Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT®

Information Sciences 158 (2004) 263–275

INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL

www.elsevier.com/locate/ins

Fuzzy linear programming technique for multiattribute group decision making in fuzzy environments

Deng-Feng Li *, Jian-Bo Yang

Manchester School of Management, University of Manchester Institute of Science and Technology, P.O. Box 88, Manchester M60 1QD, UK

Abstract

The aim of this paper is to develop a linear programming technique for multidimensional analysis of preferences in multiattribute group decision making under fuzzy environments. Fuzziness is inherent in decision data and group decision making processes, and linguistic variables are well suited to assessing an alternative on qualitative attributes using fuzzy ratings. A crisp decision matrix can be converted into a fuzzy decision matrix once the decision makers' fuzzy ratings have been extracted. In this paper, we first define group consistency and inconsistency indices based on preferences to alternatives given by decision makers and construct a linear programming decision model based on the distance of each alternative to a fuzzy positive ideal solution which is unknown. Then the fuzzy positive ideal solution and the weights of attributes are estimated using the new decision model based on the group consistency and inconsistency indices. Finally, the distance of each alternative to the fuzzy positive ideal solution is calculated to determine the ranking order of all alternatives. A numerical example is examined to demonstrate the implementation process of the technique.

Keywords: Fuzzy multiattribute group decision making; Linear programming technique for multidimensional analysis of preference; Linguistic variable; Fuzzy number; Linear programming

* Corresponding author.
E-mail addresses: d.li@umist.ac.uk, dengfengli@sina.com (D.-F. Li).

0020-0255/$ - see front matter
doi:10.1016/j.ins.2003.08.007

1. Introduction

Multiple attribute decision making (MADM) problems are widespread in real life decision situations [3,4,8–11]. A MADM problem is to find a best compromise solution from all feasible alternatives assessed on multiple attributes, both quantitative and qualitative. Suppose the decision makers have to choose one of or rank n alternatives: $A_1, A_2, \ldots, A_n$ based on m attributes: $C_1, C_2, \ldots, C_m$. Denote an alternative set by $A = \{A_1, A_2, \ldots, A_n\}$ and an attribute set by $C = \{C_1, C_2, \ldots, C_m\}$. Let x_{ij} be the score of alternative A_i ($i = 1, 2, \ldots, n$) on attribute C_j ($j = 1, 2, \ldots, m$), and suppose ω_j is the relative weight of attribute C_j, where $\omega_j \geqslant 0$ ($j = 1, 2, \ldots, m$) and $\sum_{j=1}^{m} \omega_j = 1$. Denote a weight vector by $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^{\mathrm{T}}$. A MADM problem can then be expressed as the following decision matrix:

$$D = (x_{ij})_{n\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix} \end{array}$$

The above MADM problem can be dealt with using several existing methods such as the technique for order preference by similarity to ideal solution (TOPSIS) developed by Hwang and Yoon [8], the linear programming technique for multidimensional analysis of preference (LINMAP) developed by Srinivasan and Shocker [13] and the nonmetric multidimensional scaling (MDS). The TOPSIS and LINMAP methods are two well-known MADM methods, though they require different types of information. In the TOPSIS method, the decision matrix $\boldsymbol{D}$ and the weight vector $\boldsymbol{\omega}$ are given as crisp values a priori; a positive ideal solution (PIS) and a negative ideal solution (NIS) are generated from $\boldsymbol{D}$ directly; the best compromise alternative is then defined as the one that has the shortest distance to PIS and the farthest from NIS. However, in the LINMAP method, the weight vector $\boldsymbol{\omega}$ and the positive ideal solution are unknown a priori. The LINMAP method is based on pairwise comparisons of alternatives given by decision makers and generates the best compromise alternative as the solution that has the shortest distance to the positive ideal solution.

Under many conditions, however, crisp data are inadequate or insufficient to model real-life decision problems [1,2,5,6,12]. Indeed, human judgments are vague or fuzzy in nature and as such it may not be appropriate to represent them by accurate numerical values. A more realistic approach could be to use linguistic variables to model human judgments [3–5,9,12]. In this paper, we further extend the LINMAP method to develop a new methodology for solving multiattribute group decision making problems in a fuzzy environment [6,7]. In this methodology, linguistic variables are used to capture fuzziness in decision information and group decision making processes by means of a fuzzy decision matrix. A new vertex method is proposed to calculate the distance between triangular fuzzy scores. Group consistency and inconsistency indices are defined on the basis of preferences between alternatives given by decision makers. Each alternative is assessed on the basis of its distance to a fuzzy positive ideal solution (FPIS) which is unknown. The fuzzy positive ideal solution and the weights of attributes are then estimated using a new linear programming model based upon the group consistency and inconsistency indices defined. Finally, the distance of each alternative to FPIS can be calculated to determine the ranking order of all alternatives. The lower value of the distance for an alternative indicates that the alternative is closer to FPIS.

