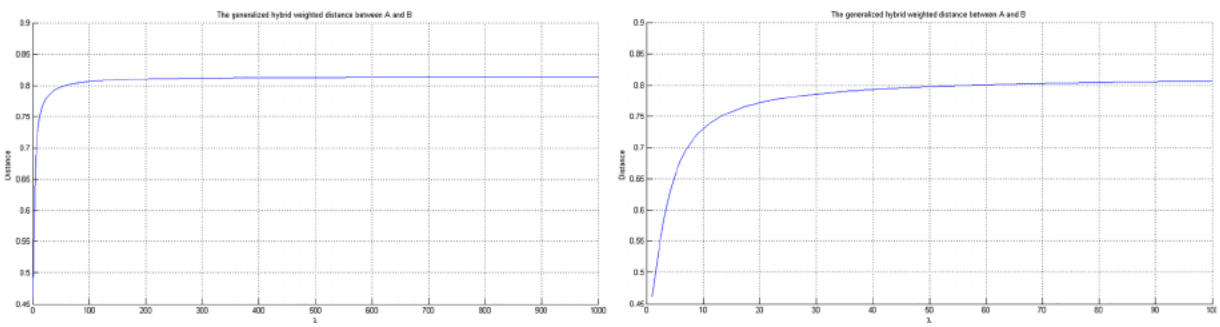
become gradually larger as the value of λ increases, and they ultimately converge at approximately 0.82. When $\lambda = 1$ and $\lambda = 100$, we obtain the same distance values, that is, 0.4619 and 0.82, representing the minimum and maximum values, respectively. It should be noted that both the generalized weighted Hausdorff distance and the weighted Minkowski distance make up the generalized hybrid weighted distance. Calculation of the generalized hybrid weighted distance is more complex and hence more time-consuming than calculation of the generalized weighted Hausdorff distance. Hybrid distance measures are more robust than single distances. We can choose which distance measure to use based on the demands of a particular DM.

Example 2. Let $A = \{(5,1/7),(6,1/7),(1/5,4)\}$, $B = \{(2,1/7),(1/7,5),(1/6,5)\}$, and $C = \{(6,1/6),(4,1/8),(1/7,2)\}$ be three IMSs, and $w_i = (1/3,1/6,1/2)^T$ be their weight vector. Four distance measures—the Minkowski distance, the weighted Minkowski distance, the generalized hybrid distance, and the generalized hybrid weighted distance—are used to determine the values of $d(A,B)$, $d(A,C)$, and $d(B,C)$, and the results are shown in Figures 3–5, respectively.

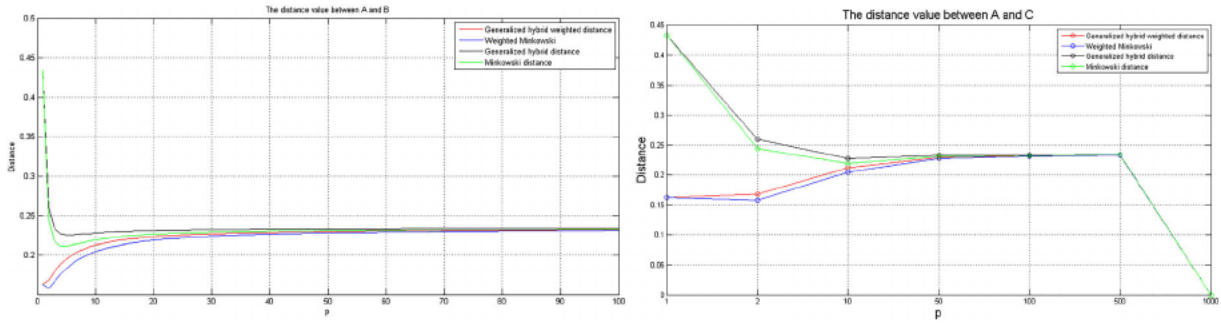
The left side of Figure 3 shows the distance between *A* and *B* when $0 \le \lambda \le 100$, whereas the right side shows the corresponding distance with $\lambda = 1,2,10,50,100,500,1,000$. Corresponding values for the distances between *A* and *C* and *B* and *C* are shown in Figures 4 and 5 respectively. Based on these figures, we can draw the following conclusions:



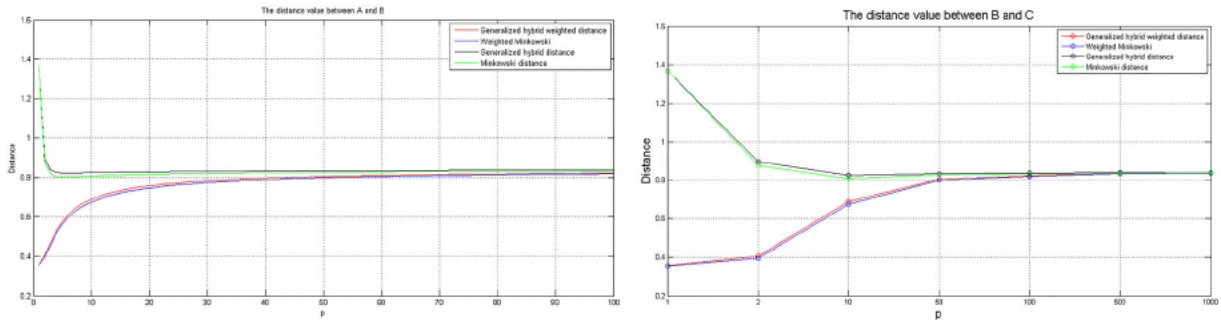
The distance between *A* and *B*



The generalized hybrid weighted distance between *A* and *B*



The distance between *A* and *C*



The distance between *B* and *C*

1 Using different distance measures, we obtain different values for the distance between IMSs. For instance, the generalized hybrid distance and the weighted Minkowski distance provide the biggest and smallest values, respectively.
2 The values of weighted distance measures gradually increase with λ, whereas values of unweighted distance measures decrease. Note also that the values of the weighted distances are smaller than those of the unweighted distances.
3 The right sides of Figures 3–5 show that the differences between the values of the different distance measures gradually decrease as the values of λ increase and finally tend to zero. Meanwhile, when $\lambda = 1$ and $\lambda = 2$, we have $d(B,C) > d(A,B) > d(A,C)$, whereas for $\lambda = 10$, $\lambda = 50$, $\lambda = 100$, $\lambda = 500$, and $\lambda = 1000$, we have $d(A,B) > d(B,C) > d(A,C)$. This means that the proposed distance measures, that is, the generalized hybrid distance and the generalized hybrid weighted distance, provide effective measurements of the difference between IMSs.

