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A belief-rule-based inventory control method under nonstationary and uncertain demand

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ABSTRACT

This paper is devoted to investigating inventory control problems under nonstationary and uncertain demand. A belief-rule-based inventory control (BRB-IC) method is developed, which can be applied in situations where demand and demand-forecast-error (DFE) do not follow certain stochastic distribution and forecasting demand is given in single-point or interval styles. The method can assist decision-making through a belief-rule structure that can be constructed, initialized and adjusted using both manager's knowledge and operational data. An extended optimal base stock (EOBS) policy is proved for initializing the belief-rule-base (BRB), and a BRB-IC inference approach with interval inputs is proposed. A numerical example and a case study are examined to demonstrate potential applications of the BRB-IC method. These studies show that the belief-rule-based expert system is flexible and valid for inventory control. The case study also shows that the BRB-IC method can compensate DFE by training BRB using historical demand data for generating reliable ordering policy.

1. Introduction

Most inventory control problems occur in industrial, distribution and service situations where demand is nonstationary and uncertain (Treharne & Sox, 2002). The nonstationarities may arise due to the following reasons: (1) a multi-stage of product life-cycle (Metan & Thiele, 2008; Song & Zipkin, 1993), (2) technological innovation and reduced product life (Bertsimas & Thiele, 2006a; Giannoccaro, Pontrandolfo, & Scozzi, 2003; Song & Zipkin, 1993), (3) seasonal effects (Karlin, 1960; Zipkin, 1989), (4) volatile customer tastes (Bertsimas & Thiele, 2006a), (5) changes in economic conditions (Song & Zipkin, 1993), and (6) exchange rate fluctuations (Scheller-Wolf & Tayur, 1997). It is inevitable that future demand will come from a distribution that differs from what governs historical demand (Scarf, 1958).

Previous studies in inventory control with nonstationary demand mainly focused on stochastic methodologies with specialized demand models. For instance, demands in successive periods were characterized by different known distributions (Bollapragada & Morton, 1999; Gavirneni & Tayur, 2001; Karlin, 1960; Morton, 1978; Tarim & Kingsman, 2006; Veinott, 1966), nonstationary Markov decision processes (Iida, 1999; Song & Zipkin, 1993; Treharne & Sox, 2002) and autoregressive, moving average or mixed autoregressive-moving average processes (Johnson & Thompson, 1975; Lee, Padmanabhan, & Whang, 1997; Lee, So, & Tang, 2000; Raghunathan, 2001). Besides, Kurawarwala and Matsuo (1996) proposed an integrated framework for forecasting and inventory management for short-cycle products. Demand-price related problems were also formulated using stochastic methodologies (Federgruen & Heching, 1999; Gallego & van Ryzin, 1994).

However, it is not always realistic to get accurate knowledge about demand such as stochastic distribution and time series characteristics in real-life inventory problem (Bertsimas & Thiele, 2006a; Petrovic, Petrovic, & Vujosevic, 1996). Inventory control strategies generated on the basis of unrealistic assumptions can result in poor performances. There is therefore a need to investigate alternative non-probabilistic inventory control strategies with uncertain and limited information about demand. Fuzzy mathematical programming (Dey & Chakraborty, 2009; Li, Kabadi, & Nair, 2002; Petrovic et al., 1996; Roy & Maiti, 1997; Yao & Su, 2000) and robust counterpart optimization (RCO) (Bertsimas & Thiele, 2006a, 2006b) have been studied to solve uncertain inventory problems by completely discarding the stochastic premise. In these studies, a series of forecasting demands is modeled in forms of fuzzy sets or intervals, and optimal policies are obtained on the basis of the finite future planning periods without taking into account historical demand. Moreover, fuzzy logical systems are proposed to solve inventory control problems with fuzzy forecasting demand (Hung, Fang, Nuttle, & King, 1997; Kamal & Sculfort, 2007; Leung, Lau, &

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Kwong, 2003), but the fuzzy rules reported in the literature are set by only taking into account qualitative expert knowledge without considering quantitative expert knowledge and historical demand. Another traditional approximation approach is to transform uncertain forecasting demand into single-point value based on the manager's knowledge and preferences (Gen, Tsujimura, & Zheng, 1997), then use stochastic approximation to generate "optimal" inventory control policies. This approach does not take full account of demand uncertainty, so the derived "optimal" policies can be hardly reliable.

Besides the nonstationary and uncertain concerns, most of the models are based on a definite planning horizon into the future, which results in that the order quantity for the forthcoming period can be significantly affected by forecasts for distant periods. However, the forecasting for distant periods is hardly reliable (Mellichamp & Love, 1978). Based on these considerations, we propose a belief-rule-based inventory control (BRB-IC) method according to current inventory, historical demand data and necessary short-term forecasting demand. The method is developed from the decision support mechanism of belief-rule-based inference methodology – RIMER (Yang, Liu, Wang, Sii, & Wang, 2006; Yang, Liu, Xu, Wang, & Wang, 2007) which is derived on the basis of evidential reasoning (ER) approach (Yang, 2001; Yang & Sen, 1994; Yang & Singh, 1994; Yang & Xu, 2002a; Yang & Xu, 2002b) and rule-based expert system. RIMER is a modeling and inference scheme under uncertainty with a belief-rule structure. It has been applied to graphite content detection (Yang et al., 2006, 2007), pipeline leak detection (Chen, Yang, Xu, Zhou, & Tang, 2011; Xu et al., 2007; Zhou, Hu, Xu, Yang, & Zhou, 2011; Zhou, Hu, Yang, Xu, & Zhou, 2009), clinical guideline (Kong, Xu, Liu, & Yang, 2009), nuclear safeguards evaluation (Liu, Ruan, Wang, & Martinez, 2009), consumer preference prediction (Wang, Yang, Xu, & Chin, 2009), new product development (Tang, Yang, Chin, Wong, & Liu, 2011), system reliability prediction (Hu, Si, & Yang, 2010), and gyroscopic drift prediction (Si, Hu, Yang, & Zhang, 2011). The BRB-IC method provides a transparent knowledge-based framework for inventory control, and can handle various kinds of uncertain information. It enables experts and decision-makers (DMs) to intervene the construction and updating of the belief-rule-base (BRB) using their judgmental knowledge. The method is easy to understand and implement and requires little computational efforts.

The paper is organized as follows: In Section 2, the basic model formulations for backorder and lost sales cases are briefly described. In Section 3, the BRB-IC method is described in terms of information transformation in ER framework, constructing BRB, initializing BRB, BRB-IC inference approaches and training BRB. In Section 4, a numerical example and a case study are examined to demonstrate potential applications of the BRB-IC method. The paper is concluded in Section 5.

2. Inventory model formulations

We consider a single-echelon periodic review inventory problem, the objective of which is to determine order quantity that maximizes total profit (equivalent to minimize total cost). The cost elements include purchase, holding and shortage (or backorder) costs, and the setup cost is negligible relative to other factors. Review period T and lead time L are assumed to be constant. One cycle of the order and arrival process requires $L + T$ periods of time. Demand forecasting, arrival of goods, review, determining order quantity, and placing an order are assumed to take place at the beginning of the period sequentially. Customer demand is assumed to take place during the period. Cost and profit are calculated at the end of the period. Inventory equations for both backorder and lost sales cases are given as follows (Axsater, 2007; Zipkin, 2000).

(A) Backorder case.

Inventory level IL_n is decided by period $n-1$'s inventory level IL_{n-1} and real demand D_{n-1} as well as period $n-L$'s order quantity Q_{n-L}.

$$IL_n = \begin{cases} IL_{n-1} - D_{n-1} + Q_{n-L}, & n > L \\ IL_{n-1} - D_{n-1}, & n \leqslant L \end{cases} \tag{1}$$

Backlogged quantity BL_n is given as following

$$BL_n = \begin{cases} 0, & n \leqslant L \vee (n > L \wedge IL_{n-1} - D_{n-1} > 0) \\ D_{n-1} - IL_{n-1}, & n > L \wedge IL_{n-1} - D_{n-1} < 0 \wedge IL_{n-1} \\ & -D_{n-1} + Q_{n-L} > 0 \\ Q_{n-L}, & IL_{n-1} - D_{n-1} + Q_{n-L} \leqslant 0 \end{cases} \tag{2}$$

In-transit inventory IT'_n before period n's order quantity Q_n is placed equals to the last $L-1$ periods' total order quantity, or

$$IT'_n = \begin{cases} \sum_{i=n-L+1}^{n-1} Q_i, & n > L \\ \sum_{i=1}^{n-1} Q_i, & n \leqslant L \end{cases} \tag{3}$$

In-transit inventory IT_n after period n's order quantity Q_n is placed equals to the sum of IT'_n and Q_n, or

$$IT_n = IT'_n + Q_n = \begin{cases} \sum_{i=n-L+1}^{n} Q_i, & n > L \\ \sum_{i=1}^{n} Q_i, & n \leqslant L \end{cases} \tag{4}$$

Inventory position IP_n is the sum of IL_n and IT_n, or

$$IP_n = IL_n + IT_n = IL_n + IT'_n + Q_n \tag{5}$$

Total cost TC is the sum of products' purchase cost, holding cost and shortage penalty cost. Total profit TP is the difference between total revenue TR and total cost TC,

$$TC = \sum_{n=1}^{N} (p_2 \cdot Q_n + h \cdot [IL_n - D_n]^+ + p \cdot [IL_n - D_n]^-) \tag{6}$$

$$TR = \sum_{n=1}^{N} p_1 \cdot (\min([IL_n]^+, D_n) + BL_n) \tag{7}$$

$$TP = TR - TC = \sum_{n=1}^{N} (p_1 \cdot (\min([IL_n]^+, D_n) + BL_n) - p_2 \cdot Q_n - h \cdot [IL_n - D_n]^+ - p \cdot [IL_n - D_n]^-) \tag{8}$$

where p_1 is the sales price per item, p_2 the purchasing price per item, h the holding cost per item per period, and p the backorder penalty cost per item per period.

