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A reference ideal model with evidential reasoning for probabilistic-based expressions

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Abstract
Due to experts' different cognitions, experiences, and knowledge backgrounds, their evaluations may be different, and none of them can be ignored, which leads to the development of the probabilistic linguistic term set (PLTS) and the probabilistic hesitant fuzzy set (PHFS). In practical situations, sometimes the optimal alternative exists in a reference ideal interval instead of the maximum or the minimum. This paper constructs a reference ideal model with evidential reasoning for the PLTS and the PHFS. At first, a maximum deviation method based on two hierarchical attributes is proposed, aiming at determining the attribute weights in a multi-attribute decision-making problem. Then, since the evaluations are provided with different forms and principles, a normalisation process can help to make the evaluations unified. Moreover, the evidential reasoning process is introduced to aggregate evaluation grades based on the probabilities in the probabilistic-based expressions. And the final decision results are obtained by applying the distance between the aggregated evaluation grades and the extreme values. Then, we use the proposed model for the potential chronic obstructive pulmonary disease patient evaluation to verify the operability. Besides, a comparative analysis is also conducted to prove the rationality of the model.

Keywords Reference ideal method · Evidential reasoning · Probabilistic linguistic term set · Probabilistic hesitant fuzzy set

1 Introduction

Multi-attribute decision-making (MADM) aims to choose the optimal alternative or rank the alternatives based on multiple attributes. Traditional MADM takes the maximal and minimal values as the ideal or non-ideal solution in a decision-making problem. But in practical situations, the judgment principle may be more flexible. For instance, the most suitable temperature for most plants to grow is between 20℃ and 40℃. And the normal pH value of the human body is between [7.35, 7.45], with an interval between 0 and 14. In these cases, the optimal solution should not be the maximum or the minimum but exist in some intervals between the maximum and minimum, which are called reference ideal intervals. A typical MADM problem usually contains attributes, alternatives and decision-makers. In a practical MADM situation, not all the attributes can be evaluated by the same judgment principle. Some may be assessed by the absolute principle, while others are evaluated by the reference ideal principle.

As another essential factor of MADM, decision-makers and their expression habits are also interesting content for investigation and discussion. Considering the more and more complex environments for decision-making, the probabilistic-based expressions, such as probabilistic linguistic term sets (PLTSs) and probabilistic hesitant fuzzy sets (PHFSs), can be applied to

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express the evaluations provided by decision-makers because they can take the hesitancy and the accuracy of assessments into consideration at the same time. Due to these apparent advantages and research value, the related properties and methods based on the PLTS and the PHFS have been deeply investigated in recent years. Zhu [1] first proposed the concept of the PHFS and the representation of the probabilistic hesitant fuzzy preference relation (PHFPR), which became a firm foundation for the follow-up studies. Then, the decision methods for the PHFS based on different decision-making situations have been studied, such as consensus building with decision groups [2], interactive decision-making approach [3], etc. It also has been applied to solve practical problems widely, such as site selection for carbon capture, utilisation and storage [4], logistics company assessment [5], sharing accommodation recommendations, etc. Recently, Liu et al. [6] obtained complete PHFPR by a modified probability calculation method for hesitant fuzzy preference relations. Then, they combined PHFPR with data envelopment analysis to yield the priority vector. Considering that PHFS can model uncertainty more effectively, Liu et al. [7] proposed a nonlinear programming model with the aim of maximizing entropy to calculate the probabilities of elements in a probabilistic hesitant fuzzy element. Meanwhile, they also established a new model inspired by the water-filling theory to calculate the probability of risk status. Additionally, by considering interdependence and variation to aggregate the different preferences expressed in PHFS, Grag et al. [8] proposed a Muirhead average operator based on variance. As for the PLTS, it was first proposed by Pang et al. [9], and they also investigated the aggregation-based decision-making methods and the TOPSIS based on PLTS. Later, Zhang et al. [10] proposed the concepts of probabilistic linguistic preference relations (PLPRs) and studied their consistency and the consensus-reaching model. Recently, Yu et al. [11] dealt with a large-scale consensus-reaching problem in the PLTS environment, in which they proposed a new distance-to-centre clustering algorithm, a new weight-determining method and a punishment-driven consensus model. Also in the consensus problem, You and Hou [12] took into account that the existing feedback mechanism used fixed boundary parameters and ignored individual psychological behaviour would result in false outcomes, and they developed a personalised feedback mechanism based on self-confidence. In order to solve the problem that the multi-granular fuzzy linguistic model enables experts to express their opinions with multiple linguistic term sets, but cannot simultaneously reflect the degree of hesitation and preference, Du and Liu [13] introduced PLTS with the multi-granular fuzzy linguistic model and combined with prospect theory to reflect the experts' attitudes. From the perspective that making decisions is not instantaneous behaviour, Zhang et al. [14] proposed a process-oriented method to deal with the MADM problem with uncertain attributes' weights.

For a MADM problem, the most critical issues are aggregating the evaluations and getting the ranking of alternatives. Evidential reasoning (ER), as a helpful tool for aggregation, can deal with these issues by integrating the belief degrees of evaluation grades based on different attributes. ER was developed based on the Dempster-Shafer evidence theory [15] and decision theory. By introducing the belief structure, Yang and Xu [16, 17] brought uncertainty to the ER and made it a complete methodology for MADM. Then, the linguistic terms were used to replace the original evaluation grades, i.e., the crisp terms [18]. And the belief degree in the ER was extended to the intervals by Wang et al. [19, 20], and so were the attribute weights [21]. Then, Zhou et al. [22] constructed several ER models with interval weights and reliabilities. Besides, Wang and Zhang [23] introduced ER approach to the environment with triangular or trapezoidal fuzzy sets. Fang and Liao [24] introduced ER to the PLTS environment. Meanwhile, they constructed the decision-making framework to adapt to the risky situation by combining it with prospect theory. Additionally, to handle the uncertain environment better, Ma et al. [25] required experts to provide PLTSs to express their evaluations of alternatives and linguistic terms to describe the degree of familiarity of the problem. By combining the familiarity degree and the group similarity degree, the reliability degree can be derived, and then the ER approach is conducted. Recently, ER approach was also combined with interval type-2 fuzzy sets [26], hesitant fuzzy belief structure [27], interval-valued intuitionistic fuzzy sets [28], etc.

From the above literature review, we can see that there are lots of studies that use PLTS or PHFS to solve MADM problems, and the ER approach has been combined with many other fuzzy sets. However, there are still some points to be improved, which are also the motivations of our study. Firstly, PLTS and PHFS, in fact, can be seen as probabilistic-based expressions, therefore it is probable that these two kinds of sets exist simultaneously in a specific situation. However, there are fewer researchers who deal with PLTS and PHFS together. Hence, in this paper, one of the tasks is to deal with probabilistic-based expressions, i.e., PHFS and PLTS, together. Besides, in most of the studies about PLTS and PHFS, the value of an alternative under each attribute is usually in the largest level or the smallest lever, and then the performance of the alternative is the best. However, in some cases, the best value exists in some intervals, and that is what existing studies ignore. Hence, in this paper, we also need to deal with some attributes with reference ideal intervals.

Therefore, in this paper, we are going to combine the PLTS and the PHFS with ER. The PHFS and the PLTS both consist of the evaluation values and the related probabilities, which can be seen as the evaluation grades and the belief degrees in ER. Therefore, the PHFS and the PLTS share the same evaluation structures as ER, but their expression forms are clearer and simpler. Based on that, we will propose a novel method for dealing with a reference ideal decision-making problem with heterogeneous expression structures. The main contributions of this paper are shown below:

(1) Propose a processing method for PLTS and the PHFS. Since the knowledge backgrounds, cognitions and expression habits of decision-makers are different from each other, they may not be willing to choose the same expression form to evaluate objects. Therefore, it is reasonable to simultaneously apply the PLTS and the PHFS in a decision-making problem, and decision-makers can use either expression form according to their needs. In this paper, the PLTS and the PHFS can be unified according to their meanings of preference degrees.
(2) Provide a maximum deviation method for two-level attributes. The maximum deviation method is used to obtain the weights of attributes based on the evaluations provided by the decision-makers. The more the evaluations among alternatives different from each other based on the attributes, the more effective information the attributes contain. In this paper, by applying the nonlinear programming model, we extend the maximum deviation method to deal with two levels of attributes.
(3) Develop a decision method with ER for reference ideal decision-making problem. In this paper, the attributes can be evaluated based on different principles, such as the reference ideal principle or the non-reference ideal principle. The evaluations should be integrated into a uniform principle, and the probabilities of evaluations are aggregated based on different attributes through ER process. Finally, the ranking of alternatives can be obtained by the unified evaluations and the aggregated probabilities.