Example 3. (Continued from Example 2) Assume that $w = (1/6,1/3,1/2)^T$ is the weight vector of the ordered positions of the distances $\Delta(A(x_i), B(x_i))$, where

$$\Delta(A(x_i),B(x_i)) = \left(\left|\log_9 \frac{\rho_A\left(x_{\eta(i-1)}\right)}{\rho_B\left(x_{\eta(i-1)}\right)}\right|^{\lambda} + \left|\log_9 \frac{\sigma_A\left(x_{\eta(i-1)}\right)}{\sigma_B\left(x_{\eta(i-1)}\right)}\right|^{\lambda} + \left|\log_9 \frac{\tau_A\left(x_{\eta(i-1)}\right)}{\tau_B\left(x_{\eta(i-1)}\right)}\right|^{\lambda}\right), \text{ for } i = 1,2,3.$$

1 Using Equation (17), we can obtain the generalized ordered weighted distance between *A* and *B*. When $\lambda = 1$ and $\lambda = 2$, we have $d_{owhd}(A,B) = 0.2366$ and $d_{owed}(A,B) = 0.3613$.
2 Using Equation (20), we can obtain the generalized ordered weighted Hausdorff distance between *A* and *B*. When $\lambda = 1$ and $\lambda = 2$, we have $d_{owhhaud}(A,B) = 0.2367$ and $d_{owehaud}(A,B) = 0.3693$.
3 Using Equation (23), we can obtain the hybrid generalized ordered weighted distance between *A* and *B*. When $\lambda = 1$ and $\lambda = 2$, we have $d_{howhd}(A,B) = 0.2367$ and $d_{howed}(A,B) = 0.3653$.

Examples 2 and 3 show that the values of the weighed distance measures, that is, those given by Equations (17), (20), and (23), are smaller than those of the unweighted distance measures, that is, Equations (2) and (10). As mentioned above, a prominent characteristic of ordered weighted distance measures is that they can modify the influence of unduly large or small deviations on the aggregation results by assigning them low or high weights (Xu & Chen, 2008). In such cases, the obtained results are closer to DMs' preferences and more consistent with the real situation. In addition, this section further investigates the use of continuous weighted distance with IMSs, which provides a solution to the problem of the continuity of weight vector for attributes. The strength, characteristics, and applicability of these proposed distance measures are further summarized in Table 1. On this basis, the choice of distance measure can be based on the demands of the DMs.

Comparisons of the proposed IM distance measures

Distance measures		Strength	Characteristics	Applicability
Unweighted form	Generalized Hausdorff distance	Moderate	Regardless of weight	MCDM problems with equal weights of the DMs or criteria; Regardless of weight
Weighted form	Generalized weighted Hausdorff distance	Strong	Both the weight vectors of DMs and criteria are considered	It can be widely used to measure the weighted distance between IMSs/IMNs
	Ordered weighted Hausdorff distance	Moderate	It can diminish or enhance the influence of unduly large or small deviations on aggregation results	The case considering the orders of criteria and DMs
	Continuous weighted Hausdorff distance	Weak	Continuous values	It is mainly used to tackle the case that the universe of discourse and the weights of elements are continuous

Note. The term "strength" in Table 1 denotes the strength of practicality, that is, the frequency of application in real case. Moreover, the hybrid distance measures for IMSs/IMNs have stronger robustness compared with single forms such as the Hausdorff distance, Hamming distance, and Euclidean distance.

Abbreviations: DM, decision-maker; IM, intuitionistic multiplicative; IMS, intuitionistic multiplicative set; IMN, intuitionistic multiplicative number; MCDM, multiple criteria decision-making.

4 | TWO DISTANCE-BASED METHODS FOR MCDM PROBLEMS WITH INTUITIONISTIC MULTIPLICATIVE INFORMATION

A MCDM can be described as follows: let $X = \{X_1, X_2, \cdots, X_m\}$ be a discrete set of options, where the DMs need to choose the best option from X according to the criterion set $C = \{C_1, C_2, \cdots, C_n\}$. Suppose that the values for the criteria, given by the DMs, are represented by the IMNs $\{(\rho_{X_i}(C_j), \sigma_{X_i}(C_j)) \mid i = 1,2,\cdots,m, j = 1,2,\cdots,n\}$, where $\rho_{X_i}(C_j)$ and $\sigma_{X_i}(C_j)$ are the degrees of membership and non-membership, respectively, of option X_i based on criterion C_j, and satisfy: $1/9 \le \rho_{X_i}(C_j), \sigma_{X_i}(C_j) \le 9$ and $0 < \rho_{X_i}(C_j)\sigma_{X_i}(C_j) \le 1$. Together these IMNs constitute an intuitionistic multiplicative decision matrix (IMDM) $R = (r_{ij})_{m \times n} = (\rho_{X_i}(C_j), \sigma_{X_i}(C_j))_{m \times n}$. In this section, based on the proposed measures of the distance between IMSs, we propose two methods of solving MCDM problems with intuitionistic multiplicative information.

4.1 | The distance-based IM-TOPSIS method

TOPSIS (Chen & Hwang, 1992) is a well-known method and is widely used in solving MCDM problems. With this method, the chosen option should be furthest from the negative ideal solution and closest to the positive ideal solution. In other words, the distance measure plays an important role in this method. There has been a lot of work done on the TOPSIS method, and it can be divided into two categories: (a) extensions of the TOPSIS method to multiple FSs (Liao & Xu, 2015), including IFSs, interval-valued IFSs, Pythagorean FSs, HFSs, and HFLTSs; (b) applications in a wide variety of fields, such as the healthcare industry, manufacturing, business investment, military training, financial management of government, and national policies.

By combining the extant research results with the proposed measures of distance between IMSs, we can develop a distance-based, IM-TOPSIS method. The general procedures of the IM-TOPSIS are shown below.

Algorithm 1

Step 1. Build the IMDM based on the initial intuitionistic multiplicative evaluation values given by the DM.

Step 2. Determine the intuitionistic multiplicative positive ideal solution (IMPIS) and the intuitionistic multiplicative negative ideal solution (IMNIS). Note that the evaluation criteria can be classified into benefit criteria and cost criteria. We need to do some unification before carrying out further calculations. The values of the cost criterion C_j can be translated to their opposites as

$$\left(\rho'_{X_i}(C_j),\sigma'_{X_i}(C_j)\right) = \left(\sigma_{X_i}(C_j),\rho_{X_i}(C_j)\right), \tag{44}$$

with C_j being the cost criterion.

In this way, we can obtain a unified IMDM R'. To facilitate the presentation, we will assume that all the criteria are benefit criteria; if not, they could be translated as illustrated above. In this case, the IMPIS X^+ and the IMNIS X^- can be defined as

$$X^+ = \left[C_1^+, C_2^+, \cdots, C_j^+, \cdots C_n^+\right] X^- = \left[C_1^-, C_2^-, \cdots, C_j^-, \cdots C_n^-\right], \tag{45}$$

where $C_j^+ = \max_i C_{ij} = \left(\rho_j^+, \sigma_j^+\right)$, $C_j^- = \min_i C_{ij} = \left(\rho_j^-, \sigma_j^-\right)$ ($j = 1,2,\cdots,n$). Given that the criteria are sometimes conflicting, it is realistic to assume that such an IMPIS does not exist; if it did, the decision would be trivial.