(B) Lost sales case.

On-hand inventory OH_n is given by

$$OH_n = \begin{cases} OH_{n-1} - D_{n-1}, & n \leqslant L \ \wedge \ OH_{n-1} > D_{n-1} \\ 0, & n \leqslant L \ \wedge \ OH_{n-1} \leqslant D_{n-1} \\ OH_{n-1} - D_{n-1} + Q_{n-L}, & n > L \ \wedge \ OH_{n-1} > D_{n-1} \\ Q_{n-L}, & n > L \ \wedge \ OH_{n-1} \leqslant D_{n-1} \end{cases} \tag{9}$$

Shortage inventory SI_n is the step function of the difference between D_n and OH_n, or

$$SI_n = [OH_n - D_n]^- = \begin{cases} D_n - OH_n, & D_n > OH_n \\ 0, & DF_n \leqslant OH_n \end{cases} \tag{10}$$

Inventory position IP_n is the sum of OH_n and IT_n, or

$$IP_n = OH_n + IT_n = OH_n + IT'_n + Q_n \tag{11}$$

where IT_n and IT'_n are defined the same as above.

Total cost TC, total revenue TR and total cost TP are given as follows:

$$TC = \sum_{n=1}^{N} (p_2 \cdot Q_n + h \cdot [OH_n - D_n]^+ + p \cdot SI_n) \tag{12}$$

$$TR = \sum_{n=1}^{N} p_1 \cdot \min(D_n, OH_n) \tag{13}$$

$$TP = TR - TC = \sum_{n=1}^{N} (p_1 \cdot \min(D_n, OH_n) - p_2 \cdot Q_n - h \cdot [OH_n - D_n]^+ - p \cdot SI_n) \tag{14}$$

where p_1, p_2 and h are defined as above, and p is the shortage penalty cost per item.

3. Belief-rule-based inventory control method

Most of traditional models for inventory control are based on a definite planning horizon into the future. These methods consistently incorporate the tacit assumption that demand forecasts are relatively accurate, which results in that the order quantity for the forthcoming period can be significantly affected by forecasts for distant periods. However, the forecasting for distant periods is hardly reliable. Inventory strategies should well be based on the current inventory, historical demand information and necessary short-term forecasting demand. Besides, there is a need to develop an inventory control model that does not rely on the stochastic hypothesis and meanwhile can handle various types of uncertain information common in real life inventory control processes. With those considerations, we propose a belief-rule-based inventory control (BRB-IC) method which inherits the information processing ability of RIMER and allows the incorporation of human knowledge and historical demand information to induce reliable inventory control strategy. Belief-rule-based expert system is an extension of traditional rule based systems and is capable of representing more complicated causal relationships and handling different types of uncertain information. In the BRB-IC model, the manager can determine order quantity by constructing, initializing and training of the belief-rule-base (BRB) from the following five points: (1) information transformation in ER framework; (2) constructing BRB; (3) initializing BRB; (4) BRB-IC inference approaches; and (5) training BRB using historical demand data to refine the belief rules. In BRB, belief rules are representing ordering policies. The BRB-IC framework is depicted in Fig. 1. If inventory position is reviewed at the beginning of a period, order quantity can be determined using the BRB-IC model. If there is no review in this period, the dynamic inventory control process proceeds to the next period directly.

3.1. Information transformation in ER framework

Various types of input information can be handled in the ER framework (Yang, 2001; Yang et al., 2006, 2007), but only single-point and interval are considered in this paper.

(A) Single-point forecasting demand.

Rule or utility based equivalent transformation techniques can be used to transform single-point forecasting demand into belief distributions (Yang, 2001). In this paper, inventory level, on-hand inventory and in-transit inventory are all assumed to be single-point values. A single-point value v_j can be represented using the following equivalent distribution:

$$S(v_j) = \{(h_n, \gamma_n) | n = 1, \ldots, N\} \tag{15}$$

where $\gamma_n = \frac{h_{n+1} - v_j}{h_{n+1} - h_n}$, $\gamma_{n+1} = 1 - \gamma_n$ (if $h_n \leqslant v_j \leqslant h_{n+1}$) and $\gamma_k = 0$ for $k = 1, \ldots, N$; $k \neq n$, $n + 1$. h_n is referred to as assessment grade.

The value v_j can be equivalently defined as

$$v_j \Longleftrightarrow \{(h_n, \gamma_n), (h_{n+1}, \gamma_{n+1})\} \text{ with } \gamma_n + \gamma_{n+1} = 1 \tag{16}$$

(B) Interval forecasting demand.

The technique for modeling interval data by interval belief structure is also proposed (Wang, Yang, Xu, & Chin, 2006). Let $v_i \in [v_i^-, v_i^+]$ be an interval number. If this interval number contains no assessment grade and is totally included by two adjacent assessment grades, such as h_n and h_{n+1}, then v_i can be assessed to h_n and h_{n+1} with interval belief degrees, i.e. $[\gamma_n^-, \gamma_n^+]$ and $[\gamma_{n+1}^-, \gamma_{n+1}^+]$, which are defined by

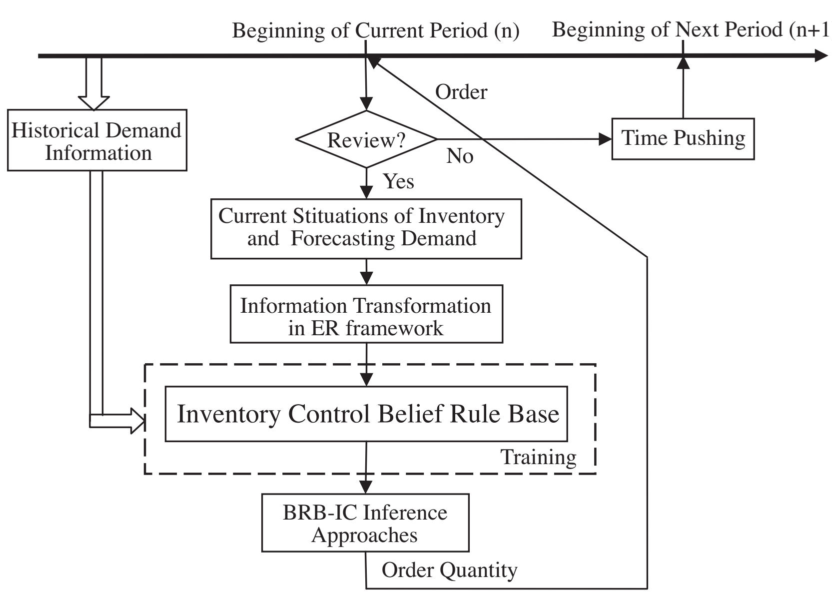

Fig. 1. BRB-IC framework.

$$\gamma_n^- = \frac{h_{n+1} - v_i^+}{h_{n+1} - h_n} \quad \text{and} \quad \gamma_n^+ = \frac{h_{n+1} - v_i^-}{h_{n+1} - h_n} \tag{17a}$$

$$\gamma_{n+1}^- = \frac{v_i^- - h_n}{h_{n+1} - h_n} \quad \text{and} \quad \gamma_{n+1}^+ = \frac{v_i^+ - h_n}{h_{n+1} - h_n} \tag{17b}$$

The interval value v_j can be equivalently defined as

$$v_j \in \left[v_j^-, v_j^+\right] \Longleftrightarrow \{(h_n, [\gamma_n^-, \gamma_n^+]), (h_{n+1}, [\gamma_{n+1}^-, \gamma_{n+1}^+])\} \quad \text{with } \gamma_n + \gamma_{n+1} = 1 \tag{18}$$

There are also situations where an interval number may contain one or more assessment grades. When only two adjacent grades, say h_n and h_{n+1}, are contained in an interval number, the formulas to calculate belief degrees are listed as follows. Interval numbers containing more assessment grades can be transformed in the same way.

$$\gamma_{n-1}^- = 0 \quad \text{and} \quad \gamma_{n-1}^+ = I_{n-1,n} \cdot \frac{h_n - v_j^-}{h_n - h_{n-1}} \tag{19a}$$

$$\gamma_n^- = 0 \quad \text{and} \quad \gamma_n^+ = I_{n-1,n} + I_{n,n+1} \tag{19b}$$

$$\gamma_{n+1}^- = 0 \quad \text{and} \quad \gamma_{n+1}^+ = I_{n,n+1} + I_{n+1,n+2} \tag{19c}$$

$$\gamma_{n+2}^- = 0 \quad \text{and} \quad \gamma_{n+2}^+ = I_{n+1,n+2} \cdot \frac{v_j^+ - h_{n+1}}{h_{n+2} - h_{n+1}} \tag{19d}$$

where $I_{k-1,k} = \begin{cases} 1, & \text{if } v_j \text{ lies between } h_{k-1} \text{ and } h_k \\ 0, & \text{otherwise} \end{cases} \quad k = n, n+1, n+2.$

The interval value v_j can be equivalently defined as

$$v_j \in \left[v_j^-, v_j^+\right] \Longleftrightarrow \{(h_{n-1}, [\gamma_{n-1}^-, \gamma_{n-1}^+]), (h_n, [\gamma_n^-, \gamma_n^+]), (h_{n+1}, [\gamma_{n+1}^-, \gamma_{n+1}^+]), (h_{n+2}, [\gamma_{n+2}^-, \gamma_{n+2}^+])\} \tag{20}$$

with $\gamma_{n-1} + \gamma_n + \gamma_{n+1} + \gamma_{n+2} = 1$ and $I_{n-1,n} + I_{n,n+1} + I_{n+1,n+2} = 1$.