The rest of this paper is organized as follows: Sect. 2 introduces the basic concepts and the distance measures of the PHFS and the PLTS and four basic axioms of evidential reasoning. In Sect. 3, at first, we provide a problem description and the congruent relationship between the fuzzy numbers and linguistic terms. Then, the maximum deviation method for two hierarchical attributes is proposed in Subsection 3.2, and the evaluations with different forms and principles are integrated into Subsection 3.3. Based on the obtained attribute weights and the normalized evaluations, the evidential reasoning algorithm is applied to aggregate the probabilities of each evaluation grade for alternatives to get the rankings and the decision results. Section 4 applies the proposed method to the evaluation problem for potential chronic obstructive pulmonary disease (COPD) patients and also makes a comparative analysis to verify the reasonability of the proposed methods. The paper ends with the concluding remarks in Sect. 5.

2 Preliminaries

In this section, the concepts of the PHFS and the PLTS are introduced, as well as their distance measures. Besides, the four axioms of evidential reasoning are also introduced.

2.1 Concepts of the PHFS and the PLTS

The concept of PHFS was first proposed by Zhu [1] to help people express their preferences in a more precise way than the hesitant fuzzy sets. Its mathematical expression is

$$H = \left\{ < x, h_x\left(p_x\right) > x \in X \right\} \tag{1}$$

where X is a reference set, and a PHFS on X is in terms of a function that when applied to X returns to a subset of $[0, 1]$. The symbols h_x and p_x are two subsets of $[0, 1]$, and h_x denotes the possible membership degrees of the element $x \in X$ to the set H, and p_x denotes the possibilities of h_x satisfying $\sum p_x = 1$. For convenience, we call $h_x\left(p_x\right)$ a P-HFE denoted by

$$h_x\left(p_x\right) = \left\{ h_x^l\left(p_x^l\right) | l = 1, 2, K, |h_x(p_x)| \right\} \tag{2}$$

where p_x^l is the probability of the possible membership degree h_x^l, satisfying $\sum_{l=1}^{|h_x(p_x)|} p_x^l = 1$.

However, in practical situations, decision makers may not provide complete information with a P-HFE, that is to say, the total probability of a P-HFE should satisfy $\sum_{l=1}^{|h_x(p_x)|} p_x^l \leq 1$. In this case, this kind of P-HFE is called the weak P-HFE [29]. The term P-HFE in the rest of this paper means the weak P-HFE.

The concept of PLTS is also proposed as the same purpose as the PHFS [9], whose definition is: Let $S = \left\{ s_{-\tau}, \ldots, s_{-1}, s_o, s_1, \ldots, s_\tau \right\}$ be a linguistic term set, where τ is a positive integer, a PLTS can be defined as

$$L(p) = \left\{ L^{(t)}\left(p^{(t)}\right) | L^{(t)} \in S, p^{(t)} \geq 0, t = 1, 2, \ldots, \#L(p), \sum_{t=1}^{\#L(p)} p^{(t)} \leq 1 \right\} \tag{3}$$

where $L^t\left(p^{(t)}\right)$ is the linguistic term $L^{(t)}$ associated with the probability $p^{(t)}$, and $\#L(p)$ is the number of all different linguistic terms in $L(p)$.

2.2 Distance measures for the PLTS and the PHFS

For the PLTS, Pang et al. [9] proposed a Euclidean distance to express the deviation degree between two PLTSs, shown as follows:

Definition 3 [9]. Let $L_1(p) = \left\{ L_1^{(k)}\left(p_1^{(k)}\right) | k = 1, 2, \ldots, \#L_1(p) \right\}$ and $L_2(p) = \left\{ L_2^{(k)}\left(p_2^{(k)}\right) | k = 1, 2, \ldots, \#L_2(p) \right\}$ be two PLTSs, we call

$$d\left(L_1(p), L_2(p)\right) = \sqrt{\frac{1}{\#L_1(p)} \sum_{k=1}^{\#L_1(p)} \left(p_1^{(k)} r_1^{(k)} - p_2^{(k)} r_2^{(k)} \right)^2} \tag{4}$$

the deviation degree between $L_1(p)$ and $L_2(p)$, where $r_1^{(k)}$ and $r_2^{(k)}$ are the subscripts of linguistic terms $L_1^{(k)}$ and $L_2^{(k)}$ respectively.

As for P-HFEs, Ding et al. [3] provide the Hamming distance measures to express the difference between the P-HFEs. Motivated by that, we propose the Euclidean distance measure for P-HFEs, as follows:

Definition 4. Let $h(p)_1 = \left\{h_1^1\left(p_1^1\right), h_1^2\left(p_1^2\right), \dots, h_1^l\left(p_1^l\right)\right\}$ and $h(p)_2 = \left\{h_2^1\left(p_2^1\right), h_2^2\left(p_2^2\right), \dots, h_2^l\left(p_2^l\right)\right\}$.

Then

$$d\left(h(p)_1, h(p)_2\right) = \sqrt{\frac{1}{l}\sum_{k=1}^{l}\left(p_1^k h_1^k - p_2^k h_2^k\right)^2} \tag{5}$$

is called the Euclidean distance of P-HFEs, where the h_1^k and h_2^k is the elements in P-HFEs $h(p)_1$ and $h(p)_2$, respectively.

2.3 Axioms of evidential reasoning

An evidential reasoning framework should satisfy following synthesis axioms [16]:

(1) If no basic attribute is assessed to an evaluation grade at all, then the general attribute should not be assessed to the same grade either.
(2) If all basic attributes are precisely assessed to an individual grade, then the general attribute should also be precisely assessed to the same grade.
(3) If all basic attributes are completely assessed to a subset of grades, then the general attribute should be completely assessed to the same subset as well.
(4) If any basic assessment is incomplete, then a general assessment obtained by aggregating the incomplete and complete basic assessments should also be incomplete with the degree of incompleteness properly assigned.

3 Reference ideal model based on evidential reasoning for PHFS and PLTS

Evidential reasoning was first proposed as a useful tool to analyse and integrate distributed assessment information in 1994 [30, 31]. Then, it was further developed and applied to solve MADM problems [16]. Moreover, the relationship between evidential reasoning and Bayes' rule is explored [32], in which the Bayes' rule can be seen as a particular case of evidential reasoning under certain conditions. In general, evidential reasoning can express and use uncertain information to obtain reasonable decision results.

In practical situations, not all the best solutions are extreme cases. Some of the optimal solutions exist in an interval between the minimum and maximum, depending on different decision-making environments. In order to solve such decision-making problems, Cables et al. [33] proposed the reference ideal method (RIM). Moreover, determining the attribute weights is also an important issue in MADM, especially when the weights are unknown. Therefore, an extension of the maximum deviation method will be proposed in this section.