Step 3. Calculate the distance between each option and the IMPIS, $d(X_i, X^+)$, and the distance between each option and the IMNIS, $d(X_i, X^-)$, using the distance measures introduced in Section 3.

Step 4. Calculate the relative closeness coefficients $c(X_i)$ ($i = 1,2,\cdots,m$) for all options, where

$$c(X_i) = d(X_i, X^-)/\left(d(X_i, X^+) + d(X_i, X^-)\right). \tag{46}$$

Step 5. Sort $c(X_i)$ ($i = 1,2,\cdots,m$) in descending order. A large value for the relative closeness coefficient $c(X_i)$ implies that the option X_i is close to the IMPIS and relatively distant from the IMNIS. Thus, all options can be ranked in descending order of their relative closeness coefficient values. The optimal option is the one with the highest closeness coefficient value.

4.2 | The distance-based IM-VIKOR method

The VIKOR (in Serbian, meaning multicriteria optimization and compromise solution) method, which yields compromise solution(s) based on the ideal solution, was first proposed by Opricovic (1998). Mardani, Zavadskas, Govindan, Senin, and Jusoh (2016) gave a comprehensive overview of the VIKOR method based on the papers published from 2004 to 2015. Our review of the literature indicated that the VIKOR method has been applied to MCDM problems in many different areas, including material selection, supplier selection, renewable energy plan selection, and information security risk control assessment. In addition, the VIKOR method has been extended to other information representation environments, such as IFSs, HFSs, HFLTSs, and interval neutrosophic sets (Bausys & Zavadskas, 2015; Liao & Xu, 2013; Liao & Xu, 2015; Liao, Xu, & Zeng, 2015).

However, there has been no research on the VIKOR method with IMSs. Hence, we have developed a distance-based IM-VIKOR method, by combining the existing research results with the proposed measures of distance between IMSs. This method involves the following steps:

Algorithm 2

- Steps 1–2. As for Algorithm 1.
- Step 3. Calculate the values of the intuitionistic multiplicative group utility measure $IMGU_i$ and the intuitionistic multiplicative individual regret measure $IMIR_i$ for all options, defined in terms of the measures of distance between IMSs introduced in Section 3, and shown below (here, we choose the Euclidean distance as an example):

$$IMGU_i = \sum_{j=1}^{n} w_j \frac{C_j^+ - r_{ij}}{C_j^+ - C_j^-} = \sum_{j=1}^{n} w_j \left(\sqrt{\frac{1}{8}\left(\left|\log_9 \frac{\rho_j^+}{\rho_j^i}\right|^2 + \left|\log_9 \frac{\sigma_j^+}{\sigma_j^i}\right|^2 + \left|\log_9 \frac{\tau_j^+}{\tau_j^i}\right|^2\right)} \Big/ \sqrt{\frac{1}{8}\left(\left|\log_9 \frac{\rho_j^+}{\rho_j^-}\right|^2 + \left|\log_9 \frac{\sigma_j^+}{\sigma_j^-}\right|^2 + \left|\log_9 \frac{\tau_j^+}{\tau_j^-}\right|^2\right)} \right), \tag{47}$$

$$IMIR_i = \max_j \left\{ w_j \frac{C_j^+ - r_{ij}}{C_j^+ - C_j^-} \right\} = \max_j \left\{ w_j \left(\sqrt{\frac{1}{8}\left(\left|\log_9 \frac{\rho_j^+}{\rho_j^i}\right|^2 + \left|\log_9 \frac{\sigma_j^+}{\sigma_j^i}\right|^2 + \left|\log_9 \frac{\tau_j^+}{\tau_j^i}\right|^2\right)} \Big/ \sqrt{\frac{1}{8}\left(\left|\log_9 \frac{\rho_j^+}{\rho_j^-}\right|^2 + \left|\log_9 \frac{\sigma_j^+}{\sigma_j^-}\right|^2 + \left|\log_9 \frac{\tau_j^+}{\tau_j^-}\right|^2\right)} \right) \right\}. \tag{48}$$

Desirable options should have values of these two measures that are as small as possible. Minimization of $IMGU_i$ implies maximization of the group intuitionistic multiplicative group utility, whereas to minimize $IMIR_i$ is to minimize individual regret.

- Step 4. Calculate the value of the intuitionistic multiplicative compromise measure IMC_i of option X_i:

$$IMC_i = v\frac{IMGU_i - IMGU^-}{IMGU^+ - IMGU^-} + (1-v)\frac{IMIR_i - IMIR^-}{IMIR^+ - IMIR^-}, \tag{49}$$

where $IMGU^- = \min_i IMGU_i$, $IMGU^+ = \max_i IMGU_i$, $IMIR^- = \min_i IMIR_i$, and $IMIR^+ = \max_i IMIR_i$. The coefficient denotes the trade-off the DM makes between group utility and individual regret over the conflicting criteria: if $0.5 < v < 1$, then the DM prefers to maximize the group utility; if $0 < v < 0.5$, then the DM pays more attention to minimization of individual regret; if $v = 0.5$, then the DM give equal weight to group utility and individual regret. The smaller the value of IMC_i is, the better the option should be.

- Step 5. Rank the options in decreasing order of the values of $IMGU_i$, $IMIR_i$ and IMC_i, producing three ranking lists.
- Step 6. Propose compromise solution(s) for the MCDM problem.

1. There is only one compromise solution $X^{(1)}$, which is the first when it is ranked in ascending order of the values of IMC_i, and satisfying the following two conditions:

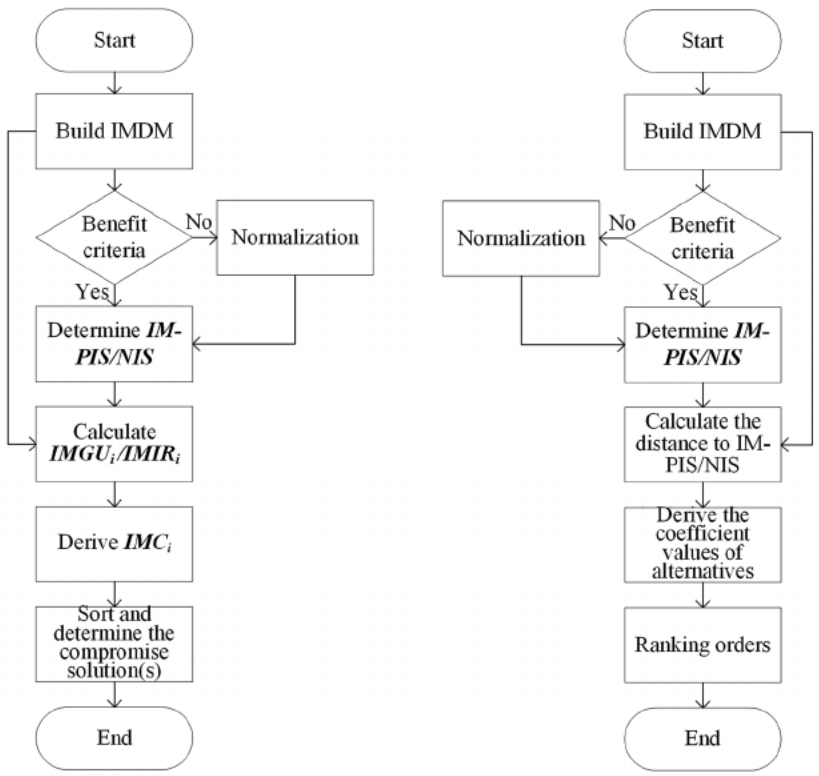
C1: $IMC(X^{(2)}) - IMC(X^{(1)}) \geq$, where $X^{(2)}$ is the second-ranked option in the list of values of the intuitionistic multiplicative compromise measure IMC_i; $= 1/(m-1)$ is the threshold, showing the acceptable advantage of the compromise option $X^{(1)}$, with m being the number of options. If $m \leq 4$, then we let $= 0.25$ in order to identify the unique compromise solution (Opricovic & Tzeng, 2004).

