Random Permutation Set Reasoning

Jixiang Deng, Yong Deng, *Member, IEEE*, and Jian-Bo Yang

***Abstract*—In artificial intelligence, it is crucial for pattern recognition systems to process data with uncertain information, necessitating uncertainty reasoning approaches such as evidence theory. As an orderable extension of evidence theory, random permutation set (RPS) theory has received increasing attention. However, RPS theory lacks a suitable generation method for the element order of permutation mass function (PMF) and an efficient determination method for the fusion order of permutation orthogonal sum (POS). To solve these two issues, this paper proposes a reasoning model for RPS theory, called random permutation set reasoning (RPSR). RPSR consists of three techniques, including RPS generation method (RPSGM), RPSR rule of combination, and ordered probability transformation (OPT). Specifically, RPSGM can construct RPS based on Gaussian discriminant model and weight analysis; RPSR rule incorporates POS with reliability vector, which can combine RPS sources with reliability in fusion order; OPT is used to convert RPS into a probability distribution for the final decision. Besides, numerical examples are provided to illustrate the proposed RPSR. Moreover, the proposed RPSR is applied to classification problems. An RPSR-based classification algorithm (RPSRCA) and its hyperparameter tuning method are presented. The results demonstrate the efficiency and stability of RPSRCA compared to existing classifiers.**

***Index Terms*—Classification, pattern recognition, random permutation set (RPS), random permutation set reasoning (RPSR), uncertainty reasoning.**

I. Introduction

In THE field of artificial intelligence, pattern recognition is the cornerstone of interpreting data structures and identifying complex patterns. This process, however, often encounters uncertain and incomplete information, leading to the rise of uncertainty reasoning. Uncertainty reasoning is a multifaceted field that employs various theories and methods to deal with different types of uncertainty present in data, including randomness, vagueness, imprecision, etc. Notably, Bayesian inference [1], fuzzy set theory [2], and possibility theory [3] address these specific types of uncertainty and have proven indispensable in a wide range of applications.

Manuscript received 12 November 2022; revised 18 April 2024; accepted 22 July 2024. Date of publication 5 August 2024; date of current version 5 November 2024. This work was supported by the National Natural Science Foundation of China under Grant 62373078. The work of Jian-Bo Yang was supported in part by the European Union Horizon 2020 Research and Innovation Programme RISE (REMESH) under Grant 823759. Recommended for acceptance by W. Hu. *(Corresponding authors: Yong Deng; Jian-Bo Yang.)*

Jixiang Deng and Yong Deng are with the Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu 610054, China (e-mail: dengentropy@uestc.edu.cn).

Jian-Bo Yang is with the Alliance Manchester Business School, The University of Manchester, M15 6PB Manchester, U.K. (e-mail: jian-bo.yang@manchester.ac.uk).

Digital Object Identifier 10.1109/TPAMI.2024.3438349

As one of the approaches for uncertainty reasoning, Dempster-Shafer evidence theory [4], [5], also known as evidence theory, stands out as a robust method that relies less on prior probabilities, efficiently aggregates different pieces of information, and assesses the probability of events based on belief and plausibility. Since evidence theory was first proposed in 1967, many researchers have been promoting the development of evidence theory. Cuzzolin proposed a geometric approach to handle the uncertainty in evidence theory [6]. To measure the uncertainty of a mass function, Deng proposed Deng entropy [7], also called belief entropy, which is further developed into information volume of mass function [8]. Deng and Jiang presented a quantum representation of mass function, which provides a connection between mass function and mixed quantum states [9]. By incorporating mass function with complex value, complex evidence theory has been reported in [10], [11]. Deng introduced the concept of order and permutation into evidence theory and proposed random permutation set theory [12]. Due to the efficiency in processing uncertain information, evidence theory has many applications, such as decision making [11], [13], data fusion [14], [15], uncertainty measures [16], [17], fractals [18], [19], target classification [20], expert systems [21], and incomplete pattern clustering [22].

In 1994, Yang and Singh proposed the evidential reasoning (ER) approach [23], which can be used for knowledge representation and weighted evidence fusion in hybrid decision making problems. Then, Yang and Sen developed a general multi-level evaluation process based on ER approach [24], in which the general form of ER algorithm were explored. In 2013, Yang and Xu extended ER algorithm into ER rule [25], which combines weighted belief distribution with reliability by considering both evidence weight and evidence reliability. Besides, ER rule can well solve the counter-intuitive problem of Dempster's rule of combination in the situation of high conflict. Also, ER rule establishes a generic conjunctive probabilistic reasoning process, which is a generalized form of Dempster's rule [4] and the original ER algorithm [23], [24]. ER approach and ER rule has been applied in many fields, like data fusion [25], [26], [27], [28], classification [29], [30], [31], decision making [23], [32], [33], [34], fault diagnosis [35], [36], medical diagnosis [37], expert system [38], [39], [40], and state assessment [41], [42].

Nevertheless, the concept of *order* is not considered in evidence theory and ER approach, which involves two aspects [12]. (i) *Order in data representation (element order):* In evidence theory and ER approach, because all propositions in a power set are defined by sets, the hypotheses within a proposition of a power set have no concept of order. For example, the propositions $\{\theta_2, \theta_3\}$ and $\{\theta_3, \theta_2\}$ are completely equivalent, because there is no order of precedence between the hypotheses θ_2 and θ_3. (ii) *Order in fusion rule (fusion order):* Since most of the original evidence fusion rules, such as Dempster's rule [4], Yager's rule [43], ER algorithm [23], and ER rule [25], are based on the orthogonal sum operation, they satisfy the commutative and associative laws. In other words, changing the order of evidence combination does not affect the final fusion results, which means that most evidence fusion rules do not take concept of order into consideration.

The concept of order is useful in some cases, especially when there exists relative importance between hypotheses or relative reliability between information sources [12]. Taking the scenario of product sales as an example, a company has developed three prototypes for a product, denoted by $\Theta = \{\theta_1, \theta_2, \theta_3\}$, which have different profits and degrees of importance. Four product managers with different professional levels need to choose the most profitable prototype from the three as the final release version of the product. Given two prototypes θ_2 and θ_3, if a product manager is uncertain about which product is more suitable, then according to the idea of evidence theory, the associated belief degree can be assigned to the proposition $\{\theta_2, \theta_3\}$. If the product θ_3 is more important than θ_2, how would this situation be represented? Actually, this relative importance can be represented by the order of the hypotheses, which involves the concept of sets with order information, namely orderable sets [44], [45]. In mathematics, a tuple is a ordered sequence of elements and can be view as an orderable set, based on which, the situation can be represented by (θ_3, θ_2). The reports of the four product managers need to be fused to facilitate the final decision. Because each manager has a different professional level, their reports need to be fused in a particular order. Generally speaking, the more professional reports should be considered first and have a greater impact on the final fusion results, which requires a fusion rule that does not satisfy the communicative law [12].

By considering the two aforementioned aspects of order, Deng extended evidence theory and proposed random permutation set (RPS) theory based on orderable sets and permutations [12]. An RPS consists of a permutation event space (PES) and a permutation mass function (PMF). The PES of a frame of discernment Θ considers all the possible permutations of Θ, named as permutation events. Another important concept of RPS theory is the fusion rule of RPS sources, called permutation orthogonal sum (POS), including two types of rules: right orthogonal sum (ROS) and left orthogonal sum (LOS). RPS theory can be viewed as an orderable extension of evidence theory. If the order of the element in permutation event is ignored, PES, PMF, and POS will degenerated into power set, BPA, and Dempster's rule of combination, respectively. In RPS theory, permutation serves to manage concepts related to order, which is also the difference between RPS theory and the original evidence theory. Specifically, RPS theory incorporates permutation events to model the order of hypotheses, and utilizes POS to address the issue in evidence fusion rules where the fusion order is overlooked. Although several methods have been proposed to deal with permutations [46] and orderable sets [44], [45], their direct applications in evidence theory are often compromised by compatibility issues. RPS theory successfully bridges this gap, utilizing PES, PMF, and POS to provide a streamlined and direct mechanism for managing permutations and modeling order-related concepts. Consequently, RPS theory enhances the practicality for real-world scenarios, especially when order information needs to be taken into account. Up till now, RPS theory has been applied in risk analysis [12], uncertainty measure [47], [48], and decision making [12].

