Nonlinear Information Aggregation via Evidential Reasoning in Multiattribute Decision Analysis Under Uncertainty

Jian-Bo Yang and Dong-Ling Xu

***Abstract*—In many decision situations, it is inevitable to deal with both quantitative and qualitative information under uncertainty. Evidence-based reasoning within a multiple criteria decision analysis framework provides an alternative way of handling such information systematically and consistently. In this paper, the evidential reasoning (ER) approach is introduced, which is based on a recursive ER algorithm that, in essence, constitutes a nonlinear information aggregation process. To facilitate the application of the ER approach and as an indispensable part of its development, the nonlinear features of the ER information aggregation process need to be thoroughly investigated and properly understood. This forms the theme of this paper where the nonlinear features are explored by examining typical reasoning patterns in aggregating harmonic, quasi-harmonic, and contradictory decision information. This analytical investigation provides insights into the recursive nature of the ER approach as well as valuable experience that could be useful to other researchers and practitioners interested in developing and applying operation research/artificial intelligence (OR/AI)-based approaches for decision analysis under uncertainty. The analytical study is complemented by the numerical studies of two application examples. The analysis of a quality assessment problem for motor engines is aimed to show the step-by-step process of implementing the ER approach and to illustrate its nonlinear features in a real-life decision situation. The study of a more complex assessment problem in ship design is intended to demonstrate the potential of the ER approach and its supporting software for dealing with general decision problems.**

***Index Terms*—Assessment, evidential reasoning (ER), multiple attribute decision analysis under uncertainty, nonlinear information aggregation, operation research/artificial intelligence (OR/AI) approach.**

I. Introduction

Most decision problems in management and engineering involve multiple criteria of both a quantitative and qualitative nature, which may constitute a hierarchy [5], [18]. It is essential to properly represent and aggregate such information for rational decision analysis. In the traditional operational research (OR) paradigm, a quantitative multiattribute decision analysis (MADA) problem is usually modeled using a decision matrix where an option is assessed on each attribute using a real number. Several methods have been proposed to deal with such quantitative MADA problems, among which the additive utility function approach is one of the simplest [8], [28].

Pairwise comparisons are widely used for modeling MADA problems, in particular, for generating relative weights for attributes [2], [10], [16], [18]. However, the use of pairwise comparisons to assess alternatives may lead to problems such as rank reversal, as within the AHP framework [2], [3], [4], [10], [12], [20].

In the above-mentioned modeling frameworks, an attribute is assessed at an alternative using a numerical score or ratio. However, it is difficult, if not impossible, to use a single score or ratio for capturing imprecision and vagueness inherent in a subjective assessment. One of the drawbacks of scoring a subjective judgment as an average number is the possible loss of linkage between assessment and business-planning activities in complex decision situations such as organizational self-assessment based on business excellence models [34].

Much attention has been paid to decision analysis under uncertainty in various application areas [6], [14], [15], [25], [26], [27]. Fuzzy sets and other artificial intelligence (AI) methods have been developed to cater for uncertain qualitative information [1], [17], [25]. In the hybrid OR/AI paradigm, a MADA problem having both quantitative and qualitative information with uncertainty may be modeled using a generalized and extended decision matrix, where an attribute is assessed using a belief structure represented by distributions [33].

The evidential reasoning (ER) approach has been developed for aggregating such distributions [26], [27], [33], [35]. The ER approach is based on an evaluation analysis model [36] and the Dempster–Shafer (D–S) theory of evidence [17] and can deal with multiattribute decision analysis problems having both quantitative and qualitative information under uncertainty. It has been applied to decision problems in engineering and management, e.g., motorcycle assessment [26], [27], general cargo ship design [19], marine system safety analysis and synthesis [21], [22], software safety synthesis [23], retrofit ferry design [30], and executive car assessment [31].

As reported in a comparative study [33], the ER approach is capable of generating credible results just as several well-known MADA methods for problems where all of them are applicable. These include multiattribute utility function approach [13], [8] and [24], Saaty's (left) eigenvector method (AHP) [18], Belton's normalized (left) eigenvector procedure [3], and Johnson's right eigenvector procedure [12]. In dealing with a quantitative attribute, the ER approach assumes a piecewise linear marginal utility function [11], [29], [33]. It can be used

Manuscript received December 13, 1999; revised January 25, 2001 and May 9, 2001. This work was supported by the U.K. Engineering and Physical Science Research Council (EPSRC) under Grant GR/N65615/01. This paper was recommended by Associate Editor Y. Y. Haimes.

The authors are with the Manchester School of Management, UMIST, Manchester M60 1QD, U.K. (http://www.umist.ac.uk/personal/staff/jian-bo.yang).

Digital Object Identifier 10.1109/TSMCA.2002.802809

as an alternative means in conjunction with other well-known methods to support the solution of a purely quantitative MADA problem. It should be noted that to apply the ER approach one does not have to use extra or different data from those required by the above well-known methods, though the ER approach does provide flexible ways to consistently represent various types of information within an integrated framework and employ a unique process for aggregating information [33].

The kernel of the ER approach is a recursive ER algorithm developed on the basis of decision theory and the evidence combination rule of the D–S theory. Due to its recursive nature, the ER algorithm exhibits various nonlinear features in information aggregation, which are difficult to reveal and explain in general decision situations. This may be a common issue associated with hybrid OR/AI approaches. For appropriate application of any decision analysis method, however, it is essential to understand how and why it works at least in specific cases, if it is difficult or impossible to do so in general situations.

This paper reports both analytical and numerical investigations into several typical reasoning patterns of the ER approach, including harmonic, quasi-harmonic, and contradictory reasoning patterns. Operational (nonrecursive) reasoning functions for these reasoning patterns are generated and illustrated in simple yet typical decision situations. The purpose of this analytical investigation is to provide insights into the recursive nature of the ER algorithm as well as useful operational functions for information aggregation. This research is also intended to provide valuable experience that could be useful to other researchers and practitioners interested in developing and applying OR/AI-based approaches for supporting decision analysis under uncertainty.

To put the analytical investigation in context, the ER modeling framework is discussed where both complete and incomplete assessments can be catered for in a consistent manner. By a complete or incomplete assessment it is meant that the total degree of belief assigned in the assessment is equal to or less than one, respectively. A real number could be transformed to a complete assessment using equivalence transformation techniques [33]. Incomplete assessments may result from the lack of evidence or the inability of the assessor to provide accurate judgments or the failure for some assessors to provide valid judgments in a group decision situation.

While it is complicated to produce general analytical functions for the ER information aggregation process, in this paper, numerical studies are provided to demonstrate similar nonlinear features in real-life decision situations. Two numerical studies are reported. The first one is concerned with a relatively simple decision problem of assessing the quality of motor engines based on multiple performance attributes. The problem is represented using the ER modeling framework, and the implementation procedure of the ER approach is illustrated on a step-by-step basis. Detailed numerical studies are provided for further investigation of the nonlinear reasoning patterns of the ER approach in connection with this problem.

The second example deals with a more complex assessment problem in ship design having a multilevel attribute hierarchy. The problem involves examining three retrofit design options for a short sea ferry to enhance its stability characteristics. This multilevel problem is modeled using a concise format of the ER framework. A new decision support system called *intelligent decision system (IDS)*, developed to implement the ER approach, is briefly described in support of the assessment process of the retrofit ferry design.

The ER modeling framework for MADA is discussed in the next section, followed by an introduction of the ER approach. In Section III, typical ER patterns are examined and illustrated. The engine quality and ship design assessment problems are analyzed in Section IV and Section V, respectively.

II. Introduction to the Evidential Reasoning (ER) Approach

A. Basic ER Framework for Multiattribute Decision Analysis Under Uncertainty

A simple ER modeling framework is discussed in this section. More general ER modeling frameworks are investigated by Yang [33]. For example, take a simple problem of assessing the quality of motor engines. Multiple quality attributes could be taken into account in the assessment, e.g., *responsiveness*, *fuel economy*, *quietness*, *vibration*, and *starting*, as shown in Fig. 1. Weights can be assigned to attributes to reflect their relative importance. To evaluate attributes, precise numbers or subjective judgments can be used to differentiate one option (engine) from another [33]. In this paper, we use distributions to represent assessment information.

A distribution is originally designed to represent a subjective assessment with uncertainty [26]. To evaluate the *quietness* of an *engine*, for example, an assessor may state that he is 50% sure it is *good* and 30% sure it is *excellent* [9]. In the statement, *good* and *excellent* denote distinctive evaluation standards (grades), and the percentage values of 50% and 30% are referred to as the degrees of belief, which indicate the extents to which the grades are assessed. The above assessment can be expressed as the following distribution:

$$S(\text{quietness}) = \{(\text{good},\ 0.5),\ (\text{excellent},\ 0.3)\} \tag{1a}$$

where $S(\text{quietness})$ stands for the state of the engine's *quietness* and the real numbers 0.5 and 0.3 denote the degrees of belief of 50% and 30%, respectively.

To assess the engine on other attributes or the quality of different engines, other evaluation grades may also be used such as *poor*, *indifferent*, and *average* [9]. To assess the above engine on the other four attributes, for example, the following distributions can be acquired [9], [26]:

$$S(\text{responsiveness}) = \{(\text{good},\ 1.0)\} \tag{1b}$$

$$S(\text{fuel economy}) = \{(\text{indifferent},\ 0.5),\ (\text{average},\ 0.5)\} \tag{1c}$$

$$S(\text{vibration}) = \{(\text{good},\ 0.5),\ (\text{excellent},\ 0.5)\} \tag{1d}$$

$$S(\text{starting}) = \{(\text{good},\ 1.0)\}. \tag{1e}$$

Note that distribution (1a) is an incomplete assessment as the total degree of belief in the assessment is $0.5 + 0.3 < 1$ (or 50%

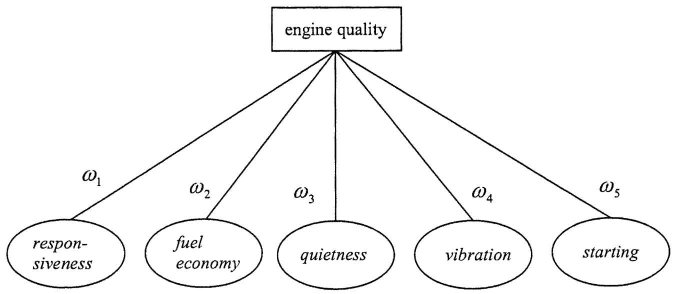

Fig. 1. Evaluation hierarchy for *engine quality*.

+ 30% < 100%), while distributions (1b)–(1e) express complete assessments. In the assessment of a quantitative attribute, numerical data may be used. Such numerical data denote precise assessments and can also be expressed as distributions using equivalence transformation techniques [33]. The ER approach summarized below provides a rational process to generate an overall assessment by aggregating the judgments, as given in (1a)–(1e).

B. The ER Computational Steps

The ER approach uses the concepts in decision theory, set theory, probability theory and the D–S theory for aggregating multiple attributes. Suppose there is a simple two level evaluation hierarchy with a general attribute y at the upper level and several associated basic attributes at the lower level. The above engine assessment problem constitutes such a hierarchy with *engine quality* as a general attribute and the five quality criteria as basic attributes. Suppose there are L basic attributes e_i $(i = 1, \ldots, L)$. The ER approach can be summarized as follows.

Step 1: Definition and representation of a multiattribute decision problem.

i) Define a set of L basic attributes e_i as follows:

$$E = \{e_1\ e_2\ \cdots\ e_i\ \cdots\ e_L\}. \tag{2}$$

Suppose the L basic attributes include all the factors influencing the assessment of the general attribute y.

ii) Estimate the relative weights of the L attributes $\omega = \{\omega_1\ \cdots\ \omega_i\ \cdots\ \omega_L\}$ where ω_i is the relative weight for basic attribute i and is normalized so that

$$0 \le \omega_i \le 1 \quad \text{and} \quad \sum_{i=1}^{L} \omega_i = 1. \tag{3}$$

iii) Define N distinctive evaluation grades H_n $(n = 1, \ldots, N)$ as a complete set of standards for assessing each option on all attributes, or

$$H = \{H_1\ H_2\ \cdots\ H_n\ \cdots\ H_N\}. \tag{4}$$

iv) Then, a multiattribute decision problem can be represented using the following distributions for an option a_l $(l = 1, \ldots, M)$ on an attribute e_i $(i = 1, \ldots, L)$:

$$S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)),\ n = 1, \ldots, N\}$$
$$i = 1, \ldots L,\ l = 1, \ldots, M \tag{5}$$

where $\beta_{n,i}(a_l)$ denotes a degree of belief and $\beta_{n,i}(a_l) \ge 0$ with $\sum_{n=1}^{N} \beta_{n,i}(a_l) \le 1$. A distribution, as shown in (5), reads that an attribute e_i at an option a_l is assessed to a grade H_n with a degree of belief of $\beta_{n,i}(a_l)$ $(n = 1, \ldots, N)$.

Step 2: Basic probability assignments for each basic attribute at an option a_l.

Let $m_{n,i}$ be a basic probability mass representing the degree to which the ith basic attribute e_i supports a hypothesis that the general attribute y at a_l is assessed to the nth evaluation grade H_n. Let $m_{H,i}$ be a remaining probability mass unassigned to any individual grade after e_i has been assessed. $m_{n,i}$ and $m_{H,i}$ are calculated as follows:

$$m_{n,i} = \omega_i \beta_{n,i}(a_l) \qquad n = 1, \ldots, N \tag{6}$$

$$m_{H,i} = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - \omega_i \sum_{n=1}^{N} \beta_{n,i}(a_l)$$
$$i = 1, \ldots, L. \tag{7a}$$

Decompose $m_{H,i}$ into $\overline{m}_{H,i}$ and $\tilde{m}_{H,i}$ as follows:

$$\overline{m}_{H,i} = 1 - \omega_i \quad \text{and} \quad \tilde{m}_{H,i} = \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}\right) \tag{7b}$$

with

$$m_{H,i} = \overline{m}_{H,i} + \tilde{m}_{H,i}.$$

$\overline{m}_{H,i}$ is the first part of the remaining probability mass that is not yet assigned to individual grades due to the fact that attribute i (denoted by e_i) only plays one part in the assessment relative to its weight. $\tilde{m}_{H,i}$ is the second part of the remaining probability mass unassigned to individual grades, which is caused due to the incompleteness in the assessment $S(e_i)$.