The paper is organized as follows. In next section, the basic definitions and notations of fuzzy numbers and linguistic variables are defined as well as the fuzzy distance formula and the normalization method. Section 3 defines group consistency and inconsistency indices between preferences of alternatives given by decision makers and the results of the decision making model, and presents a new linear programming model to solve such multiattribute group decision making problems. The developed method is also illustrated with a real life example in Section 4. The paper is concluded in Section 5.

2. Basic concepts and definitions

2.1. Notations of fuzzy numbers

A fuzzy number $\tilde{m}$ is a special fuzzy subset on the set R of real numbers which satisfy the following conditions [9,14]:

(1) There exists a $x_0 \in R$ so that the degree of its membership $u_{\tilde{m}}(x_0) = 1$.
(2) Membership function $u_{\tilde{m}}(x)$ is left and right continuous.

Generally, a fuzzy number $\tilde{m}$ can be written as

$$u_{\tilde{m}}(x) = \begin{cases} L(x) & (l \leqslant x \leqslant m) \\ R(x) & (m \leqslant x \leqslant r) \end{cases}$$

where $L(x)$ is an increasing function of $x \in [l, m]$ and right continuous, $0 \leqslant L(x) \leqslant 1$; $R(x)$ is a decreasing function of $x \in [m, r]$ and left continuous, $0 \leqslant R(x) \leqslant 1$. m is called a mode of $\tilde{m}$, and l and r are called the low and upper limits of $\tilde{m}$, respectively.

For the sake of simplicity and without loss of generality, assume that all fuzzy numbers are triangular fuzzy numbers throughout the paper unless otherwise stated.

Let $\tilde{m} = (l, m, r)$ be a triangular fuzzy number, where the membership function $\mu_{\tilde{m}}$ of $\tilde{m}$ is given by

$$\mu_{\tilde{m}}(x) = \begin{cases} \dfrac{x-l}{m-l} & (l \leqslant x \leqslant m) \\ \dfrac{r-x}{r-m} & (m \leqslant x \leqslant r) \end{cases}$$

It is easy to see that a triangular fuzzy number $\tilde{m} = (l, m, r)$ is reduced to a real number m if $l = m = r$. Conversely, a real number m can be written as a triangular fuzzy number $\tilde{m} = (m, m, m)$.

2.2. Linguistic variable

A linguistic variable is a variable whose values are linguistic terms.

The concept of linguistic variable is very useful in situations where decision problems are too complex or too ill-defined to be described properly using conventional quantitative expressions. For example, the performance ratings of alternatives on qualitative attributes could be expressed using linguistic variable such as very poor, poor, fair, good, very good, etc. Such linguistic values can be represented using positive triangular fuzzy numbers. For example, "poor" and "very good" can be represented by positive triangular fuzzy number (0.2, 0.3, 0.4) and (0.8, 0.9, 1.0), respectively.

2.3. Distance between two triangular fuzzy numbers

Let $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ be two triangular fuzzy numbers. Then the vertex method is defined to calculate the distance between them as follows:

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{3}[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2]} \tag{1}$$

Eq. (1) is an effective and simple method to calculate the distance between two triangular fuzzy numbers [2,9].

Note that if both $\tilde{m}$ and $\tilde{n}$ are real numbers then the distance measurement $d(\tilde{m}, \tilde{n})$ is identical to the Euclidean distance. In fact, suppose both $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ are two real numbers and let $m_1 = m_2 = m_3 = m$ and $n_1 = n_2 = n_3 = n$. The distance measurement $d(\tilde{m}, \tilde{n})$ can be calculated as follows:

$$\begin{aligned} d(\tilde{m}, \tilde{n}) &= \sqrt{\frac{1}{3}[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2]} \\ &= \sqrt{\frac{1}{3}[(m - n)^2 + (m - n)^2 + (m - n)^2]} = \sqrt{(m - n)^2} \end{aligned}$$

Furthermore, we can see that two triangular fuzzy numbers $\tilde{m}$ and $\tilde{n}$ are identical if and only if the distance measurement $d(\tilde{m}, \tilde{n}) = 0$.