As it stands however, RPS theory suffers from two major deficiencies: (i) Although the existing RPS theory takes element order into consideration, it lacks an efficient mechanism to convert this order into PMFs; (ii) The existing RPS theory requires a predefined fusion order to combine information sources based on the POS, but this is often impractical since the fusion order is usually unknown. The idea of evidence weight and evidence reliability from Smarandache et al.'s work [49] and ER rule [25], [50] imply a sense of relative order and may provide potential solutions to the above two problems, whereas they cannot be directly and effectively used in RPS theory.

In this paper, a reasoning model of RPS theory, called random permutation set reasoning (RPSR), is proposed to address the two aforementioned problems. RPSR includes RPS generation method (RPSGM), RPSR rule of combination, and ordered probability transformation (OPT). Specifically, RPSGM efficiently handles element order and constructs RPSs using Gaussian discriminant model and weight analysis, which determines the relative importance of elements to notably enhance RPS generation; RPSR rule incorporates POS with reliability vector, which can combine RPS sources with reliability in the optimal fusion order, thus facilitating the information fusion process; OPT is used to convert RPS into a probability distribution for the final decision. In addition, three numerical examples are shown to illustrate the proposed RPSR. Furthermore, the proposed RPSR is applied to classification problems and an RPSR-based classification algorithm (RPSRCA) is presented. The experimental results show RPSRCA's superiority over other classifiers in terms of accuracy and stability.

The rest of this article is as follows. Section II introduces some preliminaries. Section III proposes RPSR, including RPSGM, RPSR rule, and OPT. Section IV presents some numerical example for illustration. Section V designs a classification algorithm based on RPSR, which is further verified by several experiments. Section VI makes a brief conclusion.

Abbreviations: For clarity, several abbreviations in this paper are summarized as follows: Frame Of Discernment (FOD), Basic Probability Assignment (BPA), Evidential reasoning (ER), Random Permutation Set (RPS), Random permutation Set Reasoning (RPSR), Permutation Mass Function (PMF), Permutation Orthogonal Sum (POS), RPS Generation Method (RPSGM), Ordered Probability Transformation (OPT), and Gaussian Discriminant Model (GDM).

II. Preliminaries

A. Dempster-Shafer Evidence Theory

Dempster-Shafer evidence theory [4], [5], also called evidence theory, is an efficient tool for dealing with uncertainty [7], [23] and combining information sources [51], [52], [53]. In evidence theory, *Frame of discernment* (FOD) is a set of N mutually exclusive and exhaustive elements, denoted by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. The *power set* of Θ consists of all possible subsets of Θ, indicated by $2^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, \ldots, \{\theta_N\}, \{\theta_1, \theta_2\}, \ldots, \{\theta_1, \theta_N\}, \ldots, \Theta\}$. A *basic probability assignment* (BPA) is a mapping function $m : 2^\Theta \to [0, 1]$, which is constrained by $m(\emptyset) = 0$ and $\sum_{A\in 2^\Theta} m(A) = 1$. Given two BPAs m_1 and m_2, *Dempster's rule of combination* $m_1 \oplus m_2$ is defined by

$$m(A) = \begin{cases} \frac{1}{1-K} \sum_{B\cap C=A} m_1(B)m_2(C), & A \neq \emptyset \\ 0, & A = \emptyset \end{cases} \tag{1}$$

where $A, B, C \in 2^\Theta$ and $K = \sum_{B\cap C=\emptyset} m_1(B)m_2(C)$.

B. Evidential Reasoning Rule

Evidential reasoning (ER) approach and ER rule are powerful methods for decision making [23], [24], [54] and expert system [55], [56]. In this subsection, several basic concepts of evidential reasoning (ER) rule [25] are introduced.

Assume the FOD is denoted by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. A piece of evidence is indicated by *belief distribution:*

$$e_j = \left\{(\theta, p_{\theta,j}) \mid \forall\theta \subseteq \Theta, \sum\nolimits_{\theta\subseteq\Theta} p_{\theta,j} = 1\right\} \tag{2}$$

where θ is a subset of Θ, $p_{\theta,j}$ is the *belief degree* on proposition θ, and $(\theta, p_{\theta,j})$ is an element of evidence e_j.

The ER rule takes evidence weight and evidence reliability into consideration [25]. Let the weight and reliability of evidence e_j be represented by w_j and r_j, respectively. The *weighted belief distribution with reliability* is defined by:

$$m_j = \left\{(\theta, \tilde{m}_{\theta,j}) \,|\forall\theta \subseteq \Theta; \left(P\left(\Theta\right), \tilde{m}_{P(\Theta),j}\right)\right\} \tag{3}$$

where $\tilde{m}_{\theta,j}$ is referred to as the *basic probability mass* for θ from e_j with weight w_j and reliability r_j, calculated by:

$$\tilde{m}_{\theta,j} = \begin{cases} 0, & \theta = \emptyset \\ c_{rw,j} m_{\theta,j}, & \theta \subseteq \Theta, \theta \neq \emptyset \\ c_{rw,j}\left(1 - r_j\right), & \theta = P\left(\Theta\right) \end{cases} \tag{4}$$

where $m_{\theta,j} = w_j p_{\theta,j}$ is called the *weighted belief distribution* and $c_{rw,j} = (1 + w_j - r_j)^{-1}$ the normalization factor.

Given two pieces of independent evidence e_1 and e_2, the combined belief degree $p_{\theta,e(2)}$ that e_1 and e_2 jointly support θ can be established based on:

$$p_{\theta,e(2)} = \begin{cases} 0, \theta = \emptyset \\ \frac{\hat{m}_{\theta,e(2)}}{\sum_{D\subseteq\Theta} \hat{m}_{D,e(2)}}, \theta \subseteq \Theta, \theta \neq \emptyset \end{cases}$$

$$\hat{m}_{\theta,e(2)} = \left[\left(1 - r_2\right) m_{\theta,1} + \left(1 - r_1\right) m_{\theta,2}\right] + \sum_{B\cap C=\theta} m_{B,1} m_{C,2} \tag{5}$$

where r_j is the reliability of e_j.

C. Random Permutation Set Theory

Random permutation set theory is distinct from the original evidence theory, principally owing to its integration of permutations to effectively handle concepts related to order. In the process of data representation and information fusion, there may exist implicit order-related information, such as the relative importance among different elements and the relative reliability of different information sources. These concepts of order may facilitate more nuanced reasoning and enhance the practicality of classification. RPS theory utilizes permutation events to model the order of hypotheses, and employs POS to address the issue in the original evidence fusion rules where the fusion order is overlooked. By deploying PES, PMF, and POS, RPS theory presents a comprehensive approach for managing permutations and modeling order-related concepts, thereby enhancing its practicality, especially in situations when order information is critical. This subsection briefly reviews some definitions about random permutation set theory [12].