3.1 Problem description

Suppose that there are n alternatives $X_i (i = 1, 2, \dots, n)$, m attributes $A_j (j = 1, 2, \dots, m)$. The evaluations provided by decision makers for alternative X_i based on the attribute A_j can be represented by the PLTS $L(p)_{ij}$ or the P-HFE $h(p)_{ij}$. The decision problem is to rank the alternatives and select the optimal solution based on the attributes and their evaluations. In order to make it easier to understand, all the evaluations are display in a decision matrix, which is shown in Table 1 (where the PLTS $(p)_{ij} = \left\{L^{(k)}\left(p^{(k)}\right)_{ij} | L^{(k)}{}_{ij} \in S, p^{(k)}{}_{ij} \geq 0, k = 1,2, \dots, \#L(p), \sum_{k=1}^{\#L(p)} p^{(k)}{}_{ij} \leq 1\right\}$, and the P-HFE $h(p)_{ij} = \left\{h^l\left(p^l\right)_{ij} | l = 1, 2, K, |h(p)|, \sum_{l=1}^{|h(p)|} p^l \leq 1\right\}$).

Remark 1. The probability p in the PLTS and the P-HFE represents the degree of belief in evidential reasoning rule. Note that the evaluations sometimes are incomplete as the total belief degrees are less than 1.

The PLTS and the P-HFE are two different information expressions and have their own scales. But the fuzzy numbers and linguistic terms can be mutually transformed because each of their evaluation grades share the same meanings. For P-HFEs, decision makers usually use the 0.1–0.9 scale, which is symmetrically distributed around the median 0.5. In order to match the 0.1–0.9 scale for P-HFEs and be convenient for understanding and calculation, the parameter τ in the linguistic term set should be satisfy $\tau = 4$. Therefore, the linguistic term set in this paper is $S = \{s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4, \}$. Each fuzzy number or linguistic term in the scale has specific meaning, which are shown in Table 2.

3.2 Maximum deviation method for two hierarchical attributes

In MADM problem, the determination of attribute weights is one of the important steps in the calculation process. In some cases, the attribute weights are provided by decision makers or experts. But a more common situation is

Table 1 The evaluations for MADM problem

	A_1	A_2	A_m
X_1	$L(p)_{11}$ or $H(p)_{11}$	$L(p)_{12}$ or $H(p)_{12}$	$L(p)_{1m}$ or $H(p)_{1m}$
X_2	$L(p)_{21}$ or $H(p)_{21}$	$L(p)_{22}$ or $H(p)_{22}$	$L(p)_{2m}$ or $H(p)_{2nm}$
X_n	$L(p)_{n1}$ or $H(p)_{n1}$	$L(p)_{n2}$ or $H(p)_{n2}$	$L(p)_{nm}$ or $H(p)_{nm}$

Table 2 The relationship between the fuzzy number and the linguistic term

Serial number	Linguistic term scale	Meanings	0.1–0.9 scale
1	s_{-4}	Extremely not preferred	0.1
2	s_{-3}	Strongly not preferred	0.2
3	s_{-2}	Moderately not preferred	0.3
4	s_{-1}	A little bit not preferred	0.4
5	s_0	Equally preferred	0.5
6	s_1	A little bit preferred	0.6
7	s_2	Moderately preferred	0.7
8	s_3	Strongly preferred	0.8
9	s_4	Extremely preferred	0.9

that the attribute weights are unknown. Maximum deviation method was first proposed by Wang, which can obtain the attribute weights based on the evaluations provided [34] by experts. According to this method, the more the evaluations among alternatives different from each other based on the attributes, the more effective information the attributes contain. Therefore, these attributes should be allocated with larger weights than others. Motivated by this idea, we propose the maximum deviation method for two level attributes with probabilistic-based information. Since the evaluation grades of the PHFS and the PLTS are equivalent and can be mutually transformed, they can be seen as the same expression forms in a MADM problem, which is the Harmonic judgment, i.e., all attributes are assessed to the same subset of evaluation grades with different degrees of belief [17]. For convenience, we mainly use the PHFS to denote the evaluations in this paper.

In Subsection 3.1, we have provided the decision matrix with attributes $\{A_1, A_2, \ldots, A_m\}$ and alternatives $\{X_1, X_2, \ldots, X_n\}$. For an attribute A_j, it may be composed of several basic attributes $A_{jk}(k = 1, 2, \ldots, K)$. The structure of the attributes in a MADM problem can be shown in Fig. 1.

In Sect. 2, the previous researches [3, 9] have proposed some distance measures for P-HFEs. However, they ignore the fact that the numbers of elements in P-HFEs may not always be the same. So, we propose a novel distance measure to fix that problem.

Definition 3. Let $h(p)_1 = \left\{h_1^1\left(p_1^1\right), h_1^2\left(p_1^2\right), \ldots, h_1^{l_1}\left(p_1^{l_1}\right)\right\}$ and $h(p)_2 = \left\{h_2^1\left(p_2^1\right), h_2^2\left(p_2^2\right), \ldots, h_2^{l_2}\left(p_2^{l_2}\right)\right\}$

Then, we can get

$$d\left(h(p)_1, h(p)_2\right) = \sqrt{\frac{2}{l_1 + l_2}\left(\sum_{k=1}^{l_1} p_1^k h_1^k - \sum_{k=1}^{l_2} p_2^k h_2^k\right)^2} \tag{6}$$

is a distance measure for the P-HFEs $h(p)_1$ and $h(p)_2$.

In order to express the deviation between the alternatives X_i and X_s based on the basic attribute A_{jk}, we apply the distance measures of the PHFS, which are shown as follows:

$$d_{jk}\left(X_i, X_s\right) = d\left(h(p)_{x_{ijk}}, h(p)_{x_{sjk}}\right), i \neq s \tag{7}$$

And the deviation based on the basic attribute A_{jk} is to sum up all the deviation between the alternatives X_i and X_s, which are

$$D_{jk} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{s=1, s>i}^{n} d_{jk}\left(X_i, X_s\right), j = 1, 2, \ldots, m; k = 1, 2, \ldots, K_j \tag{8}$$

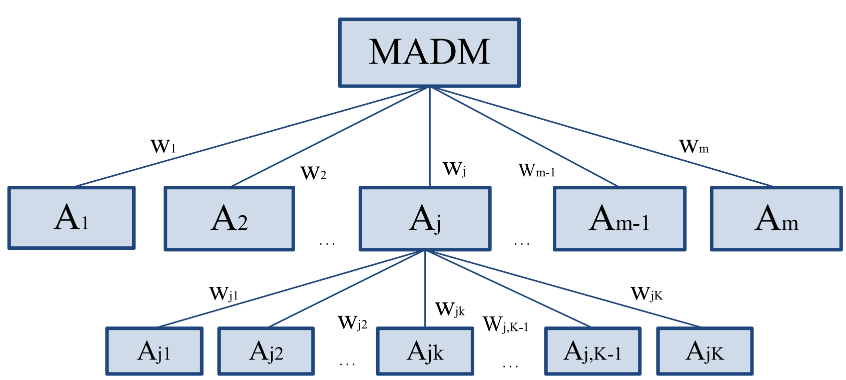
Therefore, the deviation based on the attribute A_j is expressed as

$$D_j = \sum_{k=1}^{K_j} \omega_{jk} D_{jk} \tag{9}$$

where ω_{jk} is the weight of the basic attribute A_{jk}.

Thus, we construct a nonlinear programming model for the maximum deviation based on all the attributes and obtain the weights of each attribute and basic attribute. The total deviation based on all the attributes can be expressed as $= \sum_{j=1}^{m} \omega_j D_j$. And the weights of attributes and basic attributes should satisfy $\omega_j = \sum_{k=1}^{K_j} \omega_{jk}$. Then, the nonlinear programming model is shown as follows:

Fig. 1 The structure of attributes



(M1)

$$\max\ D=\sum_{j=1}^{m}\omega_j D_j s.t.\begin{cases}\sum_{j=1}^{m}\omega_j^2=1\\ \sum_{k=1}^{k_j}\omega_{jk}=\omega_j\\ 0\le\omega_j\le1\\ 0\le\omega_{jk}\le1\\ D_j\sum_{k=1}^{K_j}\omega_{jk}=D_{jk}\\ j=1,2,\dots,m;k=1,2,\dots K_j\end{cases}$$

The solution of this nonlinear programming model is $\omega=\left(\omega_1,\omega_2,\dots,\omega_m\right)$ and $\omega_j=\left(\omega_{j1},\omega_{j2},\dots,\omega_{jK_j}\right)$. which are the weights of attributes and the basic attributes, respectively.