2.4. The normalization method

In this paper, we discuss the following fuzzy multiattribute group decision making problem. Suppose there exist n possible alternatives $A_1, A_2, \ldots, A_n$ from which P decision makers P_p $(p = 1, 2, \ldots, P)$ have to choose on the basis of m attributes $C_1, C_2, \ldots, C_m$. Suppose the rating of alternative A_i $(i = 1, 2, \ldots, n)$ on attribute C_j $(j = 1, 2, \ldots, m)$ given by decision maker P_p $(p = 1, 2, \ldots, P)$ is $\tilde{x}_{ij}^p = (a_{ij}^p, b_{ij}^p, c_{ij}^p)$. Hence, a multiattribute group decision making problem can be concisely expressed in matrix format as follows:

$$\widetilde{D}^p = (\tilde{x}_{ij}^p)_{n\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{pmatrix} \tilde{x}_{11}^p & \tilde{x}_{12}^p & \cdots & \tilde{x}_{1m}^p \\ \tilde{x}_{21}^p & \tilde{x}_{22}^p & \cdots & \tilde{x}_{2m}^p \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{n1}^p & \tilde{x}_{n2}^p & \cdots & \tilde{x}_{nm}^p \end{pmatrix} \end{array} \quad p = 1, 2, \ldots, P$$

which are referred to as fuzzy decision matrices usually used to represent the fuzzy multiattribute group decision making problem.

Denote

$$a_j^{\max} = \max\{a_{ij}^p | a_{ij}^p \in \tilde{x}_{ij}^p = (a_{ij}^p, b_{ij}^p, c_{ij}^p), \quad i = 1, 2, \ldots, n; p = 1, 2, \ldots, P\}$$

and

$$a_j^{\min} = \min\{a_{ij}^p | a_{ij}^p \in \tilde{x}_{ij}^p = (a_{ij}^p, b_{ij}^p, c_{ij}^p), \quad i = 1, 2, \ldots, n; p = 1, 2, \ldots, P\}$$

In the same way, we can explain the meaning of $b_j^{\max}$, $b_j^{\min}$, $c_j^{\max}$ and $c_j^{\min}$.

In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set C can be divided into two subsets: C^1 and C^2, where C^k $(k = 1, 2)$ is the subset of benefit attributes and cost attributes, respectively. Furthermore, $C = C^1 \cup C^2$ and $C^1 \cap C^2 = \emptyset$, where $\emptyset$ is empty set. Since the m objectives may be measured in different ways, the decision matrix $\widetilde{D}^p$ needs to be normalized.

In this paper, we choose the following normalization formula

$$\tilde{r}_{ij}^p = \left(\frac{a_{ij}^p}{c_j^{\max}}, \frac{b_{ij}^p}{b_j^{\max}}, \frac{c_{ij}^p}{a_j^{\max}} \wedge 1 \right) \quad \text{for } j \in C^1 \tag{2}$$

and

$$\tilde{r}_{ij}^p = \left(\frac{a_j^{\min}}{c_{ij}^p}, \frac{b_j^{\min}}{b_{ij}^p}, \frac{c_j^{\min}}{a_{ij}^p} \wedge 1 \right) \quad \text{for } j \in C^2 \tag{3}$$

The normalization method mentioned above is to preserve the property that the range of a normalized triangular fuzzy number $\tilde{r}_{ij}^p$ belongs to the closed interval $[0,1]$. Hence, the fuzzy decision matrices $\widetilde{D}^p$ $(p = 1, 2, \ldots, P)$ are transformed into the normalized fuzzy decision matrix $\widetilde{R}^p$ as follows:

$$\widetilde{R}^p = (\tilde{r}^p_{ij})_{n\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_m \end{array} \\ \begin{pmatrix} \tilde{r}^p_{11} & \tilde{r}^p_{12} & \cdots & \tilde{r}^p_{1m} \\ \tilde{r}^p_{21} & \tilde{r}^p_{22} & \cdots & \tilde{r}^p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{r}^p_{n1} & \tilde{r}^p_{n2} & \cdots & \tilde{r}^p_{nm} \end{pmatrix} \end{array} \quad p = 1, 2, \ldots, P$$

where $\tilde{r}^p_{ij} = (a^p_{ijL}, a^p_{ijM}, a^p_{ijR})$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ p = 1, 2, \ldots, P)$ are normalized triangular fuzzy numbers.