Let the FOD be denoted as $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. The *permutation event space* (PES) of Θ is defined as:

$$\begin{aligned} PES\left(\Theta\right) &= \{\, A_{ij} \,|\, i = 0, \ldots, N;\ j = 1, \ldots, P(N, i) \,\} \\ &= \{\emptyset, \left(\theta_1\right), \left(\theta_2\right), \ldots, \left(\theta_N\right), \left(\theta_1, \theta_2\right), \left(\theta_2, \theta_1\right), \ldots, \left(\theta_{N-1}, \theta_N\right), \\ &\left(\theta_N, \theta_{N-1}\right), \ldots, \left(\theta_1, \theta_2, \ldots, \theta_N\right), \ldots, \left(\theta_N, \theta_{N-1}, \ldots, \theta_1\right)\} \end{aligned} \tag{6}$$

which contains all possible permutations of Θ. $P(N, i) = \frac{N!}{(N-i)!}$ is the i-permutation of N. In PES, the element A_{ij} is a tuple called *permutation event* (PE).

Given a FOD Θ, its *random permutation set* (RPS) is a set of pairs defined by:

$$RPS\left(\Theta\right) = \{\langle A, \mathscr{M}(A)\rangle \,|\, A \in PES\left(\Theta\right)\} \tag{7}$$

where $\mathscr{M}$ is called the *permutation mass function* (PMF), which is mapping function $\mathscr{M} : PES(\Theta) \to [0, 1]$, satisfying $\mathscr{M}(\emptyset) = 0$ and $\sum_{A\in PES(\Theta)} \mathscr{M}(A) = 1$. An RPS source $RPS_i(\Theta)$ can be written as RPS_i when the FOD is defined and there is no ambiguity.

Permutation orthogonal sum (POS) is the fusion rule of PMFs, including two types of rules: *right orthogonal sum* (ROS) and *left orthogonal sum* (LOS). Let $X\backslash\backslash Y$ represent removing Y from X. Given two PMFs $\mathscr{M}_1$ and $\mathscr{M}_2$, ROS and LOS are defined as follows:

- ROS is denoted as $\mathscr{M}_1 \overrightarrow{\oplus} \mathscr{M}_2$ and calculated by:

$$\mathscr{M}^R(A) = \begin{cases} \frac{1}{1-\overrightarrow{K}} \sum\limits_{B\overrightarrow{\sqcap}C=A} \mathscr{M}_1(B)\mathscr{M}_2(C), & A \neq \emptyset \\ 0, & A = \emptyset \end{cases} \tag{8}$$

where $\overrightarrow{K} = \sum_{B\overrightarrow{\sqcap}C=\emptyset} \mathscr{M}_1(B)\mathscr{M}_2(C)$; $A, B, C \in PES(\Theta)$; $\overrightarrow{\sqcap}$ indicates *right intersection:* $B\overrightarrow{\sqcap}C = C\backslash\backslash \bigcup_{\theta\in C,\theta\notin B} \{\theta\}$.

- LOS is denoted as $\mathscr{M}_1 \overleftarrow{\oplus} \mathscr{M}_2$ and calculated by:

$$\mathscr{M}^L(A) = \begin{cases} \frac{1}{1-\overleftarrow{K}} \sum\limits_{B\overleftarrow{\sqcap}C=A} \mathscr{M}_1(B)\mathscr{M}_2(C), & A \neq \emptyset \\ 0, & A = \emptyset \end{cases} \tag{9}$$

where $\overleftarrow{K} = \sum_{B\overleftarrow{\sqcap}C=\emptyset} \mathscr{M}_1(B)\mathscr{M}_2(C)$; $A, B, C \in PES(\Theta)$; $\overleftarrow{\sqcap}$ indicates *left intersection:* $B\overleftarrow{\sqcap}C = B\backslash\backslash \bigcup_{\theta\in B,\theta\notin C} \{\theta\}$.

III. Random Permutation Set Reasoning

In this section, several basic concepts of random permutation set reasoning (RPSR) are proposed, including RPS generation method, RPSR rule of combination, and ordered probability transformation.

A. RPS Generation Method

The reasoning based on RPS theory starts with generating RPS source. In particular, a suitable generation method for element order of PMF is needed. This subsection proposes RPS generation method (RPSGM), which is based on Gaussian discriminant model and weight analysis.

Assume that the original dataset, containing N classes, is denoted by FOD $\Theta = \{\theta_i \,|\, i = 1, 2, \ldots, N\}$. The original dataset is divided into training set and test set. In the training set, the training sample of class θ_i with K attributes is represented by vector $\mathbf{x}_i = [x_{ij} \,|\, j = 1, 2, \ldots, K]$. The test set comprises multiple test samples. For the purposes of our analysis, a single test sample is represented by a vector denoted as $\mathbf{x}_0 = [x_{0j} \,|\, j = 1, 2, \ldots, K]$. Then, RPSGM is detailed as follows.

Step 1: Establish Gaussian discriminant model (GDM), and then construct membership vector based on the GDM.

i) Let the number of training samples in the class θ_i be N_i, and the l-th training sample in the class θ_i be $\mathbf{x}_i^{(l)}$. Calculate the mean value and the standard deviation for the j-th attribute of the training samples in class θ_i:

$$\bar{x}_{ij} = \frac{1}{N_i}\sum_{l=1}^{N_i} x_{ij}^{(l)} \tag{10}$$

$$\sigma_{ij} = \sqrt{\frac{1}{N_i - 1}\sum\nolimits_{l=1}^{N_i}\left(x_{ij}^{(l)} - \bar{x}_{ij}\right)^2} \tag{11}$$

ii) For the j-th attribute of training samples in the training set, build the corresponding GDM based on $\bar{x}_{ij}$ and σ_{ij}:

$$\mathbf{f}_j(x) = [f_{ij}(x) \,|\, i = 1, 2, \ldots, N] \tag{12}$$

where $j = 1, 2, \ldots, K$ and $f_{ij}(x)$ is Gaussian distribution:

$$f_{ij}(x) = \frac{1}{\sqrt{2\pi{\sigma_{ij}}^2}}\exp\left(-\frac{(x - \bar{x}_{ij})^2}{2{\sigma_{ij}}^2}\right) \tag{13}$$

iii) For the j-th attribute of a given test sample $\mathbf{x}_0 = [x_{0j} \,|\, j = 1, 2, \ldots, K]$, the associated membership vector (MV) is denoted as:

$$\mathbf{f}_j = [f_{ij} \,|\, i = 1, 2, \ldots, N] \tag{14}$$

where $j = 1, 2, \ldots, K$ and $f_{ij} \triangleq f_{ij}(x)|_{x=x_{0j}}$ is the membership degree between the GDM and the test sample.

Step 2: Perform weight analysis for the test sample.

i) Normalize the MV based on $\tilde{f}_{ij} = \frac{f_{ij}}{\sum_i f_{ij}}$, and then obtain the normalized normalized membership (NMV):

$$\tilde{\mathbf{f}}_j = [\tilde{f}_{ij} \,|\, i = 1, 2, \ldots, N] \tag{15}$$

where $j = 1, 2, \ldots, K$.

ii) Rank the elements $\tilde{f}_{ij}|_{i=1}^{N}$ of the j-th NMV in decreasing order $\tilde{f}_{i_u j}^{Ord}|_{u=1}^{N}$, and then get ordered normalized membership vector (ONMV):

$$\tilde{\mathbf{f}}_j^{Ord} = [\tilde{f}_{i_u j}^{Ord} \,|\, u = 1, 2, \ldots, N] \tag{16}$$

where $j = 1, 2, \ldots, K$.

iii) For the elements of ONMV $\tilde{f}_{i_u j}^{Ord}|_{u=1}^{N}$, let the corresponding GDMs be $\tilde{f}_{i_u j}^{Ord}(x)|_{u=1}^{N}$ and the associated mean values of GDMs be $\bar{x}_{i_u j}^{Ord}|_{u=1}^{N}$. Calculate supporting degree of the u-th GDM to the j-th attribute of the test sample:

$$s_{i_u j} = \exp\left(-\left|x_{0j} - \bar{x}_{i_u j}^{Ord}\right|\right) \tag{17}$$

where $u = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, K$. In principle, a higher supporting degree indicates a closer match between the GDM's value and the observed attribute value in the test sample, suggesting stronger support by the GDM for this particular attribute.

