On the Evidential Reasoning Algorithm for Multiple Attribute Decision Analysis Under Uncertainty

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***Abstract*—In multiple attribute decision analysis (MADA), one often needs to deal with both numerical data and qualitative information with uncertainty. It is essential to properly represent and use uncertain information to conduct rational decision analysis. Based on a multilevel evaluation framework, an evidential reasoning (ER) approach has been developed for supporting such decision analysis, the kernel of which is an ER algorithm developed on the basis of the framework and the evidence combination rule of the Dempster–Shafer (D–S) theory. The approach has been applied to engineering design selection, organizational self-assessment, safety and risk assessment, and supplier assessment.**

In this paper, the fundamental features of the ER approach are investigated. New schemes for weight normalization and basic probability assignments are proposed. The original ER approach is further developed to enhance the process of aggregating attributes with uncertainty. Utility intervals are proposed to describe the impact of ignorance on decision analysis. Several properties of the new ER approach are explored, which lay the theoretical foundation of the ER approach. A numerical example of a motorcycle evaluation problem is examined using the ER approach. Computation steps and analysis results are provided in order to demonstrate its implementation process.

***Index Terms*—Assessment, evidential reasoning, multiple attribute decision analysis (MADA), uncertainty, utility interval.**

I. INTRODUCTION

MANY decision problems in engineering and management involve multiple attributes of both a quantitative and qualitative nature. A decision may not be properly made without fully taking into account all attributes in question [2], [7], [10], [11], [14], [23]. It is the rational handling of qualitative attributes and uncertain or missing information that causes complexity in multiple attribute assessment. There is a growing need to develop theoretically sound methods and tools for dealing with multiple attribute decision analysis (MADA) problems under uncertainty in a way that is rational, reliable, repeatable, and transparent.

Over the past two decades, considerable research has been conducted on integrating techniques from artificial intelligence (AI) and operational research (OR) for handling uncertain information [1], [3]–[5], [8], [19], [21], [22], [32], [34]. Following this line of research, an evidential reasoning (ER) approach has been developed for MADA under uncertainty [23], [24], [28], [31]. This approach is based on an evaluation analysis model [33] and the Dempster–Shafer (D–S) theory of evidence [9]. In recent years, the ER approach has been applied to decision problems in engineering design, safety and risk assessment, organizational self-assessment, and supplier assessment, e.g., motorcycle assessment [23], general cargo ship design [12], marine system safety analysis and synthesis [15], [16], software safety synthesis [17], [18], retrofit ferry design [26], executive car assessment [27], and organizational self-assessment [29].

The kernel of the ER approach is an evidential reasoning algorithm developed on the basis of a multiattribute evaluation framework and the evidence combination rule of the D–S theory. The algorithm can be used to aggregate attributes of a multilevel structure [24]. A rational aggregation process needs to satisfy certain common sense or self-evident rules, referred to as synthesis axioms. It can be shown that the original ER approach only satisfies the following synthesis axioms approximately.

Suppose there are two levels of attributes with a general attribute at the top level and a number of basic attributes at the bottom level. Each basic attribute may be assessed with reference to a set of evaluation grades. An attribute can be assessed to individual or a subset of the evaluation grades with different degrees of belief. Within this ER assessment framework, the following four synthesis axioms are proposed.

- If no basic attribute is assessed to an evaluation grade at all, then the general attribute should not be assessed to the same grade either.
- If all basic attributes are precisely assessed to an individual grade, then the general attribute should also be precisely assessed to the same grade.
- If all basic attributes are completely assessed to a subset of grades, then the general attribute should be completely assessed to the same subset as well.
- If any basic assessment is incomplete, then a general assessment obtained by aggregating the incomplete and complete basic assessments should also be incomplete with the degree of incompleteness properly assigned.

Incomplete assessments may result from the lack of data or the inability of assessors to provide precise judgments or the failure for some assessors to provide judgments in a group decision situation. This paper is aimed to investigate the features of the ER approach based on the above synthesis axioms. As a result of this investigation, the original ER approach is evolved through the development of a new aggregation process that satisfies all the above axioms precisely. Utility intervals are proposed to characterize the degrees of incompleteness present in original assessments and describe the impact of incomplete information on decision analysis. Several properties of the new ER approach are explored, which provide the theoretical basis

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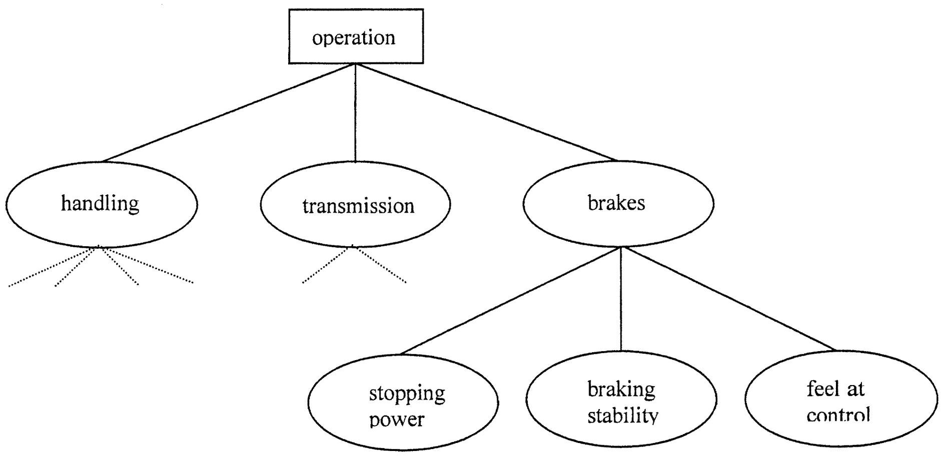


Fig. 1. Evaluation hierarchy for *operation*.

of the approach. A numerical example of a motorcycle evaluation problem is examined using the ER approach. Both complete and incomplete assessments are involved in the problem. To illustrate the implementation process of the ER approach, some of the computation steps and results are provided.

The original ER approach is revisited in the next section. As a result of this investigation, the ER approach is further developed in Section II. In Section III, the properties of the new ER approach are explored. In Section IV, a motorcycle performance assessment problem is examined. The paper is concluded in Section V.

II. The Evidential Reasoning Algorithm

A. Problem Description

Subjective judgments may be used to differentiate one alternative from another on qualitative attributes. To evaluate the quality of the *operation* of a motorcycle, for example, typical judgments may be that "the *operation* of a motor cycle is *poor, good,* or *excellent* to certain degrees." In such judgments, *poor, good,* and *excellent* denote distinctive evaluation grades. In a motorcycle evaluation problem, a set of evaluation grades is defined by [23]

$$H = \{poor\,(H_1) \quad indifferent\,(H_2) \quad average\,(H_3) \quad good\,(H_4) \quad excellent\,(H_5)\}. \tag{1}$$

Operation is a general technical concept and difficult to assess directly. It needs to be decomposed into detailed concepts, such as *handling, transmission,* and *brakes*. If a detailed concept is still too abstract to assess directly, it may be further broken down to more detailed concepts. For instance, the concept of *brakes* (y) may be measured by *stopping power* (e_1), *braking stability* (e_2), and *feel at control* (e_3), which can be directly assessed and therefore referred to as basic attributes in this paper. Assessment attributes often constitute a multilevel hierarchy [24]. An evaluation hierarchy for assessing the operation of a motorcycle is shown in Fig. 1.

In hierarchical assessment, a high level attribute is assessed through associated lower level attributes. For example, if the *stopping power, braking stability,* and *feel at control* of a motorcycle are all assessed to be exactly *good,* then its brakes should also be *good*. In evaluation of qualitative attributes, uncertain judgments could be used. In assessment of the *brakes* of a motorcycle, for example, assessors may be

1) 30% sure that its *stopping power* is at *average* level and 60% sure that it is *good;*
2) absolutely sure that its *braking stability* is *good*;
3) 50% sure that its *feel at control* is *good* and 50% sure that it is *excellent*.

In the above assessments, 30%, 50%, 60%, and 100% (absolutely sure) are referred to as degrees of belief and sometimes used in decimal format: 0.3, 0.5, 0.6, and 1, respectively. Note that assessment 1) is incomplete as the total degree of belief is $0.3 + 0.6 = 0.9 < 1$ while assessments 2) and 3) are complete. The missing 0.1 in assessment 1) represents the degree of ignorance or uncertainty. A problem arises as to how to generate an overall assessment about the *brakes* of the motorcycle by aggregating the above three judgments in a rational way. The evidential reasoning approach provides a means for dealing with the aggregation problem. The basic ER model and the ER algorithm are discussed in the next two subsections, and the synthesis axioms are defined in the following subsection where the shortcomings of the original ER algorithm are also discussed.

B. Basic Evaluation Framework

To begin with, suppose there is a simple two-level hierarchy of attributes with a general attribute at the top level and a number of basic attributes at the bottom level. Suppose there are L basic attributes $e_i\ (i = 1, \ldots, L)$ associated with a general attribute y. Define a set of L basic attributes as follows:

$$E = \{e_1 \;\; e_2 \;\; \cdots \;\; e_i \;\; \cdots \;\; e_L\}. \tag{2}$$

Suppose the weights of the attributes are given by $\omega = \{\omega_1 \;\; \omega_2 \;\; \cdots \;\; \omega_i \;\; \cdots \;\; \omega_L\}$ where ω_i is the relative weight of the ith basic attribute (e_i) with $0 \le \omega_i \le 1$. Weights play an important role in assessment. They may be estimated using existing methods such as simple rating methods or more elaborate methods based on the pairwise comparisons of attributes [7], [11], [30].

Suppose N distinctive evaluation grades are defined that collectively provide a complete set of standards for assessing an attribute, as represented by

$$H = \{H_1 \;\; H_2 \;\; \cdots \;\; H_n \;\; \cdots \;\; H_N\} \tag{3}$$

where H_n is the nth evaluation grade. Without loss of generality, it is assumed that H_{n+1} is preferred to H_n.

A given assessment for $e_i\,(i = 1, \ldots, L)$ of an alternative may be mathematically represented as the following distribution:

$$S(e_i) = \{(H_n, \beta_{n,i}),\, n = 1, \ldots, N\} \qquad i = 1, \ldots, L \tag{4}$$

where $\beta_{n,i} \geq 0$, $\sum_{n=1}^{N} \beta_{n,i} \leq 1$, and $\beta_{n,i}$ denotes a degree of belief. The above distributed assessment reads that the attribute e_i is assessed to the grade H_n with the degree of belief of $\beta_{n,i}$, $n = 1, \ldots, N$. An assessment $S(e_i)$ is complete if $\sum_{n=1}^{N} \beta_{n,i} = 1$ and incomplete if $\sum_{n=1}^{N} \beta_{n,i} < 1$. A special case is $\sum_{n=1}^{N} \beta_{n,i} = 0$ (or $\beta_{n,i} = 0$ for all $n = 1, \ldots, N$), which denotes a complete lack of information on e_i. Such partial or complete ignorance is not rare in many decision making problems. In the new ER approach to be investigated in this and next sections, ignorance will be handled using the upper and lower bounds of degrees of belief and utility intervals.

Let β_n be a degree of belief to which the general attribute y is assessed to the grade H_n. The aggregation problem is to generate $\beta_n\,(n = 1, \ldots, N)$ by aggregating the assessments for all the associated basic attributes $e_i\,(i = 1, \ldots, L)$ as given in (4). The following evidential reasoning algorithm can be used for this purpose.