Then, we can normalize the obtained weights by the normalization formulas:

$$w_{jk}=\frac{\omega_{jk}}{\sum_{j=1}^{m}\sum_{k=1}^{Kj}\omega_{jk}} \tag{10}$$

$$w_j=\sum_{k=1}^{K_j}w_{jk} \tag{11}$$

where w_{jk} is the normalized weight of the basic attributes A_{jk}, and w_j is the normalized weight of attribute A_j

3.3 Normalization for reference ideal evaluations

In practical situations, sometimes the optimal solution may not be the maximum or the minimum, but exist in an interval between the maximum and the minimum. For example, when the temperature of environment is between 18℃ and 25℃, and the relative humidity is between 40%-70%, human feel most comfortable. Therefore, the reference ideal method was first proposed by Cables et al. [33], in which the reference ideal interval or point between the maximal and minimal values is seen as the optimal solution. However, not all the attributes have the reference ideal solutions in a decision problem. Thus, all the evaluations based on different attributes should be normalized based on the idea of the reference ideal method.

In fuzzy environment or probabilistic hesitant fuzzy environment, the best possible value is 1 and the worst is 0. Therefore, in this situation, the reference ideal interval should between [0,1], which can be represented as $\left[A_j^R,B_j^R\right]$ for attributes $A_j(j=1,2,\dots,m)$. The evaluations of alternatives based on the attributes have been obtained and expressed as

$h(p)_{ij}=\left\{h^l\left(p^l\right)_{ij}|l=1,2,K,|h(p)|,\ \sum_{l=1}^{|h(p)|}p^l\le1\right\}$, where the element h^l is a fuzzy number, and the symbol l represents the serial number in Table 2. Therefore, the distance between the element h^l and the reference ideal interval $\left[A_j^R,B_j^R\right]$ is

$$d\left(h^l,\left[A_j^R,B_j^R\right]\right)=\begin{cases}0 & A_j^R\le h^l\le B_j^R\\ \left|h^l-A_j^R\right| & A\le h^l<A_j^R\\ \left|\left|h^l-B_j^R\right|\right| & B_j^R<h^l\le B\end{cases} \tag{12}$$

What we discuss above only considers that the ideal situation exists in the reference ideal interval. However, there is another possible situation, i.e., the non-ideal situation exists in the middle of all the evaluation interval. In this case, where the non-ideal situation may exist is called the reference non-ideal interval. And for a P-HFE $h(p)_{ij}=\left\{h^l\left(p^l\right)_{ij}|l=1,2,K,|h(p)|,\ \sum_{l=1}^{|h(p)|}p^l\le1\right\}$, it should be adjusted to a new P-HFE $h'(p)_{ij}=\left\{h'^l\left(p^l\right)_{ij}|l=1,2,K,|h(p)|,\ \sum_{l=1}^{|h(p)|}p^l\le1\right\}$, where $h'^l=1-h^l$.

Considering the differences between the probabilistic-based expressions and the traditional expressions, i.e., the fuzzy numbers or the linguistic terms, the original normalization function for fuzzy reference ideal decision problem cannot be applied in this situation directly. Thus, the rebuilt normalization function for a P-HFE is defined as follows:

Definition 5. Let a P-HFE $h(p)_{ij}=\left\{h^l\left(p^l\right)_{ij}|l=1,2,K,|h(p)|,\ \sum_{l=1}^{|h(p)|}p^l\le1\right\}$, a possible evaluation interval $[A,B]$ and a reference ideal interval $\left[A^R,B^R\right]$, where $\left[A^R,B^R\right]\subseteq[A,B]$. Then the normalization function for the P-HFE $h(p)_{ij}$ should satisfy

$$f^*:h(p)_{ij}[A,B]\left[A^R,B^R\right]\to[0,1]f^*\left(h(p)_{ij},[A,B],\left[A^R,B^R\right]\right)=\begin{cases}1-\frac{d\left(h^l,\left[A_j^R,B_j^R\right]\right)}{d(A,A^R)}\left(p^l\right), & A\le h^l<A_j^R\\ 1-\frac{d\left(h^l,\left[A_j^R,B_j^R\right]\right)}{d(B,B^R)}\left(p^l\right), & B_j^R<h^l\le B\\ 1, & A_j^R\le h^l\le B_j^R\end{cases},l=1,2,\dots,L \tag{13}$$

In fuzzy environment, the possible evaluation interval $[A, B] = [0, 1]$. Hence, the normalization function can be rewritten as follows:

$$f^{*}\left(h(p)_{ij}, [A, B], \left[A^{R}, B^{R}\right]\right) = \left\{1 - \frac{d\left(h^{l}, \left[A_{j}^{R}, B_{j}^{R}\right]\right)}{max\left\{d\left(0, A^{R}\right), d\left(1, B^{R}\right)\right\}}\left(p^{l}\right), l = 1,2,\ldots,L\right\} \tag{14}$$

The normalized evaluations for alternatives $X_i (i = 1, 2, \ldots, n)$ based on attributes $A_j (j = 1, 2, \ldots, m)$ is still a P-HFE. For convenience, it can be rewritten as follows:

$$f(p)_{ij} = \left\{f^{l}\left(p^{l}\right)_{ij}, l = 1, 2, \ldots, L\right\} \tag{15}$$

3.4 Evidential reasoning process with the PHFS and the PLTS

Evidential reasoning is an aggregation method for multiple attributes by using probability theory, decision theory and fuzzy set theory, etc. It can be used to deal with MADM problem with multiple hierarchical attributes. In evidential reasoning process, an evaluation for an alternative X_i based on an attribute A_j should be represented by $\left(A_j\left(X_i\right)\right) = \left\{\left(H_n, \beta_{n,j}\left(X_i\right)\right), n = 1,2,\ldots,N; i = 1,2,\ldots,n; j = 1,2,\ldots,m\right\}$, where H_n is a pre-defined distinctive evaluation grade. However, it can be easily found that $S\left(A_j\left(X_i\right)\right)$ can be replaced by the PHFS and the PLTS, because they share the same meaning with the original expression and simpler expression forms in a MADM problem. Both of the PHFS and PLTS have been deeply and systematically investigated. Their operational laws, properties, distance measures and similarity measures have already been proposed as important tools for decision-making [3, 9, 10] Moreover, the PHFS and the PLTS are proposed in order to help decision makers precisely express their preferences with multiple evaluation values and probabilities, which satisfy the original intention of evidential reasoning.

Referring to the evidential reasoning approach [16, 23, 24, 29-32], for each attribute and each alternative, the probability mass of an evaluation grade $m_{l,j}$ should be represented by the product of the belief degree, i.e., the probability of an element in a P-HFE, and the attribute weight. And the remaining probability mass $m_{H,j}$ which is not assigned to any evaluation grade should be represented by the difference value between the probability 1 and the all the probability masses $m_{l,j}$. Thus, the calculation formulas of $m_{l,j}$ and $m_{H,j}$ should be shown as follows:

$$m_{l,j} = \omega_j p_{ij}^{l} \quad l = 1, 2, \ldots, L$$

$$and\ m_{H,j} = 1 - \sum_{l=1}^{L} m_{l,j} = 1 - \omega_j \sum_{l=1}^{L} p_{ij}^{l} \tag{16}$$

where p_{ij}^{l} is the probability of an element h_{ij}^{l} in a P-HFE $h(p)_{ij}$ or a normalized P-HFE $f(p)_{ij}$, and l represent the serial number of an evaluation in the 0.1–0.9 scale.

According to the evidential reasoning approach, the remaining probability mass can be decomposed into two parts: 1) $\overline{m}_{H,j} = 1 - \omega_j$ and 2) $\widetilde{m}_{H,j} = \omega_j\left(1 - \sum_{l=1}^{L} p_{ij}^{l}\right)$, with $m_{H,j} = \overline{m}_{H,j} + \widetilde{m}_{H,j}$. The first part is determined by the weight of the attribute A_j and the second part of unassigned probability is due to the incompleteness during the assessment process.