C2: The option $X^{(1)}$ is also the best in terms of the ranking of the intuitionistic multiplicative group utility measure $IMGU_i$ and the intuitionistic multiplicative individual regret measure $IMIR_i$. This condition is used to guarantee that the unique compromise solution is stable.

2. There is more than one compromise solution if the above two conditions are not fully satisfied. If the first condition does not hold, then $X^{(1)}, X^{(2)}, \cdots, X^{(M)}$ are compromise solutions, where $X^{(M)}$ meets the condition $IMC(X^{(M)}) - IMC(X^{(1)}) <$. In this case, all the options $X^{(1)}, X^{(2)}, \cdots, X^{(M)}$ are very close to each other in terms of the intuitionistic multiplicative compromise measure IMC_i. If only the second condition does not hold, then options $X^{(1)}$ and $X^{(2)}$ are the compromise solutions.

The flow charts of the distance-based IM-TOPSIS method and IM-VIKOR method are shown in Figure 6a,b, respectively.

In this section, we extend the TOPSIS method and VIKOR method into the IMS environment. The new parts between the proposed algorithms and their traditional forms can be respectively justified as follows: (a) With the new method, DMs can express their evaluations in IMSs rather than real numbers or other forms. In this sense, it has a stronger ability to depict MCDM problems with unbalanced or asymmetric situations;



(a) Distance-based IM-TOPSIS method (b) Distance-based IM-VIKOR method

Flow charts of the intuitionistic multiplicative technique for order preference by similarity to ideal solution and Vlsekriterijumska Optimizacija I Kompromisno Resenje methods

(b) To measure the distances in the TOPSIS method or VIKOR method (Step 3 in both of them), all the proposed distance measures for IMSs can be used. In this regard, DMs can choose an appropriate distance measure of IMSs according to their actual demand in real case.

5 | A CASE STUDY: HEALTHCARE MANAGEMENT AT WCH

In this section, a case study concerning hospital management is used to illustrate the application of the proposed MCDM methods as well as the distance measures between IMSs.

5.1 | Case introduction

WCH, located in the southwest of China, is not only the largest single-site hospital in the world but also a leading medical centre that treats various complicated and severe cases especially in the fields of living donor liver transplantation, severe acute pancreatitis, and clinical anaesthesia. It is a comprehensive hospital classed as Grade A Class Three in China. It was the first hospital in China to be accredited by the American Society of Pathologists. The WCH is also seeking to become a national centre for medical scientific research and technology innovation. The WCH takes a large number of patients from Sichuan province and the surrounding provinces and cities due to its advanced medical technology, equipment, and wealth of medical resources. According to the hospital's own statistics, there are about 18,000 outpatients and more than 6,000 patients waiting for a bed on any given day.

In recent years, the continuing development of the WCH has led to the emergence of some management problems. We know that beds are an extremely scarce hospital resource and that the main goal of hospital bed management is to maximize utilization of resources. Every year, the WCH admits nearly 200,000 inpatients, but at least 30% of this number (about 60,000 inpatients) do not need to be treated in the WCH. A small proportion of the inpatients suffers from severe disease or special value of clinical pathology, but most of them suffer from less serious conditions, such as toothache and minor trauma. These patients are taking up high-quality medical resources, which means that there are no beds for other patients needing hospital treatment; on the other hand, treating patients with simple diseases (appendicitis and gallbladder problems) allows the WCH to participate in development of medical technology and in scientific research. The conflict between the large number of patients seeking to benefit from the excellent resources of the WCH despite having no clinical need for them and the scarcity of medical resource has become a more and more serious problem for the WCH.

The WCH admission centre was established in 2011 to deal with this tricky problem. The centre has two main objectives: (a) to optimize the hospitalization admission process and select patients; (b) to improve the efficiency with which resources are used and ensure the fair and intelligent use of medical resources. The first stage of the inpatient admission process is that the patient is diagnosed in an outpatient department and then registered by the admission centre to reserve waiting outside the hospital. The system is designed to prioritize admission of patients who have the greatest need of a bed or the most serious clinical need. The triage of patients should resolve the conflict between the supply of hospital resources and the considerable demand of patients. Every day, there are many patients in admission centre. Ranking the patients is an onerous task, based on highly subjective criteria. We therefore need an efficient, scientifically based decision support technique to support the works and decide which patients should be prioritized, which patients should wait for admission, and which patients should not be admitted.

A review of the relevant literature showed that many scholars in other countries have considered this issue. Solans-Domènech et al. (2013) developed a priority scoring system for patients waiting for elective surgery, based on the condition of the patient and other situational factors. Rahimi et al. (2015) proposed a risk-based framework for patient prioritization that involved using soft FSs to help hospitals to select high-risk patients. This not only ensures that the risks associated with long waiting times and delayed treatment are borne by the patients least likely to be adversely affected but also increases the overall quality of medical service. Rahimi et al. (2016) presented a new approach to decisions where the information and risks are uncertain, which takes into account the opinions of surgery team members and patients. Ashour and Kremer (2016) developed a dynamic grouping and prioritization algorithm based on existing triage algorithms, which can not only reduce emergency room waiting times and the average length of stay but also reduce the observed risks of delayed treatment. It can satisfy the demands of emergency departments. There is, however, a significant difference between the medical systems for which these tools were designed and the Chinese medical system, which means that many frameworks or prioritization approaches are not suitable for use in the Chinese healthcare context. Hence, we further view the relevant literatures regarding China's hospital management. Zhang et al. (2016) used a hesitant fuzzy linguistic VIKOR method to solve the inpatient admission assessment problem at the WCH, based on the following criteria: "urgency," "need for hospitalization," and "value of clinical pathology." However, these authors did not consider other risks, limitations on the patient's activity or time spent on the waiting list. After consulting experts and staff in the admission centre and referring to the previous literatures, we developed an approach that would address these shortcomings, specifying the following admission criteria and corresponding weights.

- C_1 (Value of clinical pathology, w_1 = 0.15). Dealing with patients who suffer from difficult or complicated disease can promote the development of iatrology and have high value of clinical pathology.