Step 3: Combined probability assignments for a general attribute at an option a_l.

Let $m_{n,I(1)} = m_{n,1}$ $(n = 1, \ldots, N)$, $\overline{m}_{H,I(1)} = \overline{m}_{H,1}$, $\tilde{m}_{H,I(1)} = \tilde{m}_{H,1}$ and $m_{H,I(1)} = m_{H,1}$. The combined probability assignments $m_{n,I(L)}$ $(n = 1, \ldots, N)$, $\overline{m}_{H,I(L)}$, $\tilde{m}_{H,I(L)}$, and $m_{H,I(L)}$ can be generated by aggregating all the basic probability assignments using the following recursive ER algorithm [35]:

$\{H_n\}$:

$$m_{n,I(i+1)} = K_{I(i+1)} \left[m_{n,I(i)} m_{n,i+1} + m_{H,I(i)} m_{n,i+1} + m_{n,I(i)} m_{H,i+1}\right]$$
$$n = 1, 2, \ldots, N \tag{8a}$$

$\{H\}$:

$$m_{H,I(i)} = \tilde{m}_{H,I(i)} + \overline{m}_{H,I(i)} \tag{8b}$$

$$\tilde{m}_{H,I(i+1)} = K_{I(i+1)} \left[\tilde{m}_{H,I(i)} \tilde{m}_{H,i+1} + \overline{m}_{H,I(i)} \tilde{m}_{H,i+1} + \tilde{m}_{H,I(i)} \overline{m}_{H,i+1}\right] \tag{8c}$$

$$\overline{m}_{H,I(i+1)} = K_{I(i+1)} \left[\overline{m}_{H,I(i)} \overline{m}_{H,i+1}\right] \tag{8d}$$

$$K_{I(i+1)} = \left[1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \ne t}}^{N} m_{t,I(i)} m_{j,i+1}\right]^{-1}$$
$$i = \{1, 2, \ldots, L-1\}. \tag{8e}$$

For $i = 1$, the terms of the above algorithm can be interpreted as follows. In (8a), the term $m_{n,1}m_{n,2}$ measures the degree of both attributes e_1 and e_2 supporting the general attribute y to be assessed to H_n; $m_{n,1}m_{H,2}$ the degree of only e_1 supporting y to be assessed to H_n; and $m_{H,1}m_{n,2}$ the degree of only e_2 supporting y to be assessed to H_n. In (8c), the term $\tilde{m}_{H,1}\tilde{m}_{H,2}$ measures the degree to which y cannot be assessed to any individual grades due to the incomplete assessments for both e_1 and e_2; $\tilde{m}_{H,1}\overline{m}_{H,2}$ the degree to which y cannot be assessed due to the incomplete assessments for e_1 only; and $\overline{m}_{H,1}\tilde{m}_{H,2}$ the degree to which y cannot be assessed due to the incomplete assessments for e_2 only. In (8d), the term $\overline{m}_{H,1}\overline{m}_{H,2}$ measures the degree to which y has not yet been assessed to individual grades due to the relative importance of e_1 and e_2 after e_1 and e_2 have been aggregated. $K_{I(2)}$ as calculated by (8c) is used to normalize $m_{n,I(2)}$ and $m_{H,I(2)}$ so that $\sum_{n=1}^{N} m_{n,I(2)} + m_{H,I(2)} = 1$.

Step 4: Calculation of the combined degrees of belief for a general attribute at an option a_l.

Let β_n denote a degree of belief that the general attribute y at a_l is assessed to the grade H_n, which is generated by combining the assessments for all the associated basic attributes e_i $(i = 1, \ldots, L)$ at a_l. β_n is then calculated by

$$\{H\}: \quad \beta_H = \frac{\tilde{m}_{H,I(L)}}{1-\overline{m}_{H,I(L)}} \tag{9a}$$

$$\{H_n\}: \quad \beta_n = \frac{m_{n,I(L)}}{1-\overline{m}_{H,I(L)}} \qquad n = 1, 2, \ldots, N. \tag{9b}$$

Step 5: Representation of the distributed overall assessment and calculation of the expected utility of an option a_l.

The distributed overall assessment of a_l is given by the following distribution:

$$S(y(a_l)) = \{(H_n, \beta_n(a_l)), n = 1, \ldots, N\}. \tag{10}$$

Suppose the utility of a grade H_n, denoted by $u(H_n)$, is estimated using for example the certainty monetary equivalence method [24] with $u(H_{n+1}) > u(H_n)$ if H_{n+1} is preferred to H_n. If the overall assessment is complete, the expected utility of a_l on y is then calculated by

$$u(a_l) = \sum_{n=1}^{N} \beta_n(a_l)u(H_n). \tag{11}$$

Step 6: Calculation of the utility interval for an option a_l.

Complementary to the distributed assessment [(10)], a utility interval can be established if the overall assessment is incomplete. The maximum, minimum, and average utilities of a_l are calculated by

$$u_{\max}(a_l) = \sum_{n=1}^{N-1} \beta_n(a_l)u(H_n) + (\beta_N(a_l) + \beta_H(a_l))u(H_N) \tag{12a}$$

$$u_{\min}(a_l) = (\beta_1(a_l) + \beta_H(a_l))u(H_1) + \sum_{n=2}^{N} \beta_n(a_l)u(H_n) \tag{12b}$$

$$u_{\text{avg}}(a_l) = \frac{u_{\max}(a_l) + u_{\min}(a_l)}{2}. \tag{12c}$$

Note that if all original assessments $S(e_i(a_l))$ in the generalized decision matrix are complete, then $\beta_H(a_l) = 0$ and $u(S(y(a_l))) = u_{\max}(a_l) = u_{\min}(a_l) = u_{\text{avg}}(a_l)$. It should also be noted that the above utilities are only used for characterizing an assessment but not for attribute aggregation.

Step 7: The ranking of the options.

The ranking of two alternatives a_l and a_k is based on their utility intervals. a_l is said to be preferred to a_k if and only if $u_{\min}(a_l) > u_{\max}(a_k)$; a_l is said to be indifferent to a_k if and only if $u_{\min}(a_l) = u_{\min}(a_k)$ and $u_{\max}(a_l) = u_{\max}(a_k)$. Otherwise, the average utility may be used to generate a ranking.

III. Investigation of Typical Evidential Reasoning Patterns

The above ER approach establishes a relationship between a combined degree of belief and basic degrees of belief. Several fundamental features of the approach, which must satisfy certain common sense rules, have already been analyzed [35]. For instance, it is proved that the combined degrees of belief as calculated using (9a) and (9b) are always summed to be equal to one with $\beta_H = 0$ or > 0, depending upon whether or not the basic assessments are all complete. It is also shown that if all basic attributes are precisely assessed to an individual grade, then the associated general attribute will be precisely assessed to the same grade. Furthermore, if all basic attributes are completely assessed to a subset of grades, then the general attribute is completely assessed to the same subset as well.

In this and the following sections, several reasoning patterns of the ER approach will be investigated, which are related to its nonlinear features in information aggregation. In this section, analytical investigations will be conducted for simple yet typical decision situations. In next section, numerical investigations will be conducted to reveal similar features in real-life decision situations.

A. Combined Degree of Belief Versus Weight in Contradictory Evaluations

Suppose a general attribute y is associated with two basic attributes e_1 and e_2 having weights ω_1 and ω_2, respectively, with $\omega_1 + \omega_2 = 1$. If e_1 and e_2 are completely assessed to two different evaluation grade, say H_k and H_l $(l \neq k)$, respectively, or

$$\beta_{k,1} = 1, \quad \beta_{n,1} = 0 \qquad \text{for } n = 1, \ldots, N, n \neq k$$
$$\beta_{l,2} = 1, \quad \beta_{n,2} = 0 \qquad \text{for } n = 1, \ldots, N, n \neq l$$

from (6)–(7b) we have

$$m_{k,1} = \omega_1, \quad m_{n,1} = 0 \qquad \text{for } n = 1, \ldots, N, n \neq k$$
$$m_{l,2} = \omega_2, \quad m_{n,2} = 0 \qquad \text{for } n = 1, \ldots, N, n \neq l$$
$$m_{H,1} = \overline{m}_{H,1} = 1 - \omega_1 \quad \text{and} \quad \tilde{m}_{H,1} = 0$$
$$m_{H,2} = \overline{m}_{H,2} = 1 - \omega_2 \quad \text{and} \quad \tilde{m}_{H,2} = 0.$$

From (8e), we have

$$K_{I(2)} = (1 - m_{k,1}m_{l,2})^{-1} = (1 - \omega_1\omega_2)^{-1}.$$

From (8a)–(8d), the combined probability assignments are obtained by

$$m_{n,I(2)} = 0 \qquad \text{for } n = 1, \ldots, N;\ n \neq k, l,\ \tilde{m}_{H,I(2)} = 0$$
$$m_{k,I(2)} = (1-\omega_1\omega_2)^{-1}\omega_1(1-\omega_2)$$
$$m_{l,I(2)} = (1-\omega_1\omega_2)^{-1}\omega_2(1-\omega_1)$$
$$\overline{m}_{H,I(2)} = (1-\omega_1\omega_2)^{-1}(1-\omega_1)(1-\omega_2).$$

Let $\omega_1 = \omega$ and note that $\omega_2 = 1-\omega_1$. From (9a) we have $\beta_H = 0$. From (9b), we calculate the combined degrees of belief as follows:

$$\beta_n = 0 \qquad \text{for } n = 1, \ldots, N;\ n \neq k, l$$
$$\beta_k = \frac{m_{k,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{\omega^2}{\omega^2+(1-\omega)^2}$$
$$\beta_l = \frac{m_{l,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{(1-\omega)^2}{\omega^2+(1-\omega)^2}.$$

Fig. 2 shows the curves of β_k and β_l with respect to ω. With the increase of ω, β_k increases and β_l decreases monotonically. Furthermore, when ω increases from 0 to 0.5, the derivative of β_k increases monotonically from 0 to 2 and then decreases from 2 to 0 when ω increases further from 0.5 to 1. This pattern implies that the two basic attributes are of similar influence if they are of equivalent importance. If one attribute becomes more important than the other, then the relative influence of the former will increase quickly, more quickly than proportionally as $\beta_k/\beta_l = (\omega/(1-\omega))^2$.

It is of interest to examine how the above pattern changes when more basic attributes are involved in evaluation of y. Suppose e_1 and e_2 are replaced by two groups of attributes E_1 and E_2, with each group having the same number of attributes and with the attributes of the same group of equal importance. Suppose all the attributes in E_1 are absolutely assessed to H_k and those in E_2 to H_l $(l \neq k)$. Let $\omega_1 = \omega$ be the total weight of all the attributes in E_1 and $\omega_2 = 1-\omega$ in E_2.

When the number of attributes in each group is one, we have already obtained a curve for β_k, as shown in Case 1 in Fig. 3. When the number is four, we can generate another curve, shown in Case 2 by running the ER algorithm. When the number is 20, we have a new curve, shown in Case 3. It is clear from Fig. 3 that with the increase of attributes in each group the relationship between β_k and ω approaches a linear function and β_k changes more slowly around $\omega = 0.5$. Perhaps this feature could be interpreted as an inertia effect of group assessments.

B. Harmonic Judgments

By harmonic judgments, it is meant that all attributes are assessed to the same subset of evaluation grades with different degrees of belief.

Suppose a general attribute y is associated with two basic attributes e_k and e_l having relative weights of ω_k and ω_l, respectively, with $\omega_k + \omega_l = 1$. Typical complete harmonic judgments would be that e_k is assessed by

$$e_k\colon\ \{(H_n, \beta_{n,k}),\ (H_{n+1}, \beta_{n+1,k})\} \ \text{ with } \ \beta_{n,k}+\beta_{n+1,k} = 1$$

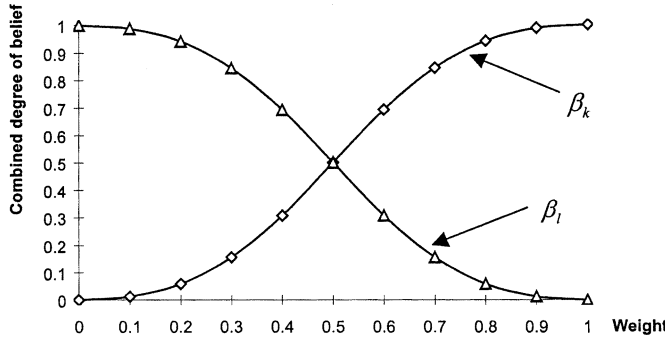

Fig. 2. Reasoning pattern for combined degree of belief versus weight.