For an alternative X_i, in order to obtain the general evaluation, the probability mass based on all the attributes should be aggregated by the following recursive evidential reasoning algorithm.

Let $m_{l,I(1)} = m_{l,1}(l = 1,2,\ldots,L)$ $m_{l,I(1)} = m_{l,1}(l = 1, 2, \ldots, L)$, $\overline{m}_{H,I(1)} = \overline{m}_{H,1}$, $\widetilde{m}_{H,I(1)} = \widetilde{m}_{H,1}$ and $m_{H,I(1)} = m_{H,1}$. Then, the probability assignments should be obtained as follows:

$$m_{l,I(j+1)} = K_{I(j+1)}\left[m_{l,I(j)}m_{l,j+1} + m_{H,I(j)}m_{l,j+1} + m_{l,I(j)}m_{H,j+1}\right], l = 1, 2, \ldots, L \tag{17}$$

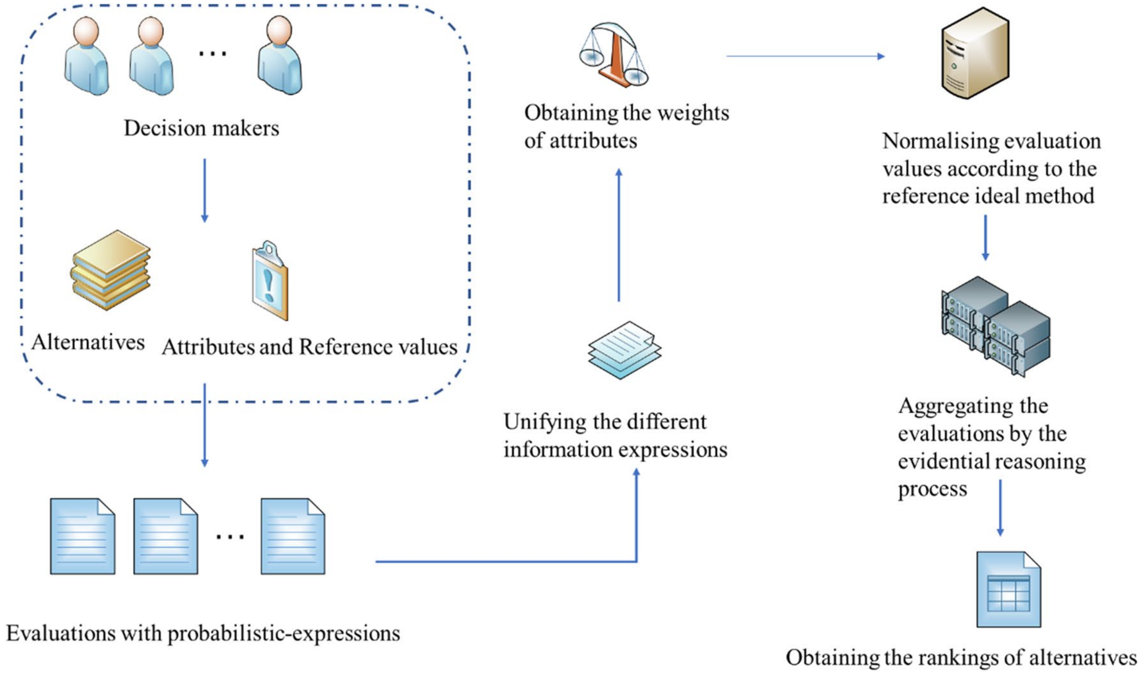


Fig. 2 The whole procedure of the reference ideal model for probabilistic-based expressions

Table 3 Indicator system

Main factors	Factors	Scale	Relation
Environment A_1	Air quality A_{11}	0.1–0.9	Negative
	Climate A_{12}	0.1–0.9	Reference non-ideal
Living habit A_2	Smoking A_{21}	$\{s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4\}$	Positive
	Cooking fume A_{22}	$\{s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4\}$	Positive
Individual susceptibility A_3	α_1-antitrypsin deficiency A_{31}	0.1–0.9	Positive
	Age A_{32}	$\{s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4\}$	Reference ideal

Table 4 The evaluations based on indicators

	A	B	C	D
Air quality	$\{0.2(0.5), 0.3(0.5)\}$	$\{0.5(0.5), 0.7(0.5)\}$	$\{0.6(0.5), 0.8(0.5)\}$	$\{0.4(0.5), 0.8(0.5)\}$
Climate	$\{0.3(0.5), 0.7(0.5)\}$	$\{0.4(0.3), 0.6(0.7)\}$	$\{0.3(0.3), 0.7(0.7)\}$	$\{0.4(0.5), 0.6(0.5)\}$
Smoking	$\{s_{-3}(0.3), s_2(0.7)\}$	$\{s_{-2}(0.3), s_4(0.7)\}$	$\{s_{-4}(1)\}$	$\{s_{-4}(0.9), s_{-3}(0.1)\}$
Cooking fume	$\{s_{-2}(0.5), s_2(0.5)\}$	$\{s_{-2}(0.3), s_2(0.7)\}$	$\{s_{-1}(0.4), s_3(0.6)\}$	$\{s_{-2}(0.2), s_2(0.8)\}$
α_1-antitrypsin deficiency	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$
Age	$\{s_{-2}(0.7), s_{-1}(0.3)\}$	$\{s_0(0.3), s_1(0.7)\}$	$\{s_0(0.4), s_1(0.6)\}$	$\{s_3(0.4), s_4(0.6)\}$

Table 5 Unified evaluations

	A	B	C	D
Air quality	$\{0.2(0.5), 0.3(0.5)\}$	$\{0.5(0.5), 0.7(0.5)\}$	$\{0.6(0.5), 0.8(0.5)\}$	$\{0.4(0.5), 0.8(0.5)\}$
Climate	$\{0.3(0.5), 0.7(0.5)\}$	$\{0.4(0.3), 0.6(0.7)\}$	$\{0.3(0.3), 0.7(0.7)\}$	$\{0.4(0.5), 0.6(0.5)\}$
Smoking	$\{0.2(0.3), 0.7(0.7)\}$	$\{0.3(0.3), 0.9(0.7)\}$	$\{0.1(1)\}$	$\{0.1(0.9), 0.2(0.1)\}$
Cooking fume	$\{0.3(0.5), 0.7(0.5)\}$	$\{0.3(0.3), 0.7(0.7)\}$	$\{0.4(0.4), 0.8(0.6)\}$	$\{0.3(0.2), 0.7(0.8)\}$
$\boldsymbol{\alpha}_1$-antitrypsin deficiency	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$
Age	$\{0.3(0.7), 0.4(0.3)\}$	$\{0.5(0.3), 0.6(0.7)\}$	$\{0.5(0.4), 0.6(0.6)\}$	$\{0.8(0.4), 0.9(0.6)\}$

$$m_{H,I(j)} = \overline{m}_{H,I(j)} + \widetilde{m}_{H,I(j)} \tag{18}$$

$$\widetilde{m}_{H,I(j+1)} = K_{I(j+1)}\left[\widetilde{m}_{H,I(j)}\widetilde{m}_{H,j+1} + \overline{m}_{H,I(j)}\widetilde{m}_{H,j+1} + \widetilde{m}_{H,I(j)}\overline{m}_{H,j+1}\right] \tag{19}$$

$$\overline{m}_{H,I(j+1)} = K_{I(j+1)}\left[\overline{m}_{H,I(j)}\overline{m}_{H,j+1}\right] \tag{20}$$

$$K_{I(j+1)} = \left[1 - \sum_{t=1}^{L} \sum_{\substack{l=1 \\ l \neq t}}^{L} m_{t,I(j)} m_{l,j+1}\right]^{-1}, j = 1, 2, \dots, m-1 \tag{21}$$

Then, the aggregated evaluation for an alternative X_i should be represented by $h(X_i) = \{h_l(\beta_l), l = 1, 2, \dots, L\}$, which can be seen as a P-HFE. The aggregated probability mass β_l and β_H are calculated by

$$\beta_l = \frac{m_{l,I(m)}}{1 - \overline{m}_{H,I(m)}} \tag{22}$$

$$\beta_H = \frac{\overline{m}_{H,I(m)}}{1 - \overline{m}_{H,I(m)}}, l = 1, 2, \dots, L \tag{23}$$

3.5 Reference ideal model for probabilistic-based expressions

In what follows, we develop a decision-making approach with evidential reasoning for reference ideal decision problem based on the PLTS and the PHFS. The whole procedure of the model is shown as Fig. 2.