- C_2 (Clinical features, w_2 = 0.32). Represents disease severity, pain and limitations on activity, etc.
- C_3 (Related risks, w_3 = 0.20) Delay could exacerbate a patient's condition or even lead to death, due to infection, comorbidity etc.
- C_4 (Urgency w_4 = 0.22) Patients who are seriously ill or have a condition requiring emergency treatment are classified as having high urgency.
- C_5 (Patient information, w_5 = 0.11). Represents medical insurance type, time on waiting list, VIP status (yes or no) etc.

For simplicity of presentation, we will use five patients $X = \{X_1, X_2, X_3, X_4, X_5\}$ to illustrate the proposed method. We invited hospital experts (such as the outpatient doctor and staff in the admission centre) to diagnose these patients and provide their assessments in IMSs. For example, the expert may consider that the patient value (based on admission criteria) is between extreme acceptance and very strong acceptance, so the patient value can be described as a number between 7 and 9. The other values can be deduced in analogous fashion, and then we can derive the intuitionistic multiplicative judgement matrix as shown in Table 2.

5.2 | Applications of the methods

As we know, MCDM methods, such as TOPSIS and VIKOR, can be applied to rank options, and they are all based on a measure of distance from some reference (ideal) points or options. Below, we use the distance-based IM-TOPSIS and IM-VIKOR methods to solve this inpatient admission assessment problem.

5.2.1 | Solving the problem with the distance-based IM-TOPSIS method

Step 1. Derive the IMDM from the values given by the outpatient doctor and the staff in admission service centre for clinical and administrative criteria, respectively (see Table 1).

Step 2. Determine IMPIS X^+ and IMNIS X^-. In our example, the criteria C_1, C_2, C_4 and C_5 are benefit criteria, whereas C_3 is the cost criterion. Thus, we get the following IMPIS A and IMNIS B.

$$X^+ = ((7,1/8,8/7),(8,1/9,9/8),(1,1/8,8),(1/3,2,3/2),(7,1/8,8/7)),$$

$$X^- = ((1,1/4,4),(3,1/5,5/3),(7,1/8,8/7),(1/8,6,4/3),(3,1/4,4/3)).$$

Step 3. The distance between each option and the IMPIS (denoted as PIS-D in Tables 3–5) and the distance between each option and the IMNIS (denoted as NIS-D in Tables 3–5) are given by Equation (11) and (16), respectively. The results are shown in Tables 3 and 4.

Step 4. We can obtain different relative closeness coefficients for different values of parameter λ from Equation (46); the results are shown in Tables 6–8.

Step 5. The relative closeness coefficient values are sorted in descending order, as shown in the final rows of Tables 6–8. Note that in these tables, the notation "31524" represents the following ranking order:X_3 X_1 X_5 X_2 X_4

Step 6. Rank all patients in ascending order of the values of $IMGU_i$, $IMIR_i$, and IMC_i and propose compromise solution(s) for the MCDM problem, based on which the ranking list can be derived. The final rankings with respect to various parameters are shown in Tables 6–8.

5.2.2 | Solving the problem with the distance-based IM-VIKOR method

- Steps 1–2. As described in 5.2.1.

The intuitionistic multiplicative judgement matrix

Patients	1	2	3	4	5
X_1	(6,1/8,4/3)	(3,1/5,5/3)	(4,1/7,7/4)	(1/3,2,3/2)	(5,1/6,6/5)
X_2	(1,1/4,4)	(8,1/9,9/8)	(3,1/5,5/3)	(1/7,2,7/2)	(3,1/4,4/3)
X_3	(3,1/6,2)	(4,1/5,5/4)	(1,1/8,8)	(1/5,5,1)	(7,1/8,8/7)
X_4	(7,1/8,8/7)	(3,1/6,2)	(5,1/6,6/5)	(1/8,6,4/3)	(6,1/7,7/6)
X_5	(4,1/6,3/2)	(7,1/8,8/7)	(7,1/8,8/7)	(1/6,4,3/2)	(5,1/9,9/5)

Distances between each option and the IMPIS or IMNIS for different values of λ. (Based on generalized weighted distances)

	= 1		= 2		= 3		= 4		= 5		= 6		= 7	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.1543	0.1544	0.1867	0.188	0.2125	0.2149	0.2317	0.2372	0.2462	0.256	0.2573	0.272	0.2662	0.2856
X_2	0.2014	0.165	0.2334	0.1732	0.257	0.1824	0.2768	0.1899	0.2936	0.1959	0.308	0.2007	0.3202	0.2045
X_3	0.1253	0.1918	0.1359	0.2306	0.144	0.2704	0.1504	0.3006	0.1557	0.3229	0.1603	0.3395	0.1642	0.3523
X_4	0.2166	0.1124	0.239	0.163	0.263	0.2078	0.2832	0.2429	0.2998	0.2699	0.3135	0.2911	0.3249	0.3079
X_5	0.1635	0.1496	0.2185	0.1586	0.2646	0.1739	0.2977	0.1888	0.3214	0.2019	0.3388	0.2133	0.3519	0.223
	= 8		= 9		= 10		= 20		= 50		= 100		= 400	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.2735	0.2972	0.2795	0.307	0.2845	0.3154	0.3105	0.3582	0.3304	0.3871	0.338	0.3973	0.3439	0.4051
X_2	0.3306	0.2077	0.3395	0.2104	0.3472	0.2128	0.3894	0.2262	0.4205	0.2392	0.4315	0.2445	0.44	0.2486
X_3	0.1676	0.3623	0.1706	0.3704	0.1733	0.377	0.188	0.4086	0.1996	0.4288	0.204	0.4357	0.2074	0.441
X_4	0.3344	0.3215	0.3424	0.3328	0.3493	0.3422	0.3855	0.389	0.4123	0.4205	0.4219	0.4315	0.4292	0.44
X_5	0.3621	0.2313	0.3703	0.2385	0.377	0.2447	0.4086	0.2772	0.4288	0.2995	0.4357	0.3074	0.441	0.3134

Abbreviations: IMNIS, intuitionistic multiplicative negative ideal solution; IMPIS, intuitionistic multiplicative positive-ideal solution.