C. *Summary of the Original ER Algorithm*

In this subsection, the original ER algorithm will be briefly discussed and its shortcomings will be analyzed. Let $m_{n,i}$ be a basic probability mass representing the degree to which the ith basic attribute e_i supports the hypothesis that the attribute y is assessed to the nth grade H_n. Let $m_{H,i}$ be a remaining probability mass unassigned to any individual grade after all the N grades have been considered for assessing the general attribute as far as e_i is concerned. $m_{n,i}$ is calculated as follows:

$$m_{n,i} = \omega_i \beta_{n,i} \qquad n = 1, \ldots, N \tag{5}$$

where ω_i need be normalized as discussed later. $m_{H,i}$ is given by

$$m_{H,i} = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - \omega_i \sum_{n=1}^{N} \beta_{n,i}. \tag{6}$$

Define $E_{I(i)}$ as the subset of the first i basic attributes as follows:

$$E_{I(i)} = \{e_1 \;\; e_2 \;\; \cdots \;\; e_i\}. \tag{7}$$

Let $m_{n,I(i)}$ be a probability mass defined as the degree to which all the i attributes in $E_{I(i)}$ support the hypothesis that y is assessed to the grade H_n. $m_{H,I(i)}$ is the remaining probability mass unassigned to individual grades after all the basic attributes in $E_{I(i)}$ have been assessed. $m_{n,I(i)}$ and $m_{H,I(i)}$ can be generated by combining the basic probability masses $m_{n,j}$ and $m_{H,j}$ for all $n = 1, \ldots, N$, $j = 1, \ldots, i$.

Given the above definitions and discussions, the original recursive evidential reasoning algorithm can be summarized as follows [24]:

$$\begin{aligned} m_{n,I(i+1)} = {} & K_{I(i+1)} \big(m_{n,I(i)} m_{n,i+1} + m_{n,I(i)} m_{H,i+1} \\ & + m_{H,I(i)} m_{n,i+1}\big) \qquad n = 1, \ldots, N \end{aligned} \tag{8a}$$

$$m_{H,I(i+1)} = K_{I(i+1)} m_{H,I(i)} m_{H,i+1} \tag{8b}$$

$$\begin{aligned} K_{I(i+1)} = {} & \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)} m_{j,i+1}\right]^{-1} \\ & i = 1, \ldots, L-1 \end{aligned} \tag{8c}$$

where $K_{I(i+1)}$ is a normalizing factor so that $\sum_{n=1}^{N} m_{n,I(i+1)} + m_{H,I(i+1)} = 1$. Note that $m_{n,I(1)} = m_{n,1}$ $(n = 1, \ldots, N)$ and $m_{H,I(1)} = m_{H,1}$. Also note that the basic attributes in E are numbered arbitrarily. This means that the results $m_{n,I(L)}$, $(n = 1, \ldots, N)$, and $m_{H,I(L)}$ do not depend on the order in which the basic attributes are aggregated.

In the original ER approach, the combined degree of belief β_n is directly given by

$$\begin{aligned} \beta_n &= m_{n,I(L)}, \qquad n = 1, \ldots, N \\ \beta_H &= m_{H,I(L)} = 1 - \sum_{n=1}^{N} \beta_n \end{aligned} \tag{9}$$

where β_H is the degree of belief unassigned to any individual evaluation grade after all the L basic attributes have been assessed. It denotes the degree of incompleteness in the assessment generated.

D. *Synthesis Axioms and Issues Associated With the Original ER Algorithm*

The aggregation process shown in the previous section may not be rational or meaningful if it does not follow certain synthesis axioms. Suppose E as defined in (2) contains a complete set of basic attributes for evaluation of y. The four synthesis axioms to be investigated in this paper are defined as follows.

Axiom 1: y must not be assessed to a grade H_n if none of the basic attributes in E is assessed to H_n, which is referred to as the independency axiom. It means that if $\beta_{n,i} = 0$ for all $i = 1, \ldots, L$, then $\beta_n = 0$.

Axiom 2: y should be precisely assessed to a grade H_n if all the basic attributes in E are precisely assessed to H_n, which is referred to as the consensus axiom. It means that if $\beta_{k,i} = 1$ and $\beta_{n,i} = 0$ for all $i = 1, \ldots, L$ and $n = 1, \ldots, N$, $n \neq k$, then $\beta_k = 1$ and $\beta_n = 0$ $(n = 1, \ldots, N,\, n \neq k)$.

Axiom 3: If all basic attributes in E are completely assessed to a subset of evaluation grades, then y

should be completely assessed to the same subset of grades, which is referred to as the completeness axiom.

Axiom 4: If an assessment for any basic attribute in E is incomplete, then the assessment for y should be incomplete to certain degree, which is referred to as the incompleteness axiom.

The first independency axiom is naturally followed in the original ER algorithm as from (8) and (9) we have $\beta_n = 0$ if $m_{n,i} = 0$ for all $i = 1, \ldots, L$.

Regarding the second synthesis axiom, we have the following conclusion.

Theorem 1: If β_n and β_H are calculated using (9), then to satisfy the consensus axiom it is necessary and sufficient to hold the following condition:

$$\prod_{i=1}^{L}(1-\omega_i) = 0. \tag{10}$$

Proof: From (5) and (6), the basic probability assignments are given by

$$m_{k,i} = \omega_i\beta_{k,i} = \omega_i \qquad i = 1, \ldots, L \tag{11}$$

$$m_{n,i} = \omega_i\beta_{n,i} = 0 \qquad i = 1, \ldots, L; \quad n = 1, \ldots, N,\ n \neq k \tag{12}$$

$$m_{H,i} = 1 - \sum_{n=1}^{N}\omega_i\beta_{n,i} = 1-\omega_i \qquad i = 1, \ldots, L. \tag{13}$$

Since $m_{n,I(1)} = m_{n,1} = 0$ for $n \neq k$, from (12) and (13), at $i = 1$ we have

$$\left.\begin{array}{l} m_{n,I(i)}m_{n,i+1} = 0, \\ m_{n,I(i)}m_{H,i+1} = 0, \\ m_{H,I(i)}m_{n,i+1} = 0 \end{array}\right. \qquad \text{for any } n = 1, \ldots, N,\ n \neq k.$$

Using (8a), we then obtain at $i = 2$

$$m_{n,I(i)} = 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k.$$

Suppose $m_{n,I(i)} = 0$ at i with $2 \leq i < L$ for $n = 1, \ldots, N$, $n \neq k$. Using (8a) again, we generate

$$\begin{aligned} m_{n,I(i+1)} &= K_{I(i+1)}\left(m_{n,I(i)}m_{n,i+1} + m_{n,I(i)}m_{H,i+1} + m_{H,I(i)}m_{n,i+1}\right) \\ &= K_{I(i+1)}\left(0\times 0 + 0\times m_{H,i+1} + m_{H,I(i)}\times 0\right) \\ &= 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k. \end{aligned}$$

Thus, we deduce that at any $i = 1, \ldots, L$

$$m_{n,I(i)} = 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k. \tag{14}$$

From (11)–(14), we have

$$m_{t,I(i)}m_{j,i+1} = 0 \qquad \text{for any } i = 1, \ldots, L-1; \quad t, j = 1, \ldots, N,\ t \neq j. \tag{15}$$

Combining (8c) and (15), we get

$$K_{I(i+1)} = 1 \qquad \text{for } i = 1, \ldots, L-1. \tag{16}$$

From (8b) and (13), we get

$$\begin{aligned} &m_{H,I(L)} \\ &\quad = K_{I(L)}m_{H,L}m_{H,I(L-1)} \\ &\quad = K_{I(L)}K_{I(L-1)}m_{H,L}m_{H,L-1}m_{H,I(L-2)} \\ &\quad \cdots\ \cdots \\ &\quad = K_{I(L)}K_{I(L-1)}\cdots K_{I(2)}m_{H,L}m_{H,L-1}\cdots m_{H,2}m_{H,1} \\ &\quad = \prod_{j=2}^{L}K_{I(j)}\prod_{i=1}^{L}m_{H,i} = \prod_{i=1}^{L}m_{H,i} = \prod_{i=1}^{L}(1-\omega_i). \end{aligned} \tag{17}$$

Since $\sum_{n=1}^{N} m_{n,I(L)} + m_{H,I(L)} = 1$ [24], combining (14) and (17) we get

$$m_{k,I(L)} + \prod_{i=1}^{L}(1-\omega_i) = 1 \ \text{ or } \ m_{k,I(L)} = 1 - \prod_{i=1}^{L}(1-\omega_i).$$

The consensus axiom and (9) require that

$$\beta_k = m_{k,I(L)} = 1 - \prod_{i=1}^{L}(1-\omega_i) = 1.$$

Therefore, we must have

$$\prod_{i=1}^{L}(1-\omega_i) = 0.$$

On the other hand, if $\prod_{i=1}^{L}(1-\omega_i) = 0$, the consensus axiom will be satisfied as

$$\beta_k = 1 - \prod_{i=1}^{L}(1-\omega_i) = 1 \ \text{ and } \ m_{H,I(L)} = \prod_{i=1}^{L}(1-\omega_i) = 0.$$

Q.E.D.

Unfortunately, (10) cannot be exactly satisfied unless ω_i is normalized to one for the most important basic attribute e_i, or $\omega_i = 1$. If $\omega_i = 1$, however, the ith attribute e_i would dominate the assessment. In other words, other basic attributes with smaller weights would play no role in the assessment of y, which is obviously unacceptable. This is the dilemma of applying the Dempser–Shafer theory to aggregating multiple criteria.

To resolve the dilemma, in the original ER approach the weights were normalized by [23], [24], [26]

$$\overline{\omega}_i = \alpha\frac{\omega_i}{\max\limits_i\{\omega_i,\ i = 1, \ldots, L\}} \tag{18}$$

and α is a constant determined by satisfying

$$\prod_{i=1}^{L}\left(1-\alpha\frac{\omega_i}{\max\limits_i\{\omega_i,\ i = 1, \ldots, L\}}\right) \leq \delta \tag{19}$$

where δ is a small constant, representing the degree of approximation in aggregation. The above normalization of weights means that the consensus axiom could only be satisfied approximately, as shown using an example in the next section. Another shortcoming of the above normalization technique is that the most important attribute may play a dominating role in the assessment of y. It will also be shown in the next section that the original ER approach does not precisely satisfy the completeness axiom either.

E. An Example to Illustrate the Original ER Algorithm

In summary, the original ER algorithm is composed of (4) for original information acquisition and representation, (18) and (19) for weight normalization, (5) and (6) for basic probability assignments where ω_i is replaced by $\overline{\omega}_i$, recursive (8a)–(8c) for attribute aggregation, and (9) for generating the combined degrees of belief.

To help understand the original ER algorithm and illustrate its shortcomings discussed above, we show the calculation steps for aggregating the following two assessments for two attributes e_1 and e_2:

$$S(e_1) = \{(H_1, 0), (H_2, 0), (H_3, 0), (H_4, 1), (H_5, 0)\}$$
$$S(e_2) = \{(H_1, 0), (H_2, 0.5), (H_3, 0.5), (H_4, 0), (H_5, 0)\}$$

which are two complete assessments. e_1 is completely assessed to H_4 and e_2 completely assessed to H_2 and H_3. From (4), we have

$$\beta_{1,1} = 0,\ \ \beta_{2,1} = 0,\ \ \beta_{3,1} = 0,\ \ \beta_{4,1} = 1,\ \ \beta_{5,1} = 0$$
$$\beta_{1,2} = 0,\ \ \beta_{2,2} = 0.5,\ \ \beta_{3,2} = 0.5,\ \ \beta_{4,2} = 0,\ \ \beta_{5,2} = 0.$$

Suppose e_1 is twice as important as e_2, or $\omega_1 = 2\omega_2$ and $\delta = 0.05$. Then the weights ω_1 and ω_2 are normalized using (18) as follows:

$$\overline{\omega}_1 = \alpha \quad \text{and} \quad \overline{\omega}_2 = \frac{\alpha}{2}.$$

Since $(1-\alpha)(1-\alpha/2) = 0.05$ leads to $\alpha = 0.9083$ [see (19)], we have

$$\overline{\omega}_1 = 0.9083 \quad \text{and} \quad \overline{\omega}_2 = 0.4542.$$