iv) Conduct weight analysis for the test sample, and then obtain weight vector for the element order in the j-th attribute:

$$\mathbf{w}_j = \left[w_{(i_1\ldots i_u\ldots i_q)j} \;\middle|\; \begin{matrix} q = 1, 2, \ldots, N \\ u = 1, 2, \ldots, q \\ (i_1 \ldots i_u \ldots i_q) \in APS(q) \end{matrix} \right] \tag{18}$$

where $j = 1, 2, \ldots, K$ and $w_{(i_1\ldots i_u\ldots i_q)j}$ is the weight factor describing the relative importance of all the permutation events in the j-th RPS:

$$w_{(i_1\ldots i_u\ldots i_q)j} = \prod_{u=1}^{q}\frac{s_{i_u j}}{\sum_{t=u}^{q} s_{i_t j}} \tag{19}$$

$APS(\cdot)$ is the all permutation space (APS) defined by:

$$APS(X) \triangleq \{(12\ldots X), \ldots, (21\ldots X), \ldots, (X\ldots 21)\} \tag{20}$$

which contains all the possible permutation from 1 to $X \geq 1$.

Step 3: Construct weighted PMF based on weight vector and ONMV, and then generate weighted RPS.

i) Let the corresponding classes for the elements of ONMV $\tilde{f}_{i_u j}^{Ord}|_{u=1}^{N}$ be denoted as $\theta_{i_u}^{Ord}|_{u=1}^{N}$. For the j-th attribute of the test sample, build weighted PMF $\mathscr{M}_j^{\mathbf{w}}$ based on the weight vector $\mathbf{w}_j$ and the ONMV $\tilde{\mathbf{f}}_j^{Ord}$:

$$\mathscr{M}_j^{\mathbf{w}}\left(\theta_{i_1}^{Ord}, \ldots, \theta_{i_u}^{Ord}, \ldots, \theta_{i_q}^{Ord}\right) =$$

$$\begin{cases} w_{(i_1\ldots i_u\ldots i_q)j}\cdot \tilde{f}^{Ord}_{i_qj}, & (i_1\ldots i_u\ldots i_q)\in APS(q)\\ 0, & (i_1\ldots i_u\ldots i_q)\notin APS(q)\end{cases} \tag{21}$$

where $j=1,2,\ldots,K$; $q=1,2,\ldots,N$; $u=1,2,\ldots,q$; $APS(\cdot)$ is defined in (20).

ii) For the j-th attribute of the test sample, generate weighted RPS based on the j-th weighted PMF:

$$RPS^{\mathbf{w}}_j=\left\{\langle A,\mathscr{M}^{\mathbf{w}}_j(A)\rangle\,|\,A\in PES\,(\Theta)\right\} \tag{22}$$

where $j=1,2,\ldots,K$.

Our RPSGM is based on the GDM due to its efficacy in different scenarios, especially when data is normally distributed. For non-normal data, it is recommended to use the Student-t distribution model. In network science, the Power-law distribution model could be a viable alternative.

B. RPSR Rule of Combination

As mentioned in the introduction part, the original RPS theory has not treated the fusion order of POS in detail, and its reliance on a predefined fusion order hinders its practicality when such order is not explicitly known. Additionally, POS assumes that all RPS sources are fully reliable, which can be problematic in the practical use of RPS theory. To resolve these practical challenges, we propose RPSR rule of combination, which integrates a reliability vector into the POS and employs a novel hyperparameter tuning method to determine the fusion order, thus enabling the combination of RPS sources with reliability in the fusion order.

Given K RPS sources $RPS_j|_{j=1}^K$ defined on FOD Θ, the RPSR rule of combination is proceed as follows.

Step 1: Let fusion order of the RPS sources be:

$$\mathscr{F}=(j_1\ldots j_v\ldots j_K)\in APS(K) \tag{23}$$

Fusion order indicates the order in which all RPS sources are combined. Rank the K RPS sources $RPS_j|_{j=1}^K$ according to $\mathscr{F}$, and then obtain the ranked RPS sources $RPS_{j_v}|_{v=1}^K$. The fusion order is one of the hyperparameters of RPSR rule, which will be optimized in Section V-B.

Step 2: Let reliability vector be denoted as

$$\mathbf{r}=[r_v\,|\,v=1,\,2,\ldots,K] \tag{24}$$

$$=\begin{cases}[r_v\,|r_1\geq r_2\geq\ldots\geq r_v\geq\ldots\geq r_K]\\ \quad(descending\ order)\\ [r_v\,|r_1\leq r_2\leq\ldots\leq r_v\leq\ldots\leq r_K]\\ \quad(ascending\ order)\end{cases} \tag{25}$$

where $r_v\in[0,1]$ is called the reliability factor, representing the reliability of the v-th ranked RPS source RPS_{j_v}. The higher the reliability factor, the more reliable the RPS source is. It can be seen from (25), the reliability factors of $\mathbf{r}$ can be in two types of order, *i.e.*, descending order and ascending order. The reliability vector is also a hyperparameter of RPSR rule, which will be further determined in Section V-B.

Step 3: For each RPS source RPS_{j_v}, produce RPS with reliability $RPS^{\mathbf{r}}_{j_v}=\{\langle A,\mathscr{M}^{\mathbf{r}}_{j_v}(A)\rangle\,|\,A\in PES(\Theta)\}$ based on PMF with reliability, calculated by:

$$\mathscr{M}^{\mathbf{r}}_{j_v}(A)=\begin{cases} r_v\cdot\mathscr{M}_{j_v}(A), & |A|=1\\ r_v\cdot\mathscr{M}_{j_v}(A)+\frac{(1-r_v)}{F(|\Theta|)-|\Theta|-1}, & |A|>1\end{cases} \tag{26}$$

where $v=1,\,2,\ldots,K$; r_v is the v-th reliability factor of $\mathbf{r}$; $F(x)\triangleq\sum_{k=0}^{x}\frac{x!}{(x-k)!}$ is the sum of all the permutation number of x elements.

Step 4: According to the order of the reliability factors in $\mathbf{r}$, combine the K PMFs with reliability $\mathscr{M}^{\mathbf{r}}_{j_v}|_{v=1}^K$ based on LOS $\overleftarrow{\oplus}$ or ROS $\overrightarrow{\oplus}$ [12], and then obtain the fused PMF:

$$\widehat{\mathscr{M}}=\begin{cases}\overleftarrow{\bigoplus}_{v=1}^K\mathscr{M}^{\mathbf{r}}_{j_v} & (descending\ order)\\ \overrightarrow{\bigoplus}_{v=1}^K\mathscr{M}^{\mathbf{r}}_{j_v} & (ascending\ order)\end{cases} \tag{27}$$

where LOS and ROS correspond to the fusion method of descending order and ascending order, respectively.

Step 5: Based on the fused PMF, output the fused RPS of the K RPS sources:

$$\widehat{RPS}=\left\{\langle A,\widehat{\mathscr{M}}(A)\rangle\,|\,A\in PES\,(\Theta)\right\} \tag{28}$$

It should be noted that the fusion order $\mathscr{F}$ in (23) and the reliability vector $\mathbf{r}$ in (24) are two hyperparameters to be determined. In Section V-B, a hyperparameter tuning method for optimizing $\mathbf{r}$ and $\mathscr{F}$ will be presented.

C. Ordered Probability Transformation

In the evidence theory-based classification algorithm [14] and decision-making model [11], [20], the final step needs to transform the mass function to facilitate discrimination or decision-making. Typically, the mass function is converted into a probability distribution.