Distances between each option and the IMPIS or IMNIS for different values of λ. (Based on generalized weighted Hausdorff distances)

	= 1		= 2		= 3		= 4		= 5		= 6		= 7	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.1543	0.1544	0.2017	0.2084	0.2282	0.2421	0.246	0.2669	0.2591	0.286	0.2693	0.301	0.2777	0.313
X_2	0.2014	0.165	0.2592	0.1927	0.2904	0.2035	0.312	0.2096	0.3281	0.2137	0.3406	0.2167	0.3506	0.2191
X_3	0.1253	0.1918	0.152	0.2378	0.1629	0.2753	0.1693	0.3033	0.1737	0.3243	0.177	0.3402	0.1796	0.3526
X_4	0.2166	0.1124	0.259	0.184	0.2848	0.2385	0.3045	0.2765	0.3207	0.3033	0.334	0.3229	0.345	0.3377
X_5	0.1635	0.1496	0.2204	0.1801	0.2653	0.1989	0.2979	0.2133	0.3215	0.2252	0.3388	0.2351	0.3519	0.2434
	= 8		= 9		= 10		= 20		= 50		= 100		= 400	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.2845	0.3228	0.2902	0.3309	0.295	0.3376	0.3191	0.3708	0.3349	0.3926	0.3403	0.4001	0.3142	0.3445
X_2	0.3589	0.2211	0.3657	0.2228	0.3716	0.2243	0.4031	0.2335	0.4263	0.2426	0.4345	0.2462	0.4407	0.4407
X_3	0.1818	0.3625	0.1837	0.3705	0.1854	0.3771	0.1946	0.4086	0.2023	0.4288	0.2054	0.4357	0.2077	0.2077
X_4	0.354	0.3493	0.3615	0.3587	0.3678	0.3663	0.3983	0.4027	0.418	0.4263	0.4248	0.4345	0.3648	0.43
X_5	0.3621	0.2504	0.3703	0.2564	0.377	0.2614	0.4086	0.2869	0.4288	0.3037	0.4357	0.3095	0.441	0.441

Abbreviations: IMNIS, intuitionistic multiplicative negative ideal solution; IMPIS, intuitionistic multiplicative positive-ideal solution.

The values of $IMGU_i$ and $IMIR_i$

	1	2	3	4	5
$IMGU_i$	0.5615	0.7031	0.4530	0.6550	0.5149
$IMIR_i$	0.3013	0.3278	0.228	0.3009	0.2487

Abbreviations: IMGU, intuitionistic multiplicative group utility measure; IMIR, intuitionistic multiplicative individual regret measure.

Patient rankings for different values of parameter v

	1	2	3	4	5	Ranking orders
0.0	0.7345	1	0	0.7305	0.2074	35412
0.1	0.7044	1	0	0.7382	0.2114	35142
0.2	0.6743	1	0	0.7459	0.2154	35142
0.3	0.6443	1	0	0.7536	0.2194	35142
0.4	0.6142	1	0	0.7613	0.2234	35142
0.5	0.5841	1	0	0.7691	0.2275	35142
0.6	0.5541	1	0	0.7768	0.2315	35142
0.7	0.524	1	0	0.7845	0.2355	35142
0.8	0.494	1	0	0.7922	0.2395	35142
0.9	0.4639	1	0	0.8	0.2435	35142

Abbreviation: IMC, intuitionistic multiplicative compromise measure.

- Step 3. Calculate the values of $IMGU_i$ and $IMIR_i$ for all patients with Equation (47) and (48), respectively. The results are shown in Table 9.
- Step 4. Calculate the values of IMC_i for all patients using Equation (49). The results are shown in Table 10.
- Steps 5 and 6. Rank all patients in ascending order of the values of $IMGU_i$, $IMIR_i$, and IMC_i and propose the compromise solution(s) for the MCDM problem. The final rankings for different parameter values are shown in Table 10.

5.3 | Analyses of results

We can derive some interesting conclusions from Tables 7 9.

1. In the case of the IM-TOPSIS method, we use generalized weighted distance, generalized weighted Hausdorff distance, and generalized hybrid weighted distance to calculate the corresponding distances, respectively. These different distance measures produce different relative closeness coefficient results but produced similar ranking results, as listed in Tables 7 9. When = 2, the ranking result is $_3$ $_1$ $_2$ $_5$ $_4$ or $_3$ $_1$ $_5$ $_2$ $_4$. However, if 3, the rankings are all $_3$ $_1$ $_4$ $_5$ $_2$. In such cases, ranking result does not change as the value of increase, regardless of whether generalized weighted distance, generalized weighted Hausdorff distance, or generalized hybrid weighted distance are used. In addition, the results derived from these three distance measures become more stable as parameter increases. This result is consistent with the conclusion presented in Section 3.

2. The variability of the values of the relative closeness coefficients differs with respect to . For $_2$ calculated from the above three distance measures, the values of the relative closeness coefficients decrease as parameter increases. For $_1$ and $_4$, the variability in relative closeness coefficients follows a different pattern from that applying to $_2$, which always increases as parameter increases. For $_3$ and $_5$, however, the variation tendencies of the values of the relative closeness coefficients are not always the same.

As mentioned above, using the IM-TOPSIS method, the relative closeness coefficient slightly varies with the selected distance measures for IMSs, including the generalized weighted distance, generalized weighted Hausdorff distance, and generalized hybrid weighted distance. However, when we consider the values of the relative closeness coefficient from the perspective of final results, we find that they converge on an instant number, such as 0.5409, 0.3610, 0.6801, 0.5062, and 0.4154 in Table 7. Therefore, in real case, the DMs only need to compute the generalized hybrid distance measures of IMS with = 1 or 2. In addition, as an integrated form of the Minkowski distance and Hausdorff distance, the generalized hybrid weighted distance measure for IMS has stronger robustness than the generalized weighted form.

Using the IM-VIKOR method, we can calculate distances using the Euclidean distance measure and obtain some meaningful results. When is changed, the rankings also change. For example, when = 0, the rank order is $_3$ $_5$ $_4$ $_1$ $_2$. However, if $\neq 0$, the rank order becomes $_3$ $_5$ $_1$ $_4$ $_2$ based on which, the compromise solution(s) can be derived as $_3$ and $_5$. The DMs can determine the value of , and we should note

Patient rankings for different values of λ (Based on generalized weighted Hausdorff distances)

	= 1	= 2	= 3	= 4	= 5	= 6	= 7	= 8	= 9	= 10	= 20	= 50	= 100	= 400
X_1	0.5002	0.5082	0.5148	0.5204	0.5247	0.5278	0.5299	0.5315	0.5328	0.5337	0.5375	0.5397	0.5404	0.5409
X_2	0.4503	0.4264	0.4120	0.4018	0.3944	0.3888	0.3846	0.3812	0.3786	0.3764	0.3668	0.3627	0.3617	0.3611
X_3	0.6049	0.6101	0.6283	0.6418	0.6512	0.6578	0.6625	0.6660	0.6685	0.6704	0.6774	0.6794	0.6796	0.6798
X_4	0.3416	0.4153	0.4558	0.4759	0.4861	0.4916	0.4947	0.4967	0.4981	0.4990	0.5027	0.5049	0.5056	0.5061
X_5	0.4778	0.4497	0.4285	0.4173	0.4119	0.4097	0.4089	0.4088	0.4091	0.4095	0.4125	0.4146	0.4153	0.4159
Ranking orders	31524	31524	31452	31452	31452	31452	31452	31452	31452	31452	31452	31452	31452	31452

that is the weight of the decision-making strategy the majority of criteria , or saying the maximum group utility (Opricovic & Tzeng, 2004), based on the DMs' preference in the given MCDM problem. The VIKOR method is more useful and flexible in our real-life situation because it allows the DMs to try to both maximize group utility and minimize individual regret of other DMs ( ).