From (5) and (6), we can calculate basic probability masses $m_{n,i}$ as follows:

$$m_{1,1} = 0,\ \ m_{2,1} = 0,\ \ m_{3,1} = 0,\ \ m_{4,1} = 0.9083$$
$$m_{5,1} = 0,\ \ m_{H,1} = 0.0917$$
$$m_{1,2} = 0\ \ m_{2,2} = 0.2271,\ \ m_{3,2} = 0.2271,\ \ m_{4,2} = 0,$$
$$m_{5,2} = 0,\ \ m_{H,2} = 0.5458.$$

The recursive (8a)–(8c) can then be used to calculate the combined probability masses as follows. Let $m_{n,I(1)} = m_{n,1}$ for $n = 1, \ldots, 5$. Since

$$\begin{aligned} K_{I(2)} &= \left[1 - \sum_{t=1}^{5} \sum_{\substack{j=1 \\ j \neq t}}^{5} m_{t,I(1)} m_{j,2}\right]^{-1} \\ &= [1 - (0 + \cdots + 0 + m_{4,1} \times m_{2,2} + m_{4,1} \\ &\quad \times m_{3,2} + 0 + \cdots + 0)]^{-1} \\ &= [1 - (0.9083 \times 0.2271 + 0.9083 \times 0.2271)]^{-1} \\ &= 1.7023 \end{aligned}$$

we then have

$$\begin{aligned} m_{1,I(2)} &= K_{I(2)}(m_{1,1}m_{1,2} + m_{1,1}m_{H,2} + m_{H,1}m_{1,2}) = 0 \\ m_{2,I(2)} &= K_{I(2)}(m_{2,1}m_{2,2} + m_{2,1}m_{H,2} + m_{H,1}m_{2,2}) \\ &= 1.7023(0 \times 0.2271 + 0 \times 0.5458 \\ &\qquad + 0.0917 \times 0.2271) = 0.035\,45 \\ m_{3,I(2)} &= K_{I(2)}(m_{3,1}m_{3,2} + m_{3,1}m_{H,2} + m_{H,1}m_{3,2}) \\ &= 1.7023(0 \times 0.2271 + 0 \times 0.5458 \\ &\qquad + 0.0917 \times 0.2271) = 0.035\,45 \\ m_{4,I(2)} &= K_{I(2)}(m_{4,1}m_{4,2} + m_{4,1}m_{H,2} + m_{H,1}m_{4,2}) \\ &= 1.7023(0.9083 \times 0 + 0.9083 \times 0.5458 \\ &\qquad + 0.0917 \times 0) = 0.8439 \\ m_{5,I(2)} &= K_{I(2)}(m_{5,1}m_{5,2} + m_{5,1}m_{H,2} + m_{H,1}m_{5,2}) = 0 \\ m_{H,I(2)} &= K_{I(2)} m_{H,1} m_{H,2} \\ &= 1.7023 \times 0.0917 \times 0.5458 = 0.0852. \end{aligned}$$

The combined degrees of belief are given using (9) as follows:

$$\beta_1 = m_{1,I(2)} = 0,\ \ \beta_2 = m_{2,I(2)} = 0.035\,45$$
$$\beta_3 = m_{3,I(2)} = 0.035\,45,\ \ \beta_4 = m_{4,I(2)} = 0.8439$$
$$\beta_5 = m_{5,I(2)} = 0,\ \ \beta_H = m_{H,I(2)} = 0.0852.$$

The above results show that the combined assessment is to a large extent focused on β_4. This is because the first attribute e_1 is completely assessed to β_4, it is twice as important as e_2, and the weighs are normalized using (19). It can also be seen that the combined assessment is not precisely complete as $\beta_H = 0.0852$, though the two basic assessments $S(e_1)$ and $S(e_2)$ are complete. It can be shown that β_H will reduce if a smaller δ is selected. However, this will make e_1 even more dominant in the assessment. In the extreme, setting $\delta = 0$ will lead to $\omega_1 = 1$ and $\omega_2 = 0.5$, resulting in $\beta_4 = 1$, $\beta_H = 0$, and $\beta_i = 0$ for $i = 1, 2, 3, 5$.

F. New Schemes for Weight Normalization and Basic Probability Assignment

The original ER approach is revised in this and next two subsections so that the four synthesis axioms can be satisfied precisely, as will be proven in Section III. The revision includes three main parts:

1) the renormalization of weights and the re-assignments of basic probability masses including the decomposition of the remaining degree of belief;
2) the development of a new ER algorithm;
3) the generation of combined degrees of belief through the normalization of combined probability masses.

The original ER approach is in essence aimed to establish certain relationships between β_n and $\beta_{n,i}$ ($i = 1, \ldots, L$; $n = 1, \ldots, N$). Such relationships are generally represented by

$$\beta_n = f_n(\omega_i \beta_{n,i},\ i = 1, \ldots, L,\ n = 1, \ldots, N), \qquad n = 1, \ldots, N \tag{20}$$

where f_n is in general a nonlinear function of $\beta_{n,i}$. If f_n is assumed to behave linearly at a specific point with $\beta_{n,i} = 1$ and $\beta_{j,i} = 0$ for all $j = 1, \ldots, N$, $j \neq n$, and $i = 1, \ldots, L$,

β_n would be expressed as a linear combination of $\beta_{n,i}$ ($i = 1, \ldots, L$) as follows:

$$\beta_n = \sum_{i=1}^{L} \omega_i \beta_{n,i}. \tag{21}$$

To satisfy the consensus axiom at this point, we would have $\beta_n = 1$ and $\beta_k = 0$ ($k = 1, \ldots, N$, $k \neq n$). Thus

$$1 = \sum_{i=1}^{L} \omega_i. \tag{22}$$

Although in general f_n is a nonlinear function, (22) does suggest an alternative way of normalizing weights, different from what was suggested in (18) and (19). Using the new weight normalization, the assignment of basic probability masses as given by (5) and (6) is appropriate in the sense that each basic attribute can play a proportionally important role relative to its weight.

The linear combination equation [(21)] is simple but not suitable for attribute aggregation in MADA under uncertainty within the ER evaluation framework. This is because not only its underlying linearity assumption is questionable but also it is incapable of properly handling incomplete information. In the rest of this section, a new evidential reasoning algorithm will be developed for aggregating both complete and incomplete information using the new weight normalization given by (22).

Different from the original ER algorithm as shown in (8a)–(8c) and (9), in the new ER algorithm, the remaining probability mass initially unassigned to any individual evaluation grades will be treated separately in terms of the relative weights of attributes and the incompleteness in an assessment. In this way, the upper and lower bounds of the belief degrees can be generated using the concepts of the belief measure and the plausibility measure in the D–S theory of evidence. This is one of the distinctive features of the new ER approach from other MADA approaches.

Note that $m_{H,i}$, given in (6), is the remaining probability mass initially unassigned to any individual grades. In the new ER algorithm, it is decomposed into two parts: 1) $\overline{m}_{H,i}$ and 2) $\tilde{m}_{H,i}$, where

$$\overline{m}_{H,i} = 1 - \omega_i \quad \text{and} \quad \tilde{m}_{H,i} = \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) \tag{23}$$

with $m_{H,i} = \overline{m}_{H,i} + \tilde{m}_{H,i}$.

$\overline{m}_{H,i}$ is the first part of the remaining probability mass that is not yet assigned to individual grades due to the fact that attribute i (denoted by e_i) only plays one part in the assessment relative to its weight. $\overline{m}_{H,i}$ is a linear decreasing function of ω_i. $\overline{m}_{H,i}$ will be one if the weight of e_i is zero or $\omega_i = 0$; $\overline{m}_{H,i}$ will be zero if e_i dominates the assessment or $\omega_i = 1$. In other words, $\overline{m}_{H,i}$ represents the degree to which other attributes can play a role in the assessment. $\overline{m}_{H,i}$ should eventually be assigned to individual grades in a way that is dependent upon how all attributes are weighted and assessed.

$\tilde{m}_{H,i}$ is the second part of the remaining probability mass unassigned to individual grades, which is caused due to the incompleteness in the assessment $S(e_i)$. $\tilde{m}_{H,i}$ will be zero if $S(e_i)$ is complete, or $\sum_{n=1}^{N} \beta_{n,i} = 1$; otherwise, $\tilde{m}_{H,i}$ will be positive. $\tilde{m}_{H,i}$ is proportional to ω_i and will cause the subsequent assessments to be incomplete. In Yang [28], $\tilde{m}_{H,i}$ ($i = 1, \ldots, L$) were linearly combined, which would not be appropriate for the generation of a plausibility measure, as discussed in Section II-H.

G. New Evidential Reasoning Algorithm

As used in Yang and Singh [23], an attribute aggregation table (Table I) is again used to deduce a new ER algorithm for combining two assessments $S(e_i)$ and $S(e_j)$. Note that in Table I, $H \cap H_n = H_n$ for all $n = 1, 2, \ldots, N$.

The combined probability masses are generated by aggregating (denoted by $\oplus$) the assessments $S(e_i)$ and $S(e_j)$ as shown in Table I as follows:

$$\begin{aligned}
\{H_n\}\colon m_{n,I(2)} &= K_{I(i+1)}[m_{n,i}m_{n,j} + \tilde{m}_{H,i}m_{n,j} \\
&\quad + \overline{m}_{H,i}m_{n,j} + m_{n,i}\tilde{m}_{H,j} + m_{n,i}\overline{m}_{H,j}] \\
&= K_{I(i+1)}[m_{n,i}m_{n,j} + (\tilde{m}_{H,i} + \overline{m}_{H,i})m_{n,j} \\
&\quad + m_{n,i}(\tilde{m}_{H,j} + \overline{m}_{H,j})] \\
&= K_{I(i+1)}[m_{n,i}m_{n,j} + m_{H,i}m_{n,j} \\
&\quad + m_{n,i}m_{H,j}] \quad n = 1, 2, \ldots, N \\
\{H\}\colon \tilde{m}_{H,I(2)} &= K_{I(i+1)}[\tilde{m}_{H,i}\tilde{m}_{H,j} + \overline{m}_{H,i}\tilde{m}_{H,j} \\
&\quad + \tilde{m}_{H,i}\overline{m}_{H,j}] \\
\{H\}\colon \overline{m}_{H,I(2)} &= K_{I(i+1)}[\overline{m}_{H,i}\overline{m}_{H,j}]
\end{aligned}$$

$$K_{I(i+1)} = \left[1 - \sum_{t=1}^{N} \sum_{\substack{l=1 \\ l \neq t}}^{N} m_{t,i}m_{l,j}\right]^{-1}$$

where $m_{n,I(2)}$ is the combined probability mass for the grade H_n generated by aggregating the two assessments $S(e_i)$ and $S(e_j)$; $\tilde{m}_{H,I(2)}$ the combined probability mass for H due to the possible incompleteness in $S(e_i)$ and $S(e_j)$, and $\overline{m}_{H,I(2)}$ for H due to the combined relative importance of e_i and e_j.