In the model of RPSR, it is needed to design a transformation method of RPS source. In this subsection, ordered probability transformation (OPT) is proposed to convert PMF of RPS source into a probability distribution. Given an RPS source $RPS(\Theta)=\{\langle A,\mathscr{M}(A)\rangle\,|\,A\in PES(\Theta)\}$ defined on FOD Θ, OPT is defined as:

$$P_{OPT}\,(\theta)=\mathscr{M}\,(\theta)+\sum_{\substack{\theta\in A\in PES\,(\Theta)\\ Last(A)\neq\theta\\ |A|>1}}\frac{\mathscr{M}(A)}{|A|-1} \tag{29}$$

where $\theta\in\Theta$ and $Last(A)$ denotes the last element of the permutation event A. For example, assume a permutation event be $A=(\theta_2,\theta_3,\theta_1,\theta_4)$. Then, $Last(A)$ is the element θ_4.

To summarize, RPSR is constructed by the above three techniques: RPSGM, RPSR rule of combination, and OPT, whose flowcharts are illustrated in Fig. 1(a), (b), and (c) respectively.

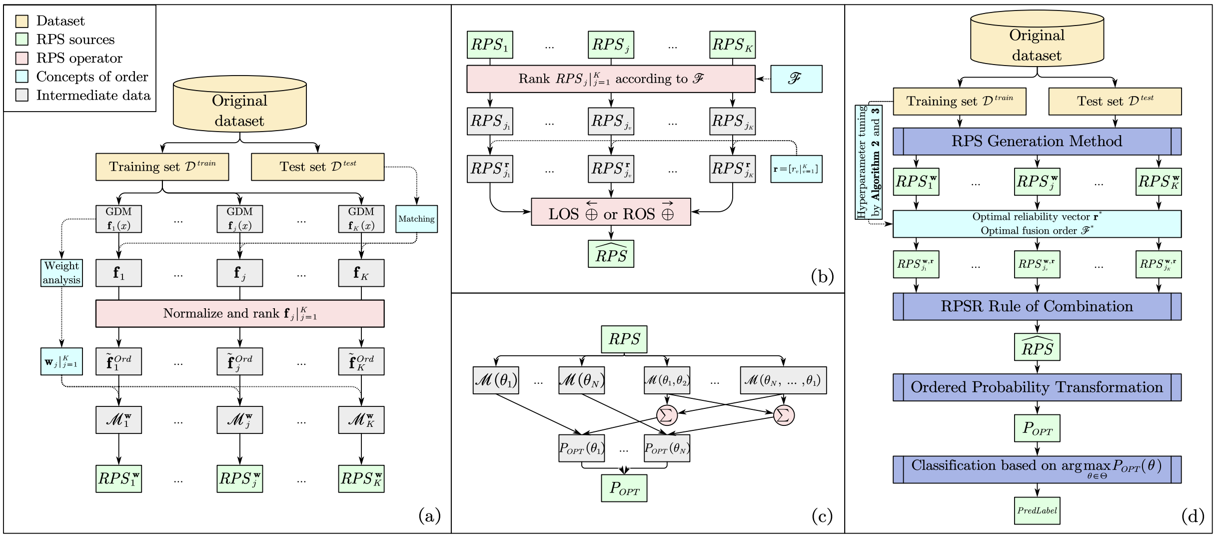

Fig. 1. Illustration of the proposed random permutation set reasoning (RPSR) framework, where (a) is the RPS generation method (RPSGM), (b) is the RPSR rule of combination, (c) is the Ordered probability transformation (OPT), and (d) is the RPSR classification algorithm (RPSRCA).

TABLE I
MEAN VALUE AND STANDARD DEVIATION OF EACH GDM

$\bar{x}_{ij}$	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	5.0118	3.3941	1.4853	0.2500
$i=2$	5.9559	2.7765	4.2794	1.3324
$i=3$	6.5882	2.9735	5.5147	2.0176
σ_{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0.3073	0.3626	0.1828	0.1108
$i=2$	0.4807	0.2955	0.4740	0.1965
$i=3$	0.6009	0.3078	0.5338	0.2938

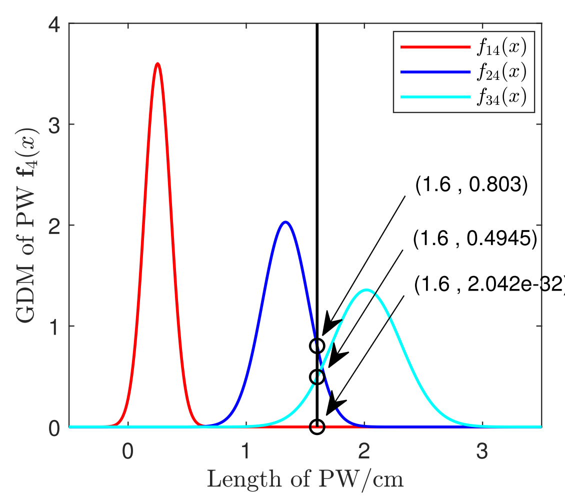

Fig. 2. Illustration of the 4-th GDM.

IV. NUMERICAL EXAMPLES

In this section, some numerical examples based on the Iris dataset are provided to demonstrate the calculation procedure of the proposed RPSR. In the Iris dataset, there are three classes, namely Setosa, Versicolour, and Virginica, which can be denoted by $\Theta = \{\theta_i \,|\, i = 1, 2, 3\}$. Each sample of the dataset has four attributes, *i.e.*, sepal length (SL), sepal width (SW), petal length(PL), and petal width (PW). The sample of the i-th class is indicated by $\mathbf{x}_i = [x_{ij} \,|\, j = 1, 2, 3, 4]$. The original dataset is randomly split into training set and test set.

Example 1: Assume a test sample from the Iris dataset be $\mathbf{x}_0 = [6.3,\ 3.3,\ 4.7,\ 1.6]$. The corresponding RPS sources of the test sample can be generated based on RPSGM:

Step 1: Based on the training set, calculate the mean value and standard deviation of the j-th attribute in class θ_i, which are listed in Table I. Then, establish GDMs $\mathbf{f}_j(x)|_{j=1}^4$ for every attributes. For each attribute of the test sample, obtain the associated MV. Take the 4-th attribute of the test sample $x_{04} = 1.6$ cm as an example. Based on the 4-th GDM (shown in Fig. 2), the MV can be acquired: $\mathbf{f}_4 = [2.04\text{e-}32,\ 0.8030,\ 0.4945]$.

Step 2: Take the 4-th attribute of the test sample as an example. Normalize the MV and calculate NMV: $\tilde{\mathbf{f}}_4 = [\tilde{f}_{i4} \,|\, i = 1, 2, 3] = [1.57\text{e-}32,\ 0.6189,\ 0.3811]$. Then, rank $\tilde{f}_{i4}|_{i=1}^3$ in decreasing order and get ONMV: $\tilde{\mathbf{f}}_4^{Ord} = [\tilde{f}_{i_u4}^{Ord} \,|\, u = 1, 2, 3] = [0.6189,\ 0.3811,\ 1.57\text{e-}32]$. Based on ONMV, the associated classes $\theta_{i_u}^{Ord}|_{u=1}^3$ and mean values of GDMs $\bar{x}_{i_u4}^{Ord}|_{u=1}^3$ can be obtained. Next, calculate the supporting degree to the 4-th attribute: $s_{i_u4}|_{u=1}^3 = [0.7652,\ 0.6586,\ 0.2592]$. Furthermore, perform weight analysis and acquire the weight vector based on (18) to (20). For the Iris dataset, the equations can be simplified as:

$$w_{(i_1\ldots i_u\ldots i_q)j} = \begin{cases} w_{(i_1)j} = \dfrac{s_{i_1j}}{s_{i_1j}}, & (i_1) \in APS(1) \\ w_{(i_1i_2)j} = \dfrac{s_{i_1j}}{\sum_{t=1}^{2} s_{i_tj}} \cdot \dfrac{s_{i_2j}}{s_{i_2j}}, & (i_1i_2) \in APS(2) \\ w_{(i_1i_2i_3)j} = \dfrac{s_{i_1j}}{\sum_{t=1}^{3} s_{i_tj}} \cdot \dfrac{s_{i_2j}}{\sum_{t=2}^{3} s_{i_tj}} \cdot \dfrac{s_{i_3j}}{s_{i_3j}}, & (i_1i_2i_3) \in APS(3) \end{cases} \tag{30}$$