Analysis of our example case shows that both the IM-TOPSIS method and the IM-VIKOR method can handle the inpatient admission assessment problem effectively. The results explain the difference between these two MCDM methods. It also shows that the IM-TOPSIS method and the IM-VIKOR method produce different rankings when dealing with the inpatient admission assessment problem. Using the IM-TOPSIS method, the best ranking is $_{3}$ $_{1}$ $_{4}$ $_{5}$ $_{2}$, whereas using the IM-VIKOR method, the compromise ranking result is $_{3}$ $_{5}$ $_{1}$ $_{4}$ $_{2}$. In real life, DMs can choose any one of them, the IM-TOPSIS or the IM-VIKOR method, according to the characteristics of the actual inpatient admission assessment problem.

6 | CONCLUSIONS

As we know, distance measures are a widely used to deal with MCDM problems. In this paper, we have developed some distance measures for IMSs, including the generalized Hausdorff distance, weighted distance measures, ordered weighted distance measures, and continuous weighted distance measures. On this basis, we have proposed a distance-based IM-TOPSIS method and a distance-based IM-VIKOR method, which can be used to handle MCDM problems with IMSs. Compared with the IM-VIKOR method, the IM-TOPSIS method produces more stable and accurate results when used to handle an admission sequencing problem. We have used a practical example in the field of hospital management to illustrate these proposed approaches. As to the studied case of inpatient admission service centre in WCH, we have proposed the evaluation indices, or saying criteria, for the admission sequencing model. It can provide some references for the hospital managers in China.

In the future, we may investigate the differences between generalized continue weighted distance and generalized ordered weighted distance as applied to IMSs. The issue of how to handle MCDM problems when criterion weights are unknown or incomplete remains open. We will try to develop more measures of the distance between IMSs, for example, projection-based distance measures and psychological distance measures. Furthermore, we may apply our distance measures to other decision-making problems.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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APPENDIX

Proof of Property .

Proof. (1) Necessary: If $A = B$, then the equality $\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\} = 0$ holds. On this basis, we have $\left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)^{1/\lambda} = 0$ Thus, the equality $d_{gwhaud}(A,B) = 0$ holds.

Sufficient: Because $\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\} \geq 0$, we have $\left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)^{1/\lambda} \geq 0$. If $d_{gwhaud}(A,B) = \left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)^{1/\lambda} = 0$, then $\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\} = 0$. On this basis, we can show that $\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda = \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda = \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda = 0$ holds, namely, $A = B$.

(2) It is apparent that (2) holds, and thus the proof is omitted here.

(3) Because

$$d_{ghaud}(A,B) = \left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)^{1/\lambda}$$
$$= \left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{|\log_9 \rho_A(x_i) - \log_9 \rho_B(x_i)|^\lambda, |\log_9 \sigma_A(x_i) - \log_9 \sigma_B(x_i)|^\lambda, |\log_9 \tau_A(x_i) - \log_9 \tau_B(x_i)|^\lambda\right\}\right)^{1/\lambda}$$

for three IMSs $A \subseteq B \subseteq C$, we have $|\log_9 \rho_A(x_i) - \log_9 \rho_C(x_i)| \geq |\log_9 \rho_A(x_i) - \log_9 \rho_B(x_i)|$, $|\log_9 \sigma_A(x_i) - \log_9 \sigma_C(x_i)| \geq |\log_9 \sigma_A(x_i) - \log_9 \sigma_B(x_i)|$, and $|\log_9 \tau_A(x_i) - \log_9 \tau_C(x_i)| \geq |\log_9 \tau_A(x_i) - \log_9 \tau_B(x_i)|$. It is easy to obtain that $\left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{|\log_9 \rho_A(x_i) - \log_9 \rho_C(x_i)|^\lambda, |\log_9 \sigma_A(x_i) - \log_9 \sigma_C(x_i)|^\lambda, |\log_9 \tau_A(x_i) - \log_9 \tau_C(x_i)|^\lambda\right\}\right)^{1/\lambda} \geq \left(\frac{1}{2^\lambda}\sum_{i=1}^n \max\left\{|\log_9 \rho_A(x_i) - \log_9 \rho_B(x_i)|^\lambda, |\log_9 \sigma_A(x_i) - \log_9 \sigma_B(x_i)|^\lambda, |\log_9 \tau_A(x_i) - \log_9 \tau_B(x_i)|^\lambda\right\}\right)^{1/\lambda}$. Thus, we have $d_{ghaud}(A,C) \geq d_{ghaud}(A,B)$. Similarly, we can prove $d_{ghaud}(A,C) \geq d_{ghaud}(B,C)$ holds. The proof of Property is completed.

Proof of Property .

Proof. (1) Necessary: If $A = B$, then $\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda = 0$. It is easy to get $\left(\sum_{i=1}^n \frac{1}{2^{\lambda+2}}\left(\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda + \frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)\right)^{1/\lambda} = 0$. Thus, $d_{ghd}(A,B) = 0$.

(Sufficient:)

Because $d_{ghd}(A,B) = \left(\sum_{i=1}^n \left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right) + \frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\}\right)\right)^{1/\lambda} \geq 0$, we have $\sum_{i=1}^n \left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda + \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right)\right) = 0$ and $\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda, \left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda\right\} = 0$. It is easy to show that $\left|\log_9 \frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^\lambda = 0$, $\left|\log_9 \frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^\lambda = 0$, and $\left|\log_9 \frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^\lambda = 0$. Thus, $A = B$ holds.

(2) It is obviously that (2) holds, and therefore the proof is omitted here.

(3) Because

$$d_{ghd}(A,B)=\left(\sum_{i=1}^{n}\left(\frac{1}{2^{\lambda+2}}\left(\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda}+\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right)+\frac{1}{2^{\lambda+1}}\max\left\{\left|\log_9\frac{\rho_A(x_i)}{\rho_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\sigma_A(x_i)}{\sigma_B(x_i)}\right|^{\lambda},\left|\log_9\frac{\tau_A(x_i)}{\tau_B(x_i)}\right|^{\lambda}\right\}\right)\right)^{1/\lambda}$$
$$=\left(\sum_{i=1}^{n}\left(\frac{1}{2^{\lambda+2}}\left(|\log_9\rho_A(x_i)-\log_9\rho_B(x_i)|^{\lambda}+|\log_9\sigma_A(x_i)-\log_9\sigma_B(x_i)|^{\lambda}+|\log_9\tau_A(x_i)-\log_9\tau_B(x_i)|^{\lambda}\right)+\right.\right.$$
$$\left.\left.\frac{1}{2^{\lambda+1}}\max\left\{|\log_9\rho_A(x_i)-\log_9\rho_B(x_i)|^{\lambda},|\log_9\sigma_A(x_i)-\log_9\sigma_B(x_i)|^{\lambda},|\log_9\tau_A(x_i)-\log_9\tau_B(x_i)|^{\lambda}\right\}\right)\right)^{1/\lambda}$$