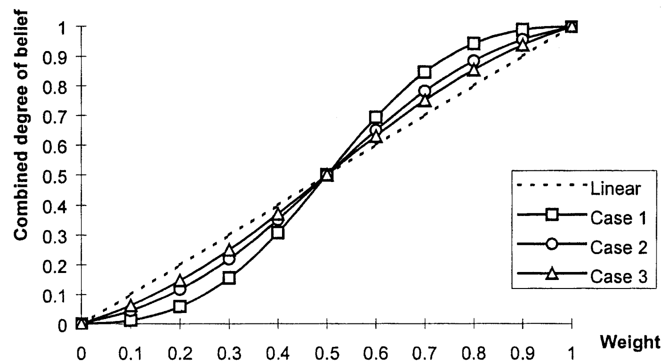

Fig. 3. Reasoning pattern for combined degree of belief versus weight with two groups of attributes.

and e_l is assessed by

$$e_l\colon\ \{(H_n, \beta_{n,l}),\ (H_{n+1}, \beta_{n+1,l})\} \ \text{ with } \ \beta_{n,l}+\beta_{n+1,l} = 1.$$

From (6)–(7b), we have

$$m_{n,k} = \omega_k\beta_{n,k}; \quad m_{n+1,k} = \omega_k(1-\beta_{n,k}) \tag{13a}$$
$$\tilde{m}_{H,k} = 0; \quad m_{H,k} = \overline{m}_{H,k} = 1-\omega_k \tag{13b}$$
$$m_{n,l} = (1-\omega_k)\beta_{n,l}$$
$$m_{n+1,l} = (1-\omega_k)(1-\beta_{n,l}) \tag{14a}$$
$$\tilde{m}_{H,l} = 0; \quad m_{H,l} = \overline{m}_{H,l} = 1-\omega_l = \omega_k. \tag{14b}$$

Using (8a)–(9b), we can obtain the following combined degrees of belief for assessment of y:

$$\beta_i = 0, \qquad i = 1, \ldots, N;\ i \neq n, n+1,\ \beta_H = 0$$
$$\beta_n = \frac{m_{n,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{m_{n,k}m_{n,l}+m_{H,k}m_{n,l}+m_{n,k}m_{H,l}}{1-(m_{n,k}m_{n+1,l}+m_{n+1,k}m_{n,l}+m_{H,k}m_{H,l})}$$
$$\beta_{n+1} = \frac{m_{n+1,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{m_{n+1,k}m_{n+1,l}+m_{H,k}m_{n+1,l}+m_{n+1,k}m_{H,l}}{1-(m_{n,k}m_{n+1,l}+m_{n+1,k}m_{n,l}+m_{H,k}m_{H,l})}.$$

Using (13a)–(14b), β_n and β_{n+1} are calculated by the equations shown at the bottom of the next page which exhibit strongly nonlinear relationships between β_n (or β_{n+1}) and ω_k, $\beta_{n,k}$, $\beta_{n,l}$.

To illustrate the relationship between the combined degrees of belief for y and the basic degrees of belief for e_k (or e_l), let us fix $\beta_{n,l}$ and ω_k. Let $\beta_{n,l} = 0$, which means that e_l is precisely evaluated to H_{n+1}. Consider three cases: 1) $\omega_k = \omega_l$; 2) $\omega_k = 2\omega_l$; and 3) $\omega_k = \omega_l/2$. In Case 1, since from (3) $\omega_k = \omega_l = 1/2$, we then have

$$\beta_n = \frac{\beta_{n,k}}{3-\beta_{n,k}}; \qquad \beta_{n+1} = \frac{3-2\beta_{n,k}}{3-\beta_{n,k}}.$$

The curves of β_n and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 4. When $\beta_{n,k} = 0$, then $\beta_{n+1} = 1$ and $\beta_n = 0$, which corresponds to the consensus evaluation as both e_k and e_l are precisely assessed to H_{n+1}. In Fig. 4, with the increase of $\beta_{n,k}$, β_n monotonically increases and β_{n+1} monotonically decreases. The change patterns are close to linear functions. When $\beta_{n,k} = 1$, then $\beta_n = \beta_{n+1} = 0.5$, which corresponds to the contradictory evaluation as e_k is precisely assessed to H_n but e_l to H_{n+1}.

In Case 2, since $\omega_k = 2\omega_l$, we have $\omega_k = 2/3$ and then

$$\beta_n = \frac{4\beta_{n,k}}{7-2\beta_{n,k}}; \qquad \beta_{n+1} = \frac{7-6\beta_{n,k}}{7-2\beta_{n,k}}.$$

The curves of β_n and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 5. As mentioned above, $\beta_{n,k} = 0$ and 1 represent the consensus evaluation and contradictory evaluation, respectively. Since the basic attributes e_k is twice as important as e_l, β_n increases quickly with $\beta_{n,k}$ and $\beta_n = 4\beta_{n+1}$ at $\beta_{n,k} = 1$, or $\beta_n/\beta_{n+1} = (\omega_k/\omega_l)^2 = 4$.

In Case 3, since $\omega_k = \omega_l/2$, we have $\omega_k = 1/3$ and then

$$\beta_n = \frac{\beta_{n,k}}{7-2\beta_{n,k}}; \qquad \beta_{n+1} = \frac{7-3\beta_{n,k}}{7-2\beta_{n,k}}.$$

The curves of β_n and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 6. Similar to Cases 1 and 2, $\beta_{n,k} = 0$ and 1 represent the consensus evaluation and contradictory evaluation, respectively. At $\beta_{n,k} = 1$, $\beta_n/\beta_{n+1} = (\omega_k/\omega_l)^2 = 0.25$.

C. Quasi-Harmonic Judgments

By quasi-harmonic judgments, it is meant that one attribute is assessed to a subset of evaluation grades and another attribute to a different subset of evaluation grades with the intersection of the two subsets being not empty. In other words, the two attributes are assessed to some common evaluation grades. Harmonic judgments are obviously a special case of quasi-harmonic judgments.

Suppose a general attribute y is associated with two basic attributes e_k and e_l with relative weights of ω_k and ω_l, respectively, and $\omega_k + \omega_l = 1$. Typical complete quasi-harmonic judgments would be that e_k is assessed by

$$e_k\colon\ \{(H_n,\, \beta_{n,k}),\, (H_{n+1},\, \beta_{n+1,k})\} \text{ with } \beta_{n,k}+\beta_{n+1,k} = 1$$

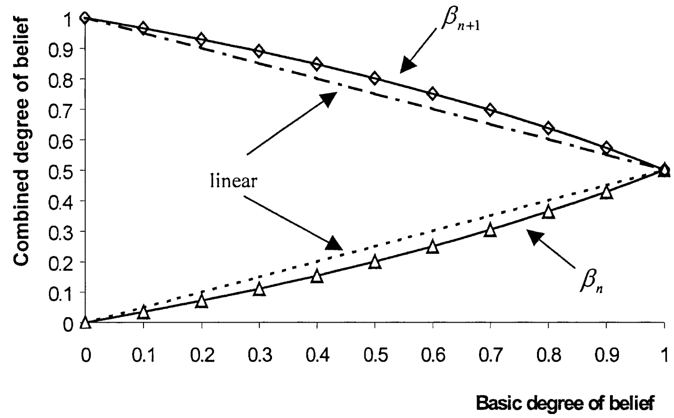

Fig. 4. Harmonic reasoning pattern with $\omega_k = \omega_l$.

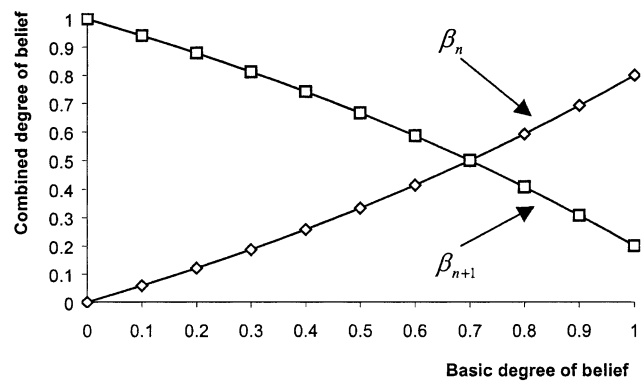

Fig. 5. Harmonic reasoning pattern with $\omega_k = 2\omega_l$.

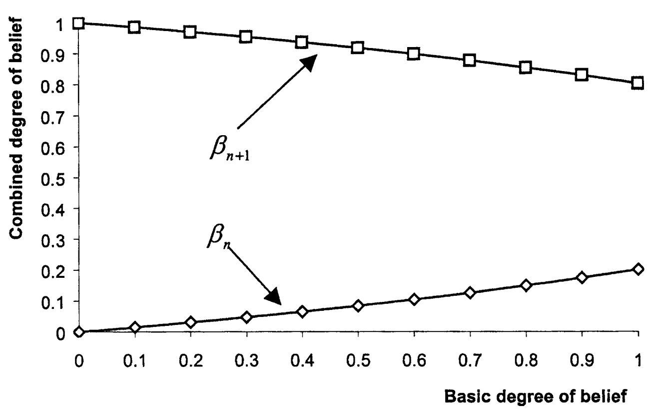

Fig. 6. Harmonic reasoning pattern with $\omega_k = \omega_l/2$.

and e_l is assessed by

$$e_l\colon\ \{(H_{n-1},\, \beta_{n-1,l}),\, (H_n,\, \beta_{n,l})\} \text{ with } \beta_{n-1,l}+\beta_{n,l} = 1.$$

$$\beta_n = \frac{\omega_k(1-\omega_k)\beta_{n,k}\beta_{n,l} + (1-\omega_k)^2\beta_{n,l} + \omega_k^2\beta_{n,k}}{1-\omega_k(1-\omega_k)[\beta_{n,k}(1-\beta_{n,l}) + (1-\beta_{n,k})\beta_{n,l} + 1]}$$

$$\beta_{n+1} = \frac{\omega_k(1-\omega_k)(1-\beta_{n,k})(1-\beta_{n,l}) + (1-\omega_k)^2(1-\beta_{n,l}) + \omega_k^2(1-\beta_{n,k})}{1-\omega_k(1-\omega_k)[\beta_{n,k}(1-\beta_{n,l}) + (1-\beta_{n,k})\beta_{n,l} + 1]}$$

From (6)–(7b), the basic probability assignments are given by

$$m_{n,k} = \omega_k \beta_{n,k}; \quad m_{n+1,k} = \omega_k(1-\beta_{n,k}) \tag{15a}$$

$$m_{H,k} = \overline{m}_{H,k} = 1-\omega_k; \quad \tilde{m}_{H,k} = 0 \tag{15b}$$

and

$$m_{n-1,l} = (1-\omega_k)(1-\beta_{n,l})$$

$$m_{n,l} = (1-\omega_k)\beta_{n,l} \tag{16a}$$

$$m_{H,l} = \overline{m}_{H,l} = 1-\omega_l; \quad \tilde{m}_{H,l} = 0. \tag{16b}$$

Using (8a)–(9b), we can obtain the combined degrees of belief for assessment of y as in the equations shown at the bottom of the page. From (15a)–(16b), we then have

$$\beta_{n-1} = \frac{(1-\omega_k)^2(1-\beta_{n,l})}{1-\omega_k(1-\omega_k)(2-\beta_{n,k}\beta_{n,l})}$$

$$\beta_n = \frac{\omega_k(1-\omega_k)\beta_{n,l}\beta_{n,k} + (1-\omega_k)^2\beta_{n,l} + \omega_k^2\beta_{n,k}}{1-\omega_k(1-\omega_k)(2-\beta_{n,k}\beta_{n,l})}$$

$$\beta_n = \frac{\omega_k^2(1-\beta_{n,k})}{1-\omega_k(1-\omega_k)(2-\beta_{n,k}\beta_{n,l})}$$

which exhibit strongly nonlinear relationships between β_n (β_{n-1} or β_{n+1}) and ω_k, $\beta_{n,k}$, $\beta_{n,l}$.

To illustrate the relationships between the combined degrees of belief for y and the basic degrees of belief for e_k (or e_l), let us fix $\beta_{n,l}$ and ω_k. Let $\beta_{n,l} = 1/2$, which means that e_l is equally evaluated to H_{n-1} and H_n. Then, consider three cases: 1) $\omega_k = \omega_l$; 2) $\omega_k = 2\omega_l$; 3) and $\omega_k = \omega_l/2$.

In Case 1, since from (3) $\omega_k = \omega_l = 1/2$, we then have

$$\beta_{n-1} = \frac{1}{4+\beta_{n,k}}; \quad \beta_n = \frac{3\beta_{n,k}+1}{4+\beta_{n,k}}; \quad \beta_{n+1} = \frac{2(1-\beta_{n,k})}{4+\beta_{n,k}}.$$

The curves of β_{n-1}, β_n, and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 7. At $\beta_{n,k} = 0$, e_k is assessed to H_{n+1} but not to H_n. In this case, quasi-harmonic judgments reduce to contradictory judgments as discussed in the next section. At this point

$$\beta_{n+1}/(\beta_n + \beta_{n-1}) = (\omega_k/\omega_l)^2$$

$$\beta_n = \beta_{n-1} \quad \text{and} \quad \beta_{n-1} + \beta_n + \beta_{n+1} = 1. \tag{17}$$

Therefore, $\beta_{n+1} = 0.5$, $\beta_n = \beta_{n-1} = 0.25$. At $\beta_{n,k} = 1$, e_k is completely assessed to H_n, leading to a high value for β_n while $\beta_{n+1} = 0$ as e_l is equally assessed to H_{n-1} and H_n but not to H_{n+1}.

In Case 2, since $\omega_k = 2\omega_l$, we have $\omega_k = 2/3$ and then

$$\beta_{n-1} = \frac{1}{10+2\beta_{n,k}}; \qquad \beta_n = \frac{10\beta_{n,k}+1}{10+2\beta_{n,k}}$$

$$\beta_{n+1} = \frac{8(1-\beta_{n,k})}{10+2\beta_{n,k}}.$$

The curves of β_{n-1}, β_n, and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 8. Similar to Case 1, quasi-harmonic judgments reduces to contradictory judgments at $\beta_{n,k} = 0$. Since at this point (17) is satisfied, we have $\beta_{n+1} = 0.8$, $\beta_n = \beta_{n-1} = 0.1$. At $\beta_{n,k} = 1$, e_k is completely assessed to H_n, resulting in a high value for β_n while $\beta_{n+1} = 0$.