TABLE II
SUMMARY OF SEVERAL TERMS FROM **EXAMPLE 1**

	$i=2$	$i=3$	$i=1$
Original class θ_i	θ_2	θ_3	θ_1
Original MV f_{i4}	0.8030	0.4945	2.04e-32
Original NMV $\tilde{f}_{i4}$	0.6189	0.3811	1.57e-32
	$u=1$	$u=2$	$u=3$
ONMV $\tilde{f}^{Ord}_{i_u 4}$	0.6189	0.3811	1.57e-33
Associated class $\theta^{Ord}_{i_u}$	$\theta^{Ord}_{i_1}$	$\theta^{Ord}_{i_2}$	$\theta^{Ord}_{i_3}$
Associated $\bar{x}^{Ord}_{i_u 4}$	1.3324	2.0176	0.2500
Associated $s_{i_u 4}$	0.7652	0.6586	0.2592

TABLE III
WEIGHT FACTORS AND PMFS OF THE 4-TH ATTRIBUTE OF THE TEST SAMPLE FROM **EXAMPLE 1**

$(i_1 \ldots i_q)$	Associated PEs	Original PEs	$w_{(\cdot)4}$	$\mathscr{M}^{\mathbf{w}}_4(\cdot)$
(i_1):	$(\theta^{Ord}_{i_1})$:			
(1)	(θ^{Ord}_1)	(θ_2)	1.0000	0.6189
$(i_1 i_2)$:	$(\theta^{Ord}_{i_1}, \theta^{Ord}_{i_2})$:			
$(1,2)$	$(\theta^{Ord}_1, \theta^{Ord}_2)$	(θ_2, θ_3)	0.5374	0.2048
$(2,1)$	$(\theta^{Ord}_2, \theta^{Ord}_1)$	(θ_3, θ_2)	0.4626	0.1763
$(i_1 i_2 i_3)$:	$(\theta^{Ord}_{i_1}, \theta^{Ord}_{i_2}, \theta^{Ord}_{i_3})$:			
$(1,2,3)$	$(\theta^{Ord}_1, \theta^{Ord}_2, \theta^{Ord}_3)$	$(\theta_2, \theta_3, \theta_1)$	0.0828	5.13e-33
$(1,3,2)$	$(\theta^{Ord}_1, \theta^{Ord}_3, \theta^{Ord}_2)$	$(\theta_2, \theta_1, \theta_3)$	0.0713	2.02e-33
$(2,1,3)$	$(\theta^{Ord}_2, \theta^{Ord}_1, \theta^{Ord}_3)$	$(\theta_3, \theta_2, \theta_1)$	0.1284	4.60e-33
$(2,3,1)$	$(\theta^{Ord}_2, \theta^{Ord}_3, \theta^{Ord}_1)$	$(\theta_3, \theta_1, \theta_2)$	0.3262	1.56e-33
$(3,1,2)$	$(\theta^{Ord}_3, \theta^{Ord}_1, \theta^{Ord}_2)$	$(\theta_1, \theta_2, \theta_3)$	0.0990	1.30e-33
$(3,2,1)$	$(\theta^{Ord}_3, \theta^{Ord}_2, \theta^{Ord}_1)$	$(\theta_1, \theta_3, \theta_2)$	0.2923	1.12e-33

According to (30), the weight factors $w_{(i_1 \ldots i_u \ldots i_q)4}$ for the 4-th attribute of the test sample are calculated in Table III. For better understanding, several terms in Step 2 are summarized and compared in Tables II and III.

Step 3: The associated classes $\theta^{Ord}_{i_u}|^3_{u=1}$ for the elements of ONMV $\tilde{f}^{Ord}_{i_u j}|^3_{u=1}$ are shown in Table II. Based on the weight vector and the ONMV shown in Table II, build weighted PMF for the 4-th attribute of the test sample according to: (31), shown at the bottom of this page. The results of the weighted PMF are listed in Table III. Based on the weighted PMF, construct RPS source for the 4-th attribute $RPS^{\mathbf{w}}_4$. Next, based on Step 1 and Step 2, generate weighted RPS sources $RPS^{\mathbf{w}}_j|^4_{j=1}$ for all the attribute of the test sample, which are summarized in Table IV(a).

Example 2: Suppose RPSGM has generated four weighted RPS sources, as shown in Table IV(a). These RPS sources can be fused based on the RPSR rule of combination:

Step 1: Assume that the fusion order of the RPS sources is $\mathscr{F} = (j_1 \ldots j_v \ldots j_K) = (4,3,1,2)$. Then, rank the given RPS sources $RPS^{\mathbf{w}}_j|^K_{j=1}$ and acquire the ranked weighted RPS sources $RPS^{\mathbf{w}}_{j_v}|^K_{v=1}$:

$$RPS^{\mathbf{w}}_{j_1} = RPS^{\mathbf{w}}_4;\ RPS^{\mathbf{w}}_{j_2} = RPS^{\mathbf{w}}_3;$$
$$RPS^{\mathbf{w}}_{j_3} = RPS^{\mathbf{w}}_1;\ RPS^{\mathbf{w}}_{j_4} = RPS^{\mathbf{w}}_2. \tag{32}$$

Step 2: Let the reliability vector be in descending order: $\mathbf{r} = [r_v | r_v = 1 - \frac{(v-1)}{12}; v = 1,2,3,4] = [1, \frac{11}{12}, \frac{5}{6}, \frac{3}{4}]$.

Step 3: Based on weighted RPS source $RPS^{\mathbf{w}}_{j_v}$ and reliability factor r_v, construct weighted RPS with reliability $RPS^{\mathbf{w},\mathbf{r}}_{j_v} = \{\langle A, \mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_v}(A)\rangle \,|\, A \in PES(\Theta)\}$ by:

$$\mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_v}(A) = \begin{cases} r_v \cdot \mathscr{M}^{\mathbf{w}}_{j_v}(A), & |A| = 1 \\ r_v \cdot \mathscr{M}^{\mathbf{w}}_{j_v}(A) + \frac{(1-r_v)}{12}, & |A| > 1 \end{cases} \tag{33}$$

where $v = 1,2,3,4$. The ranked weighted RPS sources with reliability are shown in Table IV(b).

Step 4: Since the reliability factors are in descending order, RPSR rule should use LOS $\overleftarrow{\oplus}$ to combine RPS sources. According to (26), calculate the fused PMF based on:

$$\widehat{\mathscr{M}} = \mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_1} \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_2} \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_3} \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_{j_4}$$
$$= \mathscr{M}^{\mathbf{w},\mathbf{r}}_4 \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_3 \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_1 \overleftarrow{\oplus} \mathscr{M}^{\mathbf{w},\mathbf{r}}_2 \tag{34}$$

Step 5: Based on $\widehat{\mathscr{M}}$, obtain the fused RPS source $\widehat{RPS}$, as shown in Table IV(c).

It should be pointed that in this example, the fusion order $\mathscr{F}$ and the reliability vector $\mathbf{r}$ are assumed to be specific values. However, in practice, $\mathscr{F}$ and $\mathbf{r}$ should be determined based on hyperparameter tuning method, which will be presented in Section V-B.