Let $m_{n,I(i)}$ ($n = 1, 2, \ldots, N$), $\tilde{m}_{H,I(i)}$ and $\overline{m}_{H,I(i)}$ denote the combined probability masses generated by aggregating the first i assessments. The following new ER algorithm is then developed for combining the first i assessments with the $(i+1)$th assessment using the same process as shown in Table I in a recursive manner

$$\begin{aligned}
\{H_n\}\colon m_{n,I(i+1)} &= K_{I(i+1)}[m_{n,I(i)}m_{n,i+1} \\
&\quad + m_{H,I(i)}m_{n,i+1} + m_{n,I(i)}m_{H,i+1}]
\end{aligned} \tag{24a}$$

$$\begin{aligned}
m_{H,I(i)} &= \tilde{m}_{H,I(i)} + \overline{m}_{H,I(i)} \\
&\quad n = 1, 2, \ldots, N
\end{aligned} \tag{24b}$$

$$\begin{aligned}
\{H\}\colon \tilde{m}_{H,I(i+1)} &= K_{I(i+1)}[\tilde{m}_{H,I(i)}\tilde{m}_{H,i+1} \\
&\quad + \overline{m}_{H,I(i)}\tilde{m}_{H,i+1} + \tilde{m}_{H,I(i)}\overline{m}_{H,i+1}]
\end{aligned} \tag{24c}$$

$$\{H\}\colon \overline{m}_{H,I(i+1)} = K_{I(i+1)}[\overline{m}_{H,I(i)}\overline{m}_{H,i+1}] \tag{24d}$$

$$\begin{aligned}
K_{I(i+1)} &= \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)}m_{j,i+1}\right]^{-1} \\
&\quad i = \{1, 2, \ldots, L-1\}.
\end{aligned} \tag{24e}$$

TABLE I
AGGREGATION OF TWO ASSESSMENTS

$S(e_i) \oplus S(e_j)$		$S(e_i)$							
		$(m_{1,i})$ $\{H_1\}$	$(m_{2,i})$ $\{H_2\}$	$\cdots$	$(m_{n,i})$ $\{H_n\}$	$\cdots$	$(m_{N,i})$ $\{H_N\}$	$(\tilde{m}_{H,i})$ $\{H\}$	$(\bar{m}_{H,i})$ $\{H\}$
$S(e_j)$	$(m_{1,j})$ $\{H_1\}$	$(m_{1,i}m_{1,j})$ $\{H_1\}$	$(m_{2,i}m_{1,j})$ $\{\phi\}$	$\cdots$	$(m_{n,i}m_{1,j})$ $\{\phi\}$	$\cdots$	$(m_{N,i}m_{1,j})$ $\{\phi\}$	$(\tilde{m}_{H,i}m_{1,j})$ $\{H_1\}$	$(\bar{m}_{H,i}m_{1,j})$ $\{H_1\}$
	$(m_{2,j})$ $\{H_2\}$	$(m_{1,i}m_{2,j})$ $\{\phi\}$	$(m_{2,i}m_{2,j})$ $\{H_2\}$	$\cdots$	$(m_{n,i}m_{2,j})$ $\{\phi\}$	$\cdots$	$(m_{N,i}m_{2,j})$ $\{\phi\}$	$(\tilde{m}_{H,i}m_{2,j})$ $\{H_2\}$	$(\bar{m}_{H,i}m_{2,j})$ $\{H_2\}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	$(m_{n,j})$ $\{H_n\}$	$(m_{1,i}m_{n,j})$ $\{\phi\}$	$(m_{2,i}m_{n,j})$ $\{\phi\}$	$\cdots$	$(m_{n,i}m_{n,j})$ $\{H_n\}$	$\cdots$	$(m_{N,i}m_{n,j})$ $\{\phi\}$	$(\tilde{m}_{H,i}m_{n,j})$ $\{H_n\}$	$(\bar{m}_{H,i}m_{n,j})$ $\{H_n\}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	$(m_{N,j})$ $\{H_N\}$	$(m_{1,i}m_{N,j})$ $\{\phi\}$	$(m_{2,i}m_{N,j})$ $\{\phi\}$	$\cdots$	$(m_{n,i}m_{N,j})$ $\{\phi\}$	$\cdots$	$(m_{N,i}m_{N,j})$ $\{H_N\}$	$(\tilde{m}_{H,i}m_{N,j})$ $\{H_N\}$	$(\bar{m}_{H,i}m_{N,j})$ $\{H_N\}$

The terms $\overline{m}_{H,\,I(i)}\tilde{m}_{H,\,i+1}$ and $\tilde{m}_{H,\,I(i)}\overline{m}_{H,\,i+1}$ are assigned to $\tilde{m}_{H,\,I(i+1)}$ rather than to $\overline{m}_{H,\,I(i+1)}$ so that the incompleteness synthesis axiom can be satisfied, as proven in Section III-D.

After all L assessments have been aggregated, the combined degrees of belief are generated by assigning $\overline{m}_{H,\,I(L)}$ back to all individual grades proportionally using the following normalization process:

$$\{H_n\}\colon\ \beta_n = \frac{m_{n,\,I(L)}}{1-\overline{m}_{H,\,I(L)}} \qquad n = 1,\,2,\,\ldots,\,N \tag{25a}$$

$$\{H\}\colon\ \beta_H = \frac{\tilde{m}_{H,\,I(L)}}{1-\overline{m}_{H,\,I(L)}}. \tag{25b}$$

β_n generated above is a likelihood to which H_n is assessed. β_H is the unassigned degree of belief representing the extent of incompleteness in the overall assessment. It will be proven in the next section that the combined degrees of belief generated above satisfy all the four synthesis axioms as defined in Section II-D.

In summary, the new ER algorithm is composed of (4) for information acquisition and representation, (22) for weight normalization, (5), (6), and (23) for basic probability assignments, (24a)–(24e) for attribute aggregation, and (25a) and (25b) for generating combined degrees of belief.

Similar to (4), the generated assessment for y can be represented by the following distribution:

$$S(y) = \{(H_n,\,\beta_n),\ n = 1,\,\ldots,\,N\} \tag{26}$$

which reads that y is assessed to the grade H_n with the degree of belief of β_n ($n = 1,\,\ldots,\,N$).

H. Expected Utility and Utility Interval of the ER Approach

There may be occasions where distributed descriptions are not sufficient to show the difference between two assessments. In such cases, it is desirable to generate numerical values equivalent to the distributed assessments in a sense. The concept of expected utility is used to define such values. Suppose $u(H_n)$ is the utility of the grade H_n with

$$u(H_{n+1}) > u(H_n) \quad \text{if } H_{n+1} \text{ is preferred to } H_n. \tag{27}$$

$u(H_n)$ may be estimated using the probability assignment method [8], [20] or by constructing regression models using partial rankings or pairwise comparisons [30]. If all assessments are complete and precise, there will be $\beta_H = 0$ and the expected utility of the attribute y can be used for ranking alternatives, which is calculated by

$$u(y) = \sum_{n=1}^{N} \beta_n u(H_n). \tag{28}$$

An alternative a is preferred to another alternative b on y if and only if $u(y(a)) > u(y(b))$.

If any assessment for the basic attribute is incomplete, it will be proven that β_H is positive. Within the ER assessment framework, β_n given in (25a) represents the belief measure in the D–S theory and thus provides the lower bound of the likelihood to which y is assessed to H_n [9], [13], [23]. The upper bound of the likelihood is given by a plausibility measure [13], [32]. It can be shown that the plausibility measure for H_n within the ER evaluation framework is given by $(\beta_n + \beta_H)$. Thus the belief interval $[\beta_n,\,(\beta_n+\beta_H)]$ provides the range of the likelihood to which y may be assessed to H_n. It is obvious that the interval will reduce to a point β_n if all assessments are complete.

The above discussion shows that if any basic assessment is incomplete, the likelihood to which y may be assessed to H_n is not unique and can be anything in the interval $[\beta_n,\,(\beta_n+\beta_H)]$. In such circumstances, we define three measures to characterize the assessment for y, namely the minimum, maximum and average expected utilities.

Without loss of generality, suppose H_1 is the least preferred grade having the lowest utility and H_N the most preferred grade having the highest utility. Then the maximum, minimum and average expected utilities on y are given by

$$u_{\max}(y) = \sum_{n=1}^{N-1} \beta_n u(H_n) + (\beta_N + \beta_H)u(H_N) \quad (29)$$

$$u_{\min}(y) = (\beta_1 + \beta_H)u(H_1) + \sum_{n=2}^{N} \beta_n u(H_n) \quad (30)$$

$$u_{avg}(y) = \frac{u_{\max}(y) + u_{\min}(y)}{2}. \quad (31)$$

If all original assessments $S(e_i)$ are complete, then $\beta_H = 0$ and $u(y) = u_{\max}(y) = u_{\min}(y) = u_{avg}(y)$. Note that the above utilities are only used for characterizing an assessment but not for attribute aggregation.

The ranking of two alternatives a_l and a_k is based on their utility intervals. a_l is said to be preferred to a_k on y if and only if $u_{\min}(y(a_l)) > u_{\max}(y(a_k))$; a_l is said to be indifferent to a_k if and only if $u_{\min}(y(a_l)) = u_{\min}(y(a_k))$ and $u_{\max}(a_l) = u_{\max}(a_k)$. Otherwise, average expected utility may be used to generate a ranking, though such a ranking is inconclusive. For instance, if $u_{avg}(y(a_l)) > u_{avg}(y(a_k))$ but $u_{\max}(y(a_k)) > u_{\min}(y(a_l))$, one could say that a_l is preferred to a_k on an average basis. However, this ranking is not reliable, as there is a chance that a_k may have higher utility than a_l. In such cases, to generate a reliable ranking the quality of the original assessments must be improved by reducing incompleteness present in the original assessments associated with a_l and a_k. Note that to clarify the relationship between a_l and a_k there is no need to improve the quality of information related to other alternatives.

III. Properties of the New ER Algorithm

In this section, we prove the conclusions that we took for granted to develop the new ER approach in the last section. These include the basic synthesis theorem, the consensus synthesis theorem, the complete synthesis theorem and the incomplete synthesis theorem.

A. Basic Synthesis Theorem

In the new ER approach, the combined degrees of belief β_n ($n = 1, \ldots, N$) and β_H are generated using (24a)–(24e), (25a), and (25b). These belief degrees are between zero and one and are summed to one as proved in the following theorem.

Theorem 2: The degrees of belief generated using (25a) and (25b) possess the following property:

$$0 \le \beta_n \le 1, \qquad n = 1, \ldots, N;\ 0 \le \beta_H \le 1 \quad (32)$$

$$\sum_{n=1}^{N} \beta_n + \beta_H = 1. \quad (33)$$

Proof: First of all, we prove that (24b) is held for any $i = \{1, 2, \ldots, L-1\}$. Note that without partition of $m_{H,I(i+1)}$ into $\tilde{m}_{H,I(i+1)}$ and $\overline{m}_{H,I(i+1)}$, $m_{H,I(i+1)}$ would be calculated using (8b), or

$$m_{H,I(i+1)} = K_{I(i+1)}[m_{H,I(i)} m_{H,i+1}] \quad i = \{1, 2, \ldots, L-1\}.$$

From the definitions of $\tilde{m}_{H,l}$ and $\overline{m}_{H,l}$ shown in (23), we have

$$m_{H,l} = \overline{m}_{H,l} + \tilde{m}_{H,l} \qquad \text{for all } l = \{1, 2, \ldots, L-1\}.$$

Suppose $E_{I(2)} = \{e_1, e_2\}$ and note that $\tilde{m}_{H,I(1)} = \tilde{m}_{H,1}$ and $\overline{m}_{H,I(1)} = \overline{m}_{H,1}$. For $i = 1$, from the new ER algorithm [(24c) and (24d)] we have

$$\begin{aligned}
&\tilde{m}_{H,I(2)} + \overline{m}_{H,I(2)} \\
&\quad = K_{I(2)}[\tilde{m}_{H,1}\tilde{m}_{H,2} + \overline{m}_{H,1}\tilde{m}_{H,2} + \tilde{m}_{H,1}\overline{m}_{H,2}] \\
&\qquad + K_{I(2)}[\overline{m}_{H,1}\overline{m}_{H,2}] \\
&\quad = K_{I(2)}[(\tilde{m}_{H,1} + \overline{m}_{H,1})\tilde{m}_{H,2} + (\tilde{m}_{H,1} + \overline{m}_{H,1})\overline{m}_{H,2}] \\
&\quad = K_{I(2)}[m_{H,1}\tilde{m}_{H,2} + m_{H,1}\overline{m}_{H,2}] \\
&\quad = K_{I(2)}[m_{H,1}(\tilde{m}_{H,2} + \overline{m}_{H,2})] \\
&\quad = K_{I(2)}[m_{H,1}m_{H,2}] = m_{H,I(2)}.
\end{aligned}$$