As Fig. 2 shows, the reference ideal model for probabilistic-based expressions mainly has seven steps. Firstly, decision makers, alternatives, attributes and reference ideal intervals for a decision-making problem should be determined, as well as the structure of the attributes. They are the basic components in the decision-making problem. Meanwhile, the appropriate information expressions need to be chosen for different attributes according to their features. Then, decision makers provide the evaluations of alternatives based on the attributes by using different expression forms, i.e., P-HFS and PLTS. After that, considering that the evaluations are provided in different probabilistic-based expressions, the different probabilistic-based expressions should be transformed and unified according to Table 2 in Subsection 3.1. Followingly, the weights of attributes should be determined, which can be calculated based on the provided evaluations by using the maximum deviation method for double hierarchical attributes. Then, the evaluation values should be normalised according to the reference ideal method. The normalization process contains two parts: 1) obtain the distance between the reference ideal interval and the evaluations; 2) obtain the evaluations through the normalized function. Subsequently, we can aggregate the evaluations by the evidential reasoning process, in which the evaluations for basic attributes $A_{jk}(k = 1, 2, \dots, K)$ of an attribute A_j can be aggregated by the recursive algorithm first, and the aggregated evaluations based on attributes $A_j(j = 1, 2, \dots, m)$ can be obtained. Based on that, the aggregated evaluations for all the attributes can be obtained by the evidential reasoning process in Subsection 3.4. Finally, we can calculate the expected value of the aggregated evaluations of each alternative through Eq. (24) and obtain the rankings of alternatives (The alternatives are in a descending ranking based on E_i).

$$E_i = \sum_{l=1}^{L} \beta_l h_l \tag{24}$$

where E_i is the expected value of the aggregated evaluations of the alternative X_i

Table 6 Distances of alternatives

Factors	Air quality	Climate	Smoking	Cooking fume	$\boldsymbol{\alpha}_1$-antitrypsin deficiency	Age
Distances	0.1591	0.0330	0.2907	0.0542	0	0.1886

Table 7 Attribute weights

Factors	Air quality	Climate	Smoking	Cooking fume	α_1-antitrypsin deficiency	Age
Weights	0.0013	0.0006	0.9959	0	0.0005	0.0017
Main factors	Environment		Living habit		Individual susceptibility	
Weights	0.0019		0.9959		0.0022	

4 Illustrations

In this section, a numerical example is provided to illustrate operability and applicative value of the proposed method. Besides, the reasonability of the method is reflected by the comparative analysis with TOPSIS.

4.1 Background and indicator system

In recent years, due to air pollution, the number of respiratory disease patients has been rapidly growing. Respiratory disease mainly includes asthma, bronchitis, chronic obstructive pulmonary disease (COPD), pulmonary heart disease and tuberculosis, etc. Among them, COPD is one of the most susceptible respiratory diseases to environmental pollution. The report from World Health Organization showed that air pollution caused 3.7 million premature deaths worldwide, of which 14% died of COPD or acute respiratory infection [35]. Currently, COPD ranks fourth in the causes of global death. COPD is a slow, preventable disease. In the early stage of COPD, the treatment cost is relatively low, and the effect is good. However, when COPD develops to a severe stage, it is difficult to treat, and the effect of treatment is not as good as in the early stage. It is important and urgent to predict and evaluate the progression of respiratory disease patients, especially COPD patients. Considering that MADM skills have been applied to the evaluation of healthcare-related problems widely [36-38] we try to use the MSDM method proposed in this paper to evaluate the progression of COPD patients.

The causes of COPD are complicated, and there are still a lot of pathogenic factors remaining unknown which need further research. For now, it has been found that individual susceptibility and the environment are the three main pathogenic factors for COPD. For the environmental factor, the recent deterioration of air quality also leads to an increase in the incidence of COPD, and COPD is more likely to break out when the climate becomes cold. Thus, air quality and climate change should be considered environmental factors. Smoking is one of the most important causes of COPD. The longer time people smoke, the greater the risk of the illness. Besides, in some rural areas, cooking fume also causes COPD. Smoke from smoking and cooking contains plenty of harmful substances, which can weaken the sterilization of alveolar phagocytic cells and result in infection by bacterial invasion. But unlike air quality and climate change, the degrees of harm from smoking and the cooking fume can be reduced by patients. Hence, these two factors can also be classified into living habits. As for individual susceptibility, studies have shown that the α_1 -antitrypsin deficiency can make it easier to infect COPD. Moreover, the incident rate is also related to age, and mid-aged people are highly at risk for getting COPD.

The α_1-antitrypsin deficiency, air quality and the climate should be evaluated by experts or provided by computers after some comprehensive calculation based on some observed data, which can be expressed by the 0.1–0.9 scale. And the information of smoking, cooking fume and age can be provided by patients' description, whose evaluation values should belong to $\{s_{-4}, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3, s_4\}$.

The better the air quality, the larger the evaluation values. But better air quality has less influence on causing COPD, so the relationship between the air quality and COPD is negative. As for smoking, cooking fume and α_1-antitrypsin deficiency, they have a positive relation with causing COPD. For climate, when it is cold, a low evaluation value will be allocated to the attribute, and a high evaluation will be allocated to the hot weather. Both of cold and hot weathers are not suitable for the COPD patients. Moreover, the mid-aged people are most likely infected with COPD. Hence, the climate and the age are reference non-ideally and reference ideally related to causing COPD, respectively.

The causes of COPD, their scales and relation can be summarized in Table 3.

For the indicator "climate", both of the cold and hot weathers are positively related to the risk of getting COPD. Hence, the reference non-ideal interval of indicator "climate" is $[0.4, 0.6]$. As for the indicator "age", the mid-aged people are more likely getting COPD, so the reference ideal interval of this indicator is $[s_{-1}, s_1]$.

Table 8 The adjusted evaluations

Normalized evaluations	A	B	C	D
Air quality	$\{0.8(0.5), 0.7(0.5)\}$	$\{0.5(0.5), 0.3(0.5)\}$	$\{0.4(0.5), 0.2(0.5)\}$	$\{0.6(0.5), 0.2(0.5)\}$

Table 9 The normalized evaluations

Normalized evaluations	A	B	C	D
Climate	$\{0.75(1)\}$	$\{1(1)\}$	$\{0.75(1)\}$	$\{1(1)\}$
Age	$\{0.75(0.7), 1(0.3)\}$	$\{1(1)\}$	$\{1(1)\}$	$\{0.25(0.6), 0.5(0.4)\}$

4.2 Calculation process

Suppose that there are four potential patients, named A, B, C, D. The potential patient A is a 25-year-old man from Beijing. He has been smoking for more than three years. B is a middle-aged man who lives in a small town in north of China, and he has smoked for ten years. C is housewife, aged 40 and lives in countryside in the south of China. D is an old woman who lives in a small town in south of China. All of the potential patients are without the α_1-antitrypsin deficiency. Considering above information, the evaluations for the potential patients based on the indicators for causing COPD are shown in Table 4.

Since the evaluations are provided with different expression forms, they should be unified according to the relationships between the linguistic terms and the fuzzy numbers in Table 2, and the unified evaluations can be shown in Table 5.