In Case 3, since $\omega_k = \omega_l/2$, we have $\omega_k = 1/3$ and then

$$\beta_{n-1} = \frac{4}{10+2\beta_{n,k}}; \qquad \beta_n = \frac{4(1+\beta_{n,k})}{10+2\beta_{n,k}}$$

$$\beta_{n+1} = \frac{2(1-\beta_{n,k})}{10+2\beta_{n,k}}.$$

The curves of β_{n-1}, β_n, and β_{n+1} with respect to $\beta_{n,k}$ are shown in Fig. 9. Since in this case (17) is satisfied at $\beta_{n,k} = 0$, we then have $\beta_{n+1} = 0.2$, $\beta_n = \beta_{n-1} = 0.4$. With the increase of $\beta_{n,k}$, it becomes more likely that e_k is assessed to H_n. In Fig. 9, this means that β_{n-1} and β_{n+1} monotonically decrease and β_n monotonically increases. At $\beta_{n,k} = 1$, $\beta_{n+1} = 0$ and β_n becomes quite large as now e_k is completely assessed to H_n.

D. *Contradictory Judgments*

By contradictory judgments it is meant that one attribute is assessed to a subset of evaluation grades and another attribute to a completely different subset of evaluation grades with the intersection of the two subsets being empty.

Suppose a general attribute y is associated with two basic attributes e_k and e_l with relative weights of ω_k and ω_l, respectively, and $\omega_k + \omega_l = 1$. Typical complete contradictory judgments would be that e_k is assessed by

$$e_k\colon \{(H_n, \beta_{n,k}), (H_{n+1}, \beta_{n+1,k})\} \text{ with } \beta_{n,k} + \beta_{n+1,k} = 1$$

and e_l is assessed by

$$e_l\colon \{(H_{i-1}, \beta_{i-1,l}), (H_i, \beta_{i,l})\} \text{ with } \beta_{i-1,l} + \beta_{i,l} = 1$$

with $n > i$. From (6)–(7b), we have

$$m_{n,k} = \omega_k\beta_{n,k}; \quad m_{n+1,k} = \omega_k(1-\beta_{n,k})$$

$$\tilde{m}_{H,k} = 0; \quad m_{H,k} = \overline{m}_{H,k} = 1-\omega_k$$

and

$$m_{i-1,l} = (1-\omega_k)(1-\beta_{i,l}); \quad m_{i,l} = (1-\omega_k)\beta_{i,l}$$

$$\tilde{m}_{H,l} = 0; \quad m_{H,l} = \overline{m}_{H,l} = \omega_k.$$

$$\beta_i = 0, \qquad i = 1, \ldots, N;\ i \neq n-1, n, n+1,\ \beta_H = 0$$

$$\beta_{n-1} = \frac{m_{n-1,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{m_{n-1,l}m_{H,k}}{1-(m_{n-1,l}m_{n,k} + m_{n-1,l}m_{n+1,k} + m_{n,l}m_{n+1,k} + m_{H,k}m_{H,l})}$$

$$\beta_n = \frac{m_{n,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{m_{n,l}m_{n,k} + m_{n,l}m_{H,k} + m_{H,l}m_{n,k}}{1-(m_{n-1,l}m_{n,k} + m_{n-1,l}m_{n+1,k} + m_{n,l}m_{n+1,k} + m_{H,k}m_{H,l})}$$

$$\beta_{n+1} = \frac{m_{n+1,I(2)}}{1-\overline{m}_{H,I(2)}} = \frac{m_{n+1,k}m_{H,l}}{1-(m_{n-1,l}m_{n,k} + m_{n-1,l}m_{n+1,k} + m_{n,l}m_{n+1,k} + m_{H,k}m_{H,l})}$$

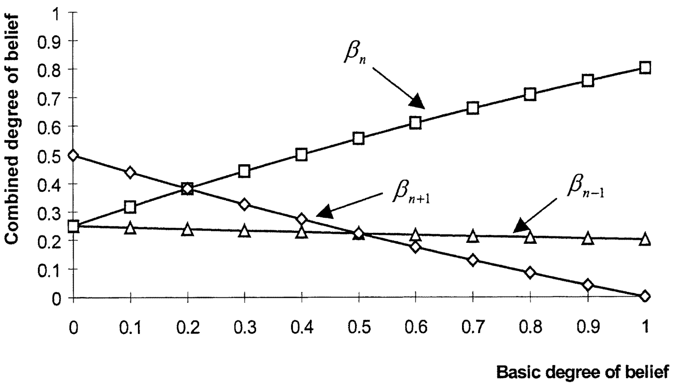

Fig. 7. Quasi-harmonic reasoning pattern with $\omega_k = \omega_l$.

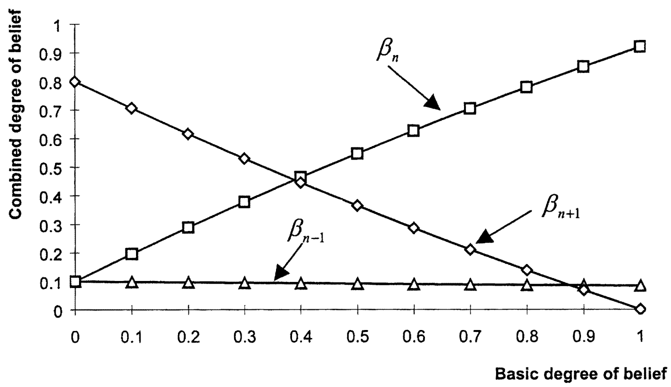

Fig. 8. Quasi-harmonic reasoning pattern with $\omega_k = 2\omega_l$.

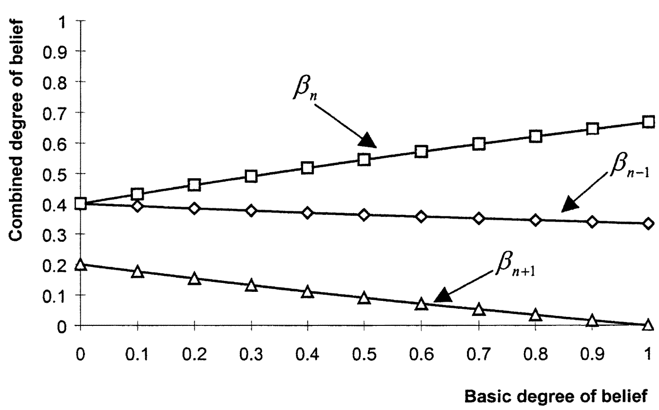

Fig. 9. Quasi-harmonic reasoning pattern with $\omega_k = \omega_l/2$.

Using (8a)–(9b), the combined degrees of belief for assessment of y are generated as follows:

$$\beta_j = 0, \qquad j = 1, \ldots, N; \quad j \neq i-1,\, i,\, n,\, n+1,\, \beta_H = 0$$

$$\beta_{i-1} = \frac{(1-\omega_k)^2}{\omega_k^2 + (1-\omega_k)^2}\,(1-\beta_{i,l})$$

$$\beta_i = \frac{(1-\omega_k)^2}{\omega_k^2 + (1-\omega_k)^2}\,\beta_{i,l}$$

$$\beta_n = \frac{\omega_k^2}{\omega_k^2 + (1-\omega_k)^2}\,\beta_{n,k}$$

$$\beta_{n+1} = \frac{\omega_k^2}{\omega_k^2 + (1-\omega_k)^2}\,(1-\beta_{n,k}).$$

For a given ω_k, β_{i-1} is obviously proportional to $(1-\beta_{i,l})$ only, β_i to $\beta_{i,l}$, β_n to $\beta_{n,k}$, and β_{n+1} to $(1-\beta_{n,k})$. This seems to be a logical conclusion in a sense that β_{i-1} should not be affected by any basic degrees of belief other than $(1-\beta_{i,l})$.

IV. Motor Engine Assessment

In the previous section, the typical nonlinear reasoning patterns of the ER approach were analyzed in aggregation of two attributes and two groups of attributes only. Theoretically, similar vigorous analyses could be conducted for aggregation of three or more attributes, though this could be rather complicated. Alternatively, numerical simulation can be conducted to demonstrate similar reasoning patterns.

In this section, we apply the ER approach to dealing with a relatively simple decision problem of assessing the quality of four motor engines. The purpose of this study is to illustrate how to implement the ER approach step-by-step and to conduct sensitivity analyses for demonstrating its nonlinear features for aggregating multiple attributes. Imprecise assessments are included in this example.

A. Computational Steps

Step 1: The assessment attributes are as given in Fig. 1. The set of basic attributes is defined by

$$E = \{\text{responsiveness}(e_1),\ \text{fuel economy}(e_2)\ \text{quietness}(e_3),\ \text{vibration}(e_4),\ \text{starting}(e_5)\}. \tag{18}$$

We first assume equal weights for all the five basic attributes, or $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_5 = 0.2$. Five evaluation grades are defined as follows:

$$H = \{\text{poor}(H_1),\ \text{indifferent}(H_2),\ \text{average}(H_3)\ \text{good}(H_4),\ \text{excellent}(H_5)\}.$$

The assessments for the first engine are given by (1a)–(1e). For the second engine, the assessments are expressed by the following distributions:

$$S(\text{responsiveness}) = \{(\text{excellent},\ 0.8)\} \tag{19a}$$

$$S(\text{fuel economy}) = \{(\text{average},\ 1.0)\} \tag{19b}$$

$$S(\text{quietness}) = \{(\text{indifferent},\ 0.5),\ (\text{average},\ 0.5)\} \tag{19c}$$

$$S(\text{vibration}) = \{(\text{good},\ 1.0)\} \tag{19d}$$

$$S(\text{starting}) = \{(\text{good},\ 1.0)\}. \tag{19e}$$

For the third engine

$$S(\text{responsiveness}) = \{(\text{good},\ 0.3)(\text{excellent},\ 0.6)\} \tag{20a}$$

$$S(\text{fuel economy}) = \{(\text{indifferent},\ 1.0)\} \tag{20b}$$

$$S(\text{quietness}) = \{(\text{average},\ 1.0)\} \tag{20c}$$

$$S(\text{vibration}) = \{(\text{indifferent},\ 1.0)\} \tag{20d}$$

$$S(\text{starting}) = \{(\text{average},\ 0.6),\ (\text{good},\ 0.3)\}. \tag{20e}$$

For the fourth engine

$$S(\text{responsiveness}) = \{(\text{indifferent}, 1.0)\} \tag{21a}$$

$$S(\text{fuel economy}) = \{(\text{excellent}, 1.0)\} \tag{21b}$$

$$S(\text{quietness}) = \{(\text{excellent}, 1.0)\} \tag{21c}$$

$$S(\text{vibration}) = \{(\text{poor}, 1.0)\} \tag{21d}$$

$$S(\text{starting}) = \{(\text{average}, 1.0)\}. \tag{21e}$$

Step 2: Calculate the basic probability masses for the first engine. From (1a)–(1e) and (5), we have

$$\begin{array}{lllll}
\beta_{1,1}=0, & \beta_{2,1}=0, & \beta_{3,1}=0, & \beta_{4,1}=1.0, & \beta_{5,1}=0 \\
\beta_{1,2}=0, & \beta_{2,2}=0.5, & \beta_{3,2}=0.5, & \beta_{4,2}=0, & \beta_{5,2}=0 \\
\beta_{1,3}=0, & \beta_{2,3}=0, & \beta_{3,3}=0, & \beta_{4,3}=0.5, & \beta_{5,3}=0.3 \\
\beta_{1,4}=0, & \beta_{2,4}=0, & \beta_{3,4}=0, & \beta_{4,4}=0.5, & \beta_{5,4}=0.5 \\
\beta_{1,5}=0, & \beta_{2,5}=0, & \beta_{3,5}=0, & \beta_{4,5}=1.0, & \beta_{5,5}=0.
\end{array}$$

From (6)–(7b), we calculate the basic probability assignments as follows:

$$\begin{array}{llll}
m_{1,1}=0, & m_{2,1}=0, & m_{3,1}=0, & m_{4,1}=0.2 \\
m_{5,1}=0, & \tilde{m}_{H,1}=0, & \overline{m}_{H,1}=0.8 & m_{1,2}=0 \\
m_{2,2}=0.1, & m_{3,2}=0.1, & m_{4,2}=0, & m_{5,2}=0 \\
\tilde{m}_{H,2}=0, & \overline{m}_{H,2}=0.8 & m_{1,3}=0, & m_{2,3}=0 \\
m_{3,3}=0, & m_{4,3}=0.1, & m_{5,3}=0.06, & \tilde{m}_{H,3}=0.04 \\
\overline{m}_{H,3}=0.8 & m_{1,4}=0, & m_{2,4}=0, & m_{3,4}=0 \\
m_{4,4}=0.1, & m_{5,4}=0.1, & \tilde{m}_{H,4}=0, & \overline{m}_{H,4}=0.8 \\
m_{1,5}=0, & m_{2,5}=0, & m_{3,5}=0, & m_{4,5}=0.2 \\
m_{5,5}=0, & \tilde{m}_{H,5}=0, & \overline{m}_{H,5}=0.8. &
\end{array}$$

Steps 3 and 4: Using (8a)–(9b), the combined degrees of belief are generated by

$$\begin{aligned}
&\beta_1=0, \quad \beta_2=0.085, \quad \beta_3=0.085\\
&\beta_4=0.6579, \quad \beta_5=0.1397.
\end{aligned}$$

The step-by-step illustration of the recursive ER algorithm can be found in [33] and [35].