3.2. Constructing BRB in BRB-IC model

The order quantity Q_n at the beginning of period n should be decided on the basis of the current inventory level IL_n (or on-hand inventory OH_n) and in-transit inventory IT'_n. IL_n should be considered in the backorder case and OH_n in the lost sales case. Meanwhile, Q_n should be determined by taking account of certain future periods' forecasting demand $D^f_{n,FP}$. If $FP < L + T$, the $(L + T) - FP$ periods' demands from period $n + FP$ to period $n + L + T - 1$ will not be satisfied. Otherwise, if $FP > L + T$, the $FP - (L + T)$ periods' demands from period $n + L + T$ to period $n + FP - 1$ will be unnecessarily considered, because they can be satisfied by placing order in period $n + T$. Thereby the antecedents of an BRB-IC model for the backorder model are IL_n, IT'_n and $D^f_{n,L+T}$, those for the lost sales model are OH_n, IT'_n and $D^f_{n,L+T}$, and the consequent is Q_n uniformly.

Suppose the numbers of evaluation grades for IL_n (or OH_n), IT'_n, $D^f_{n,L+T}$ and Q_n are J_1, J_2, J_3 and J_4 respectively. The referential evaluations of belief-rules' antecedents and consequents can be represented as follows:

$$G_n^a\left(IL_n, IT'_n, D^f_{n,L+T}\right) = \{(H_{i,j_i}, \alpha_{i,j_i}) | j_i = 1, \ldots, J_i;\ i = 1, \ldots, 3\}, \quad \text{or} \tag{21}$$

$$G_n^a\left(OH_n, IT'_n, D^f_{n,L+T}\right) = \{(H_{i,j_i}, \alpha_{i,j_i}) | j_i = 1, \ldots, J_i;\ i = 1, \ldots, 3\} \tag{22}$$

$$G_n^c(Q_n) = \{(H_{j_4}, \beta_{r,j_4}) | j_4 = 1, \ldots, J_4;\ r = 1, \ldots, R\} \tag{23}$$

where H_{i,j_i} and H_{j_4} represent the evaluation grades of the antecedents and consequent respectively. α_{i,j_i} and β_{r,j_4} are belief degrees assessed to H_{i,j_i} and H_{j_4} respectively. R is the total number of rules in the BRB, and may be decided by $R = J_1 * J_2 * J_3$, although this does not have to be strictly satisfied. The utility function for each grade is defined by $\mu(H_{i,j_i}) = U_{i,j_i}$ $(j_i = 1, \ldots, J_i;\ i = 1, \ldots, 3)$ and $\mu(H_{j_4}) = U_{j_4}$ $(j_4 = 1, \ldots, J_4)$.

The belief-rule formulations, $Q_n = F_{RIMER}\left(IL_n, IT'_n, D^f_{n,L+T}\right)$ for the backorder case and $Q_n = F_{RIMER}\left(OH_n, IT'_n, D^f_{n,L+T}\right)$ for the lost sales case, are stated as follows,

$$\text{If } IL_n \text{ is } H_{1,j_1} \text{ and } IT'_n \text{ is } H_{2,j_2} \text{ and } D^f_{n,L+T} \text{ is } H_{3,j_3}, \text{ then } Q_n \text{ is } \{(H_1, \beta_{r,1}), (H_2, \beta_{r,2}), \ldots, (H_{J_4}, \beta_{r,J_4})\} \text{ with } \theta_r \tag{24}$$

$$\text{If } OH_n \text{ is } H_{1,j_1} \text{ and } IT'_n \text{ is } H_{2,j_2} \text{ and } D^f_{n,L+T} \text{ is } H_{3,j_3}, \text{ then } Q_n \text{ is } \{(H_1, \beta_{r,1}), (H_2, \beta_{r,2}), \ldots, (H_{J_4}, \beta_{r,J_4})\} \text{ with } \theta_r \tag{25}$$

where $j_1 = 1, \ldots, J_1$, $j_2 = 1, \ldots, J_2$, $j_3 = 1, \ldots, J_3$, $r = 1, \ldots, R$, and θ_r is the belief-rule weight.

3.3. Initializing BRB in BRB-IC model

(A) Qualitative expert knowledge.

The BRB must be initialized after its formulation is determined. This can be done using qualitative expert knowledge for inventory control. For example, a grocer manager may have the following knowledge for certain beverage sales:

IF on-hand inventory is low, in-transit inventory is low and forecasting demand is high,
THEN order quantity is high;
IF on-hand inventory is medium, in-transit inventory is high and forecasting demand is low,
THEN order quantity is low.

Another example is the car sales at an Auto 4S store. The sales managers of the store interviewed by us have the following knowledge:

IF inventory level is rare and forecasting demand is rare,
THEN order quantity is rare with 0.5 belief degree and moderate with 0.5 belief degree;
IF inventory level is moderate and forecasting demand is many,
THEN*order quantity is moderate with 0.8 belief degree and many with 0.2 belief degree*.

(B) Quantitative expert knowledge.

Nevertheless, there are situations where the DM's qualitative knowledge may not always be available or credible. It is therefore necessary to use possible quantitative prior knowledge as well to assist initializing BRB. In Appendix A, we prove an extended optimal base stock (EOBS) policy as a kind of quantitative expert knowledge for initializing BRB.

Traditionally, without a given batch quantity and setup costs are negligible relative to other factors, the optimal policy will be base stock policy with order-up-to level being $(L + T)\mu + \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L + T}\sigma$ under the postulation that the stochastic demand is normally distributed with mean μ and standard deviation $\sigma(D_n \sim N(\mu,\sigma)$ for $\forall n)$ (Axsater, 2007). In order to relax this direct stochastic constraint on demand, we assume now that the demand is nonstationary and forecasted in single-point value but the demand-forecast-error (DFE) follows certain stochastic distribution. In this context, the adaptive (s_n,S_n) policy $\left((s_n, S_n) = \left(\sum_{i=n}^{n+L-1} D_i^f, \sum_{i=n}^{n+L+T-1} D_i^f\right)\right)$ (Ellon & Elmaleh, 1970), myopic base stock (T,S_n) policy $\left(Q_n = D_n^f + L \cdot \left(d_{n+1}^f - d_n^f\right)\right)$ (Graves, 1999), and (T,S_n) policies $\left(S_n = \sum_{j=1}^{L+T} D^f_{n+j-1} + \Phi^{-1}(CSL)\sigma_{CFU_{L+T}}\right)$ (Babai & Dallery, 2009) have been proposed. But none of them is proved to be optimal. As an extension, we prove an EOBS policy with order-up-to level being

$$S_n = D^f_{n,L+T} - (L+T)\mu^{\mathrm{DFE}} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{\mathrm{DFE}}, \quad \text{if } D^f_n = D_n + \varepsilon_n; \ \text{or} \tag{26}$$

$$S_n = D^f_{n,L+T} + (L+T)\mu^{\mathrm{DFE}} + \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{\mathrm{DFE}}, \quad \text{if } D^f_n = D_n - \varepsilon_n \tag{27}$$

where D^f_n is forecasting demand for period n and ε_n is DFE for period n.

When $D^f_n = D_n + \varepsilon_n$, we can deduce from the EOBS policy that

$$Q_n = F_{EOBS}\left(IL_n, IT'_n, D^f_{L+T}\right) = IP^*_n - IL_n - IT'_n = D^f_{n,L+T} - IL_n - IT'_n - (L+T)\mu^{\mathrm{DFE}} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{\mathrm{DFE}}, \quad \text{or} \tag{28}$$

$$Q_n = F_{EOBS}\left(OH_n, IT'_n, D^f_{L+T}\right) = IP^*_n - OH_n - IT'_n = D^f_{n,L+T} - OH_n - IT'_n - (L+T)\mu^{\mathrm{DFE}} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{\mathrm{DFE}} \tag{29}$$

Then, the functions $Q_n = F_{EOBS}\left(IL_n, IT'_n, D^f_{L+T}\right)$ and $Q_n = F_{EOBS}\left(OH_n, IT'_n, D^f_{L+T}\right)$ can be used to initialize $Q_n = F_{RIMER}\left(IL_n, IT'_n, D^f_{n,L+T}\right)$ and $Q_n = F_{RIMER}\left(OH_n, IT'_n, D^f_{n,L+T}\right)$ respectively.

3.4. BRB-IC inference approaches

After a BRB for inventory control is constructed and initialized, it can be utilized to infer order quantity. Yang et al. (2006) proposed a BRB inference methodology (RIMER), based on which we can deduce a BRB-IC inference approach for single-point inputs in the next subsection (1). Then, in subsection (2), a BRB-IC inference approach for interval inputs is developed on the basis of the interval ER approaches (Guo, Yang, Chin, & Wang, 2007; Wang et al., 2006).

(1) Single-point inputs.

Let $STATE_n\left(IL_n, IT'_n, D^f_{n,L+T}\right)$ (or $STATE_n\left(OH_n, IT'_n, D^f_{n,L+T}\right)$) be the state of inventory and forecasting demand at the beginning of period n, where IL_n (or OH_n), IT'_n and $D^f_{n,L+T}$ are all given in single-point style. The forecasting demand can be transformed into evaluation distribution using the transformation techniques discussed in Section 3.1. Then, the belief-rule-based inference engine can be implemented as follows (Yang et al., 2007).

The activation weight ω_r of the antecedent state $STATE_n$ for the rth rule is calculated by

$$\omega_r = \frac{\theta_r \prod_{i=1}^3 \alpha_i^r}{\sum_{j=1}^R \left[\theta_j \prod_{i=1}^3 \alpha_i^j\right]}, \quad r = 1, \ldots, R; \ i = 1, \ldots, 3 \tag{30}$$

where α_i^r represents the matching degree of the ith input to the rth rule. The 1st, 2nd and 3rd inputs refer to IL_n (or OH_n), IT'_n and $D^f_{n,L+T}$ respectively.