Suppose for $i = r-1$ there is $m_{H,I(r)} = \tilde{m}_{H,I(r)} + \overline{m}_{H,I(r)}$. For $i = r$, from the new ER algorithm we then get

$$\begin{aligned}
&\tilde{m}_{H,I(r+1)} + \overline{m}_{H,I(r+1)} \\
&\quad = K_{I(r+1)}[\tilde{m}_{H,I(r)}\tilde{m}_{H,r+1} + \overline{m}_{H,I(r)}\tilde{m}_{H,r+1} \\
&\qquad + \tilde{m}_{H,I(r)}\overline{m}_{H,r+1}] + K_{I(r+1)}[\overline{m}_{H,I(r)}\overline{m}_{H,r+1}] \\
&\quad = K_{I(r+1)}[(\tilde{m}_{H,I(r)} + \overline{m}_{H,I(r)})\tilde{m}_{H,r+1} \\
&\qquad\qquad + (\tilde{m}_{H,I(r)} + \overline{m}_{H,I(r)})\overline{m}_{H,r+1}] \\
&\quad = K_{I(r+1)}[m_{H,I(r)}\tilde{m}_{H,r+1} + m_{H,I(r)}\overline{m}_{H,r+1}] \\
&\quad = K_{I(r+1)}[m_{H,I(r)}(\tilde{m}_{H,r+1} + \overline{m}_{H,r+1})] \\
&\quad = K_{I(r+1)}[m_{H,I(r)}m_{H,r+1}] = m_{H,I(r+1)}.
\end{aligned}$$

Having proven (24b), (24a)–(24e) is then the straightforward implementation of the evidence combination rule of the D–S theory within the ER framework and thus ensure that

$$\sum_{n=1}^{N} m_{n,I(L)} + \overline{m}_{H,I(L)} + \tilde{m}_{H,I(L)} = 1. \quad (34)$$

Therefore, from (25a) and (25b), we have

$$\begin{aligned}
&\sum_{n=1}^{N} \beta_n + \beta_H \\
&\quad = \sum_{n=1}^{N} \frac{m_{n,I(L)}}{1 - \overline{m}_{H,I(L)}} + \frac{\tilde{m}_{H,I(L)}}{1 - \overline{m}_{H,I(L)}} \\
&\quad = \frac{1}{1 - \overline{m}_{H,I(L)}} \left(\sum_{n=1}^{N} m_{n,I(L)} + \tilde{m}_{H,I(L)} \right) = 1. \quad (35)
\end{aligned}$$

As $K_{I(i+1)}$ calculated by (24e) is always positive, we must have

$$m_{n,I(L)} \ge 0 \qquad \text{for all } n = 1, 2, \ldots, N,\ \overline{m}_{H,I(L)} \ge 0 \text{ and } \tilde{m}_{H,I(L)} \ge 0. \quad (36)$$

With criteria weights normalized using (22) and from (34) and (36), there must be $\overline{m}_{H,I(L)} < 1$. From (25a), (25b), and (36), we then have

$$\beta_n \ge 0, \qquad n = 1, \ldots, N \text{ and } \beta_H \ge 0. \quad (37)$$

From (35) and (37), we therefore conclude

$$\beta_n \le 1, \qquad n = 1, \ldots, N \text{ and } \beta_H \le 1.$$

Q.E.D.

B. Consensus Synthesis Theorem

In this section, we prove that the combined degrees of belief generated using (24a)–(24e), (25a), and (25b) satisfy the consensus axiom, as shown by the following theorem.

Theorem 3: If in (4) $\beta_{k,i} = 1$ and $\beta_{n,i} = 0$ for all $n = 1, \ldots, N$ with $n \neq k$ and $i = 1, \ldots, L$, then β_n and β_H calculated using (25a) and (25b) satisfy that $\beta_k = 1$, $\beta_n = 0$ for all $n = 1, \ldots, N$ with $n \neq k$ and $\beta_H = 0$.

Proof: From (5) and (6), the basic probability assignments are given by

$$m_{k,i} = \omega_i \beta_{k,i} = \omega_i \qquad i = 1, \ldots, L \tag{38}$$

$$m_{n,i} = \omega_i \beta_{n,i} = 0 \qquad i = 1, \ldots, L; \quad n = 1, \ldots, N,\ n \neq k \tag{39}$$

$$\overline{m}_{H,i} = 1 - \omega_i \qquad i = 1, \ldots, L \tag{40}$$

$$\tilde{m}_{H,i} = \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) = 0 \qquad i = 1, \ldots, L. \tag{41}$$

Since $\tilde{m}_{H,I(1)} = \tilde{m}_{H,1} = 0$, from (24c) and (41), we have

$$\begin{aligned}\tilde{m}_{H,I(2)} &= K_{I(2)}[\tilde{m}_{H,1}\tilde{m}_{H,2} + \overline{m}_{H,1}\tilde{m}_{H,2} + \tilde{m}_{H,1}\overline{m}_{H,2}] \\ &= K_{I(2)}[0 \times 0 + \overline{m}_{H,1} \times 0 + 0 \times \overline{m}_{H,2}] = 0.\end{aligned}$$

Suppose $\tilde{m}_{H,I(i)} = 0$ for $i = r > 2$. We then get

$$\begin{aligned}&\tilde{m}_{H,I(r+1)} \\ &\quad = K_{I(r+1)}[\tilde{m}_{H,I(r)}\tilde{m}_{H,r+1} + \overline{m}_{H,I(r)}\tilde{m}_{H,r+1} \\ &\qquad\qquad + \tilde{m}_{H,I(r)}\overline{m}_{H,r+1}] \\ &\quad = K_{I(r+1)}[0 \times 0 + \overline{m}_{H,I(r)} \times 0 + 0 \times \overline{m}_{H,r+1}] = 0.\end{aligned}$$

Therefore, there must be

$$\tilde{m}_{H,I(i+1)} = 0 \qquad \text{for any } i = 1, 2, \ldots, L-1. \tag{42}$$

Thus

$$\beta_H = \frac{\tilde{m}_{H,I(L)}}{1 - \overline{m}_{H,I(L)}} = 0. \tag{43}$$

Since $m_{n,I(1)} = m_{n,1} = 0$ for $n \neq k$, from (39) and (40), we have at $i = 1$

$$\left.\begin{aligned}&m_{n,I(i)}m_{n,i+1} = 0, \\ &m_{n,I(i)}m_{H,i+1} = 0, \\ &m_{H,I(i)}m_{n,i+1} = 0\end{aligned}\right. \qquad \text{for any } n = 1, \ldots, N,\ n \neq k.$$

Using (24a), we then get at $i = 2$

$$m_{n,I(i)} = 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k.$$

Suppose $m_{n,I(i)} = 0$ at i with $2 < i < L$ for $n = 1, \ldots, N, n \neq k$. Using (24a)–(24e) again, we generate

$$\begin{aligned}m_{n,I(i+1)} &= K_{I(i+1)}\left(m_{n,I(i)}m_{n,i+1} + m_{n,I(i)}m_{H,i+1} + m_{H,I(i)}m_{n,i+1}\right) \\ &= K_{I(i+1)}\left(0 \times 0 + 0 \times m_{H,i+1} + m_{H,I(i)} \times 0\right) \\ &= 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k.\end{aligned}$$

Thus, we deduce

$$m_{n,I(i)} = 0 \text{ at any } i = 1, \ldots, L \text{ for } n = 1, \ldots, N, n \neq k.$$

We therefore conclude that

$$\beta_n = \frac{m_{n,I(L)}}{1 - \overline{m}_{H,I(L)}} = 0 \qquad \text{for } n = 1, \ldots, N,\ n \neq k. \tag{44}$$

Combining (33), (43), and (44), we have

$$\sum_{n=1}^{N} \beta_n + \beta_H = 1 \quad \text{or} \quad \beta_k + \left\{ \sum_{\substack{n=1 \\ n \neq k}}^{N} \beta_n + \beta_H \right\} = 1.$$

Therefore, $\beta_k = 1$. Q.E.D.

C. Complete Synthesis Theorem

We now prove that if all basic attributes are completely assessed to a set of evaluation grades then the associated general attribute is also completely assessed to the same subset of the grades using the new ER approach.

Theorem 4: Let H^+ be a subset of H defined by (3) and H^- the negation of H^+, or $H^+, H^- \subseteq H$, $H^+ \cup H^- = H$, $H^+ \cap H^- = \phi$. Define $I^+ = \{n | H_n \in H^+\}$ and $I^- = \{j | H_j \in H^-\}$. If $\beta_{n,i} > 0$ ($n \in I^+$), $\sum_{n \in I^+} \beta_{n,i} = 1$ and $\beta_{j,i} = 0$ ($j \in I^-$) for all $i = 1, \ldots, L$, then the combined degrees of belief obtained using (24a)–(24e), (25a), and (25b) satisfy $\sum_{n \in I^+} \beta_n = 1$ and $\beta_j = 0$ ($j \in I^-$).

Proof: From (5), (6), and (23), the basic probability assignments are given by

$$\begin{aligned}m_{n,i} &= \omega_i \beta_{n,i} > 0 \qquad \text{for } n \in I^+,\ i = 1, \ldots, L \\ m_{j,i} &= \omega_i \beta_{j,i} = 0 \qquad \text{for } j \in I^-,\ i = 1, \ldots, L \\ \overline{m}_{H,i} &= 1 - \omega_i \\ \tilde{m}_{H,i} &= \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) = 0, \\ &\qquad \text{for } i = 1, \ldots, L.\end{aligned}$$

In the same way of proving (42), we can get

$$\tilde{m}_{H,I(i+1)} = 0 \qquad \text{for all } i = 1, 2, \ldots, L-1.$$

Therefore

$$\beta_H = \frac{\tilde{m}_{H,I(L)}}{1 - \overline{m}_{H,I(L)}} = 0. \tag{45}$$

Since $m_{j,I(1)} = m_{j,1} = 0$ for $j \in I^-$, we then have at $i = 1$

$$\left.\begin{aligned}&m_{j,I(i)}m_{j,i+1} = 0, \\ &m_{j,I(i)}m_{H,i+1} = 0, \\ &m_{H,I(i)}m_{j,i+1} = 0\end{aligned}\right. \qquad \text{for } j \in I^-.$$

Using (24a)–(24e), we then get $m_{j,I(i)} = 0$ for $j \in I^-$ at $i = 2$.

Suppose $m_{j,I(r)} = 0$ for $j \in I^-$ at $i = r$. Using (24a)–(24e), we generate

$$\begin{aligned}&m_{j,I(r+1)} \\ &\quad = K_{I(r+1)}\left(m_{j,I(r)}m_{j,r+1} + m_{j,I(r)}m_{H,r+1} \right. \\ &\qquad\qquad \left. + m_{H,I(r)}m_{j,r+1}\right) \\ &\quad = K_{I(r+1)}\left(0 \times 0 + 0 \times m_{H,r+1} + m_{H,I(r)} \times 0\right) \\ &\quad = 0 \qquad \text{for } j \in I^-.\end{aligned}$$

Thus, we deduce $m_{j,I(i)} = 0$ for $j \in I^-$ at any $i = 1, \ldots, L$. We therefore conclude that

$$\beta_n = \frac{m_{n,I(L)}}{1 - \overline{m}_{H,I(L)}} = 0 \qquad \text{for } j \in I^-.$$

Thus

$$\sum_{j\in I^-} \beta_j = 0. \tag{46}$$

From Theorem 1, (45) and (46), we finally conclude

$$1 = \sum_{n=1}^{N} \beta_n + \beta_H = \sum_{n\in I^+} \beta_n + \sum_{j\in I^-} \beta_j + \beta_H = \sum_{n\in I^+} \beta_n.$$

Q.E.D

D. Incomplete Synthesis Theorem

If any basic assessment is not complete, we have the following conclusion.

Theorem 5: Suppose no basic attribute is given a weight of zero (completely ignored) or a weight of one (dominating the assessment). If the assessment of a basic attribute is not complete, then the assessment for the associated general attribute will not be complete either, or $\beta_H > 0$.