Step 5: The quality of the first engine can then be assessed by

$$\begin{aligned}
S(\text{engine 1}) = \{&(\text{indifferent}, 0.085), (\text{average}, 0.085)\\
&(\text{good}, 0.6579), (\text{excellent}, 0.1397)\}.
\end{aligned}$$

Step 6: Repeating Steps 2–5, we can generate the overall assessments for the quality of the other three engines as follows:

$$\begin{aligned}
S(\text{engine 2}) = \{&(\text{indifferent}, 0.0939), (\text{average}, 0.305)\\
&(\text{good}, 0.4223), (\text{excellent}, 0.143)\}
\end{aligned}$$

$$\begin{aligned}
S(\text{engine 3}) = \{&(\text{indifferent}, 0.4193), (\text{average}, 0.3227)\\
&(\text{good}, 0.1131), (\text{excellent}, 0.1091)\}
\end{aligned}$$

$$\begin{aligned}
S(\text{engine 4}) = \{&(\text{poor}, 0.1905), (\text{indifferent}, 0.1905)\\
&(\text{average}, 0.1905), (\text{excellent}, 0.4286)\}.
\end{aligned}$$

B. Overall Assessments

From the above overall distributed assessments of the four engines, it is fair to say that, in quality, the first engine is better than the second engine, that is in turn better than the third engine. It is also fair to say that the quality of the first engine is on average better than that of the fourth engine, though the latter does show excellent performance in some areas. However, the precise comparisons of the fourth engine with the other three engines depend on how the evaluation grades are defined, which may be measured by utility.

Suppose the utilities of the evaluation grades are estimated by

$$\begin{aligned}
&u(H_1)=u(\text{poor})=0, \quad u(H_2)=u(\text{indifferent})=0.35\\
&u(H_3)=u(\text{average})=0.55, \quad u(H_4)=u(\text{good})=0.85\\
&u(H_5)=u(\text{excellent})=1.
\end{aligned}$$

Then, the utilities of the engines can be generated using (12a)–(12c) as follows:

$$\begin{array}{ll}
u_{\max}(\text{engine 1})=0.8078, & u_{\min}(\text{engine 1})=0.7754\\
u_{\text{aver}}(\text{engine 1})=0.7905 & u_{\max}(\text{engine 2})=0.7384\\
u_{\min}(\text{engine 2})=0.7026, & u_{\text{aver}}(\text{engine 2})=0.7205\\
u_{\max}(\text{engine 3})=0.5653, & u_{\min}(\text{engine 3})=0.5294\\
u_{\text{aver}}(\text{engine 3})=0.5474 & u_{\max}(\text{engine 4})=0.6\\
u_{\min}(\text{engine 4})=0.6, & u_{\text{aver}}(\text{engine 4})=0.6.
\end{array}$$

The minimum utility of *engine* 1 is 0.7754, larger than the maximum utilities of the other three engines. Therefore, *engine* 1 should be ranked the first. This ranking is conclusive in spite of the imprecision present in the original assessment data. The ranking of the other engines can be achieved in the same way, given as follows:

$$\text{engine 1} \succ \text{engine 2} \succ \text{engine 4} \succ \text{engine 3}$$

where $\succ$ denotes "is preferred to."

Obviously, the above assessments and ranking are dependent upon the original assessment data provided, the weights assigned and the utilities estimated. To demonstrate in more realistic situations the nonlinear features of the ER approach as analyzed in Section III, the following sensitivity analyses are conducted.

C. Sensitivity Analysis I—Changing Weights

Suppose the weights of all the five attributes are normalized so that their total weights are summed to one. Suppose the weight ω_2 of *fuel economy* is changed from zero to one with the weights of the other four attributes being equal, or $\omega_1 = \omega_3 = \omega_4 = \omega_5 = (1-\omega_2)/4$. Then, four average utility curves as given by (12c) can be drawn for the four engines with respect to ω_2. To simplify the analysis and without loss of generality, the average utilities are used for drawing the utility curves. A window-based and graphically designed IDS[1] developed on the basis of the ER approach is used to support the analysis.

Fig. 10 shows the four curves of the average utilities of the quality for the four engines. It is clear that with the increase of

[1]A demo version of the IDS software is available from the authors via email: jian-bo.yang@umist.ac.uk.

ω_2, the utility of the fourth engine increases and the utilities of the other three engines decrease. The increase or decrease rates of the utilities are not constant, indicating nonlinear relationships between the utilities and ω_2. Also note that at points where all the five attributes are of roughly equivalent importance (or ω_2 is around 0.2), the utility curves change faster than at other points. These observations are consistent with the analyses reported in Section III-A.

In Fig. 10, it is also clear that the ranking of the four engines changes with ω_2. When ω_2 is small (<0.15), the fourth engine is always ranked the last. With the increase of the importance of *fuel economy*, the ranking of the fourth engine keeps improving and eventually becomes the most preferred engine when ω_2 is large (>0.35). This is because the fourth engine has the best *fuel economy* among the four engines. The ranking of the first and the second engines is swapped at around $\omega_2 = 0.45$. With the increase of ω_2, the second engine becomes relatively more attractive than the first because the former has slightly better *fuel economy*.

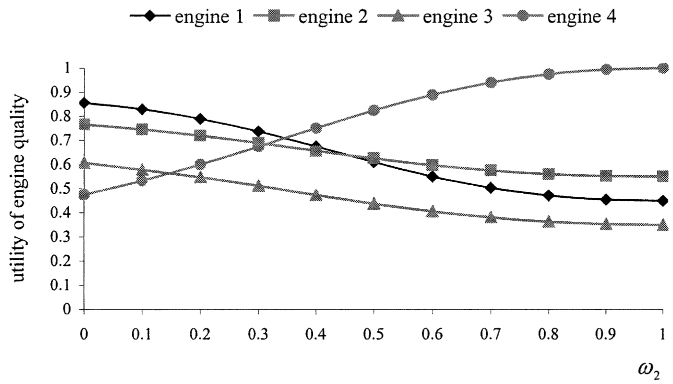

Fig. 10. Change of utilities with respect to ω_2.

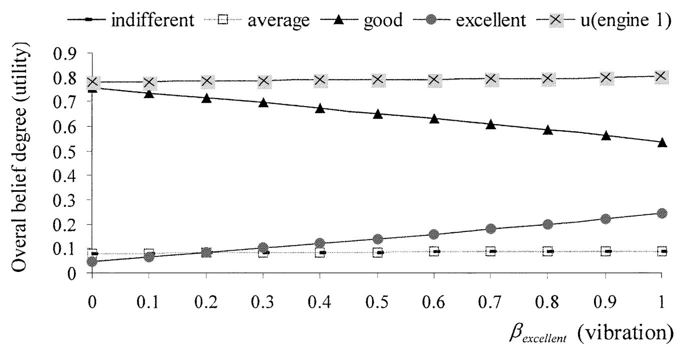

Fig. 11. Sensitivity analysis for engine 1 $(\omega_4 = \omega_1 = \omega_2 = \omega_3 = \omega_5)$.

D. Sensitivity Analysis II—Harmonic Judgments

In a decision problem of aggregating multiple attributes, most judgments may not be purely one of the following three types of judgments: 1) harmonic; 2) quasi-harmonic; or 3) contradictory. In this subsection, harmonic judgments are analyzed together with contradictory judgments. The assessments of *engine* 1 on the five attributes fall into this type of judgments and are investigated to show the sensitivity of the overall assessments with respect to the changes of basic degrees of belief.

As shown in (1a)–(1e), four attributes (*quietness*, *responsiveness*, *vibration*, and *starting*) are all assessed to be *good* and *excellent* for *engine* 1 and only its *fuel economy* is assessed to the two different grades: 1) *indifferent* and 2) *average*. Suppose the original assessment on *vibration* for *engine* 1 changes from 100% *good* up to 100% *excellent*. Let the parameter $\beta_{\text{excellent}}(\text{vibration})$ denote the degree of belief that *vibration* of *engine* 1 is assessed to the grade *excellent*. Suppose the five attributes are of equal importance.

The overall belief degrees assessed to the four grades (*indifferent*, *average*, *good*, and *excellent*) with respect to $\beta_{\text{excellent}}(\text{vibration})$ are shown in Fig. 11. As expected, the belief degree to *good* monotonically (not linearly) decreases and that to *excellent* monotonically (almost linearly) increases. The belief degrees to *indifferent* and *average* have little change with $\beta_{\text{excellent}}(\text{vibration})$. These quasi-linear patterns are generated due to the assignment of equal weights to all the five attributes. The above changes improve the attractiveness of *engine* 1 that is already the best engine of the four alternatives with the five attributes given equal importance. The change patterns of both harmonic and contradictory judgments shown in Fig. 11 are similar to those of the quasi-harmonic reasoning patterns of Section III-B, as shown in Fig. 4.

If the attributes are given different weights, the above reasoning patterns will change. Suppose *vibration* is twice as important as each of the other four attributes, or $\omega_4 = 2\omega_1 = 2\omega_3 = 2\omega_2 = 2\omega_5$. The overall belief degrees assessed to the four grades (*indifferent*, *average*, *good*, and *excellent*) with respect to $\beta_{\text{excellent}}(\text{vibration})$ is then shown in Fig. 12. Compared with Fig. 11, the belief degrees to both *good* and *excellent* change more quickly in a more strongly nonlinear fashion. This is consistent with the reasoning patterns shown in Fig. 5 of Section III-B and in Fig. 2 of Section III-A. With the changes applied, *engine* 1 is still ranked the best, though its utility is slightly reduced when its *vibration* is assumed to be 100% *good*.

The above analysis show that the ER aggregation process for harmonic judgments is nonlinear and the nonlinearity depends on the assignment of weights to attributes. With the existence of harmonic judgments, the greater the difference among attribute weights the stronger of the nonlinearity and the more dramatic the changes of the overall assessments.

E. Sensitivity Analysis III—Contradictory Judgments

In Figs. 11 and 12, the belief degrees to *indifferent* and *average* did not remain completely constant but had almost negligible changes with $\beta_{\text{excellent}}(\text{vibration})$. The reasons for this phenomenon are twofold. On the one hand, the evaluation grades *indifferent* and *average* are different from *good* and *excellent*. The change in $\beta_{\text{excellent}}(\text{vibration})$ does not have direct impact on *indifferent* or *average*, as demonstrated in Section III-D. On the other hand, there are harmonic judgments in (1a) and (1b) and the judgment given in (1a) is incomplete. These factors have indirect influence on *indifferent* and *average* in the ER aggregation process due to the treatment of remaining probability mass as defined by (7a) and (7b).

The reasoning patterns of two contradictory judgments were analyzed in Section III-D. To demonstrate similar patterns with three and more contradictory judgments, take *engine* 4, for example. As given by (21a)–(21e), four out of the five judgments

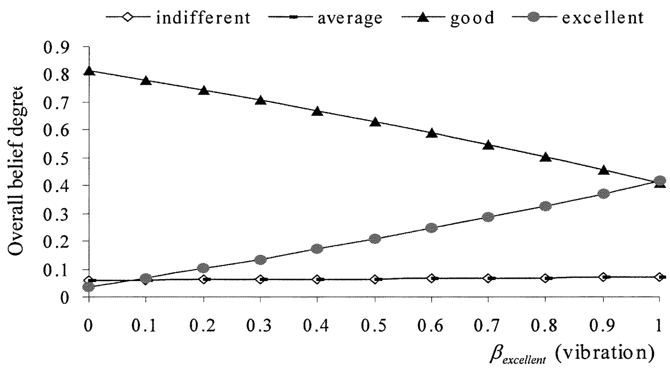

Fig. 12. Sensitivity analysis for engine 1 ($\omega_4 = 2\omega_1,\ \omega_1 = \omega_2 = \omega_3 = \omega_5$).

are in conflict. In the following analysis, the weight of *quietness* is assumed to be 0 or $\omega_3 = 0$, so that the judgment on *quietness* does not take effect. In this way, the remaining four judgments are completely contradictory to one another.

Suppose the original assessment on *fuel economy* for *engine* 4 changes from 100% good up to 100% *excellent*. Let the parameter $\beta_{\text{excellent}}(\text{fuel economy})$ denote the degree of belief that *fuel economy* for *engine* 4 is assessed to the grade *excellent*. Suppose the four remaining attributes are of equal importance. The overall belief degrees assessed to the five grades (*poor*, *indifferent*, *average*, *good*, and *excellent*) with respect to $\beta_{\text{excellent}}(\text{fuel economy})$ are shown in Fig. 13. As expected, the three belief degrees to *poor*, *indifferent*, and *average* remain completely constant at the level of 0.25 (25%), the belief degree to the grade *good* decreases from 0.25 to 0 proportionally, and that to *excellent* increases from 0 to 0.25 proportionally. These results are consistent with the analytical results reported in Section III-D.

The above results are generated by assuming that all the four remaining attributes are of equal importance. Suppose *fuel economy* is twice as important as each of the other three attributes: *responsiveness*, *vibration*, and *starting*. The new results are shown in Fig. 14. The overall belief degrees to *poor*, *indifferent*, and *average* still remain constant though at a lower level of 0.1765 and those to *good* and *excellent* still change proportionally but more dramatically between 0 and 0.4706. This dramatic change can be explained in the same way as in Section III-A and Fig. 2.

F. Sensitivity Analysis IV—Quasi-Harmonic Judgments

To investigate the quasi-harmonic reasoning pattern, *engine* 3 is selected for which all the five attributes are assessed to either *indifferent*, *average*, *good*, or *excellent*. Suppose the original assessment on *quietness* for *engine* 3 changes from 100% *indifferent* up to 100% *average* and then to 100% *good*. Let the parameter $\beta_{\text{average}}(\text{quietness})$ denote the degree of belief that *quietness* for *engine* 3 is assessed to the grade *average*. Suppose the five attributes are of equal importance. Then, *engine* 3 is the least preferred engine as originally assessed in Section IV-B.