3. Fuzzy group LINMAP model and method

3.1. Group consistency and inconsistency measurements

Let $\widetilde{\mathbf{R}}^p_i = (\tilde{r}^p_{i1}, \tilde{r}^p_{i2}, \ldots, \tilde{r}^p_{im})$ express a triangular fuzzy number vector for alternatives A_i $(i = 1, 2, \ldots, n)$ and decision maker P_p $(p = 1, 2, \ldots, P)$. Using Eq. (1), the square of the weighted Euclidean distance between $\widetilde{\mathbf{R}}^p_i$ and the fuzzy positive ideal solution $\tilde{\mathbf{a}}^* = (\tilde{a}^*_1, \tilde{a}^*_2, \ldots, \tilde{a}^*_m)$ can be calculated as

$$S^p_i = \sum_{j=1}^{m} \omega_j [d(\tilde{r}^p_{ij}, \tilde{a}^*_j)]^2$$

where $\tilde{a}^*_j = (a^*_{jL}, a^*_{jM}, a^*_{jR})$ $(j = 1, 2, \ldots, m)$ are triangular fuzzy numbers. S^p_i can be written explicitly as

$$S^p_i = \frac{1}{3} \sum_{j=1}^{m} \omega_j [(a_{ijL} - a^*_{jL})^2 + (a_{ijM} - a^*_{jM})^2 + (a_{ijR} - a^*_{jR})^2] \tag{4}$$

Assume that the decision maker P_p $(p = 1, 2, \ldots, P)$ gives the preference relations between alternatives by $\Omega^p = \{(k, l) | A_k R_p A_l, k, l = 1, 2, \ldots, n\}$, where R_p is a preference relation given by the decision maker P_p. $A_k R_p A_l$ means that either the decision maker P_p prefers A_k to A_l or the decision maker P_p is indifferent between A_k and A_l. If the weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^{\mathrm{T}}$ and the fuzzy positive ideal solution $\tilde{\mathbf{a}}^* = (\tilde{a}^*_1, \tilde{a}^*_2, \ldots, \tilde{a}^*_m)$ are chosen by the group already, then the decision maker P_p can calculate the square of the weighted Euclidean distance between each pair of alternative (k, l) and the fuzzy positive ideal solution $\tilde{\mathbf{a}}^* = (\tilde{a}^*_1, \tilde{a}^*_2, \ldots, \tilde{a}^*_m)$ as follows:

$$S^p_k = \sum_{j=1}^{m} \omega_j [d(\tilde{r}^p_{kj}, \tilde{a}^*_j)]^2$$

and

$$S_l^p = \sum_{j=1}^{m} \omega_j [d(\tilde{r}_{lj}^p, \tilde{a}_j^*)]^2$$

For each pair of alternatives $(k,l) \in \Omega^p$, the alternative A_k is closer to the fuzzy positive ideal solution than the alternative A_l if $S_l^p \geqslant S_k^p$. So the ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p based on $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ is consistent with the preference given by the decision maker P_p. Conversely, if $S_l^p < S_k^p$, then $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ is not chosen properly since it results in that ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p based on $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ is inconsistent with the preferences given by the decision maker P_p. Therefore, $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ should be chosen so that the ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p is consistent with the preference provided by the decision maker P_p.

We define an index $(S_l^p - S_k^p)^-$ to measure inconsistency between the ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p and the preferences given by the decision maker P_p preferring A_k to A_l as follows:

$$(S_l^p - S_k^p)^- = \begin{cases} 0 & (S_l^p \geqslant S_k^p) \\ S_k^p - S_l^p & (S_l^p < S_k^p) \end{cases}$$