Using the analytical ER algorithm (Yang, 2001), the combined belief degree β_{j_4} for H_{j_4} can be generated as follows:

$$\beta_{j_4} = \frac{\mu\left[\prod_{r=1}^R\left(\omega_r\beta_{r,j_4} + 1 - \omega_r\sum_{j_4=1}^{J_4}\beta_{r,j_4}\right) - \prod_{r=1}^R\left(1 - \omega_r\sum_{j_4=1}^{J_4}\beta_{r,j_4}\right)\right]}{1 - \mu\left[\prod_{r=1}^R(1-\omega_r)\right]}, \quad j_4 = 1, \ldots, J_4 \tag{31}$$

where

$$\mu = \left[\sum_{j_4=1}^{J_4}\prod_{r=1}^R\left(\omega_r\beta_{r,j_4} + 1 - \omega_r\sum_{j_4=1}^{J_4}\beta_{r,j_4}\right) - (J_4 - 1)\prod_{r=1}^R\left(1 - \omega_r\sum_{j_4=1}^{J_4}\beta_{r,j_4}\right)\right]^{-1}.$$

Then, the inferred order quantity can be represented using the following distributed formulation:

$$RIMER(STATE_n) = \{(U_{j_4}, \beta_{j_4}) | j_4 = 1, \ldots, J_4\} \tag{32}$$

With the utility function defined in the distributed assessment framework, the inferred order quantity can be aggregated using the following formula:

$$Q_n = F_{RIMER} = \sum_{j_4=1}^{J_4} U_{j_4}\beta_{j_4} \tag{33}$$

If the antecedents, i.e. IL_n (or OH_n), IT'_n and $D^f_{n,L+T}$, are complete, inference consequences will be complete as well, i.e. $\sum_{j_4=1}^{J_4}\beta_{j_4} = 1$; otherwise, inference consequences will be incomplete, i.e. $0 \leqslant \sum_{j_4=1}^{J_4}\beta_{j_4} < 1$. In the latter case, incomplete information is evaluated by $1 - \sum_{j_4=1}^{J_4}\beta_{j_4}$.

(2) Interval inputs.

Interval forecasting demand can be transformed into evaluation distribution with interval belief degrees. The interval ER (IER) approaches for interval inputs are proposed by Wang et al. (2006) and Guo et al. (2007). Based on the IER approaches, we propose a BRB-IC inference approach with interval inputs as follows.

An interval input value $y_i \in [y_i^-, y_i^+]$ can be equivalently expressed using the interval belief degrees structure, i.e. $y_i \in [y_i^-, y_i^+] \Longleftrightarrow \left\{\left(H_{n,i}, \left[\gamma_{n,i}^-, \gamma_{n,i}^+\right]\right)\right\}$ $(n = 1, \ldots, N)$, where $\gamma_{n,i}^- \leqslant \gamma_{n,i}^+$. Correspondingly, the activation weight α_i^r of the ith input to the rth rule is given by $\alpha_i^r \in \left[\alpha_{i,r}^-, \alpha_{i,r}^+\right]$. Then, order quantity inferred from BRB can be generated using the following nonlinear optimization model:

$$\text{Max/Min} \quad Q_n = F_{RIMER} = \sum_{j_4=1}^{J_4} U_{j_4} \cdot \beta_{j_4} \tag{34a}$$

s.t. Eqs. (30) and (31), and

$$\alpha_{i,r}^- \leqslant \alpha_i^r \leqslant \alpha_{i,r}^+, \quad r = 1, \ldots, R; \ i = 1, \ldots, 3 \tag{34b}$$

$$\sum_{k=1}^K I_k = 1 \tag{34c}$$

$$I_k \geqslant 0, \quad k = 1, \ldots, K \tag{34d}$$

$$\sum_{k=1}^K \left[I_k \cdot \left(\sum_{j=k+1}^K I_j\right)\right] = 0, \quad (\text{if } K \geqslant 2) \tag{34e}$$

where Eqs. (34c)–(34e) are equivalent to those shown in Eq. (19) (Guo et al., 2007).

Uncertain inputs always lead to uncertain order quantity. It is argued that you should use all quantitative data you can get, but you still have to distrust it and use your own intelligence and judgment (Gopal, 2007). Any results generated by an optimization model should be reconsidered by DMs, because good judgments usually come from experience. In practice, a DM should make ultimate decision on order quantity by taking into account external conditions as well as his experience, preference, intuition and judgment, while suggestions made from a model should be used as references. For example, market planning, assignment, consumer situation and financial status may affect the decision. Besides, the DM's subjective preference, such as optimistic, neutral and pessimistic, may affect the decision as well. For example, an optimistic DM tends to make a higher order quantity than a pessimistic DM.

3.5. Training BRB in BRB-IC model

Initialized belief-rule-base can be directly used to assist decision-making (Yang et al., 2006). But in circumstances where demand is nonstationary, intuitive expert knowledge may not be sufficient to construct a BRB with satisfactory accuracy and consistency. So, it is necessary to develop a supporting mechanism to train the BRB in BRB-IC model using known quantitative information.

The process of training BRB in BRB-IC model using historical demand information is shown in Fig. 2 which includes an initialization subprocess, an inner recursive subprocess and an outer iterative subprocess. The inference engine refers to the BRB-IC inference approaches introduced in Section 3.4. The training process can be summarized as follows:

Step 1: Initialization subprocess.
- ∘1 Construction and initialization of BRB in BRB-IC model using qualitative or quantitative expert knowledge.
- ∘2 Historical demand information, including historical real demand and forecasting demand, is sorted to configure a N-period inventory control process to assist training BRB in BRB-IC model.

Step 2: Inner recursive subprocess.
- ∘1 The N-period inventory control process is operated from period 1 to period N in a forward rolling way.
- ∘2 At the end of period N, the total profit TP is calculated using Eq. (8) in the backorder case or Eq. (14) in the lost sales case.

Step 3: Outer iterative subprocess.

The purpose of the outer iterative subprocess is to maximize TP to generate the optimum BRB for inventory control. The objective function is stated as follows.

$$\max \quad (TP) \tag{35a}$$

where TP is defined by Eq. (8) in the backorder model, which is recursively calculated using Eqs. (1)–(7), (15), (16), (21), (23), (24), (30)–(33) for single-point inputs, or Eqs. (1)–(7), (17)–(21), (23), (24), (34) for interval inputs; or Eq. (14) in the lost sales model, which is recursively calculated using Eqs. (9)–(13), (15), (16), (22), (23), (25), (30)–(33) for single-point inputs, or Eqs. (9)–(13), (17)–(20), (22), (23), (25), (34)for interval inputs.

The constraints are given by

$$0 \leqslant \beta_{r,j_4} \leqslant 1, \quad r = 1,\ldots,R;\ j_4 = 1,\ldots,J_4 \tag{35b}$$

$$\sum_{j_4=1}^{J_4} \beta_{r,j_4} = 1 \tag{35c}$$

$$0 \leqslant \theta_r \leqslant 1, \quad r = 1,\ldots,R \tag{35d}$$

where β_{r,j_4} and θ_r are defined in Section 3.2.

Yang et al. (2007) proposed to use FMINCON and FMINIMAX functions in MATLAB to train BRB. Afterwards, Chang, Yang, and Wang (2007) improved the work and developed a gradient and dichotomy algorithm, which is used and mended in this paper to solve the numerical example and case study provided in Section 4. Once the consequent belief degrees and rule weights of a BRB are tuned using the gradient and dichotomy algorithm, the inner recursive subprocess is activated again to calculate the total profit TP. If TP increases, go to the next loop of inner recursive subprocess in Step 2; otherwise, make another adjustment of the parameters to see whether TP can be increased. Repeat this outer iterative subprocess until TP cannot be increased by adjusting the parameters. Then the trained BRB is used to infer optimum order quantity Q_n, as illustrated in Fig. 1.

4. Numerical example, case study and discussion

4.1. A numerical example with single-point forecasting demand

This is a single-echelon retailer inventory problem, the purpose of which is to determine an optimal order policy that can maximize total profit. An autoregressive integrated moving average (ARIMA) process is used (Graves, 1999), i.e.

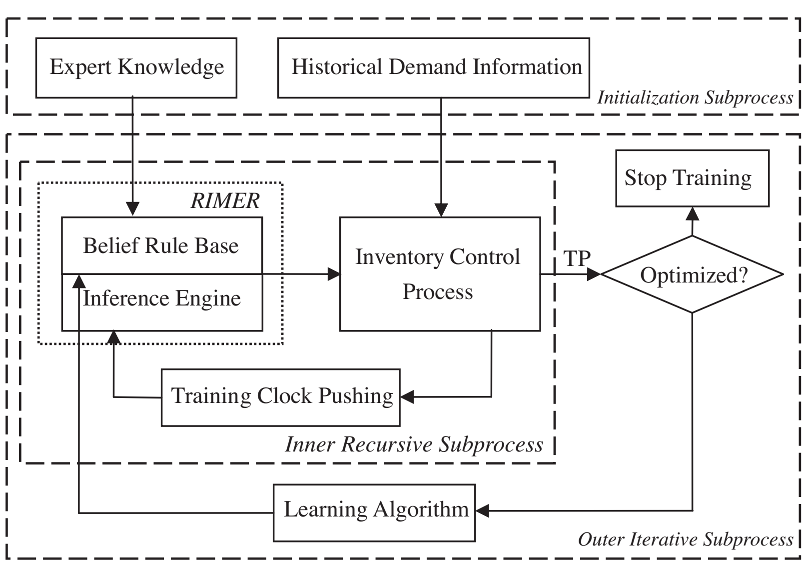

Fig. 2. Training BRB in BRB-IC model.