The simulation results are shown in Fig. 15. With the assessment of *quietness* changing from 100% *indifferent* to 100% *average*, or $\beta_{\text{average}}(\text{quietness})$ increasing from 0 to 1, the overall belief degree to *indifferent* decreases from 0.6547 to 0.4175 quasi-linearly and that to *average* increases from 0.1005

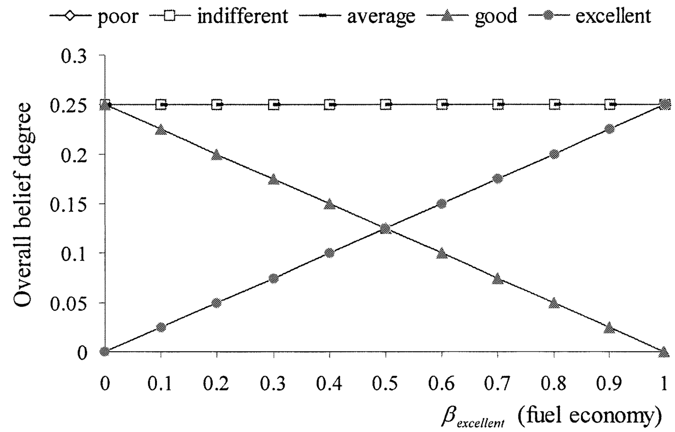

Fig. 13. Sensitivity analysis for engine 4 ($\omega_4 = \omega_1 = \omega_2 = \omega_5,\ \omega_3 = 0$).

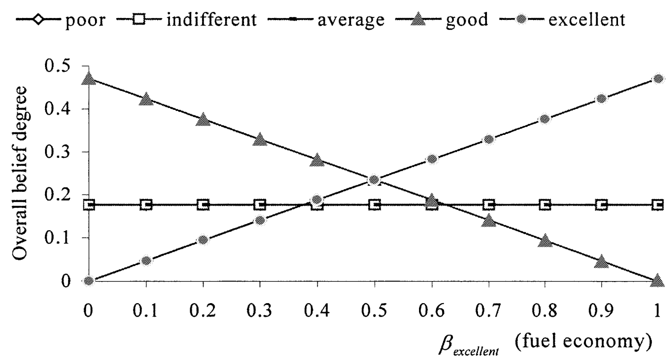

Fig. 14. Sensitivity analysis for engine 4 ($\omega_4 = 2\omega_1,\ \omega_1 = \omega_2 = \omega_5,\ \omega_3 = 0$).

to 0.3213 quasi-linearly, while those to *good* and *excellent* remain nearly constant at slightly different levels because of the incomplete assessments for *engine* 3, as shown in (20a)–(20e). With the assessment of *quietness* changing from 100% *average* to 100% *good*, or $\beta_{\text{average}}(\text{quietness})$ decreasing from 1 to 0, the overall belief degree to *good* increases from 0.1126 to 0.3259 almost linearly and that to *average* decreases from 0.3213 to 0.1085 also nearly linearly, while those to both *indifferent* and *excellent* remain almost constant at different levels. When the assessment of *quietness* is improved to 90% *good*, that is, $\beta_{\text{average}}(\text{quietness})$ reduces from 1 to 0.1 again, then the average utility of *engine* 3 is 0.6048, larger than the average utility of *engine* 4 (0.6).

If *quietness* is assumed to be twice as important as each of the other four attributes, the overall results are shown in Fig. 16. The reasoning patterns in Fig. 16 are similar to those in Fig. 15 except that the changes in Fig. 16 are more dramatic and more strongly nonlinear. Again, this is consistent with the analyses of Sections III-A and III-C.

V. Assessment of Retrofit Ship Designs

Many real-world decision problems are more complex than the above engine quality assessment problem in that more attributes need to be catered, which may constitute a multilevel hierarchy. In this section, we apply the ER approach to examine an assessment problem in ship design, provided by a Tyneside company [7]. The IDS software is used to support the analysis as illustrated later.

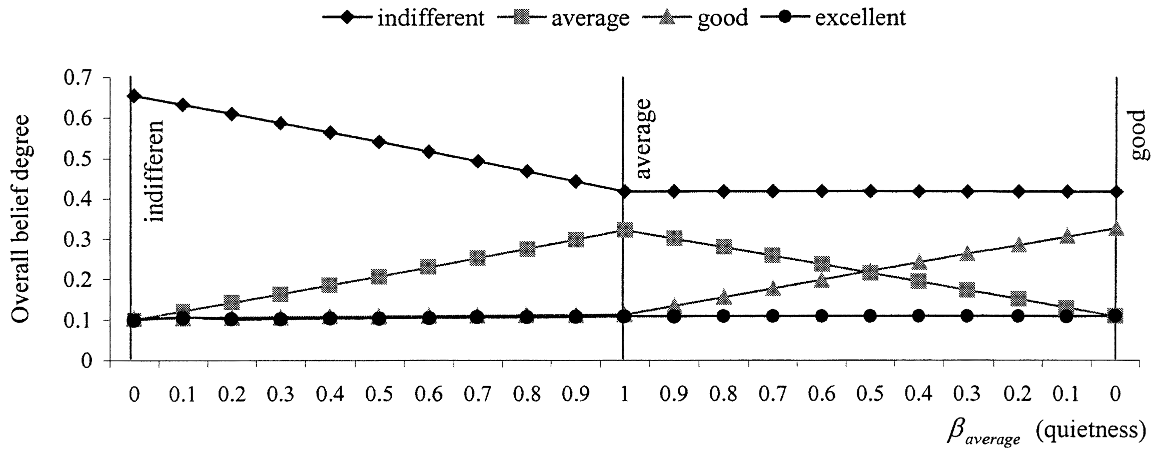

Fig. 15. Sensitivity analysis for engine 3 ($\omega_3 = \omega_1 = \omega_2 = \omega_4 = \omega_5$).

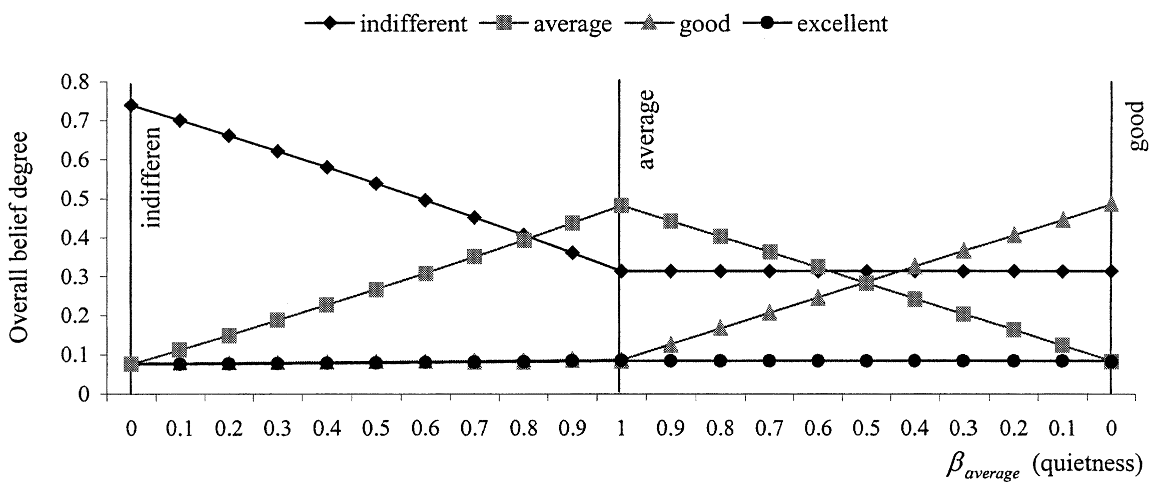

Fig. 16. Sensitivity analysis for engine 3 ($\omega_3 = 2\omega_1$, $\omega_1 = \omega_2 = \omega_4 = \omega_5$).

A. Ship Design Assessment Problem

The technical problem is to examine retrofit options for a short sea roll-on roll-off ferry to enhance its damage stability characteristics (roll-on roll-off is simply called *ro–ro*, which means that vehicles drive on/off ferries). The selection of a retrofit option demands a clear definition of the necessary attributes and their associated contributing attributes which could influence the ship's operation. These attributes include both the commercial and the technical aspects of the option. Such attributes are defined and arranged in a hierarchical structure, as shown in Fig. 17.

In Fig. 17, the selection of a retrofit option is based on the assessment of the option over two general attributes (*ship operation* and *installation*) which are broken down into lower-level attributes within a hierarchy. In the attribute hierarchy, the relative weights of the attributes at a single level with regard to the same upper-level attribute are shown by the numbers in the brackets in Fig. 17.

The influence of each retrofit option on a ship can initially be assessed on each basic (bottom level) attribute. Such an assessment may be acquired and represented using a belief structure or a numerical number [30]. With regard to an attribute such as modification required to enhance *collision resistance*, for example, it may be stated that "the modification of a particular retrofit option for a ship is *minor* or *moderate*." In the statement, *minor* and *moderate* are referred to as evaluation grades describing the degrees of modification. It is also possible that the degree of modification is something between "*minor*" and "*moderate*." For instance, it could be judged that "the modification of a retrofit option in terms of *collision resistance* is *minor* to an extent of β_1 and *moderate* to an extent of β_2 with $\beta_1,\ \beta_2 \geq 0$, and $\beta_1 + \beta_2 \leq 1$, which may be represented using a belief structure: $S(collision\ resistance) = \{(minor,\ \beta_1),\ (moderate,\ \beta_2)\}$.

Other assessment grades such as *none*, *major*, and *fundamental* may also be defined to describe the degrees of modification. A large modification for a ship leads to increase of cost and technical complexity. It may therefore be less desirable. The grade *minor* may thus be preferred to the grade *major*. A particular set of evaluation grades for this ship design assessment problem is defined by [30]

$$H = \{fundamental,\ major,\ moderate,\ minor,\ none\}.$$

The design problem is about the retrofitting of a typical short sea ferry for compliance with the requirements of Safety of Life at Sea Regulations (SOLAS) 90. The three options considered

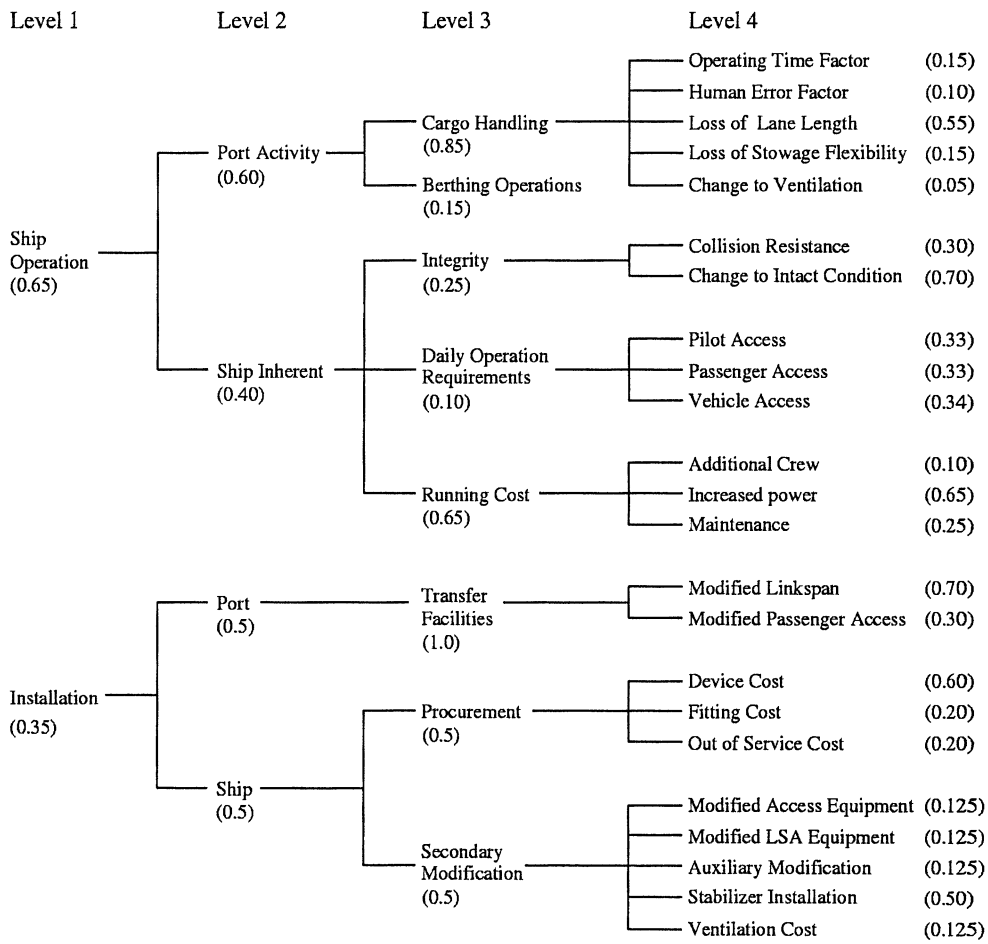

Fig. 17. Hierarchy of attributes for ship design.

to meet the increased stability requirements are the provision of sponsons (additional structures to increase the buoyancy of a ferry near the waterline), movable transverse bulkheads (internal walls used for dividing up the ferry into smaller compartments to improve safety) and the provision of buoyant wing compartments. Ro–ro ferries, in general, have large open decks and provision of additional stability in the intact condition allows them to have enhanced survivability characteristics if the water should get onto the open decks. Each of the three options is compared with the original ship. The assessment data is as shown in Table I.

In Table I, the multiattribute assessment problem is represented in a concise format. The evaluation grades *fundamental*, *major*, *moderate*, *minor*, and *none* are abbreviated by *FU*, *MA*, *MO*, *MI*, and *NO*, respectively. The real numbers in the brackets following the abbreviations denote the degrees of belief. For instance, a statement for assessment of option 1 is that "in terms of *berthing operations* the modification of the option from the original ship is *minor* to an extent of 0.1667 and *none* to a degree of 0.8333." This statement is represented by $S(berthing\ operation) = \{(minor,\ 0.1667),\ (none,\ 0.8333)\}$ or simply by *MI(0.1667), NO(0.8333)* as shown by column 2 and row 3 of Table I. It should be noted that each basic attribute at an option may be assessed to one or more than one grade defined and not all assessments in Table I are complete. More details about the assessment problem can be found in [30].