Obviously, the ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p based on $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ is consistent with the preferences given by the decision maker P_p if $S_l^p \geqslant S_k^p$. Hence, $(S_l^p - S_k^p)^-$ is defined to be 0. On the other hand, the ranking order of alternatives A_k and A_l determined by S_l^p and S_k^p based on $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ is inconsistent with the preferences given by the decision maker P_p if $S_l^p < S_k^p$. Hence, $(S_l^p - S_k^p)^-$ is defined to be $S_k^p - S_l^p$. Then, the inconsistency index can be rewritten as

$$(S_l^p - S_k^p)^- = \max\{0, S_k^p - S_l^p\}$$

Let

$$B^p = \sum_{(k,l)\in\Omega^p} (S_l^p - S_k^p)^-$$

B^p is an total inconsistency index between the ranking order of $(k,l) \in \Omega^p$ using Eq. (4) based on $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$ and the preferences given by the decision maker P_p. Then, an inconsistency index of the group is defined as

$$B = \sum_{p=1}^{P} B^p = \sum_{p=1}^{P} \sum_{(k,l)\in\Omega^p} (S_l^p - S_k^p)^- \tag{5}$$

In a similar way, a consistency index of the group is defined as

$$G = \sum_{p=1}^{P} G^p = \sum_{p=1}^{P} \sum_{(k,l)\in\Omega^p} (S_l^p - S_k^p)^+ \tag{6}$$

where

$$(S_l^p - S_k^p)^+ = \begin{cases} S_l^p - S_k^p & (S_l^p \geqslant S_k^p) \\ 0 & (S_l^p < S_k^p) \end{cases}$$

This equation can be rewritten as

$$(S_l^p - S_k^p)^+ = \max\{0, S_l^p - S_k^p\}$$

Obviously,

$$(S_l^p - S_k^p)^+ - (S_l^p - S_k^p)^- = S_l^p - S_k^p$$

3.2. Fuzzy group LINMAP model

To determine $(\boldsymbol{\omega}, \tilde{\mathbf{a}}^*)$, we construct the following mathematical programming

$$\begin{aligned} &\max \quad \left\{ \sum_{p=1}^{P} \sum_{(k,l)\in\Omega^p} \max\{0, S_l^p - S_k^p\} \right\} \\ &\text{s.t.} \quad \begin{cases} G - B \geqslant h \\ \sum_{j=1}^m \omega_j = 1 \\ \omega_j \geqslant \varepsilon, \quad j = 1, 2, \ldots, m \end{cases} \end{aligned} \tag{7}$$

where $h > 0$ is given by the group a priori and $\varepsilon > 0$ is sufficiently small.

For each $(k, l) \in \Omega^p$, let

$$\lambda_{kl}^p = \max\{0, S_l^p - S_k^p\}$$

Then, we have

$$\lambda_{kl}^p \geqslant S_l^p - S_k^p \quad \text{with } \lambda_{kl}^p \geqslant 0$$

Thus, Eq. (7) can be transformed into

$$\begin{aligned} &\max \quad \left\{ \sum_{p=1}^{P} \sum_{(k,l)\in\Omega^p} \lambda_{kl}^p \right\} \\ &\text{s.t.} \quad \begin{cases} G - B \geqslant h \\ \omega_j \geqslant \varepsilon, \quad j = 1, 2, \ldots, m \\ \sum_{j=1}^m \omega_j = 1 \\ S_k^p - S_l^p + \lambda_{kl}^p \geqslant 0, \quad (k, l) \in \Omega^p; \ \ p = 1, 2, \ldots, P \\ \lambda_{kl}^p \geqslant 0, \quad (k, l) \in \Omega^p; \ \ p = 1, 2, \ldots, P \end{cases} \end{aligned} \tag{8}$$

Using Eqs. (4)–(6), Eq. (8) can be rewritten as follows:

$$\max \quad \left\{ \sum_{p=1}^{P} \sum_{(k,l)\in\Omega^p} \lambda_{kl}^p \right\}$$