Table 1
Total profits for the above five inventory control policies.

Treatment of shortages	Policy				
	Ellon's adaptive (s_n,S_n) policy	Graves' myopic base stock policy	Babai's adaptive (T,S_n) policy	EOBS policy	BRB-IC policy
Backorder	7959	9082	9612	9741	9625
Lost sales	7651	8914	9543	9680	9556

$$\begin{cases} D_1 = \xi + \varepsilon_1 \\ D_n = d_{n-1} - (1-\alpha)\varepsilon_{n-1} + \varepsilon_n, \quad n = 2, \ldots, N \end{cases} \tag{36}$$

to simulate real demand. ξ and α $(0 \leqslant \alpha \leqslant 1)$ are known parameters, and ε_n represents the demand forecasting error. $\{\varepsilon_n\}$ is a time series of independent and identically distributed (i.i.d.) random variables with $E(\varepsilon_n) = 0$ and $Var(\varepsilon_n) = (\sigma^{\mathrm{DFE}})^2$. When $0 < \alpha \leqslant 1$, the demand process is nonstationary. The demand forecasting model of the retailer is given by

$$\begin{cases} D_1^f = \xi \\ D_n^f = \alpha d_{n-1} - (1-\alpha)F_{n-1}, \quad n = 2, \ldots, N \end{cases} \tag{37}$$

which is an unbiased forecast $\left(D_n^f = D_n - \varepsilon_n\right)$. Let $\xi = 10$, $\alpha = 0.5$, $\varepsilon_n \sim N(0,1)$, $L = 5$, $T = 1$, $h = 1$, $b = 5$, $p_1 = 20$, $p_2 = 10$, and $N = 100$.

Exactly we have proven an EOBS policy in Appendix A for this kind of problem. Herein, Ellon's adaptive (s_n,S_n) policy, Graves's myopic base stock policy, Babai's (T,S_n) policy, the EOBS policy, and the BRB-IC method are used for comparison. The trained BRB is listed in Table B.1 in Appendix B for the backorder case. The total profits generated using the above five inventory control policies are listed in Table 1. From this table we can see that the order of excellence of these policies is EOBS policy $\succ$ BRB-IC policy $\succ$ Babai's policy $\succ$ Graves' policy $\succ$ Ellon's policy. Besides, the BRB-IC policy can approximate the EOBS policy very well, which means that the BRB-IC method is credible and reliable in this numerical study.

These relations can also be reflected in the inventory level curves shown in Fig. 3 for the backorder case. The inventory level means the cumulative shortage demand when it is below the zero baseline, or the cumulative on-hand inventory when it is above the zero baseline. In Fig. 3(a) and (b), the inventory level curves fluctuate greatly and frequently around the zero baseline, which results in high holding cost and shortage cost. Furthermore, inventory in Fig. 3(a) is better controlled than in Fig. 3(b). The inventory level curves in Fig. 3(c)–(e) are very close. The small differences are that the inventory level curves in Fig. 3(c) and (d) present more on-hand inventory and more shortages than that in Fig. 3(e). For the lost sales case, we have derived similar results.

4.2. A case study with interval forecasting demand

Auto 4S is the principal marketing channel for automobile industry in Mainland China, and it has made tremendous

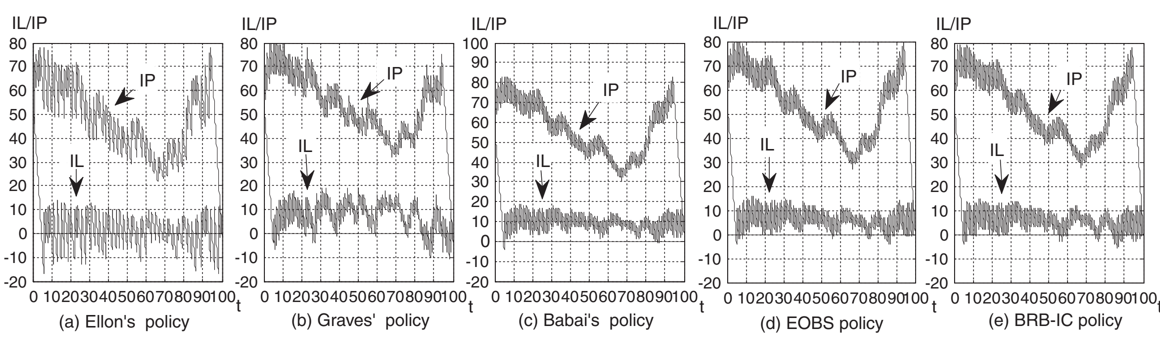

Fig. 3. IL and IP curves for the above five inventory control policies.

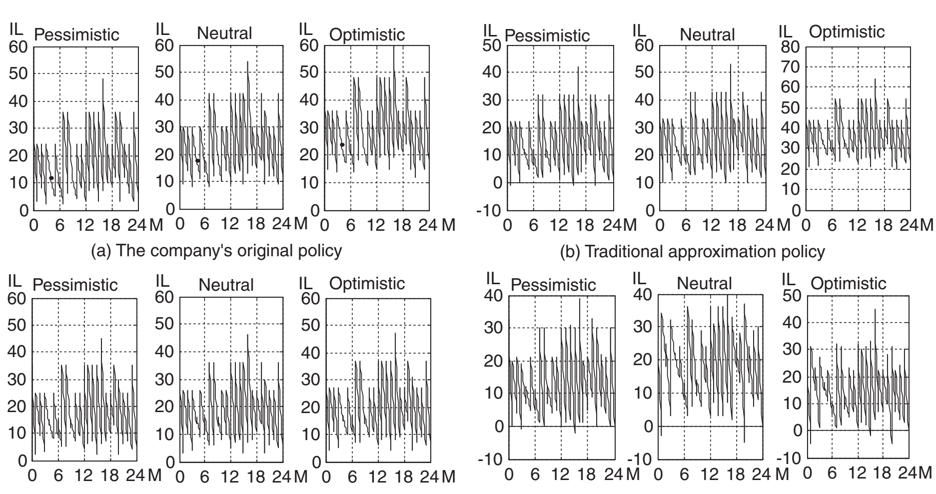

Fig. 4. IL curves for the above four inventory control policies.

contributions to China's automobile industry. As independent companies, Auto 4S stores must establish elaborate purchasing, selling and stocking plans to generate profits and gain market shares in fierce competition. Table B.2 in Appendix B lists 36 months' sales and forecasted data of an automobile model for an Auto 4S store in Wuhan, China, from 2005 to 2007, in which the real demand is nonstationary and the forecasting demand is given in interval. As the orders are replenished instantly and the decision period is one month, the lead time can be regarded to be $L = 0$ and there is no in-transit inventory. The inventory position evolves to the same curve as the inventory level. The shortage demand is backlogged, holding cost $h = 2000$, backorder cost $b = 1000$, unit sale price for a car $p_1 = 160{,}000$ and order price $p_2 = 150{,}000$.

The company's original policy, the traditional approximation policy, the robust counterpart optimization (RCO) policy and the BRB-IC policy are examined for comparison. The initial and trained BRBs for this case study are listed in Table B.3 in Appendix B. The company's original policy is to maintain the planning stock about 1.2 times as much as the forecasting demand. The traditional approximation method for dealing with interval forecasting demand is to transform it into single-point value directly (Gen et al., 1997), and then use stochastic approximation to generate the best policy. In this example, the EOBS policy is used for approximation.

The inventory level curves generated using the above four policies are shown in Fig. 4. The demand data in 2005 are used for initialization and the demand data in 2006 and 2007 are used for comparison. Three kinds of preferences, i.e. pessimistic, neutral and optimistic, are adopted to conduct the sensitivity analysis. The lower, center, and upper bounds of the intervals generated using the four polices are paralleled to pessimistic, neutral and optimistic preferences respectively. As shown in Fig. 4(a), the on-hand inventories generated from the company's original policy are very high in all the three kind of preferences. In Fig. 4(b) where the traditional approximation policy is used, the inventory improves a lot in pessimistic preference, but it has no obvious improvement in the other preferences. Fig. 4(c) shows that the inventory is relatively better controlled using RCO policy. Fig. 4(d) shows that the BRB-IC policy brings the least inventory fluctuation among all the four policies in these three kinds of preferences.

The total profits generated using the four inventory control policies are listed in Table 2. Evidently, the BRB-IC policy leads to the highest profits in all the three kinds of preferences. These results are consistent with the analyses about the inventory level curves.

It should be noted from the inventory level curves and the profits table that policies in pessimistic preference are superior to those in neutral preference which are in turn superior to those in optimistic preference using the company's original policy, tradition approximation policy and RCO policy. The reason is that these three kinds of policies only take account of the current forecasting demand but do not consider the historical demand information and the DFE. When forecasting demand is generally greater than real demand as in this case study, pessimistic preference will be better than optimistic preference using these kinds of three policies. However, this phenomenon is not shown in the BRB-IC policy, because the BRB is trained using the historical demand information, which can compensate the DFE adaptively.

Table 2
Total profits for the above four inventory control policies.

Preference	Policy			
	The company's original policy	Traditional approximation policy	Robust optimization policy	BRB-IC policy
Pessimistic	5,070,000	5,937,000	5,368,000	6,297,000
Neutral	3,894,000	3,784,000	5,172,000	6,048,000
Optimistic	2,718,000	1,616,000	4,974,000	6,166,000

5. Conclusions

In order to deal with inventory control problems under nonstationary and uncertain demand, a BRB-IC method was proposed in this paper. Unlike traditional methods that make ordering policies based on a definite planning horizon into the future whose resulting order quantity for the forthcoming period can be significantly affected by unreliable forecasts for distant periods, the BRB-IC method makes policies according to current inventory, historical demand data and necessary short-term forecasting demand. The method is developed from the decision support mechanism of belief-rule-based inference methodology – RIMER. Belief rules in the BRB-IC model can be seen as certain pivotal ordering policies. There are five essential components in the BRB-IC model, including information transformation, constructing BRB, initializing BRB, BRB-IC inference and training BRB. Specially, an extended optimal base stock (EOBS) policy, which can be used to initialize the BRB, was proven under the assumption that forecasting demand is given in single-point style and DFE follows certain stochastic distribution, and also a BRB-IC inference approach with interval inputs was proposed. This BRB-IC method was validated and shown to be feasible through an example and a case study.