Example 3: Given a RPS source as listed in Table IV(c), its associated probability distribution P_{OPT} can be calculated by OPT. For example, $P_{OPT}(\theta_2)$ can be calculated by:

$$\begin{aligned} P_{OPT}(\theta_2) &= \widehat{\mathscr{M}}(\theta_2) + \frac{\widehat{\mathscr{M}}(\theta_2\theta_1)}{2-1} + \frac{\widehat{\mathscr{M}}(\theta_2\theta_3)}{2-1} \\ &\quad + \frac{\widehat{\mathscr{M}}(\theta_1\theta_2\theta_3)}{3-1} \\ &\quad + \frac{\widehat{\mathscr{M}}(\theta_2\theta_1\theta_3)}{3-1} + \frac{\widehat{\mathscr{M}}(\theta_2\theta_3\theta_1)}{3-1} + \frac{\widehat{\mathscr{M}}(\theta_3\theta_2\theta_1)}{3-1} \\ &= 0.8757 + 3.20\text{e-}35 + 0.0283 + \frac{1}{2} \cdot 3.21\text{e-}36 \\ &\quad + \frac{1}{2} \cdot 4.97\text{e-}36 + \frac{1}{2} \cdot 1.26\text{e-}35 + \frac{1}{2} \cdot 1.13\text{e-}35 \\ &= 0.9040 \end{aligned} \tag{35}$$

$$\mathscr{M}^{\mathbf{w}}_j = \begin{cases} \mathscr{M}^{\mathbf{w}}_j\left(\theta^{Ord}_{i_1}\right) = w_{(i_1)j} \cdot \tilde{f}^{Ord}_{i_1 j}, & (i_1) \in APS(1) \\ \mathscr{M}^{\mathbf{w}}_j\left(\theta^{Ord}_{i_1}, \theta^{Ord}_{i_2}\right) = w_{(i_1 i_2)j} \cdot \tilde{f}^{Ord}_{i_2 j}, & (i_1 i_2) \in APS(2) \\ \mathscr{M}^{\mathbf{w}}_j\left(\theta^{Ord}_{i_1}, \theta^{Ord}_{i_2}, \theta^{Ord}_{i_3}\right) = w_{(i_1 i_2 i_3)j} \cdot \tilde{f}^{Ord}_{i_3 j}, & (i_1 i_2 i_3) \in APS(3) \\ 0, & others \end{cases} \tag{31}$$

TABLE IV
RPS SOURCES AND THEIR PMFS, INCLUDING WEIGHTED RPS, WEIGHTED RPS WITH RELIABILITY, AND FUSED RPS

	(θ_1)	(θ_2)	(θ_3)	(θ_1,θ_2)	(θ_2,θ_1)	(θ_1,θ_3)	(θ_3,θ_1)	(θ_2,θ_3)	(θ_3,θ_2)	$(\theta_1,\theta_2,\theta_3)$	$(\theta_1,\theta_3,\theta_2)$	$(\theta_2,\theta_1,\theta_3)$	$(\theta_2,\theta_3,\theta_1)$	$(\theta_3,\theta_1,\theta_2)$	$(\theta_3,\theta_2,\theta_1)$
(a) Weighted RPS sources $RPS_j^{\mathbf{w}}$:															
$\mathscr{M}_1^{\mathbf{w}}$	0	0.5204	0	0	0	0	0	0.2330	0.2464	1.24e-05	1.31e-05	1.76e-05	4.79e-05	1.94e-05	4.99e-05
$\mathscr{M}_2^{\mathbf{w}}$	0.5106	0	0	0	0	0.1977	0.1567	0	0	0.0249	0.0303	0.0200	0.0159	0.0265	0.0173
$\mathscr{M}_3^{\mathbf{w}}$	0	0.7089	0	0	0	0	0	0.1739	0.1172	3.96e-69	2.67e-69	9.02e-69	9.95e-68	4.22e-69	6.89e-68
$\mathscr{M}_4^{\mathbf{w}}$	0	0.6189	0	0	0	0	0	0.2048	0.1763	1.30e-33	1.12e-33	2.02e-33	5.13e-33	1.56e-33	4.60e-33
(b) Ranked weighted RPS sources with reliability $RPS_{j_v}^{\mathbf{w},\mathbf{r}}$:															
$\mathscr{M}_4^{\mathbf{w},\mathbf{r}}$	0	0.6189	0	0	0	0	0	0.2048	0.1763	1.30e-33	1.12e-33	2.02e-33	5.13e-33	1.56e-33	4.60e-33
$\mathscr{M}_3^{\mathbf{w},\mathbf{r}}$	0	0.6498	0	0.0069	0.0069	0.0069	0.0069	0.1663	0.1144	0.0069	0.0069	0.0069	0.0069	0.0069	0.0069
$\mathscr{M}_1^{\mathbf{w},\mathbf{r}}$	0	0.4337	0	0.0139	0.0139	0.0139	0.0139	0.2081	0.2192	0.0139	0.0139	0.0139	0.0139	0.0139	0.0139
$\mathscr{M}_2^{\mathbf{w},\mathbf{r}}$	0.3830	0	0	0.0208	0.0208	0.1691	0.1384	0.0208	0.0208	0.0395	0.0436	0.0359	0.0327	0.0407	0.0338
(c) Fused RPS source $\widehat{RPS}$:															
$\widehat{\mathscr{M}}$	2.50e-34	0.8757	0.0717	1.08e-35	3.20e-35	3.49e-35	8.88e-35	0.0283	0.0243	3.21e-36	2.76e-36	4.97e-36	1.26e-35	3.84e-36	1.13e-35

Hence, P_{OPT} can be obtained: $P_{OPT}(\theta_1) = 3.04\text{e-}34$, $P_{OPT}(\theta_2) = 0.9040$, $P_{OPT}(\theta_3) = 0.0960$.

V. APPLICATION OF RPSR IN TARGET CLASSIFICATION

In this section, the proposed RPSR is applied in classification problems. First, an RPSR-based classification algorithm is proposed. Then, several experiments are conducted to verify the performance of the proposed algorithm compared with existing classification methods.

A. Problem Statement

Target classification is one of the topics of pattern recognition, which involves predicting the class of a given sample. Researchers have proposed many classification algorithms based on a variety of techniques, of which probability theory is the most commonly used one [57], [58].

Due to the capabilities of uncertain information processing and data fusion, evidence theory [4], [5] is also a useful tool for target classification [59]. ER approach [23] and its variants [24], [25], [26] are extensions of evidence theory that further improve the performance of evidence theory in classification problems. Classifiers based on evidence theory and ER approach have been widely reported, such as complex mass function-based methods [14], [59], entropy-based methods [60], [61], multi-view classifier [62], ER rule-based classifiers [29], [31], [63]. and belief rule base (BRB) classifiers [64].

Typically, the framework of a classifier based on evidence theory consists of the following three parts: (i) BPA generation method for data representation, (ii) evidence combination, and (iii) probability transformation for target identification. However, as mentioned in the introduction part, evidence theory and ER approach do not consider the concept of order in terms of the order in data representation and the order in fusion rule, which leads to a certain loss of performance on classification problems [12].

To overcome the above issue and enhance the efficiency of evidential classification framework, the following subsection aims to design a classification algorithm based on RPSR, which consists of RPSGM, RPSR rule, and OPT.

TABLE V
INFORMATION OF THE DATASETS

Dataset	#Sample	#Class	#Attribute	Missing
Iris	150	3	4	No
Wine	178	3	13	No
Heart	270	2	13	No
Ionosphere	351	2	34	No
Sonar	208	2	60	No
Australian	690	2	14	Yes
Diabetes	768	2	8	Yes

B. RPSR-Based Classification Algorithm

By considering the concept of order, this subsection proposes an RPSR-based classification algorithm (RPSRCA) that includes: (i) RPSGM for data representation with element order, (ii) RPSR rule for combining RPS sources with fusion order, and (iii) OPT for transformation and identification.