$$\text{s.t.} \quad \begin{cases} \frac{1}{3}\sum_{j=1}^m \omega_j \sum_{p=1}^P \sum_{(k,l)\in\Omega^p}[(a_{ljL}^{p^2} - a_{kjL}^{p^2}) + (a_{ljM}^{p^2} - a_{kjM}^{p^2}) + (a_{ljR}^{p^2} - a_{kjR}^{p^2})] \\ \quad -\frac{2}{3}\sum_{p=1}^P \Big[\sum_{j=1}^m v_{jL} \sum_{(k,l)\in\Omega^p}(a_{ljL}^p - a_{kjL}^p) \\ \quad + \sum_{j=1}^m v_{jM} \sum_{(k,l)\in\Omega^p}(a_{ljM}^p - a_{kjM}^p) \\ \quad + \sum_{j=1}^m v_{jR} \sum_{(k,l)\in\Omega^p}(a_{ljR}^p - a_{kjR}^p) \Big] \geqslant h \\ \frac{1}{3}\sum_{j=1}^m \omega_j[(a_{kjL}^{p^2} - a_{ljL}^{p^2}) + (a_{kjM}^{p^2} - a_{ljM}^{p^2}) + (a_{kjR}^{p^2} - a_{ljR}^{p^2})] \\ \quad -\frac{2}{3}\Big[\sum_{j=1}^m v_{jL}(a_{kjL}^p - a_{ljL}^p) + \sum_{j=1}^m v_{jM}(a_{kjM}^p - a_{ljM}^p) \\ \quad + \sum_{j=1}^m v_{jR}(a_{kjR}^p - a_{ljR}^p) \Big] + \lambda_{kl}^p \geqslant 0, \quad (k,l)\in\Omega^p;\ p=1,2,\ldots,P \\ \omega_j \geqslant \varepsilon, \quad j=1,2,\ldots,m \\ \sum_{j=1}^m \omega_j = 1 \\ v_{jL}\geqslant 0, v_{jM}\geqslant 0, v_{jR}\geqslant 0, \quad j=1,2,\ldots,m \\ \lambda_{kl}^p \geqslant 0, \quad (k,l)\in\Omega^p;\ p=1,2,\ldots,P \end{cases} \tag{9}$$

where

$$\begin{cases} v_{jL} = \omega_j a_{jL}^* \\ v_{jM} = \omega_j a_{jM}^* \\ v_{jR} = \omega_j a_{jR}^* \end{cases} \tag{10}$$

ω_j, v_{jL}, v_{jM} and v_{jR} ($j = 1,2,\ldots,m$) can be obtained by solving the above linear programming (i.e., Eq. (9)) using the Simplex method. Then, the best values of a_{jL}^*, a_{jM}^* and a_{jR}^* ($j = 1,2,\ldots,m$) are computed using Eq. (10) and are denoted as the triangular fuzzy number $\tilde{a}_j^*$.

3.3. Fuzzy group decision making method

In the above, the (group) weight vector $\boldsymbol{\omega}$ and the fuzzy (group) positive ideal solution $\tilde{\mathbf{a}}^*$ can be determined. For the decision maker P_p ($p = 1,2,\ldots,P$), the ranking order of all alternatives can be obtained according to the increasing order of S_i^p ($i = 1,2,\ldots,n$) generated by Eq. (4). The group ranking order of all alternatives can be obtained using social choice functions such as Borda's function and Copeland's function [9]. The detailed process is omitted. Compared with the LINMAP method, Eqs. (9) and (10) can be used in fuzzy decision-making environments with more than one decision maker. Furthermore, to avoid the situation of $\omega_j = 0$ as it may be the case in the LINMAP method, the constraints $\omega_j \geqslant \varepsilon$ and $\sum_{j=1}^{m} \omega_j = 1$ are added to Eq. (9). It is easy to show that Eqs. (9) and (10) are reduced to the linear programming model of the LINMAP method in a crisp environment if the fuzzy ratings $\tilde{a}^p_{kj}$ and $\tilde{a}^p_{lj}$ are reduced to the crisp ratings a_{kj} and a_{lj}, respectively, and there is only one decision maker (i.e., $P = 1$).

4. A numerical example

An extended air-fighter selection problem (Hwang and Yoon [8]) is investigated in this section.

Suppose one country X plans to buy air-fighters from another country Y. The Defense Department of the country Y would provide the country X with characteristic data for four candidate air-fighters A_1, A_2, A_3 and A_4. Suppose there are three experts P_1, P_2 and P_3, who consist in an air-force purchase group and all agree to take into consideration the following six attributes in evaluating the air-fighters, including maximum speed (C_1), cruise radius (C_2), maximum loading (C_3), price (C_4), reliability (C_5) and maintenance (C_6). C_5 and C_6 are qualitative attributes and their ratings are expressed using linguistic variables.

The data and ratings of all alternatives on every attribute are given by the three experts P_1, P_2 and P_3 as in Tables 1–3, respectively.

The corresponding relations between linguistic variables and positive triangular fuzzy numbers are given in Table 4.

Assume that the experts P_p ($p = 1, 2, 3$) provide their preferences between alternatives as follows:

$$\Omega^1 = \{(1,2),(3,2),(1,3),(3,4)\}, \quad \Omega^2 = \{(1,2),(2,3),(4,2)\},$$
$$\Omega^3 = \{(1,2),(2,3),(1,3),(1,4)\}$$

Using Eqs. (2) and (3), we can obtain the normalized decision matrix $\widetilde{\mathbf{R}}^1$ of the expert P_1 according to Tables 1 and 4.