The BRB-IC method has several unique features. Firstly, it can to some extent overcome the drawback of information distortion caused by using stochastic theory alone to approximate real situations. Secondly, it can represent and handle various types of uncertain information such as interval demand data that may emerge in inventory control processes. Thirdly, it can capture complicated and continuous casual and nonlinear relationships. Fourthly, it provides an inventory control model, which is easy to understand and implement. It can incorporate human knowledge and risk preferences, and allows experts to construct and update the structure and parameters using their judgmental knowledge. Fifthly, belief rules can be trained using historical demand information and can compensate the DFE to induce reliable ordering policy.

However, the review period and lead time are assumed to be constant in this paper. In reality, they are usually variable, especially the lead time. Besides, only the parameters of the BRB are identified in this paper. If the nonstationary demand with significant change makes the referential evaluations of belief-rules' antecedents do not fit the hidden pattern, the BRB model should be constructed with variant structure. These limitations need to be investigated in future studies.

Acknowledgements

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Appendix A. The proof of the EOBS policy

Theorem. (EOBS policy): *For the single-echelon inventory problem, if the forecasting demand D_n^f is given in single-point value and the DFE*

ε_n follows a normal distribution ($\varepsilon_n \sim N$ (μ^{DFE}, σ^{DFE})) with density function $f^{DFE}(x)$, we prove an EOBS policy with order-up-to level being

$$S_n = D^f_{n,L+T} - (L+T)\mu^{DFE} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{DFE}, \text{ if } D^f_n = D_n + \varepsilon_n, \text{or} \tag{A.1}$$

$$S_n = D^f_{n,L+T} + (L+T)\mu^{DFE} + \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{DFE}, \text{ if } D^f_n = D_n - \varepsilon_n. \tag{A.2}$$

We only give the proof with $D^f_n = D_n + \varepsilon_n$, and the counterpart result can be obtained in a similar way. The definition of all the variables here is the same as that in the main text of the paper.

Proof. One cycle of the inventory control process includes $L+T$ periods, the time from the beginning of period n to the beginning of period $n+L+T$. The protection period for making an order is $L+T$, which is the same as the inventory control problems with stationary stochastic demand. As stated in the well-known reward renewal theorem, the expected long run cost is equal to the expected cost per cycle divided by the expected cycle length. So, we only need to analyze the cycle $[n, n+L+T)$.

In the backorder case, the inventory level at the beginning of period $n+L+T$ is IL_{n+L+T}, that is

$$\begin{aligned} IL_{n+L+T} &= IL_n + IT_n - \sum_{i=n}^{n+L+T-1} D_i \\ &= IL_n + IT_n - \sum_{i=n}^{n+L+T-1} D^f_i + \sum_{i=n}^{n+L+T-1} \varepsilon_i \\ &= IP_n - D^f_{n,L+T} + \varepsilon_{L+T} \end{aligned} \tag{A.3}$$

where $D^f_{n,L+T} = \sum_{i=n}^{n+L+T-1} D^f_i$ and $\varepsilon_{L+T} = \sum_{i=n}^{n+L+T-1} \varepsilon_i$.

The expected total cost at the end of period $n+L+T-1$ is $E(TC_{n+L+T})$, that is

$$\begin{aligned} E[TC_{n+L+T}] &= E[h[IL_{n+L+T}]^+ + p[IL_{n+L+T}]^-] \\ &= E[(h+p)[IL_{n+L+T}]^+ - pIL_{n+L+T}] \\ &= (h+p)E[IL_{n+L+T}]^+ - p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \\ &= (h+p)\int_{-\left(IP_n - D^f_{n,L+T}\right)}^{+\infty}\left(IP_n - D^f_{n,L+T} + x\right)f^{DFE}_{L+T}(x)dx - p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \end{aligned} \tag{A.4}$$

where $[x]^+ = \max(0, x)$, $[x]^- = \max(0, -x)$, and $[x]^+ = [x]^- + x$.

Using $f^{DFE}_{L+T}(x) = \frac{1}{\sigma^{DFE}_{L+T}}\varphi\left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right)$, we have

$$\begin{aligned} E[TC_{n+L+T}] &= (h+p)\int_{-(IP_n - D^f_{n,L+T})}^{+\infty}\left(IP_n - D^f_{n,L+T} + x\right) \\ &\times \frac{1}{\sigma^{DFE}_{L+T}}\varphi\left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right)dx - p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \\ &= (h+p)\sigma^{DFE}_{L+T}\int_{-(IP_n - D^f_{n,L+T})}^{+\infty}\left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}} - \frac{-IP_n + D^f_{n,L+T} - \mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right) \\ &\varphi \times \left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right)d\left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right) - p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \end{aligned} \tag{A.5}$$

Assuming $\upsilon = \frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}$, then we get

$$\begin{aligned} E[TC_{n+L+T}] &= (h+p)\sigma^{DFE}_{L+T} \\ &\times \int_{\frac{-(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T})}{\sigma^{DFE}_{L+T}}}^{+\infty}\left(\upsilon - \frac{-(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T})}{\sigma^{DFE}_{L+T}}\right)\varphi(\upsilon)d\upsilon \\ &- p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \end{aligned} \tag{A.6}$$

Using $G(x) = \int_x^\infty(\upsilon - x)\varphi(\upsilon)d\upsilon$, there is

$$\begin{aligned} E[TC_{n+L+T}] &= (h+p)\sigma^{DFE}_{L+T}G\left(-\frac{(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T})}{\sigma^{DFE}_{L+T}}\right) \\ &- p\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right) \end{aligned} \tag{A.7}$$

where μ^{DFE}_{L+T}, σ^{DFE}_{L+T} and $f^{DFE}_{L+T}(x)$ are mean, standard deviation and density function of ε_{L+T} respectively; $\varphi(x)(\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}})$ and $\Phi(x)$ are the density function and distribution function of the standard normal distribution; $f^{DFE}_{L+T}(x)$ and $\varphi(x)$ satisfy $f^{DFE}_{L+T}(x) = \frac{1}{\sigma^{DFE}_{L+T}}\varphi\left(\frac{x-\mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right)$; $G(x)$ is defined as $G(x) = \int_x^\infty(\upsilon - x)\varphi(\upsilon)d\upsilon = \varphi(x) - x(1-\Phi(x))$ and satisfies $G'(x) = \Phi(x) - 1$ and $G'(-x) = -(\Phi(-x) - 1)$.

The first derivative of $E[TC_{n+L+T}]$ for IP_n is given by

$$\frac{d(E[TC_{n+L+T}])}{dIP_n} = -(h+p)\left[\Phi - \frac{\left(IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}\right)}{\sigma^{DFE}_{L+T}}\right] - p \tag{A.8}$$

The second derivative of $E[TC_{n+L+T}]$ for IP_n is given by

$$\frac{d^2(E[TC_{n+L+T}])}{d(IP_n)^2} = \frac{h+p}{\sigma^{DFE}_{L+T}}\varphi\left(-\frac{IP_n - D^f_{n,L+T} + \mu^{DFE}_{L+T}}{\sigma^{DFE}_{L+T}}\right) \tag{A.9}$$

Because $\frac{d^2(E[TC_{n+L+T}])}{d(IP_n)^2} > 0, E[TC_{n+L+T}]$ will be optimized when $\frac{d(E[TC_{n+L+T}])}{dIP_n} = 0$, i.e.

$$IP^*_n = D^f_{n,L+T} - \mu^{DFE}_{L+T} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sigma^{DFE}_{L+T} \tag{A.10}$$

Then, we get the EOBS policy:

$$\begin{aligned} S_n = IP^*_n &= D^f_{n,L+T} - \mu^{DFE}_{L+T} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sigma^{DFE}_{L+T} \\ &= D^f_{n,L+T} - (L+T)\mu^{DFE} - \Phi^{-1}\left(\frac{h}{h+p}\right)\sqrt{L+T}\sigma^{DFE} \end{aligned} \tag{A.11}$$

The proof for the lost sales case is similar to the above, except that the inventory level IL_n should be replaced by the on-hand inventory OH_n, and the inventory position IP_n should be defined as Equation (11) in the main text. The EOBS policy S_n has the same format as Equation (A.9).

Appendix B. Tables for the numerical example and the case study

Table B.1
The trained BRB for the numerical example.