Proof: Suppose the assessment on basic attribute j is not complete, or $\sum_{n=1}^{N} \beta_{n,\,j} < 1$ and all other assessments are complete. If no attribute is dominant and the weights of attributes are normalized using (22), then there will be $1 > \omega_i > 0$ for all $i = \{1, \ldots, L\}$. From (5) and (6), the basic remaining probability assignments are given by

$$\bar{m}_{H,\,i} = 1 - \omega_i > 0 \qquad i = 1, \ldots, L \tag{47}$$

$$\tilde{m}_{H,\,i} = \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,\,i}\right) = 0 \qquad i = 1, \ldots, L;\ i \neq j \tag{48}$$

$$\tilde{m}_{H,\,j} = \omega_j \left(1 - \sum_{n=1}^{N} \beta_{n,\,j}\right) > 0. \tag{49}$$

Equation (24e) ensures that $K_{I(i)} \geq 1$ for any $i = \{1, \ldots, L\}$. From (24d), we then have

$$\bar{m}_{H,\,I(i)} = K_{I(i)} \bar{m}_{H,\,i} \bar{m}_{H,\,I(i-1)} = \prod_{l=2}^{i} K_{I(l)} \prod_{k=1}^{i} \bar{m}_{H,\,k} > 0 \quad \text{for any } i = \{1, \ldots, L\}$$

and thus $\bar{m}_{H,\,I(j-1)} \tilde{m}_{H,\,j} > 0$. Therefore, (24c) leads to

$$\tilde{m}_{H,\,I(j)} = K_{I(j)} \left[\tilde{m}_{H,\,I(j-1)} \tilde{m}_{H,\,j} + \bar{m}_{H,\,I(j-1)} \tilde{m}_{H,\,j} + \tilde{m}_{H,\,I(j-1)} \bar{m}_{H,\,j}\right] > 0.$$

Suppose $\tilde{m}_{H,\,I(i)} > 0$ for some $i \geq j$. Then, $\tilde{m}_{H,\,I(i)} \bar{m}_{H,\,i+1} > 0$. Thus

$$\tilde{m}_{H,\,I(i+1)} = K_{I(i+1)} \left[\tilde{m}_{H,\,I(i)} \tilde{m}_{H,\,i+1} + \bar{m}_{H,\,I(i)} \tilde{m}_{H,\,i+1} + \tilde{m}_{H,\,I(i)} \bar{m}_{H,\,i+1}\right] > 0.$$

We therefore conclude $\tilde{m}_{H,\,I(i)} > 0$ for any $i = \{j, \ldots, L\}$ and

$$\beta_H = \frac{\tilde{m}_{H,\,I(L)}}{1 - \bar{m}_{H,\,I(L)}} > 0. \qquad \text{Q.E.D.}$$

Note that β_H is the remaining degree of belief unassigned to individual grades, measuring the degree of incompleteness in the assessment. The new ER algorithm inherits the features of the evidence combination rule of the D–S theory and ensures that $(\beta_i + \beta_H)$, as the plausibility measure in the ER framework, provides the upper bound of the degree of belief to which y is assessed to the grade H_i.

IV. Numerical Study

A. Motorcycle Assessment Problem

In this section, the new evidential reasoning approach is applied to analyzing the performance of four types of motorcycles, including *Kawasaki*, *Yamaha*, *Honda*, and *BMW*. For the purpose of demonstrating the ER algorithm, only qualitative performance attributes are taken into account, though quantitative attributes may also be included [23], [24], [28].

Three major performance attributes are considered which are *quality of engine*, *operation*, and *general finish*. These attributes are general and difficult to assess directly. Lower level attributes are therefore used to facilitate the assessment of these three attributes. For instance, the quality of an engine may be assessed through the engine's *responsiveness*, *fuel economy*, *quietness*, *vibration*, and *starting*. An attribute hierarchy for evaluation of motorcycles is shown in Fig. 2, where ω_i, ω_{ij}, and ω_{ijk} denotes relative weights of relevant attributes.

Qualitative evaluation information can be expressed using statements such as 1)–3) as discussed in Section II-A. These assessments can be summarized as in Table II, where typical elements in a subjective judgment are listed, including the definitions of attributes, evaluation grades, and degrees of belief.

Using the grades defined in (1), the above three assessments can be represented using the following three distributions as defined in (4)

$$S(stopping\ power) = \{(average,\ 0.3),\ (good,\ 0.6)\} \tag{50a}$$

$$S(braking\ stability) = \{(good,\ 1.0)\} \tag{50b}$$

$$S(feel\ at\ control) = \{(good,\ 0.5),\ (excellent,\ 0.5)\}. \tag{50c}$$

Note that only grades with nonzero degrees of belief are listed in the distributions. The other assessment information in terms of the basic attributes as defined in Fig. 2 were extracted and transformed from the published sources [24]. The assessment problem is summarized as in Table III, where P, I, A, G, and E are the abbreviations of the evaluation grades *poor*, *indifferent*, *average*, *good*, and *excellent*, respectively, and a number in a bracket denotes a degree of belief to which an attribute is assessed to a grade. For instance, E(0.8) means "*excellent* to a degree of 0.8 (80%)."

In Table III, a subjective assessment is expressed in a concise format. As shown for the *brakes* of *Yamaha*, for example, statement 1) in Section II-A or (50a) is expressed by "A(0.3), G(0.6)," statement 2) or (50b) by "G(1.0)," and statement 3) or (50c) by "G(0.5), E(0.5)." Note that statement 1) is incomplete and in Table III there are a number of incomplete assessments similar to statement 1).

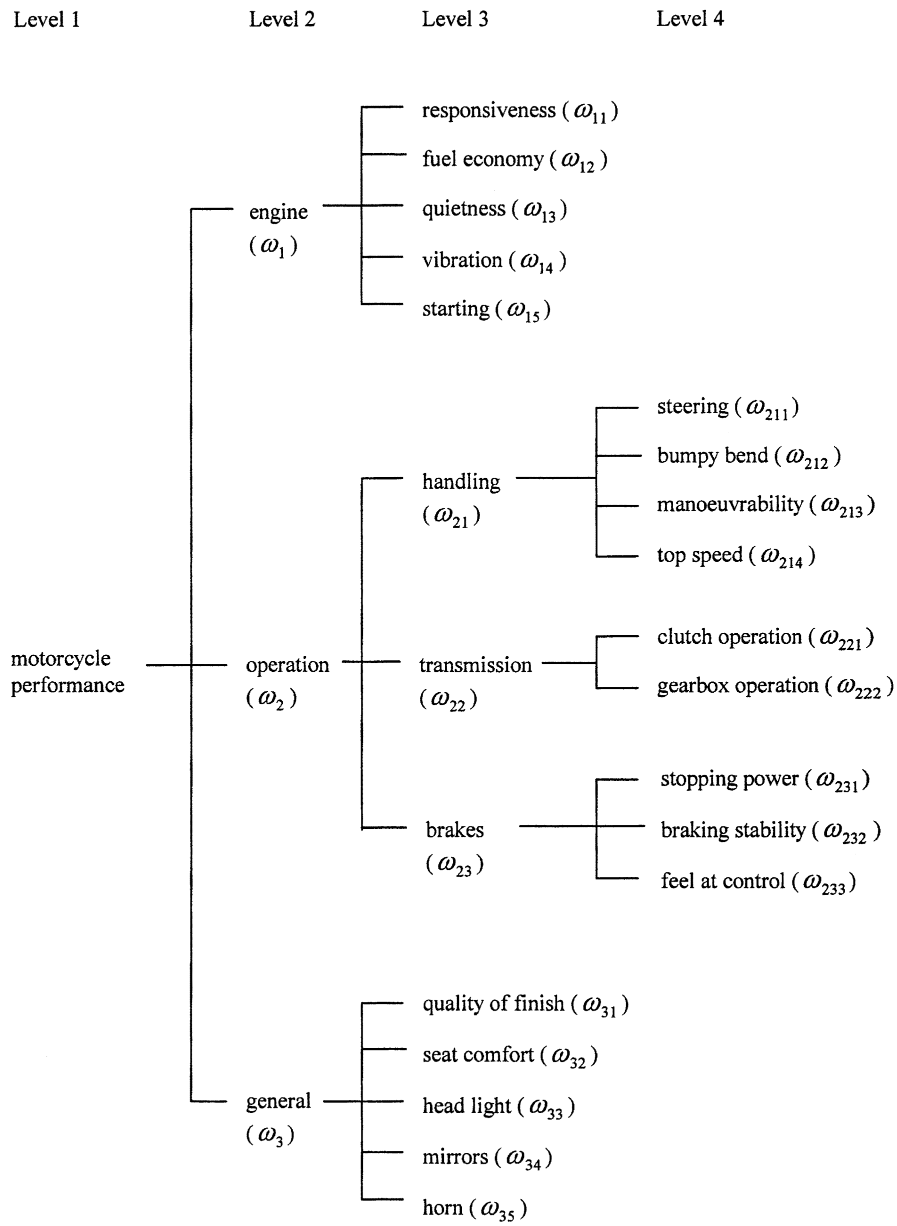


Fig. 2. Evaluation hierarchy for motorcycle performance assessment.

TABLE II
SUBJECTIVE JUDGMENTS FOR EVALUATING *BRAKES* OF *YAMAHA*

Degree of belief (β)		Evaluation grade				
		poor	*indifferent*	*average*	*good*	*excellent*
Basic attribute	*stopping power*			0.3	0.6	
	braking stability				1.0	
	feel at control				0.5	0.5

B. Aggregating Assessments via Evidential Reasoning—A Step-by-Step Illustration

A basic assessment problem is how the original judgments as given in Table II or (50a)–(50c) could be aggregated to arrive at an assessment about the quality of the *brakes* of *Yamaha*. It is intuitively clear from Table II that the quality of *Yamaha's brakes* should be *good* to a large extent. To generate a precise assessment, however, the relative importance of the three attributes needs to be assigned. Several well-known methods could be used for weight assignment [7], [11]. For the purpose of demonstrating the ER algorithm, hypothetical weights are used in this analysis. Without loss of generality, suppose the attributes are of equal importance. The evidential reasoning approach can then be applied to deal with the assessment problem.

To demonstrate the implementation procedure of the new evidential reasoning algorithm, we first show the detailed calculation steps for generating the assessment for *Yamaha*'s brakes (y) by aggregating three basic attributes: *stopping power, braking stability,* and *feel at control,* as shown in (50a)–(50c) and denoted by e_1, e_2, and e_3, respectively. The evaluation grades are as defined in (1). Let $y = e_1 \oplus e_2 \oplus e_3$, where $\oplus$ denotes the aggregation of two attributes.