Table 1
Decision information given by the decision maker P_1

Air-fighters	Attributes					
	C_1 (mach)	C_2 (mile)	C_3 (lb)	C_4 ($\$\times 10_6$)	C_5	C_6
A_1	2.0	1500	20,000	5.5	Fair	Very good
A_2	2.5	2700	18,000	6.5	Poor	Fair
A_3	1.8	2000	21,000	4.5	Good	Good
A_4	2.2	1800	20,000	5.0	Fair	Fair

Table 2
Decision information given by the decision maker P_2

Air-fighters	Attributes					
	C_1 (mach)	C_2 (mile)	C_3 (lb)	C_4 ($\$\times 10^6$)	C_5	C_6
A_1	2.0	1500	20,000	5.5	Good	Good
A_2	2.5	2700	18,000	6.5	Poor	Fair
A_3	1.8	2000	21,000	4.5	Fair	Very good
A_4	2.2	1800	20,000	5.0	Fair	Fair

Table 3
Decision information given by the decision maker P_3

Air-fighters	Attributes					
	C_1 (mach)	C_2 (mile)	C_3 (lb)	C_4 ($\$\times 10^6$)	C_5	C_6
A_1	2.0	1500	20,000	5.5	Fair	Very good
A_2	2.5	2700	18,000	6.5	Poor	Good
A_3	1.8	2000	21,000	4.5	Fair	Fair
A_4	2.2	1800	20,000	5.0	Good	Fair

Table 4
The relations between linguistic variables and triangular fuzzy numbers

Linguistic variables	Triangular fuzzy numbers
Very good	(0.8, 0.9, 1)
Good	(0.6, 0.7, 0.8)
Fair	(0.4, 0.5, 0.6)
Poor	(0.2, 0.3, 0.4)
Very poor	(0, 0.1, 0.3)

$$\widetilde{\mathbf{R}}^{1^{\mathrm{T}}} = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} (0.8, 0.8, 0.8) & (1.0, 1.0, 1.0) & (0.72, 0.72, 0.72) & (0.88, 0.88, 0.88) \\ (0.55, 0.55, 0.55) & (1.0, 1.0, 1.0) & (0.74, 0.74, 0.74) & (0.67, 0.67, 0.67) \\ (0.95, 0.95, 0.95) & (0.86, 0.86, 0.86) & (1.0, 1.0, 1.0) & (0.95, 0.95, 0.95) \\ (0.82, 0.82, 0.82) & (0.69, 0.69, 0.69) & (1.0, 1.0, 1.0) & (0.9, 0.9, 0.9) \\ (0.5, 0.71, 1.0) & (0.25, 0.43, 0.67) & (0.75, 1.0, 1.0) & (0.5, 0.71, 1.0) \\ (0.8, 1.0, 1.0) & (0.4, 0.56, 0.75) & (0.6, 0.78, 1.0) & (0.4, 0.56, 0.75) \end{pmatrix} \end{array}$$

where $\widetilde{\mathbf{R}}^{1^{\mathrm{T}}}$ is the transpose of the matrix $\widetilde{\mathbf{R}}^1$.

In a similar way, according to Tables 2–4, we can obtain the normalized decision matrices $\widetilde{\mathbf{R}}^2$ and $\widetilde{\mathbf{R}}^3$ of the experts P_2 and P_3, respectively (omitted).

Taking $\varepsilon = 0.01$ and $h = 1.0$. Substituting $\widetilde{\mathbf{R}}^p$ and Ω^p ($p = 1, 2, 3$) into Eqs. (9) and (10), we solve the linear programming problem using the Simplex method and obtain the weight vector and the fuzzy positive ideal solution as follows:

Table 5
Borda's scores of all air-fighters with respect to every experts

Air-fighters	Experts			
	P_1	P_2	P_3	Borda's scores
A_1	2	2	3	7
A_2	0	0	1	1
A_3	3	3	0	6
A_4	1	1	2	4

$$\boldsymbol{\omega} = (0.132, 0.253, 0.146, 0.247, 0.179, 0.042)^{\mathrm{T}}$$

and

$$\tilde{\mathbf{a}}^* = ((0.51, 0.51, 0.51), \quad (0.45, 0.45, 0.45), \quad (0.49, 0.49, 0.49), \\ (0.46, 0.46, 0.46), \quad (0.48, 0.48, 0.51), \quad (0.56, 0.57, 0.59))$$