N	Trained BRB			
	Q_n			θ_r
	H_1	H_2	H_3	
	β_{r1}	β_{r2}	β_{r3}	
1	0.56	0.44	0	0.1985
2	0.06	0.94	0	0.3246
3	0	0.56	0.44	0.3213
4	0	0.06	0.94	0.5047
5	0	0	1	0.6685
6	1	0	0	0.5047
7	0.56	0.44	0	0.1985
8	0.06	0.94	0	0.3903
9	0	0.56	0.44	0.2436
10	0	0.06	0.94	0.5047
11	1	0	0	0.6016
12	1	0	0	0.5047
13	0.56	0.44	0	0.1745
14	0.06	0.94	0	0.3903
15	0	0.56	0.44	0.3213
16	1	0	0	0.5
17	1	0	0	0.5116
18	1	0	0	0.5047
19	0.56	0.44	0	0.1985
20	0.06	0.94	0	0.3246
21	1	0	0	0.5
22	1	0	0	0.5
23	1	0	0	0.6016
24	1	0	0	0.5047
25	0.56	0.44	0	0.1985

N	Trained BRB			
	Q_n			θ_r
	H_1	H_2	H_3	
	β_{r1}	β_{r2}	β_{r3}	
26	1	0	0	0.6515
27	0.66	0.44	0	0.1981
28	1	0	0	0.3232
29	0	0.66	0.34	0.2436
30	0	0.16	0.84	0.5047
31	1	0	0	0.5870
32	1	0	0	0.6515
33	0.66	0.34	0	0.2385
34	0.16	0.84	0	0.3903
35	0	0.66	0.34	0.2436
36	1	0	0	0.5313
37	1	0	0	0.7766
38	1	0	0	0.7205
39	0.66	0.34	0	0.2385
40	0.16	0.84	0	0.3232
41	1	0	0	0.5
42	1	0	0	0.5
43	1	0	0	0.7766
44	1	0	0	0.6515
45	0.66	0.34	0	0.1981
46	1	0	0	0.5
47	1	0	0	0.5
48	1	0	0	0.5313
49	1	0	0	0.5870
50	1	0	0	0.6515

N	Trained BRB			
	Q_n			θ_r
	H_1	H_2	H_3	
	β_{r1}	β_{r2}	β_{r3}	
51	1	0	0	0.6023
52	1	0	0	0.7234
53	0.76	0.24	0	0.1745
54	0.26	0.74	0	0.2395
55	0	0.76	0.24	0.2436
56	1	0	0	0.5
57	1	0	0	0.8172
58	1	0	0	0.7155
59	0.76	0.24	0	0.1745
60	0.26	0.74	0	0.2395
61	1	0	0	0.5
62	1	0	0	0.7031
63	1	0	0	0.6250
64	1	0	0	0.7155
65	0.76	0.24	0	0.1745
66	1	0	0	0.5
67	1	0	0	0.5
68	1	0	0	0.7031
69	1	0	0	0.8172
70	1	0	0	0.7234
71	1	0	0	0.5
72	1	0	0	0.5
73	1	0	0	0.5
74	1	0	0	0.5
75	1	0	0	0.6023

N	Trained BRB			
	Q_n			θ_r
	H_1	H_2	H_3	
	β_{r1}	β_{r2}	β_{r3}	
76	1	0	0	0.5
77	1	0	0	0.5713
78	1	0	0	0.7484
79	0.86	0.14	0	0.1745
80	0.36	0.64	0	0.2512
81	1	0	0	0.5
82	1	0	0	0.5713
83	1	0	0	0.7141
84	1	0	0	0.7455
85	0.86	0.14	0	0.1745
86	1	0	0	0.5
87	1	0	0	0.5
88	1	0	0	0.5
89	1	0	0	0.7141
90	1	0	0	0.7484
91	1	0	0	0.5
92	1	0	0	0.5
93	1	0	0	0.5
94	1	0	0	0.5713
95	1	0	0	0.5713
96	1	0	0	0.5
97	1	0	0	0.5
98	1	0	0	0.5
99	1	0	0	0.5
100	1	0	0	0.5

N	Trained BRB			
	Q_n			θ_r
	H_1	H_2	H_3	
	β_{r1}	β_{r2}	β_{r3}	
101	1	0	0	0.5
102	1	0	0	0.5
103	1	0	0	0.5078
104	1	0	0	0.8191
105	0.96	0.04	0	0.5047
106	1	0	0	0.5
107	1	0	0	0.5
108	1	0	0	0.5078
109	1	0	0	0.5078
110	1	0	0	0.8191
111	1	0	0	0.5
112	1	0	0	0.5
113	1	0	0	0.5
114	1	0	0	0.5078
115	1	0	0	0.5078
116	1	0	0	0.5
117	1	0	0	0.5
118	1	0	0	0.5
119	1	0	0	0.5
120	1	0	0	0.5
121	1	0	0	0.5
122	1	0	0	0.5
123	1	0	0	0.5
124	1	0	0	0.5
125	1	0	0	0.5

Table B.2
Sales and forecasted data of an automobile model for an Auto 4S store.

Time	Sales	Forecast	Time	Sales	Forecast	Time	Sales	Forecast
2005.01	46	[40, 50]	2006.01	33	[30, 40]	2007.01	29	[30, 40]
2005.02	27	[30, 40]	2006.02	16	[20, 30]	2007.02	28	[30, 40]
2005.03	44	[40, 50]	2006.03	22	[20, 30]	2007.03	33	[30, 40]
2005.04	41	[40, 50]	2006.04	12	[20, 30]	2007.04	29	[30, 40]
2005.05	30	[30, 40]	2006.05	7	[10, 20]	2007.05	38	[40, 50]
2005.06	44	[40, 50]	2006.06	16	[20, 30]	2007.06	15	[20, 30]
2005.07	37	[40, 50]	2006.07	10	[10, 20]	2007.07	12	[20, 30]
2005.08	25	[30, 40]	2006.08	30	[30, 40]	2007.08	33	[30, 40]
2005.09	24	[30, 40]	2006.09	23	[30, 40]	2007.09	24	[20, 30]
2005.10	30	[30, 40]	2006.10	8	[10, 20]	2007.10	18	[20, 30]
2005.11	20	[30, 40]	2006.11	16	[20, 30]	2007.11	21	[20, 30]
2005.12	33	[30, 40]	2006.12	21	[20, 30]	2007.12	30	[30, 40]

Table B.3
The initial and trained BRBs for the case study.

N	IL_n	$D^f_{n,L+T}$	Initial BRB				Trained BRB (Pessimistic)				Trained BRB (Neutral)				Trained BRB (Optimistic)			
			Q_n			θ_r	Q_n			θ_r	Q_n			θ_r	Q_n			θ_r
			H_1	H_2	H_3		H_1	H_2	H_3		H_1	H_2	H_3		H_1	H_2	H_3	
			β_{r1}	β_{r2}	β_{r3}		β_{r1}	β_{r2}	β_{r3}		β_{r1}	β_{r2}	β_{r3}		β_{r1}	β_{r2}	β_{r3}	
1	H_{11}	H_{21}	0.45	0.55	0	1	0.45	0.55	0	0.5	0.3	0.7	0	0.5	0.15	0.85	0	0.5
2	H_{11}	H_{22}	0.15	0.85	0	1	0.15	0.85	0	0.5	0	1	0	0.5	0	0.85	0.15	0.5
3	H_{11}	H_{23}	0	0.825	0.175	1	0	0.825	0.175	0.5	0	0.8	0.2	0.5	0	0.725	0.275	0.5
4	H_{11}	H_{24}	0	0.525	0.475	1	0	0.525	0.475	0.5	0	0.525	0.475	0.5	0	0.5	0.5	0.5
5	H_{11}	H_{25}	0	0.2	0.8	1	0	0.2	0.8	0.5	0	0.2	0.8	0.5	0	0.275	0.725	0.5
6	H_{12}	H_{21}	1	0	0	1	1	0	0	0.75	0.975	0.025	0	0.5	1	0	0	0.5
7	H_{12}	H_{22}	0.775	0.225	0	1	0.825	0.175	0	0.75	0.925	0.075	0	0.25	1	0	0	0.5
8	H_{12}	H_{23}	0.45	0.55	0	1	0.45	0.55	0	0.75	0.35	0.65	0	0.25	0.675	0.325	0	0.25
9	H_{12}	H_{24}	0.15	0.85	0	1	0.175	0.825	0	0.75	0.25	0.75	0	0.25	0.45	0.55	0	0.5
10	H_{12}	H_{25}	0	0.825	0.175	1	0	0.825	0.175	0.5	0	0.9	0.1	0.25	0.025	0.975	0	0.25
11	H_{13}	H_{21}	1	0	0	1	0.975	0.025	0	0.5	1	0	0	0.5	0.975	0.025	0	0.5
12	H_{13}	H_{22}	1	0	0	1	0.975	0.025	0	0.5	1	0	0	0.75	0.975	0.025	0	0.5
13	H_{13}	H_{23}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.75	0.975	0.025	0	0.75
14	H_{13}	H_{24}	0.775	0.225	0	1	0.775	0.225	0	0.5	0.875	0.125	0	0.75	0.975	0.025	0	0.75
15	H_{13}	H_{25}	0.45	0.55	0	1	0.45	0.55	0	0.5	0.475	0.025	0	0.75	0.6	0.4	0	0.75
16	H_{14}	H_{21}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
17	H_{14}	H_{22}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
18	H_{14}	H_{23}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
19	H_{14}	H_{24}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
20	H_{14}	H_{25}	1	0	0	1	0.975	0.025	0	0.5	0.95	0.05	0	0.5	0.975	0.025	0	0.5
21	H_{15}	H_{21}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
22	H_{15}	H_{22}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
23	H_{15}	H_{23}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
24	H_{15}	H_{24}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5
25	H_{15}	H_{25}	1	0	0	1	0.975	0.025	0	0.5	0.975	0.025	0	0.5	0.975	0.025	0	0.5

References

Axsater, S. (2007). *Inventory control*. Peking: Tsinghua University Press.

Babai, M. Z., & Dallery, Y. (2009). Dynamic versus static control policies in single stage production-inventory systems. *International Journal of Production Research, 47*(2), 415–433.

Bertsimas, D., & Thiele, A. (2006a). Robust and data-driven optimization: Modern decision-making under uncertainty. *INFORMS Tutorials on Operations Research* (Chapter 4).

Bertsimas, D., & Thiele, A. (2006b). A robust optimization approach to inventory theory. *Operations Research, 54*(1), 150–168.

Bollapragada, S., & Morton, T. E. (1999). A simple heuristic for computing nonstationary (*s*,*S*) policies. *Operations Research, 47*(4), 576–584.