4.3 Attribute weights

Then, we can obtain the weights of attributes by the maximum deviation method based on the unified evaluations. At first, we calculate distances of all the alternatives based on the factors, and the results are shown in Table 6.

Then, the weights for main factors and factors can be calculated and obtained based according to Model (M1) and Eqs. (10)-(11), the results are shown in Table 7. From Table 7, we can see that the attribute "smoking" is assigned the largest weight, and the attribute "cooking fume" is assigned the weight 0. That is to say, based on the supposed data in Table 4, the attribute "smoking" is the most significant indicator, and the effect of attribute "cooking fume" can be ignored. The attribute "living habit" are more important than the attribute "environment" and the attribute "individual susceptibility" to the four potential patients in the illustrative example.

4.4 Evaluation normalization and evidential reasoning process

According to Table 3, it can be found that the factor air quality A_{11} is negative. In order to be consistent with the positive factors, the evaluations based on air quality A_{11} can be transformed by $h^{l'} = 1 - h^l$, and the adjusted evaluations are shown in Table 8.

And the factor climate A_{12} is reference non-ideal and the factor age A_{32} is reference ideal. At first, the evaluations based on climate A_{12} should be transformed by $h^{l'} = 1 - h^l$. Then, we can obtain the normalized evaluations based on formula (14) and Definition 5, which can be shown in Table 9.

Based on the formula (17–21), we can get the probability assignments of each evaluation grades. Finally, the expect values of alternatives are:

$$E_A = 0.5501, E_B = 0.7199, E_C = 0.1000, E_D = 0.1100$$

Hence, the ranking of the alternatives is $B \succ A \succ D \succ C$, which means the potential patient B is most risky to get the COPD, and the potential patient C has the least risk to get the COPD. Therefore, the potential B should pay more attention to the risk of getting COPD. Referring to the data in Table 5, we can see that both the evaluations of the attribute "smoking" of potential patient B and the potential patient A have relatively large values. Meanwhile, from the derived attribute weights, the attribute "smoking" is the most influenced indicator to cause COPD. Therefore, these two potential patients have relatively high risk to get COPD.

4.5 Comparative analysis

In this subsection, we compare the method proposed in this paper with one of the conventional methods named TOPSIS and one of the new methods named the gained and lost dominance score (GLDS) method to the rational of the proposed method.

Table 10 The complete normalized evaluations

	A	B	C	D
Air quality	$\{0.8(0.5), 0.7(0.5)\}$	$\{0.5(0.5), 0.3(0.5)\}$	$\{0.4(0.5), 0.2(0.5)\}$	$\{0.6(0.5), 0.2(0.5)\}$
Climate	$\{0.75(1)\}$	$\{1(1)\}$	$\{0.75(1)\}$	$\{1(1)\}$
Smoking	$\{0.2(0.3), 0.7(0.7)\}$	$\{0.3(0.3), 0.9(0.7)\}$	$\{0.1(1)\}$	$\{0.1(0.9), 0.2(0.1)\}$
Cooking fume	$\{0.3(0.5), 0.7(0.5)\}$	$\{0.3(0.3), 0.7(0.7)\}$	$\{0.4(0.4), 0.8(0.6)\}$	$\{0.3(0.2), 0.7(0.8)\}$
α_1-antitrypsin deficiency	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$
Age	$\{0.75(0.7), 1(0.3)\}$	$\{1(1)\}$	$\{1(1)\}$	$\{0.25(0.6), 0.5(0.4)\}$

Table 11 The distances to the best value

	A	B	C	D
Air quality	0.2041*	0.4899	0.5715**	0.4899
Climate	0.2500	0.9000*	0.2500	0*
Smoking	0.3674	0.2286*	0.9**	0.7267
Cooking fume	0.4082**	0.3429	0.2939*	0.3103
$\boldsymbol{\alpha}_1$-antitrypsin deficiency	0.9000*	0.9000*	0.9000*	0.9000*
Age	0.1429	0.0000 *	0.0000 *	0.5307 **

4.6 The comparison with TOPSIS

TOPSIS is one of the most widely used decision making methods in past several decades. It has been applied in kinds of fields, such as supply chain management [39], marketing management [40], etc. TOPSIS was firstly proposed by Hwang and Yoon [41] in 1981, which tried to choose the optimal solution with the shortest distance to the ideal solution and the farthest distance to the non-ideal solution.

In subsection 4.2, we can easily get the normalized evaluations, which are shown in Table 10.

According to the methodology of TOPSIS, the first step is to find the ideal solution and the non-ideal solution, which are the alternatives with the shortest distance to the best value and the largest distance to the best value. Hence, the distances to the best value based on each attribute are shown in Table 11. In this case, the best value is 1, and both of them should be seen as the special case of the P-HFEs, which are $\{0(1)\}$ and $\{1(1)\}$.

Based on Table 12, the ideal solution and the non-ideal solution for each attribute can be found, which are denoted by the sign * and ** in Table 11, respectively. Therefore, the distances to the ideal solution and non-ideal solution based on each attribute can be calculated, which are shown in Table 12.

Then, we aggregate the distances by $D_i^+ = \sum_{j=1}^{m} w_j d\left(h_{ij}(p_{ij}), h_j^+\right)$ and $D_i^- = \sum_{j=1}^{m} w_j d\left(h_{ij}(p_{ij}), h_j^-\right)$, and we can obtain the comprehensive distances of alternatives to the ideal solution and the non-ideal solution, which are

$$D_1^+ = 0.1201, D_2^+ = 0.0009, D_3^+ = 0.5047, D_4^+ = 0.4308$$

$$D_1^- = 0.3673, D_2^- = 0.5052, D_3^- = 0.0013, D_4^- = 0.0088$$

The smaller the distances to the ideal solution, the better the alternatives; the smaller the distances to the non-ideal solution, the worse the alternatives. Therefore, the smallest distance to the ideal values and the largest distance to the non-ideal values need to be found, which are

$$D_{max}^- = \max_{1 \le i \le n} \{D_i^-\} \tag{25}$$

$$D_{min}^+ = \min_{1 \le i \le n} \{D_i^+\} \tag{26}$$

where D_{max}^- denotes the largest distance to the non-ideal values, D_{min}^+ denotes the smallest distance to the ideal values.

Then, we use the improved closeness coefficient to help rank the alternatives, shown as follows:

$$CI's(X_i) = \frac{D_i^-}{D_{max}^-} - \frac{D_i^+}{D_{min}^-} \tag{27}$$

where $CI'(X_i)$ denotes the improved closeness coefficient of the alternative X_i.

Therefore, we can get $I'(A) = -139.225$, $CI'(B) = 0$, $CI'(C) = -588.104$, $CI'(D) = -501.946$. Rank the alternatives in descending order of the improved closeness coefficients and we can get

$B \succ A \succ D \succ C$

From the result derive from TOPSIS, the potential patient B is most risky to get the COPD, and the potential patient C has the least risk to get the COPD. That is to say, the potential patient B should pay more attention to the risk of getting COPD. From the perspective of method, the result is the same to the result of the proposed method. The main differences between the proposed method and the TOPSIS is that the evidential reasoning process, which the TOPSIS does not contain. In the TOPSIS method, the probabilities in P-HFEs are seen as a part of evaluations and used in the calculation

Table 12 The distances to the ideal solution and non-ideal solution

Distances	A		B		C		D	
	Ideal	Non-ideal	Ideal	Non-ideal	Ideal	Non-ideal	Ideal	Non-ideal
Air quality	0.0000	0.3182	0.2475	0.0707	0.3182	0.0000	0.2475	0.0707
Climate	0.2500	0.6500	0.9000	0.0000	0.2500	0.6500	0.0000	0.9000
Smoking	0.1202	0.3674	0.0000	0.5062	0.5062	0.0000	0.4313	0.0082
Cooking fume	0.0990	0.0000	0.0424	0.0566	0.0000	0.0990	0.0141	0.0849
$\boldsymbol{\alpha}_1$-antitrypsin deficiency	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Age	0.1429	0.3359	0.0000	0.5307	0.0000	0.5307	0.5307	0.0000