B. Result Analysis

The ER approach can be used to generate an overall assessment for each option by aggregating the original assessment data, as shown in Fig. 17 and Table I. Each upper-level attribute and its associated immediate lower-level attributes constitute an elementary evaluation model.

For example, take the assessment of *cargo handling* for option 2. Let y stand for *cargo handling* and E for its five associated basic attributes. For option 2, the basic degrees of belief are given in column 3 and row 2 in Table I. The generated evaluation for *cargo handling* of option 2 is shown in column 3 and row 2 of Table II, that is, *MO(0.0048), MI(0.1153), NO(0.855)*. This evaluation reads that there is only marginal difference between option 2 and the original ship in terms of *cargo handling*.

In a similar way, all the attributes at level 3 can be evaluated through the attributes at level 4. The results are shown in Table II. The attributes at level 2 can in turn be evaluated through the attributes at level 3, as shown in Table III. Table IV shows the evaluations of *ship operation* and *installation* at each option, which are generated through the attributes at level 2. Finally, the evaluations of these three options are obtained in terms of the degrees of modification, as shown by row 2 of Table V.

TABLE I
ORIGINAL ASSESSMENTS OF BOTTOM LEVEL ATTRIBUTES AT EACH OPTION

Level 4	Option S1	Option S2	Option S3
Operating Time Factor	NO(1)	MI(0.1667), NO(0.8333)	MI(0.7517), NO(0.2483)
Human Error Factor	NO(1)	MO(0.1000), MI(0.2000) NO(0.6000)	MO(0.0500), MI(0.3000) NO(0.5000)
Loss of Lane Length	NO(1)	MI(0.1333), NO(0.8667)	MI(0.4333), NO(0.5667)
Loss of Stowage Flexibility	NO(1)	MI(0.2000), NO(0.7000)	MO(0.1000), MI(0.8000) NO(0.0500)
Change to Ventilation	NO(1)	NO(1)	MI(0.3333), NO(0.6667)
Berthing Operations	MI(0.1667), NO(0.8333)	NO(1)	NO(1)
Collision Resistance	FU(0.4000), MA(0.4000) MO(0.1000)	FU(0.8000), MA(0.1500)	NO(1)
Change to Intact Condition	MI(0.8000), NO(0.1500)	NO(1)	NO(1)
Pilot Access	NO(1)	NO(1)	NO(1)
Passenger Access	MI(0.6000), NO(0.3000)	NO(1)	MO(0.1000), MI(0.4000) NO(0.4500)
Vehicle Access	NO(1)	NO(1)	MO(0.1500), MI(0.7000) NO(0.1000)
Additional Crew	NO(1)	FU(1)	NO(1)
Increased Power	MI(0.1667), NO(0.8333)	NO(1)	NO(1)
Maintenance	MI(0.2333), NO(0.7667)	MI(0.0333), NO(0.9667)	MI(0.0667), NO(0.9333)
Modified Linkspan	MI(0.0033), NO(0.9967)	NO(1)	NO(1)
Modified Passenger Access	MI(0.0003), NO(0.9997)	NO(1)	NO(1)
Device Cost	MI(0.0847), NO(0.9153)	MI(0.0100), NO(0.9900)	MI(0.0467), NO(0.9533)
Fitting Cost	MI(0.0217), NO(0.9783)	MI(0.0023), NO(0.9977)	NO(1)
Out of Service Cost	MI(0.0140), NO(0.9860)	NO(1)	MI(0.0100), NO(0.9900)
Modify Access Equipment	NO(1)	NO(1)	NO(1)
Modify LSA Equipment	MI(0.0067), NO(0.9933)	NO(1)	NO(1)
Auxiliary Modification	MI(0.0030), NO(0.9970)	NO(1)	NO(1)
Stabilizer Installation	NO(1)	NO(1)	NO(1)
Ventilation Cost	NO(1)	NO(1)	NO(1)

TABLE II
OBTAINED JUDGMENTS FOR EVALUATION OF LEVEL 3 ATTRIBUTES

Level 3	Option S1	Option S2	Option S3
Cargo Handling	NO(1.0)	MO(0.0048), MI(0.1153), NO(0.855)	MO(0.011), MI(0.5143), NO(0.4522)
Berthing Operation	MI(0.1667), NO(0.8333)	NO(1)	NO(1)
Integrity	FU(0.0653), MA(0.0653) MO(0.0163), MI(0.6637), NO(0.1244)	FU(0.121), MA(0.0227), NO(0.8413)	NO(1.0)
Daily Operation Requirements	MI(0.1463), NO(0.8207)	NO(1.0)	MO(0.0746), MI(0.3592), NO(0.5326)
Running Cost	MI(0.1387), NO(0.8613)	FU(0.0383), MI(0.0038) NO(0.9579)	MI(0.007), NO(0.993)
Transfer Facilities	MI(0.0021), NO(0.9979)	NO(1.0)	NO(1.0)
Procurement	MI(0.0487), NO(0.9513)	MI(0.0054), NO(0.9946)	MI(0.0254), NO(0.9746)
Secondary Modification	MI(0.0006), NO(0.9994)	NO(1.0)	NO(1.0)

Suppose the utility of each of the evaluation grades are estimated by

$$u(fundamental) = 0, \quad u(major) = 0.3$$

$$u(moderate) = 0.5, \quad u(minor) = 0.7, \quad u(none) = 1.$$

Then, the utilities and the ranking of the three retrofit options are shown in the last two rows of Table V. The most favorable option is the sponson, followed closely by the movable bulkheads and then by the wing compartments. The results are in harmony with the analysis conducted by Evans using other techniques.

The sponsons prove beneficial in the analysis for short sea ro–ros as the loss of *collision resistance* and the decrease in the natural roll period associated with *change to intact condition* do not play an important role in the evaluation of the whole ship. This is also because the increase in power and hence increased fuel costs have a small overall effect on the total economics. This option does not affect the ro–ro concept, which for the short sea ro–ro has a critical effect on the operating economics.

The retrofitting with buoyant wing compartments is the least preferred option in this analysis due to their disruptive effects on *loss of stowage flexibility* (the through flow of traffic) and on *cargo handling*.

C. Decision Analysis Supported by IDS

The decision making process for the ferry design problem is more complex than for the previous engine quality assess-

TABLE III
OBTAINED JUDGMENTS FOR EVALUATION OF LEVEL 2 ATTRIBUTES

Level 2	Option S1	Option S2	Option S3
Port Activities	MI(0.0044), NO(0.9956)	MO(0.004), MI(0.0971), NO(0.8776)	MO(0.0099), MI(0.4618), NO(0.5092)
Ship Inherent	FU(0.0082), MA(0.0082) MO(0.002), MI(0.213), NO(0.749)	FU(0.0369), MA(0.0024), MI(0.0023), NO(0.9546)	MO(0.0027), MI(0.0172), NO(0.9767)
Port	MI(0.0021), NO(0.9979)	NO(1.0)	NO(1.0)
Ship	MI(0.0167), NO(0.9833)	MI(0.0018), NO(0.9982)	MI(0.0085), NO(0.9915)

TABLE IV
OBTAINED JUDGMENTS FOR EVALUATION OF LEVEL 1 ATTRIBUTES

Level 1	Option S1	Option S2	Option S3
Ship Operation	FU(0.0019), MA(0.0019), MO(0.0005), MI(0.0509), NO(0.9372)	FU(0.0084), MA(0.0006), MO(0.002), MI(0.0487), NO(0.9262)	MO(0.0062), MI(0.2646), NO(0.7163)
Installation	MI(0.0063), NO(0.9937)	MI(0.0006), NO(0.9994)	MI(0.0029), NO(0.9971)

TABLE V
OVERALL EVALUATION AND RANKING OF THE THREE OPTIONS

Selection	Option S1	Option S2	Option S3
Evaluations	FU(0.001), MA(0.001), MO(0.0003), MI(0.0294), NO(0.9632)	FU(0.0047), MA(0.0003), MO(0.0011), MI(0.0272), NO(0.9575)	MO(0.0037), MI(0.158), NO(0.83)
Average utilities	0.9868	0.9818	0.9466
Ranking	1	2	3

ment problem due to the larger number of attributes. Many real-world decision problems could be much more complicated and it would be unrealistic to use the ER approach without proper software support. This recognition has led to the development of the IDS to implement the ER approach, which transforms the lengthy and tedious model building and result analysis process into an easy window-based click and design activity [32]. In this subsection, IDS is briefly described in support of the modeling and analysis of the ferry design problem.

The main window of IDS for the ferry design problem is shown in Fig. 18, which consists of a main menu bar, a tool bar, and two model display windows. In the right window, the hierarchy of attributes is displayed in a tree structure and in the left window the three ferry deign options are listed. The main window provides access to all functions for building, modifying, saving, and opening MADA models; entering numerical data and descriptive information; conducting decision analysis; and reporting analysis results using text files, bar charts, or curves.

Once the assessment framework is established and raw assessment information for the bottom level of attributes entered, IDS processes the information using the ER approach and displays the assessment results both numerically and graphically. Fig. 19 shows the final distributed assessment of the first ferry design on the top attribute: *ferry design selection*. It is clear from Fig. 19 that compared with the original design, the first retrofit ferry design only requires very small percentages of *fundamental* (0.11%), *major* (0.11%), *moderate* (0.03%), and *minor* (3%) modifications. This is why it is ranked the most preferred retrofit design in this analysis. IDS is capable of providing and displaying a similar distributed assessment for any attribute in the decision model. IDS also allows visual comparison of different design options on any selected attributes. Fig. 20 shows this for the three ferry design options on the first and second levels of subattributes.

VI. CONCLUDING REMARKS

Decision problems often involve both numerical data and subjective assessments under uncertainty that are inherently associated with imprecision and vagueness. The ER approach as discussed in this paper provides an alternative way of modeling and aggregating both complete and incomplete information using the belief structure. Due to its recursive nature, the ER approach exhibits various nonlinear features in information aggregation. In this paper, typical reasoning patterns of the ER approach are examined, establishing several nonrecursive reasoning functions. The generation and illustration of these functions revealed the nonlinear features of the ER approach in simple yet typical decision situations, though their behavioral implications need to be further investigated. Nevertheless, this analytical investigation provides valuable experience in developing a rational OR/AI approach for decision analysis under uncertainty, which could be useful to other researchers and practitioners interested in developing and applying this or similar approaches.

Complementary to the analytical investigation, the two numerical studies were conducted to apply the ER approach to multiattribute decision problems involving both complete and

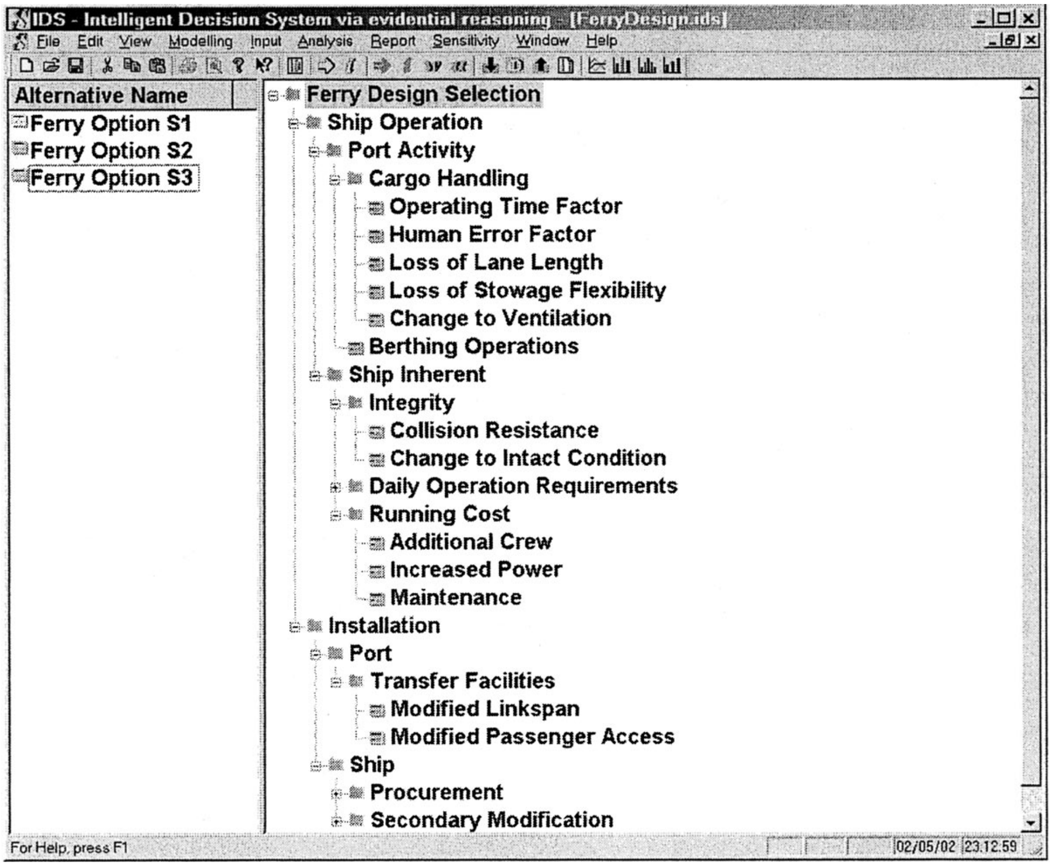

Fig. 18. IDS main window for ferry design selection.