From (50a)–(50c) and (4), we have

$$\beta_{1,1} = 0, \quad \beta_{2,1} = 0, \quad \beta_{3,1} = 0.3, \quad \beta_{4,1} = 0.6, \quad \beta_{5,1} = 0$$
$$\beta_{1,2} = 0, \quad \beta_{2,2} = 0, \quad \beta_{3,2} = 0, \quad \beta_{4,2} = 1.0, \quad \beta_{5,2} = 0$$
$$\beta_{1,3} = 0, \quad \beta_{2,3} = 0, \quad \beta_{3,3} = 0, \quad \beta_{4,3} = 0.5, \quad \beta_{5,3} = 0.5.$$

TABLE III
GENERALIZED DECISION MATRIX FOR MOTORCYCLE ASSESSMENT

General attribute			Basic attribute	Type of motorcycle: *Kawasaki*	*Yamaha*	*Honda*	*BMW*
overall performance	engine ω_1		responsiveness ω_{11}	E(0.8)	G(0.3), E(0.6)	G(1.0)	I(1.0)
			fuel economy ω_{12}	A(1.0)	I(1.0)	I(0.5), A(0.5)	E(1.0)
			quietness ω_{13}	I(0.5), A(0.5)	A(1.0)	G(0.5), E(0.3)	E(1.0)
			vibration ω_{14}	G(1.0)	I(1.0)	G(0.5), E(0.5)	P(1.0)
			starting ω_{15}	G(1.0)	A(0.6), G(0.3)	G(1.0)	A(1.0)
	operation ω_2	handling ω_{21}	steering ω_{211}	E(0.9)	G(1.0)	A(1.0)	A(0.6)
			bumpy bends ω_{212}	A(0.5), G(0.5)	G(1.0)	G(0.8), E(0.1)	P(0.5), I(0.5)
			Manoeuvrability ω_{213}	A(1.0)	E(0.9)	I(1.0)	P(1.0)
			top speed stability ω_{214}	E(1.0)	G(1.0)	G(1.0)	G(0.6), E(0.4)
		transmission ω_{22}	clutch operation ω_{221}	A(0.8)	G(1.0)	E(0.85)	I(0.2), A(0.8)
			gearbox operation ω_{222}	A(0.5), G(0.5)	I(0.5), A(0.5)	E(1.0)	P(1.0)
		brakes ω_{23}	stopping power ω_{231}	G(1.0)	A(0.3), G(0.6)	G(0.6)	E(1.0)
			braking stability ω_{232}	G(0.5), E(0.5)	G(1.0)	A(0.5), G(0.5)	E(1.0)
			feel at control ω_{233}	P(1.0)	G(0.5), E(0.5)	G(1.0)	G(0.5), E(0.5)
	general ω_3		quality of finish ω_{31}	P(0.5), I(0.5)	G(1.0)	E(1.0)	G(0.5), E(0.5)
			seat comfort ω_{32}	G(1.0)	G(0.5), E(0.5)	G(0.6)	E(1.0)
			headlight ω_{33}	G(1.0)	A(1.0)	E(1.0)	G(0.5), E(0.5)
			mirrors ω_{34}	A(0.5), G(0.5)	G(0.5), E(0.5)	E(1.0)	G(1.0)
			horn ω_{35}	A(1.0)	G(1.0)	G(0.5), E(0.5)	E(1.0)

Suppose the three attributes are of equal importance. From (22) we then have $L = 3$ and $\omega_1 = \omega_2 = \omega_3 = 1/3$. From (5), (6), and (23), we can calculate basic probability masses $m_{n,\,i}$ as follows:

$$\begin{aligned}
m_{1,\,1} &= 0,\;\; m_{2,\,1} = 0,\;\; m_{3,\,1} = 0.3/3,\;\; m_{4,\,1} = 0.6/3\\
m_{5,\,1} &= 0,\;\; \overline{m}_{H,\,1} = 2/3,\;\; \tilde{m}_{H,\,1} = 0.1/3\\
m_{1,\,2} &= 0,\;\; m_{2,\,2} = 0,\;\; m_{3,\,2} = 0,\;\; m_{4,\,2} = 1/3\\
m_{5,\,2} &= 0,\;\; \overline{m}_{H,\,2} = 2/3,\;\; \tilde{m}_{H,\,2} = 0\\
m_{1,\,3} &= 0,\;\; m_{2,\,3} = 0,\;\; m_{3,\,3} = 0,\;\; m_{4,\,3} = 0.5/3\\
m_{5,\,3} &= 0.5/3,\;\; \overline{m}_{H,\,3} = 2/3,\;\; \tilde{m}_{H,\,3} = 0.
\end{aligned}$$

We can now use the recursive equations (24a)–(24e) to calculate the combined probability masses as follows. Let $m_{n,\,I(1)} = m_{n,\,1}$ for $n = 1, \ldots, 5$. First, we aggregate *stopping power* and *braking stability*. Since

$$\begin{aligned}
K_{I(2)} &= \left[1 - \sum_{t=1}^{5} \sum_{\substack{j=1\\ j\neq t}}^{5} m_{t,\,I(1)} m_{j,\,2}\right]^{-1}\\
&= [1 - (0 + \cdots + 0 + m_{3,\,1} \times m_{4,\,2} + 0 + \cdots + 0)]^{-1}\\
&= [1 - 0.1 \times 0.3333]^{-1} = 1.0345
\end{aligned}$$

and $m_{H,\,i} = \overline{m}_{H,\,i} + \tilde{m}_{H,\,i}$ ($i = 1, 2, 3$), we then have

$$\begin{aligned}
m_{1,\,I(2)} &= K_{I(2)}(m_{1,\,1}m_{1,\,2} + m_{1,\,1}m_{H,\,2} + m_{H,\,1}m_{1,\,2}) = 0\\
m_{2,\,I(2)} &= K_{I(2)}(m_{2,\,1}m_{2,\,2} + m_{2,\,1}m_{H,\,2} + m_{H,\,1}m_{2,\,2}) = 0\\
m_{3,\,I(2)} &= K_{I(2)}(m_{3,\,1}m_{3,\,2} + m_{3,\,1}m_{H,\,2} + m_{H,\,1}m_{3,\,2})\\
&= 1.0345(0 + 0.1 \times 2/3 + 0) = 0.069\\
m_{4,\,I(2)} &= K_{I(2)}(m_{4,\,1}m_{4,\,2} + m_{4,\,1}m_{H,\,2} + m_{H,\,1}m_{4,\,2})\\
&= 1.0345(0.2 \times 1/3 + 0.2 \times 2/3 + 2.1/3 \times 1/3)\\
&= 0.4483\\
m_{5,\,I(2)} &= K_{I(2)}(m_{5,\,1}m_{5,\,2} + m_{5,\,1}m_{H,\,2} + m_{H,\,1}m_{5,\,2}) = 0\\
\tilde{m}_{H,\,I(2)} &= K_{I(2)}(\tilde{m}_{H,\,1}\tilde{m}_{H,\,2} + \overline{m}_{H,\,1}\tilde{m}_{H,\,2} + \tilde{m}_{H,\,1}\overline{m}_{H,\,2})\\
&= 1.0345(0.1/3 \times 0 + 2/3 \times 0 + 0.1/3 \times 2/3)\\
&= 0.023\\
\overline{m}_{H,\,I(2)} &= K_{I(2)}\overline{m}_{H,\,1}\overline{m}_{H,\,2}\\
&= 1.0345 \times 2/3 \times 2/3 = 0.4598.
\end{aligned}$$

Now, we combine the above results for *stopping power* and *braking stability* with *feel at control*. Since

$$\begin{aligned}
K_{I(3)} &= \left[1 - \sum_{t=1}^{5} \sum_{\substack{j=1\\ j\neq t}}^{5} m_{t,\,I(2)} m_{j,\,3}\right]^{-1}\\
&= \big[1 - (0 + \cdots + 0 + m_{3,\,I(2)}m_{4,\,3} + m_{3,\,I(2)}m_{5,\,3}\\
&\qquad + m_{4,\,I(2)}m_{5,\,3} + 0 + \cdots + 0)\big]^{-1}\\
&= [1 - (0.069 \times 0.5/3 + 0.069 \times 0.5/3\\
&\qquad + 0.4483 \times 0.5/3)]^{-1} = 1.1083
\end{aligned}$$

and $m_{H,\,I(2)} = \overline{m}_{H,\,I(2)} + \tilde{m}_{H,\,I(2)} = 0.4598 + 0.023 = 0.4828$, we then have

$$\begin{aligned}
m_{1,\,I(3)} &= K_{I(3)}\left(m_{1,\,I(2)}m_{1,\,3} + m_{1,\,I(2)}m_{H,\,3}\right.\\
&\qquad \left. + \, m_{H,\,I(2)}m_{1,\,3}\right) = 0\\
m_{2,\,I(3)} &= K_{I(3)}\left(m_{2,\,I(2)}m_{2,\,3} + m_{2,\,I(2)}m_{H,\,3}\right.\\
&\qquad \left. + \, m_{H,\,I(2)}m_{2,\,3}\right) = 0\\
m_{3,\,I(3)} &= K_{I(3)}\left(m_{3,\,I(2)}m_{3,\,3} + m_{3,\,I(2)}m_{H,\,3}\right.\\
&\qquad \left. + \, m_{H,\,I(2)}m_{3,\,3}\right)\\
&= 1.1083(0.069 \times 0 + 0.069 \times 2/3 + 0.4828 \times 0)\\
&= 0.051\\
m_{4,\,I(3)} &= K_{I(3)}\left(m_{4,\,I(2)}m_{4,\,3} + m_{4,\,I(2)}m_{H,\,3}\right.\\
&\qquad \left. + \, m_{H,\,I(2)}m_{4,\,3}\right)\\
&= 1.1083(0.4483 \times 0.5/3 + 0.4483 \times 2/3\\
&\qquad + 0.4828 \times 0.5/3) = 0.5032\\
m_{5,\,I(3)} &= K_{I(3)}\left(m_{5,\,I(2)}m_{5,\,3} + m_{5,\,I(2)}m_{H,\,3}\right.\\
&\qquad \left. + \, m_{H,\,I(2)}m_{5,\,3}\right)\\
&= 1.1083(0 \times 0.5/3 + 0 \times 2/3 + 0.4828 \times 0.5/3)\\
&= 0.0892\\
\tilde{m}_{H,\,I(3)} &= K_{I(3)}\left(\tilde{m}_{H,\,I(2)}\tilde{m}_{H,\,3} + \overline{m}_{H,\,I(2)}\tilde{m}_{H,\,3}\right.\\
&\qquad \left. + \, \tilde{m}_{H,\,I(2)}\overline{m}_{H,\,3}\right)\\
&= 1.1083(0.023 \times 0 + 0.4598 \times 0\\
&\qquad + 0.023 \times 2/3) = 0.017\\
\overline{m}_{H,\,I(3)} &= K_{I(3)}\overline{m}_{H,\,I(2)}\overline{m}_{H,\,3}\\
&= 1.1083 \times 0.4598 \times 2/3 = 0.3397.
\end{aligned}$$

From (25a) and (25b), the combined degrees of belief are calculated by

$$\begin{aligned}
\beta_n &= \frac{m_{n,\,I(3)}}{1 - \overline{m}_{H,\,I(3)}} = 0, \qquad n = 1, 2\\
\beta_3 &= \frac{m_{3,\,I(3)}}{1 - \overline{m}_{H,\,I(3)}} = \frac{0.051}{1 - 0.3397} = 0.0772\\
\beta_4 &= \frac{m_{4,\,I(3)}}{1 - \overline{m}_{H,\,I(3)}} = \frac{0.5032}{1 - 0.3397} = 0.7621\\
\beta_5 &= \frac{m_{5,\,I(3)}}{1 - \overline{m}_{H,\,I(3)}} = \frac{0.0892}{1 - 0.3397} = 0.1350\\
\beta_H &= \frac{\tilde{m}_{H,\,I(3)}}{1 - \overline{m}_{H,\,I(3)}} = \frac{0.017}{1 - 0.3397} = 0.0257.
\end{aligned}$$

The assessment for *Yamaha*'s *brakes* by aggregating *stopping power, braking stability,* and *feel at control* is therefore given by the following distribution [see (26)]

$$\begin{aligned}
S(brakes) &= S(stopping\ power \oplus braking\ stability\\
&\qquad \oplus feel\ at\ control)\\
&= \{(average,\ 0.0772),\ (good,\ 0.7621),\\
&\qquad (excellent,\ 0.135)\}.
\end{aligned}$$

Note that changing the order of combining the three basic attributes does not change the final result at all.

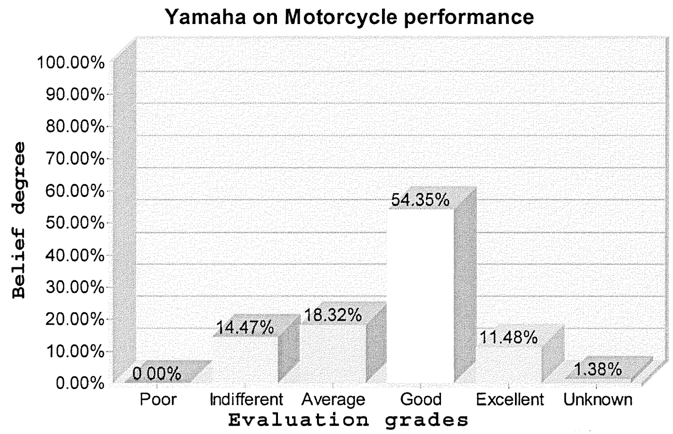


Fig. 3. Distributed assessment for *Yamaha*.