Using Eq. (4), the distances between $\widetilde{\mathbf{R}}_i^p$ ($p = 1, 2, 3$; $i = 1, 2, 3, 4$) and the fuzzy positive ideal solution $\tilde{\mathbf{a}}^*$ can be generated. Comparing these distances, the ranking orders of the four air-fighters for the three experts P_p ($p = 1, 2, 3$) are generated respectively as follows: $A_3 \succ A_1 \succ A_4 \succ A_2$, $A_3 \succ A_1 \succ A_4 \succ A_2$ and $A_1 \succ A_4 \succ A_2 \succ A_3$.

Using Borda's function, we can obtain Borda's scores of all air-fighters as in Table 5.

From Table 5, the ranking order of the four air-fighters for the air-force purchase group is generated as follows

$$A_1 \succ A_3 \succ A_4 \succ A_2$$

Therefore, the best selection is the air-fighter A_1.

5. Conclusion

Most multiattribute decision making problems include both quantitative and qualitative attributes which are often assessed using imprecise data and human judgments. Fuzzy set theory is well suited to dealing with such decision problems. In this paper, the classical LINMAP method is further developed to solve multiattribute group decision making problems in fuzzy environments. Linguistic variables as well as crisp numerical values are used to assess qualitative and quantitative attributes. In particular, triangular fuzzy numbers are used in this paper to assess alternatives with respect to qualitative attributes.

A fuzzy linear programming (FLP) model was constructed to rank alternative decisions using the pairwise comparisons between alternatives, which can be used in both crisp and fuzzy environments. In the FLP model, the normalization constraints on weights are imposed, which ensures that the weights generated are not zero. The technique can be used to generate consistent and reliable ranking order of alternatives in question.

The developed method is illustrated using an air-fighter selection problem. It is expected to be applicable to decision problems in many areas, especially in situations where multiple decision makers are involved and the weights of attributes are not provided a priori.

References

[1] C. Carlsson, R. Fuller, Multiobjective linguistic optimization, Fuzzy Sets and Systems 115 (2000) 5–10.

[2] C.T. Chen, Extensions of the TOPSIS for group decision-making under fuzzy environment, Fuzzy Sets and Systems 114 (2000) 1–9.

[3] C. Cheng, K.W. Chau, Fuzzy iteration methodology for reservoir flood control operation, Journal of the American Water Resources Association 37 (5) (2001) 1381–1388.

[4] C. Cheng, K.W. Chau, Three-person multi-objective conflict decision in reservoir flood control, European Journal of Operational Research 142 (3) (2002) 625–631.

[5] M. Delgado, J.L. Verdegay, M.A. Vila, Linguistic decision-making models, International Journal of Intelligent System 7 (1992) 479–492.

[6] F. Herrera, E. Herrera, V. Viedma, J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets and Systems 78 (1996) 73–87.

[7] H.M. Hsu, C.T. Chen, Aggregation of fuzzy opinions under group decision making, Fuzzy Sets and Systems 79 (1996) 279–285.

[8] C.L. Hwang, K. Yoon, Multiple Attributes Decision Making Methods and Applications, Springer, Berlin, 1981, pp. 95–101.

[9] D.F. Li, Fuzzy Multiobjective Many-Person Decision Makings and Games, National Defense Industry Press, Beijing, 2003, pp. 138–158.

[10] D.F. Li, C. Cheng, Fuzzy multiobjective programming methods for fuzzy constrained matrix games with fuzzy numbers, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 10 (4) (2002) 385–400.

[11] D.F. Li, Fuzzy multiattribute decision making models and methods with incomplete information, Fuzzy Sets and Systems 106 (2) (1999) 113–119.

[12] D.F. Li, A fuzzy multiobjective programming approach to solve fuzzy matrix games, International Journal of Fuzzy Mathematics 7 (4) (1999) 907–912.

[13] V. Srinivasan, A.D. Shocker, Linear programming techniques for multidimensional analysis of preference, Psychometrica 38 (1973) 337.

[14] R. Zwick, E. Carlstein, D.V. Budescu, Measures of similarity among fuzzy concepts: a comparative analysis, International Journal of Approximate Reasoning (1987) 221–242.