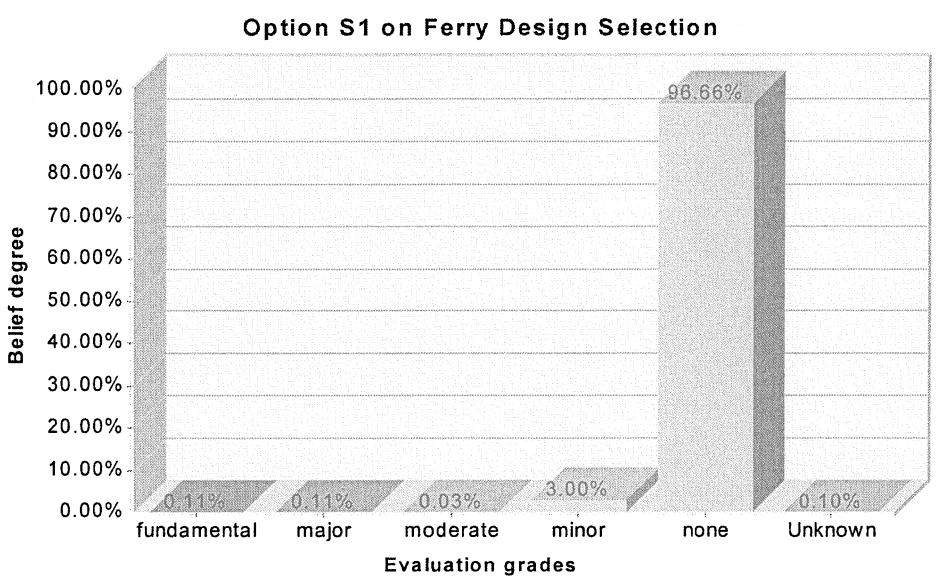

Fig. 19. IDS distributed assessment for the first retrofit ferry design.

incomplete assessments. The examination of the first engine quality assessment problem illustrated the step-by-step procedure of implementing the ER approach as well as the nonlinear features of the ER aggregation process in a real-life decision situation. The investigation of the second ship design selection problem demonstrated the potential of the ER approach and its supporting software (IDS) for general multiattribute decision analysis.

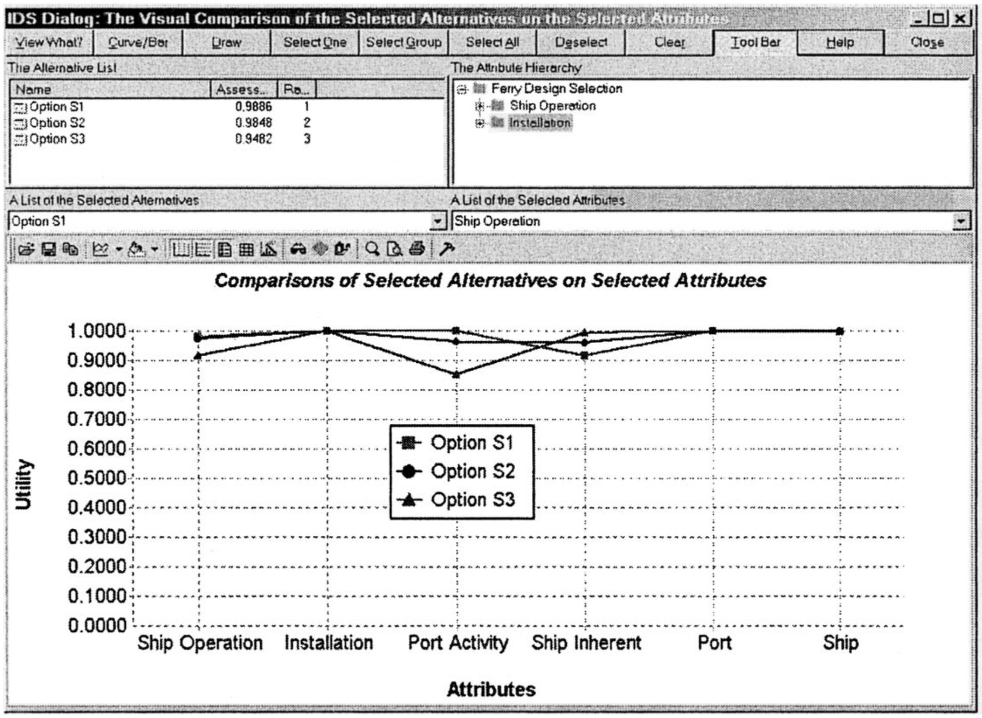

Fig. 20. IDS visual comparison window for three retrofit ferry designs.

ACKNOWLEDGMENT

The author is grateful to the two anonymous referees for their constructive and valuable comments.

REFERENCES

[1] G. Balestra and A. Tsoukias, "Multicriteria analysis represented by artificial intelligence techniques," *J. Oper. Res. Soc.*, vol. 41, no. 5, pp. 419–430, 1990.

[2] J. Barzilai, "Deriving weights from pairwise comparison matrices," *J. Oper. Res. Soc.*, vol. 48, pp. 1226–1232, 1997.

[3] V. Belton and T. Gears, "On a short-coming of Saaty's method of analytic hierarchy," *OMEGA*, vol. 11, no. 3, pp. 228–230, 1981.

[4] V. Belton, "Comparison of the analytic hierarchy process and a simple multiattribute value function," *Eur. J. Oper. Res.*, vol. 26, pp. 7–21, 1986.

[5] I. Bogardi and A. Baedossy, "Application of MADM to geological exploration," in *Essays and Surveys on Multiple Criteria Decision Making*, P. Hansen, Ed. New York: Springer-Verlag, 1983.

[6] J. L. Corner and C. W. Kirkwood, "The magnitude of errors in proximal multiattribute decision analysis with probabilistically dependent attributes," *Manage. Sci.*, vol. 42, no. 7, pp. 1033–1042, 1996.

[7] J. P. Evans, "Techno-economic analysis of retrofit options," Transmarine Ltd., Wallsend Research Station, Wallsend, Tyne & Wear, U.K., Research Rep., 1993.

[8] C. L. Huang and K. Yoon, *Multiple Attribute Decision Making Methods and Applications, A State-of-Art Survey*. New York: Springer-Verlag, 1981.

[9] T. Isitt, "The sports tourers," *Motor Cycle Int.*, vol. 64, pp. 19–27, 1990.

[10] G. Islei and A. G. Lockett, "Judgmental modeling based on geometric least square," *Eur. J. Oper. Res.*, vol. 36, pp. 27–35, 1988.

[11] E. Jacquet-Lagreze and J. Siskos, "Assessing a set of additive utility functions for multicriteria decision making, the UTA method," *Eur. J. Oper. Res.*, vol. 10, pp. 151–164, 1982.

[12] C. R. Johnson, W. B. Beine, and T. J. Wang, "Right–left asymmetry in an eigenvector ranking procedure," *J. Math. Psychol.*, vol. 19, pp. 61–64, 1979.

[13] R. L. Keeney and H. Raiffa, *Decision With Multiple Objectives: Preference and Value Tradeoffs*. New York: Wiley, 1976.

[14] C. W. Kirkwood, "Estimating the impact of uncertainty on a deterministic multiattribute evaluation," *Manage. Sci.*, vol. 38, no. 6, p. 819, 1992.

[15] ——, "An algebraic approach to formulating and solving large models for sequential decisions under uncertainty," *Manage. Sci.*, vol. 39, no. 7, p. 900, 1993.

[16] N. C. Proudlove, "The support of group decision making using judgmental modeling: An exploration of the contribution of behavioral factors," Ph.D. dissertation, Manchester Business School, Manchester, U.K., 1999.

[17] R. Lopez de Mantaras, *Approximate Reasoning Models*. Chichester, U.K.: Ellis Horwood Ltd., 1990.

[18] T. L. Saaty, *The Analytic Hierarchy Process*. Pittsburgh, PA: Univ. Pittsburgh, 1988.

[19] P. Sen and J. B. Yang, "Multiple criteria decision making in design selection and synthesis," *J. Eng. Design*, vol. 6, no. 3, pp. 207–230, 1995.

[20] T. J. Stewart, "A critical survey on the status of multiple criteria decision making theory and practice," *OMEGA Int. J. Manage. Sci.*, vol. 20, no. 5–6, pp. 569–586, 1992.

[21] J. Wang, J. B. Yang, and P. Sen, "Safety analysis and synthesis using fuzzy sets and evidential reasoning," *Reliab. Eng. Syst. Saf.*, vol. 47, no. 2, pp. 103–118, 1995.

[22] ——, "Multiperson and multiattribute design evaluations using evidential reasoning based on subjective safety and cost analysis," *Reliab. Eng. Syst. Saf.*, vol. 52, pp. 113–127, 1996.

[23] J. Wang, "A subjective methodology for safety analysis of safety requirements specifications," *IEEE Trans. Fuzzy Syst.*, vol. 5, pp. 1–13, June 1997.

[24] W. L. Winston, *Operations Research—Applications and Algorithms*. , CA: Duxbury Press, 1994.

[25] R. R. Yager, "Decision-making under various types of uncertainties," *J. Intell. Fuzzy Syst.*, vol. 3, no. 4, pp. 317–323, 1995.

[26] J. B. Yang and M. G. Singh, "An evidential reasoning approach for multiple attribute decision making with uncertainty," *IEEE Trans. Syst., Man, Cybern.*, vol. 24, no. 1, pp. 1–18, 1994.

[27] J. B. Yang and P. Sen, "A general multilevel evaluation process for hybrid MADM with uncertainty," *IEEE Trans. Syst., Man, Cybern.*, vol. 24, pp. 1458–1473, 1994.

[28] J. B. Yang, *Multiple Criteria Decision Making Methods and Applications*. Changsha, China: Hunan Publishing House, 1996, ISBN 7-5438-1508-7.

[29] J. B. Yang and P. Sen, "Preference modeling by estimating local utility functions for multiobjective optimization," *Eur. J. Oper. Res.*, vol. 95, pp. 115–138, 1996.

[30] ——, "Multiple attribute design evaluation of complex engineering products using the evidential reasoning approach," *J. Eng. Design*, vol. 8, no. 3, pp. 211–230, 1997.

[31] J. B. Yang and D. L. Xu, "Knowledge-based executive car evaluation using the evidential reasoning approach," in *Advances in Manufacturing Technology XII*, Baines, Taleb-Bendiab, and Zhao, Eds. London, U.K.: Professional Engineering Publishing, 1998, pp. 741–749.

[32] ——, "Intelligent decision system via evidential reasoning," IDSL, Cheshire, U.K., Ver. 1.1, 2000.

[33] J. B. Yang, "Rule and utility-based evidential reasoning approach for complex decision analysis," *Eur. J. Oper. Res.*, vol. 131, no. 1, pp. 31–61, 2001.

[34] J. B. Yang, B. G. Dale, and C. H. R. Siow, "Self-assessment of excellence: An application of the evidential reasoning approach," *Int. J. Prod. Res.*, vol. 39, no. 16, pp. 3789–3812, 2001.

[35] J. B. Yang and D. L. Xu, "On the evidential reasoning algorithm for multiattribute decision analysis with uncertainty," *IEEE Trans. Syst., Man, Cybern. A*, vol. 32, pp. 289–304, May 2002.

[36] Z. J. Zhang, J. B. Yang, and D. L. Xu, "A hierarchical analysis model for multiobjective decision making," in *Analysis, Design and Evaluation of Man–Machine System 1989*. Oxford, U.K.: Pergamon, 1990, pp. 13–18.

Jian-Bo Yang received the B.Eng. and M.Eng. degrees in control engineering from the North Western Polytechnic University, Xi'an, China, in 1981 and 1984, respectively, and the Ph.D. degree in systems engineering from Shanghai Jiao Tong University, Shanghai, China, in 1987.

He is currently Senior Lecturer in decision and system sciences at the Manchester School of Management of UMIST, Manchester, U.K. Prior to his current appointment, he was a Faculty Member of the University of Birmingham (1995–1997), the University of Newcastle Upon Tyne (1991–1995), UMIST (1990), and Shanghai Jiao Tong University (1987–1989). Over the last two decades, he has been conducting research in the areas of multiple criteria decision analysis using both quantitative and qualitative information with uncertainties, hybrid decision methodologies combining OR methods with AI techniques, multiple objective optimization, intelligent decision support systems, and dynamic system modeling, simulation, and control for engineering and management systems. His current applied research covers various areas such as design decision-making, quality management, risk and safety analysis, supply chain modeling and control, and supplier assessment. He has published three books and over 120 papers in journals and conferences. He has also developed several large software packages including a Windows-based intelligent decision system (IDS) via evidential reasoning.

Dr. Yang was given the Science and Technology Award for Young Scientists and Engineers by the China Association for Science and Technology in 1988 and Research Fellowships by British Council and the Alexander von Humboldt Foundation of Germany in the 1990s.

Dong-Ling Xu received the B.Eng. degree in electrical engineering from Hunan University, Changsha, China, in 1983, and the M.Eng. and Ph.D. degrees in system control engineering from Shanghai Jiao Tong University, Shanghai, China, in 1986 and 1988, respectively.

She is currently a Research Fellow in decision and system sciences at the Manchester School of Management of UMIST, Manchester, U.K. She was a Senior Software Engineer in an IT company from 1995 to early 2001, a Postdoctoral Research Associate at the University of Newcastle Upon Tyne from 1993 to 1995, and a Lecturer and an Associate Professor of East China University, Shanghai, from 1988 to 1993. As a Co-designer, she developed a Windows-based assessment tool called "intelligent decision system" or "IDS." The tool is now used in a wide range of assessment activities such as organizational self-assessment in quality management, product (e.g., cars, houses, and computers) evaluation, financial investment policy assessment, safety and risk assessment, and supplier evaluation. She has published a book and over 50 papers in computerized traffic management and control, optimization, decision analysis, control system design, statistical process control, and statistical analysis. Her current research interests are in the areas of multiple attribute decision analysis under uncertainties, decision theory, utility theory, evidential reasoning approach and their applications in quality management, environmental management, and engineering design.