for three IMSs $A \subseteq B \subseteq C$, we have $|\log_9\rho_A(x_i) - \log_9\rho_C(x_i)| \geq |\log_9\rho_A(x_i) - \log_9\rho_B(x_i)|$, $|\log_9\sigma_A(x_i) - \log_9\sigma_C(x_i)| \geq |\log_9\sigma_A(x_i) - \log_9\sigma_B(x_i)|$, and $|\log_9\tau_A(x_i) - \log_9\tau_C(x_i)| \geq |\log_9\tau_A(x_i) - \log_9\tau_B(x_i)|$. It is easy to get

$$\sum_{i=1}^{n}\left(\frac{1}{2^{\lambda+2}}\left(|\log_9\rho_A(x_i)-\log_9\rho_C(x_i)|^{\lambda}+|\log_9\sigma_A(x_i)-\log_9\sigma_C(x_i)|^{\lambda}+|\log_9\tau_A(x_i)-\log_9\tau_C(x_i)|^{\lambda}\right)\right)\geq$$
$$\sum_{i=1}^{n}\left(\frac{1}{2^{\lambda+2}}\left(|\log_9\rho_A(x_i)-\log_9\rho_B(x_i)|^{\lambda}+|\log_9\sigma_A(x_i)-\log_9\sigma_B(x_i)|^{\lambda}+|\log_9\tau_A(x_i)-\log_9\tau_B(x_i)|^{\lambda}\right)\right)$$

and

$$\frac{1}{2^{\lambda+1}}\max\left\{|\log_9\rho_A(x_i)-\log_9\rho_C(x_i)|^{\lambda},|\log_9\sigma_A(x_i)-\log_9\sigma_C(x_i)|^{\lambda},|\log_9\tau_A(x_i)-\log_9\tau_C(x_i)|^{\lambda}\right\}\geq$$
$$\frac{1}{2^{\lambda+1}}\max\left\{|\log_9\rho_A(x_i)-\log_9\rho_B(x_i)|^{\lambda},|\log_9\sigma_A(x_i)-\log_9\sigma_B(x_i)|^{\lambda},|\log_9\tau_A(x_i)-\log_9\tau_B(x_i)|^{\lambda}\right\}.$$

Thus, we have $d_{ghd}(A,C) \geq d_{ghd}(A,B)$. Similarly, we can prove that $d_{ghd}(A,C) \geq d_{ghd}(B,C)$ holds. This completes the proof of Property .

Distances between each option and the IMPIS or IMNIS for different values of λ. (Based on generalized hybrid weighted distances)

	= 1		= 2		= 3		= 4		= 5		= 6		= 7	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.1543	0.1544	0.1944	0.1985	0.2207	0.2293	0.2392	0.2534	0.2529	0.2727	0.2637	0.2883	0.2723	0.3012
X_2	0.2014	0.165	0.2466	0.1832	0.2747	0.1935	0.2959	0.2005	0.3127	0.2056	0.3263	0.2094	0.3375	0.2126
X_3	0.1253	0.1918	0.1442	0.2343	0.154	0.2728	0.1607	0.302	0.1656	0.3236	0.1696	0.3399	0.1729	0.3524
X_4	0.2166	0.1124	0.2492	0.1738	0.2743	0.2242	0.2944	0.2613	0.3109	0.2885	0.3246	0.309	0.3358	0.3248
X_5	0.1635	0.1496	0.2195	0.1697	0.2649	0.1872	0.2978	0.2022	0.3214	0.2148	0.3388	0.2255	0.3519	0.2345
	= 8		= 9		= 10		= 20		= 50		= 100		= 400	
	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D	PIS-D	NIS-D
X_1	0.2794	0.3118	0.2852	0.3207	0.2902	0.3282	0.3154	0.3655	0.333	0.3903	0.3394	0.3989	0.3442	0.4055
X_2	0.3467	0.2151	0.3545	0.2173	0.3612	0.2192	0.3973	0.2304	0.4239	0.2412	0.4332	0.2455	0.4404	0.2489
X_3	0.1757	0.3624	0.1781	0.3704	0.1802	0.3771	0.1919	0.4086	0.2012	0.4288	0.2048	0.4357	0.2076	0.441
X_4	0.3452	0.3374	0.353	0.3476	0.3596	0.3561	0.3929	0.397	0.4156	0.4239	0.4236	0.4332	0.4297	0.4404
X_5	0.3621	0.2422	0.3703	0.2487	0.377	0.2543	0.4086	0.2828	0.4288	0.302	0.4357	0.3086	0.441	0.3137

Abbreviations: IMNIS, intuitionistic multiplicative negative ideal solution; IMPIS, intuitionistic multiplicative positive-ideal solution.

Patient rankings for different values of λ (Based on generalized weighted distances)

	= 1	= 2	= 3	= 4	= 5	= 6	= 7	= 8	= 9	= 10	= 20	= 50	= 100	= 400
X_1	0.5002	0.5017	0.5028	0.5059	0.5098	0.5139	0.5176	0.5208	0.5234	0.5258	0.5357	0.5395	0.5403	0.5409
X_2	0.4503	0.4260	0.4151	0.4069	0.4002	0.3945	0.3897	0.3858	0.3826	0.3800	0.3674	0.3626	0.3617	0.3610
X_3	0.6049	0.6292	0.6525	0.6665	0.6747	0.6793	0.6821	0.6837	0.6847	0.6851	0.6849	0.6824	0.6811	0.6801
X_4	0.3416	0.4055	0.4414	0.4617	0.4738	0.4815	0.4866	0.4902	0.4929	0.4949	0.5023	0.5049	0.5056	0.5062
X_5	0.4778	0.4206	0.3966	0.3881	0.3858	0.3863	0.3879	0.3898	0.3918	0.3936	0.4042	0.4112	0.4137	0.4154
Ranking orders	31524	31254	31425	31425	31425	31425	31425	31452	31452	31452	31452	31452	31452	31452

Patient rankings for different values of $\lambda_{.}$ (Based on generalized hybrid weighted distances)

	= 1	= 2	= 3	= 4	= 5	= 6	= 7	= 8	= 9	= 10	= 20	= 50	= 100	= 400
X_1	0.5002	0.5052	0.5096	0.5144	0.5188	0.5223	0.5252	0.5274	0.5293	0.5307	0.5368	0.5396	0.5403	0.5409
X_2	0.4503	0.4262	0.4133	0.4039	0.3967	0.3909	0.3865	0.3829	0.3800	0.3777	0.3671	0.3627	0.3617	0.3611
X_3	0.6049	0.6190	0.6392	0.6527	0.6615	0.6671	0.6709	0.6735	0.6753	0.6767	0.6804	0.6806	0.6802	0.6799
X_4	0.3416	0.4109	0.4497	0.4702	0.4813	0.4877	0.4917	0.4943	0.4961	0.4976	0.5026	0.5049	0.5056	0.5061
X_5	0.4778	0.4360	0.4141	0.4044	0.4006	0.3996	0.3999	0.4008	0.4018	0.4028	0.4090	0.4132	0.4146	0.4157
Ranking orders	31524	31524	31425	31425	31452	31452	31452	31452	31452	31452	31452	31452	31452	31452