C. Results and Analysis Generated Using Intelligent Decision System (IDS)

A general assessment problem arises as to how the four types of motorcycle could be assessed and ranked on the basis of the attributes and the original assessments information related to the basic attributes as shown in Table III. The relative weights of attributes at a single level associated with the same upper level attribute are defined by ω_i, ω_{ij}, and ω_{ijk} for the attributes at levels 2, 3, and 4, respectively, as shown in Table III and Fig. 2. For the purpose of demonstrating the ER algorithm and without loss of generality, we assume that all relevant attributes are of equal relative importance, that is

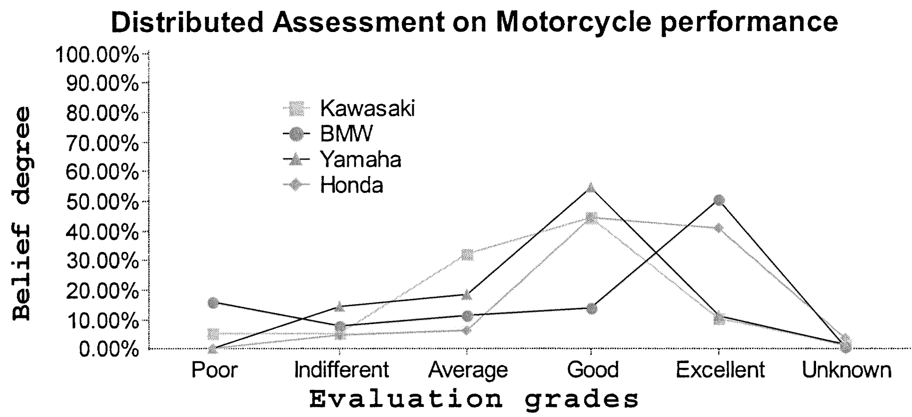
$$\omega_1 = \omega_2 = \omega_3 = 0.3333 \tag{51a}$$
$$\omega_{11} = \omega_{12} = \omega_{13} = \omega_{14} = \omega_{15} = 0.2 \tag{51b}$$
$$\omega_{21} = \omega_{22} = \omega_{23} = 0.3333 \tag{51c}$$
$$\omega_{211} = \omega_{212} = \omega_{213} = \omega_{214} = 0.25 \tag{51d}$$
$$\omega_{221} = \omega_{222} = 0.5; \quad \omega_{231} = \omega_{232} = \omega_{233} = 0.3333 \tag{51e}$$
$$\omega_{31} = \omega_{32} = \omega_{33} = \omega_{34} = \omega_{35} = 0.2. \tag{51f}$$

It should be noted, however, that weights play an important role and should be estimated with care. In general, it is advisable to estimate a range of weights to test whether the generated assessments are reliable.

A Windows-based intelligent decision system (IDS[1]) has been developed to implement the new ER approach. All of the following calculations and Figs. 3–5 were generated using IDS. Given the evaluation information about the motorcycle assessment problem as shown in Fig. 2, Table III, and (51a)–(51f), the assessment for each motorcycle can be generated using IDS. For instance, the aggregated assessments for the upper level attributes are generated for *Yamaha* as follows:

$$\begin{aligned}
&S(engine)\\
&\quad = \{(indifferent,\ 0.4193),\ (average,\ 0.3227),\\
&\qquad (good,\ 0.1131),\ (excellent,\ 0.1091)\}\\
&S(handling)\\
&\quad = \{(good,\ 0.8095),\ (excellent,\ 0.1715)\}
\end{aligned}$$

[1]A demo version of the IDS software is available from the author via e-mail: jian-bo.yang@umist.ac.uk.



	Poor	Indifferent	Average	Good	Excellent	Unknown
Kawasaki	5.47%	5.41%	32.16%	44.52%	10.58%	1.86%
BMW	15.76%	7.92%	11.24%	14.04%	50.26%	0.78%
Yamaha	0.00%	14.47%	18.32%	54.35%	11.48%	1.38%
Honda	0.00%	4.74%	6.21%	44.37%	40.68%	3.99%

Fig. 4. Distributed assessment for four motorcycle.

$$\begin{aligned}
&S(transmission)\\
&\quad = \{(indifferent, 0.25), (average, 0.25), (good, 0.5)\}\\
&S(brakes)\\
&\quad = \{(average, 0.0772), (good, 0.7621),\\
&\qquad (excellent, 0.135)\}\\
&S(operation)\\
&\quad = \{(indifferent, 0.0664), (average, 0.0892),\\
&\qquad (good, 0.7491), (excellent, 0.0836)\}\\
&S(general)\\
&\quad = \{(average, 0.1674), (good, 0.6547),\\
&\qquad (excellent, 0.1779)\}.
\end{aligned}$$

The final assessment about *Yamaha*'s performance is generated as follows:

$$\begin{aligned}
S(Yamaha) = \{&(indifferent, 0.1447), (average, 0.1832),\\
&(good, 0.5435), (excellent, 0.1148), (H, 0.0138)\}. \quad (52)
\end{aligned}$$

The degree of incompleteness in the above evaluation for *Yamaha* is 0.0138 (or 1.38%). This is incurred due to the incomplete assessments in four of the basic attributes for *Yamaha,* as shown in Table III. Compared with the four original incomplete assessments with the degree of incompleteness being each 10%, the overall incompleteness is significantly reduced due to the relatively large number of complete assessments in other basic attributes. The above distributed assessment for *Yamaha* can be shown graphically as in Fig. 3. In a similar way, the performances of the other three types of motorcycle are also assessed as shown in Fig. 4.

From Figs. 3 and 4, the differences between some of the four motorcycles can be identified and used to rank them, though it may not be a straightforward task. For instance, it is clear that *Yamaha* is preferred to *Kawasaki* as the former is to a large degree assessed to the grades *good* and *excellent* and to a smaller degree to the grades *poor, indifferent,* and *average*. Similarly, it can be seen that *Honda* is preferred to *Yamaha*. It is also fair to say that Honda is preferred to *BMW* as the former is assessed to *good* and *excellent* to a total degree of over 85% while the latter under 65%. However, it is not straightforward to tell the difference between *Yamaha* and *BMW*.

To precisely rank the four motorcycles, their utilities need to be estimated. To do so, the utilities of the five individual evaluation grades need to be estimated first. The above partial rankings of alternatives could be used to formulate regression models for estimating the utilities of grades [30]. Alternatively, the probability method could be used for utility estimation [20]. We first normalize utilities so that the worst grade is given a utility value of zero and the best given one. Thus

$$u(H_1) = u(poor) = 0, \qquad u(H_5) = u(excellent) = 1.$$

Using the probability method, the utilities of the other grades may be estimated as follows. To estimate the utility of the grade *average,* for example, two hypothetical tickets are shown to the decision-maker (DM). The first ticket offers a motorcycle with *average* performance. The second ticket contains a lottery offering one motorcycle having *poor* performance with a probability of $1 - p$ and another motorcycle having *excellent* performance with a probability of p. The DM is asked which ticket he prefers. The probability p $(0 \le p \le 1)$ is regulated until the DM is unable to differentiate between the two tickets.

Suppose the DM is indifferent to the two tickets when $p = 0.55$. Then the utility of the grade *average* is calculated by

$$\begin{aligned}
u(H_3) &= u(average)\\
&= (1-p) \times u(poor) + p \times u(excellent)\\
&= 0.45 \times 0 + 0.55 \times 1 = 0.55.
\end{aligned}$$

In a similar way, the utilities of the grades *indifferent* and *good* could be estimated. Suppose $u(H_2) = u(indifferent) = 0.35$ and $u(H_4) = u(good) = 0.85$.

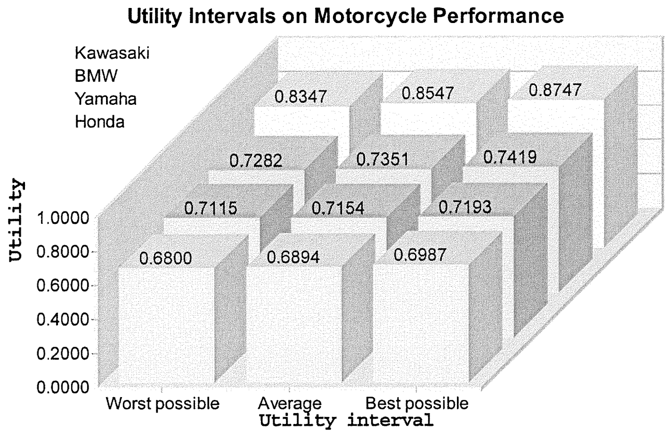


Fig. 5. Utility intervals for four motorcycles.

The degrees of belief for *Yamaha* are given as follows [see (52)]

$$\beta_1 = 0.0, \quad \beta_2 = 0.1447, \quad \beta_3 = 0.1832,$$
$$\beta_4 = 0.5435, \quad \beta_5 = 0.1148, \quad \beta_H = 0.0138.$$

Since β_H is not zero, the assessment for *Yamaha*'s performance is not unique and is characterized by the utility interval $[u_{\min}(Yamaha), u_{\max}(Yamaha)]$ where

$$\begin{aligned} u_{\min}(Yamaha) &= (\beta_1 + \beta_H)u(H_1) + \sum_{n=2}^{5} \beta_n \times u(H_n) \\ &= 0.7282 \\ u_{\max}(Yamaha) &= \sum_{n=1}^{4} \beta_n \times u(H_n) + (\beta_5 + \beta_H)u(H_5) \\ &= 0.7419. \end{aligned}$$

The above utility interval can be shown graphically as in Fig. 5. Similarly, we can generate the utility intervals for the other three motorcycles, as shown in Fig. 5.

It is clear from Fig. 5 that the minimum utility of *Honda* is larger than the maximum utilities of the other three motorcycles. This means that the overall performance of *Honda* is the most preferred among the four motorcycles. Based on the same principle, the ranking of the four motorcycles is given by

$$Honda \succ Yamaha \succ BMW \succ Kawasaki$$

where $\succ$ denotes "is preferred to." The above ranking is conclusive despite the imprecision present in the original assessments shown in Table III. Note, however, that this ranking is generated on the basis of the equal weighting assumed in (51a)–(51f). In general, a range of weights could be used for sensitivity analysis to test whether or not the rankings generated are reliable. It should also be noted that there may be overlap in utility intervals if there is greater incompleteness in original assessments. In such circumstances, it may be necessary to improve the quality of the original information to achieve a reliable ranking.

V. Concluding Remarks

Real-world decision problems are complex in that they often involve multiple attributes with uncertainty. It is essential to conduct decision analysis in a way that is rational, reliable, repeatable, and transparent. In this paper, the evidential reasoning approach is investigated and further developed. Its properties were explored and the ER aggregation process was enhanced to provide a rational means for aggregating multiple attributes. This new ER approach satisfies all the four synthesis axioms as discussed in this paper. In the new aggregation process, proper compensation among attributes is allowed so that an attribute can play a significant role relative to its weight, however small the weight may be.

The evidential reasoning algorithm essentially establishes a nonlinear relationship between an aggregated assessment for a general attribute and original assessments for basic attributes. To handle incomplete information, utility intervals were established to describe the impact of missing information on decision analysis. This provides a basis for improving the quality of original data and for conducting sensitivity analysis. In general, the theoretical results reported in this paper revealed the distinctive features of the ER approach, which should also be of interest to researchers who are developing or applying hybrid AI and OR methods to deal with complex decision problems.

The numerical analysis of the paper dealt with a product selection problem with the information taken from published sources. It demonstrated the implementation process of the new ER approach on a step-by-step basis. Using the Windows-based IDS developed for implementing the ER approach, the user is relieved from the tedious calculations and can concentrate on model building and validation, scenario generation, and sensitivity analysis. Although the example is taken from the domain of engineering design, it is clear that the ER approach can be applied to general MADA problems with or without uncertainty.

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