Table 13 The adjusted evaluations

	A	B	C	D
Air quality	$\{0.8(0.5), 0.7(0.5)\}$	$\{0.5(0.5), 0.3(0.5)\}$	$\{0.4(0.5), 0.2(0.5)\}$	$\{0.6(0.5), 0.2(0.5)\}$
Climate	$\{0.75(1)\}$	$\{1(1)\}$	$\{0.75(1)\}$	$\{1(1)\}$
Smoking	$\{0.2(0.3), 0.7(0.6), 0.7(0.1)\}$	$\{0.3(0.3), 0.9(0.6), 0.9(0.1)\}$	$\{0.1(0.3), 0.1(0.6), 0.1(0.1)\}$	$\{0.1(0.3), 0.1(0.6), 0.2(0.1)\}$
Cooking fume	$\{0.3(0.2), 0.3(0.1), 0.3(0.1), 0.3(0.1), 0.7(0.5)\}$	$\{0.3(0.2), 0.3(0.1), 0.7(0.1), 0.7(0.1), 0.7(0.5)\}$	$\{0.4(0.2), 0.4(0.1), 0.4(0.1), 0.8(0.1), 0.8(0.5)\}$	$\{0.3(0.2), 0.7(0.1), 0.7(0.1), 0.7(0.1), 0.7(0.5)\}$
α_1-antitrypsin deficiency	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$	$\{0.1(1)\}$
Age	$\{0.75(0.6), 0.75(0.1), 1(0.3)\}$	$\{1(0.6), 1(0.1), 1(0.3)\}$	$\{1(0.6), 1(0.1), 1(0.3)\}$	$\{0.25(0.6), 0.5(0.1), 0.5(0.3)\}$

process directly. However, in the proposed method, the probabilities of each evaluation grades are aggregated by the evidential reasoning process, which is more scientific and precise for dealing with the probabilities than the TOPSIS.

4.7 The comparison with GLDS method

The GLDS method is proposed by Wu and Liao [42] in 2019, which is one of the outranking decision-making methods. Compared with other outranking-based methods, the advantage of GLDS method is that the optimal alternative has good comprehensive performance under all attributes.

In this part, we conduct the GLDS method based on the normalization evaluations in the Table 10. Then, we can derive the adjusted evaluations as follows.

Based on the data in the Table 13, we can get the normalized values of the net gained dominance scores (DS_1), the normalized values of the net lost dominance scores(DS_2), the corresponding ranking values of the alternatives. The results are shown in Table 14.

Then, the collective scores can be derived, which is shown in Table 15.

Therefore, the final ranking is: $B \succ A \succ D \succ C$, which also means that the potential patient B is most risky to get the COPD. The result is also the same with the method proposed in this paper. That is to say, the result derived from the method proposed in this paper is rational. Compared with the GLDS method, the method proposed in this paper considers the reference ideal intervals, but the GLDS method is not appropriate to solve this kind of problem. In addition, the method proposed in this paper can utilize the probability information more by ER approach. Besides, in this paper, the authors consider the PHFS and PLTS as a whole entity, i.e., probabilistic-based expression, while the original GLDS method is applied in PLTS environment. From this perspective, the whole method proposed in this paper can handle more complex decision-making problems.

In addition, we also compare the proposed method with other similar methods in Table 16.

From Table 16, we can see that ER approach has been introduced to PLTSs and PHFSs environments, but the detailed methods are different, and the applicable situations are also different. The method proposed by Wang and Zhang [23] is preliminary research on combining PLTSs with ER approach, and the process of calculation is essentially based on triangular or trapezoidal fuzzy sets. Fang and Liao's study [24] also introduces ER approach to PLTSs. They aggregate the evaluation information of each alternative over all the criteria and obtain a comprehensive evaluation of the alternatives by ER approach directly. Then, they combine ER approach with prospect theory to derive the final ranking, which is different from the conventional ER approach-combined methods. Their method fits with the multi-attribute group decision-making problems in a risk environment. The method proposed by Ma et al.[25] requires experts to provide PLTSs to express their evaluations on alternatives and linguistic terms to describe the degree of familiarity with the problem. By combining the familiarity degree and the group similarity degree, the reliability degree can be derived, and then the ER approach is conducted. The measurement of the reliability degree reflects the uncertain environment more vividly. For the method in this paper, we mainly focus on the probabilistic-based expressions (PLTS and PHFS) with reference to ideal evaluations, which are common in real life. Then, to better aggregate the evaluation information on different attributes and different experts, the ER approach is combined. Hence, the method proposed in this paper can

Table 14 Some results derived from GLDS method

	A	B	C	D
Normalized (DS_1)	0.5369	0.8436	0.0008	0.0060
Ranking value based on (DS_1)	2	1	4	3
Normalized (DS_2)	0.1918	0.0005	0.6996	0.6880
Ranking value based on (DS_2)	2	1	4	3

Table 15 The collective scores of the alternatives

	A	B	C	D
Collective scores	0.1227	0.3374	-0.2800	-0.2100

Table 16 The comparisons on other similar methods

Reference number	Information background	Transformation method	Main method
[23]	Different PLTSs	Transforming PLTSs to triangular or trapezoidal fuzzy sets	Combining ER approach directly with triangular or trapezoidal fuzzy sets
[24]	PLTSs	No transformation	Using the ER approach to aggregate the evaluation information of each alternative over all criteria and obtain the comprehensive evaluation of the alternatives firstly. Then, referring to the functions in the prospect theory to derive the final ranking
[25]	PLTSs(evaluations on alternatives) and linguistic term sets(the familiarity of experts on attributes)	Transforming linguistic terms into utilities of grades by maximizing the group similarity of experts	Firstly, deriving reliability through the degree of group similarity and the familiarity degree. Then, using ER approach to aggregate the information of experts and attributes
The method in this paper	PLTSs and PHFSs	Transforming PLTSs to PHFSs	At first, normalizing PHFSs based on reference ideal evaluations. Then, combining ER approach directly

be applied to the multi-attribute group decision-making problems with reference to ideal evaluations. Meanwhile, the evaluation information can be a probabilistic-based expression.

5 Conclusion

In actual situations, some best solutions may exist in an ideal reference interval between the maximum and minimum. Moreover, the evaluations provided by experts may be in different expression forms according to the experts' knowledge backgrounds, cognitions and experiences. The PLTS and the PHFS are two useful tools for experts to express their preferences. Therefore, in this paper, we provide a reference ideal model to deal with the probabilistic-based expressions by combining the evidential reasoning approach.

Firstly, since the provided evaluations in this paper are in different expression forms, the relationships between the PLTS and the PHFS are investigated, which can be used to transfer them from each other. From this perspective, the method proposed in this paper can deal with the decision-making problems with PHFS and PLTS simultaneously, which is more flexible than the conventional methods. Besides, to get the attribute weights, we proposed a maximum deviation method for hierarchical attributes. Considering that the attributes in this paper may not be evaluated by the same principle, a normalization process is provided to make all the evaluations unified. From this aspect, the method proposed in this paper considers the decision-making situation more comprehensively, which can handle the real problem more effectively. What's more, ER is applied to aggregate the probabilities of the evaluation grades so that the aggregated evaluations for each alternative are obtained, which is the basis for ranking the alternatives. Besides, through the illustrations, the rationality of the method is presented. From this point, the method proposed in this paper can utilize the probability information in the probabilistic-based expressions effectively by the ER approach, and the combination with the ER approach improves the adaptability of the method in an uncertain environment.

However, in this paper, we do not consider the case of big data, and the method might not effective to solve the MSDM problems with a huge amount of data. In the future, considering that big data play a more and more vital role in decision-making, we should develop some models to deal with uncertain decision-making problems with big data, and try to mine the experts' preferences hidden behind the data. Besides, the applications related to the proposed model should be deeply investigated, such as medical system evaluation, disease diagnosis and medical resource allocation, etc.

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Declarations

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