Chang, R., Yang, J. B., & Wang, H. W. (2007). An algorithm for training parameters in belief rule-bases based on the gradient and dichotomy methods. *Systems Engineering* (Special Issue, pp. 287–291, in Chinese).

Chen, Y. W., Yang, J. B., Xu, D. L., Zhou, Z. J., & Tang, D. W. (2011). Inference analysis and adaptive training for belief rule based systems. *Expert Systems with Applications, 38*(10), 12845–12860.

Dey, O., & Chakraborty, D. (2009). Fuzzy period review system with fuzzy random variable demand. *European Journal of Operational Research, 198*(1), 113–120.

Ellon, S., & Elmaleh, J. (1970). Adaptive limits in inventory control. *Management Science, 16*(8), B533–B548.

Federgruen, A., & Heching, A. (1999). Combined pricing and inventory control under uncertainty. *Operations Research, 47*(3), 454–475.

Gallego, G., & van Ryzin, G. (1994). Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science, 40*(8), 999–1020.

Gavirneni, S., & Tayur, S. (2001). An efficient procedure for nonstationary inventory control. *IIE Transactions, 33*(2), 83–89.

Gen, M., Tsujimura, Y., & Zheng, D. Z. (1997). An application of fuzzy set theory to inventory control models. *Computers and Industrial Engineering, 33*(3–4), 553–556.

Giannoccaro, I., Pontrandolfo, P., & Scozzi, B. (2003). A fuzzy echelon approach for inventory management in supply chains. *European Journal of Operational Research, 149*(1), 185–196.

Gopal, T. V. (2007). Ethics in deploying data to make wise decisions. *International Review of Information Ethics, 7*, 1–7.

Graves, S. C. (1999). A single-item inventory model for a nonstationary demand process. *Manufacturing and Service Operations Management, 1*(1), 50–61.

Guo, M., Yang, J. B., Chin, K. S., & Wang, H. W. (2007). Evidential reasoning based preference programming for multiple attribute decision analysis under uncertainty. *European Journal of Operational Research, 182*(3), 1294–1312.

Hu, C. H., Si, X. S., & Yang, J. B. (2010). System reliability prediction model based on evidential reasoning algorithm with nonlinear optimization. *Expert Systems with Applications, 37*(3), 2550–2562.

Hung, T. W., Fang, S. C., Nuttle, H. L. W., & King, R. E. (1997). A fuzzy-control-based quick response reorder scheme for retailing of seasonal apparel. In *Proceedings of the international conference of information sciences* (Vol. 2, pp. 300–303).

Iida, T. (1999). The infinite horizon non-stationary stochastic inventory problem: Near myopic policies and weak ergodicity. *European Journal of Operational Research, 116*, 405–422.

Johnson, G. D., & Thompson, H. E. (1975). Optimality of myopic inventory policies for certain dependent demand processes. *Management Science, 21*(11), 1303–1307.

Kamal, L., & Sculfort, J. L. (2007). Fuzzy modeling of inventory control system in uncertain environment. *International Symposium on Logistics and Industrial Informatics*, 53–57.

Karlin, S. (1960). Dynamic inventory policy with varying stochastic demands. *Management Science, 6*(3), 231–258.

Kong, G. L., Xu, D. L., Liu, X. B., & Yang, J. B. (2009). Applying a belief rule-base inference methodology to a guideline-based clinical decision support system. *Expert Systems, 26*(5), 391–408.

Kurawarwala, A. A., & Matsuo, H. (1996). Forecasting and inventory management of short life-cycle products. *Operations Research, 44*(1), 131–150.

Lee, H. L., Padmanabhan, V., & Whang, S. (1997). Information distortion in a supply chain: The bullwhip effect. *Management Science, 43*(4), 546–558.

Lee, H. L., So, K. C., & Tang, C. S. (2000). The value of information sharing in a two-level supply chain. *Management Science, 46*(5), 626–643.

Leung, R. W. K., Lau, H. C. W., & Kwong, C. K. (2003). On a responsive replenishment system: A fuzzy logic approach. *Expert Systems, 20*(1), 20–32.

Li, L. S., Kabadi, S. N., & Nair, K. P. K. (2002). Fuzzy models for single-period inventory problem. *Fuzzy Sets and Systems, 132*(3), 273–289.

Liu, J., Ruan, D., Wang, H., & Martinez, L. (2009). Improving nuclear safeguards evaluation through enhanced belief rule-based inference methodology. *International Journal of Nuclear Knowledge Management, 3*(3), 312–339.

Mellichamp, J. M., & Love, R. M. (1978). Production switching heuristics for the aggregate planning problem. *Management Science, 24*(12), 1242–1251.

Metan, G., & Thiele, A. C. (2008). The Value of information in inventory management. *INFORMS annual meeting*, Washington.

Morton, T. E. (1978). The nonstationary infinite horizon inventory problem. *Management Science, 24*(14), 1474–1482.

Petrovic, D., Petrovic, R., & Vujosevic, M. (1996). Fuzzy models for the newsboy problem. *International Journal of Production Economics, 45*(1-3), 435–441.

Raghunathan, S. (2001). Information sharing in a supply chain: A note on its value when demand is nonstationary. *Management Science, 47*(4), 605–610.

Roy, T. K., & Maiti, M. (1997). A fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. *European Journal of Operational Research, 99*(2), 425–432.

Scarf, H. (1958). A min–max solution of an inventory problem. *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press (Chapter 12).

Scheller-Wolf, A., & Tayur, S. (1997). Reducing international risk through quantity contracts. *GSIA Working Paper*, Pittsburgh: Carnegie Mellon University.

Si, X. S., Hu, C. H., Yang, J. B., & Zhang, Q. (2011). On the dynamic evidential reasoning algorithm for fault prediction. *Expert Systems with Applications, 38*(5), 5061–5080.

Song, J. S., & Zipkin, P. (1993). Inventory control in a fluctuating demand environment. *Operations Research, 41*(2), 351–370.

Tang, D. W., Yang, J. B., Chin, K. S., Wong, Z. S. Y., & Liu, X. B. (2011). A methodology to generate a belief rule base for customer perception risk analysis in new product development. *Expert Systems with Applications, 38*(5), 5373–5383.

Tarim, S. A., & Kingsman, B. G. (2006). Modelling and computing (R_n,S_n) policies for inventory systems with non-stationary stochastic demand. *European Journal of Operational Research, 174*(1), 581–599.

Treharne, J. T., & Sox, C. R. (2002). Adaptive inventory control for nonstationary demand and partial information. *Management Science, 48*(2), 607–624.

Veinott, A. F. (1966). On the optimality of (s,S) inventory policies: New conditions and new proof. *SIAM Journal on Applied Mathematics, 14*(5), 1067–1083.

Wang, Y. M., Yang, J. B., Xu, D. L., & Chin, K. S. (2006). The evidential reasoning approach for multiple attribute decision analysis using interval belief degrees. *European Journal of Operational Research, 175*(1), 35–66.

Wang, Y. M., Yang, J. B., Xu, D. L., & Chin, K. S. (2009). Consumer preference prediction by using a hybrid evidential reasoning and belief rule-based methodology. *Expert Systems with Applications, 36*(4), 8421–8430.

Xu, D. L., Liu, J., Yang, J. B., Liu, G. P., Wang, J., Jenkinson, I., et al. (2007). Inference and learning methodology of belief-rule-based expert system for pipeline leak detection. *Expert Systems with Applications, 32*(1), 103–113.

Yang, J. B. (2001). Rule and utility based evidential reasoning approach for multiple attribute decision analysis under uncertainty. *European Journal of Operational Research, 131*(1), 31–61.

Yang, J. B., Liu, J., Wang, J., Sii, H. S., & Wang, H. W. (2006). A generic rule-base inference methodology using the evidential reasoning approach-RIMER. *IEEE Transactions on System, Man and Cybernetics – Part A: Systems and Humans, 36*(2), 266–285.

Yang, J. B., Liu, J., Xu, D. L., Wang, J., & Wang, H. W. (2007). Optimization models for training belief rule based systems. *IEEE Transactions on systems, Man, and Cybernetics – Part A: Systems and Humans, 37*(4), 569–585.

Yang, J. B., & Sen, P. (1994). A general multi-level evaluation process for hybrid MADM with uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics, 24*(10), 1458–1473.

Yang, J. B., & Singh, M. G. (1994). An evidential reasoning approach for multiple attribute decision making with uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics, 24*(1), 1–18.

Yang, J. B., & Xu, D. L. (2002a). On the evidential reasoning algorithm for multiattribute decision analysis under uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans, 32*(3), 289–304.

Yang, J. B., & Xu, D. L. (2002b). Nonlinear information aggregation via evidential reasoning in multiattribute decision analysis under uncertainty. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans, 32*(3), 376–393.

Yao, J. S., & Su, J. S. (2000). Fuzzy inventory with backorder for fuzzy total demand based on interval-valued fuzzy set. *European Journal of Operational Research, 124*, 390–408.

Zhou, Z. J., Hu, C. H., Xu, D. L., Yang, J. B., & Zhou, D. H. (2011). Bayesian reasoning approach based recursive algorithm for online updating belief rule based expert system of pipeline leak detection. *Expert Systems with Applications, 38*(4), 3937–3943.

Zhou, Z. J., Hu, C. H., Yang, J. B., Xu, D. L., & Zhou, D. H. (2009). Online updating belief-rule-based system for pipeline leak detection under expert intervention. *Expert Systems with Applications, 36*(4), 7700–7709.

Zipkin, P. H. (1989). Critical number policies for inventory models with periodic data. *Management Science, 35*(1), 71–80.

Zipkin, P. H. (2000). *Foundations of inventory management*. New York: McGraw-Hill.