Assume that the original dataset has N classes, which is indicated by FOD $\Theta = \{\theta_i \,|\, i = 1, 2, \ldots, N\}$. In the dataset, each sample has K attributes. The original dataset is split into training set $\mathcal{D}^{train}$ and test set $\mathcal{D}^{test}$. The proposed RPSRCA aims to predict the label of each test sample, and its procedure is listed in Algorithm 1.

In Algorithm 1, the fusion order $\mathscr{F}$ and the reliability vector $\mathbf{r}$ are two hyperparameters. Their default values are assumed to be $\mathscr{F} = [1, 2, \ldots, K]$ (original order of attributes) and $\mathbf{r} = [r_v \,| r_v = 1 - \frac{0.5(v-1)}{K-1}; v = 1, 2, \ldots, K]$ (descending order), respectively. For better classification performance of RPSRCA, the hyperparameters should be further optimized. Hence, we present a hyperparameter tuning method for searching the optimal values of $\mathscr{F}$ and $\mathbf{r}$, as shown in Algorithms 2 and 3. In practical use, it is recommended to use Algorithms 2 and 3 to determine the optimal $\mathscr{F}^*$ and $\mathbf{r}^*$ first, and then apply Algorithm 1 to the classification problems according to the optimal hyperparameters.

To better understand, Fig. 1(d) shows the overall procedure of the proposed RPSRCA (Algorithm 1) with the hyperparameter tuning method (Algorithms 2 and 3).

TABLE VI
PERCENTAGE OF CLASSIFICATION ACCURACY AND STANDARD DEVIATION BASED ON MACHINE LEARNING CLASSIFIERS AND RPSRCA UNDER FIVE-FOLD CROSS-VALIDATION

	Iris	Wine	Heart	Ionosphere	Sonar	Australian	Diabetes	Average
NB	95.49±3.45	97.20±2.47	84.07±4.58	82.56±4.57	68.15±7.01	77.77±2.96	75.47±3.30	82.96±4.05
BayesNet	93.67±3.61	**98.50**±1.91	82.60±4.69	89.62±3.00	75.75±5.91	85.94±2.93	75.18±3.67	85.89±3.67
SVM	**96.33**±3.02	98.43±1.88	83.67±4.58	87.94±3.21	77.05±5.55	84.99±2.69	76.87±2.98	86.47±3.42
SVM-RBF	90.85±7.83	40.26±1.68	82.92±4.50	74.20±2.79	67.32±4.56	85.51±2.58	67.67±1.59	72.68±3.65
DT	94.33±3.99	92.71±4.73	78.08±5.50	89.51±3.35	73.42±7.35	85.21±2.63	73.56±3.41	83.83±4.42
k-NN	95.33±3.37	95.32±3.08	75.64±5.35	86.75±3.11	**85.80**±5.25	81.97±2.88	70.45±3.02	84.47±3.72
AdaBoost	94.63±3.81	89.04±5.20	82.08±4.68	90.39±3.47	75.38±6.03	84.97±2.81	74.58±3.18	84.44±4.17
LR	96.25±3.77	96.96±2.51	83.41±4.84	87.22±3.81	71.65±6.30	84.72±2.86	77.18±3.08	85.34±3.88
MLP	96.07±3.31	97.56±2.46	79.25±5.40	**90.40**±3.36	82.52±5.39	83.18±3.12	75.13±3.00	86.30±3.72
RPSRCA	95.99±0.13	98.04±0.65	**85.70**±1.02	87.97±0.51	75.39±1.46	**87.65**±0.26	**77.31**±0.47	**86.86**±0.64

The best results are shown in bold.

TABLE VII
PERCENTAGE OF CLASSIFICATION ACCURACY AND STANDARD DEVIATION BASED ON EVIDENTIAL CLASSIFIERS AND RPSRCA UNDER FIVE-FOLD CROSS-VALIDATION

	Iris	Wine	Heart	Ionosphere	Sonar	Australian	Diabetes	Average
DRCM	94.65±0.51	96.86±0.71	84.66±0.95	81.73±0.63	70.76±1.66	80.14±0.34	73.84±0.54	83.23±0.76
ER rule	**96.06**±0.19	97.00±0.36	84.27±0.96	85.78±0.53	70.46±1.86	79.98±0.43	73.63±0.53	83.88±0.69
MRCM	93.79±0.64	90.60±0.85	83.13±1.05	76.41±0.68	70.12±1.29	81.48±0.37	73.21±0.52	81.25±0.77
EDM	92.19±0.81	87.36±0.90	82.81±1.14	75.03±0.68	69.90±1.25	80.55±0.35	72.77±0.48	80.09±0.80
BJSM	92.49±0.60	87.34±0.96	82.71±1.15	74.02±0.75	69.66±1.41	79.80±0.38	72.86±0.52	79.84±0.83
RPSRCA	95.99±0.13	**98.04**±0.65	**85.70**±1.02	**87.97**±0.51	**75.39**±1.46	**87.65**±0.26	**77.31**±0.47	**86.86**±0.64

The best results are shown in bold.

C. Experiment and Result

In this subsection, RPSRCA is applied to classification problems and compared with several classifiers. The experiment description is summarized as follows.

i) *Datasets:* We consider seven real-world datsets from the UCI machine learning repository,[1] including Iris, Wine, Heart (Statlog), Ionosphere, Sonar (Sonar, Mines vs. Rocks), Australian (Australian Credit Approval), and Diabetes (Pima Indians Diabetes), whose information is listed in Table V. Missing values are replaced by mode values (for categorical attributes) or mean values (for continuous attributes).

ii) *Machine learning classifiers:* Nine well-known machine learning classifiers are selected: Naive Bayes (NB), Bayes Network (BayesNet), Support Vector Machine (SVM), SVM with Radial Basis Function (SVM-RBF), C4.5 Decision Tree (DT), k-Nearest Neighborhood (k-NN), AdaBoost with decision stump (AdaBoost), Logistic Regression (LR), and Multilayer Perception (MLP).

iii) *Evidential classifiers:* Five evidential classifiers are chosen: Dempster's rule of combination-based model (DRCM) [4], ER rule-based model (ER rule) [25], Murphy's rule of combination-based model (MRCM) [65], evidential distance-based model (EDM) [66], and belief JS divergence-based model (BJSM) [60]. The reliability and the weight of ER rule-based model are determined by the evaluation method presented in [29]. For better comparison, the following experiments use the same BPA generation method and probability transformation for each evidential classifier, *i.e.*, Xu et al.'s normal distribution-based method [67] and Smet's pignistic probability transformation (PPT) [68], respectively.

iv) *Implementations:* The proposed RPSRCA is compared with the selected machine learning classifiers and evidential classifiers. All classifiers are performed on the datasets and evaluated in terms of classification accuracy by five-fold cross-validation for 100 times.

The experimental results of the classifiers are summarized in Tables VI and VII, in which the results of the best performance are shown in bold. It can be observed in Table VI that the proposed RPSRCA is better than the machine learning classifiers on the Heart dataset (85.7%), the Australian dataset (87.65%), and the Diabetes dataset (77.31%). Especially on the Heart dataset, RPSRCA surpasses the worst classifier (75.64%) by 10.06% and the second-best classifier (84.07%) by 1.63%. Besides, the best classifiers on Iris, Wine, Ionosphere, and Sonar are as follows: SVM, BayesNet, MLP, and k-NN, respectively. Admittedly, on the aforementioned four datasets, the performance of RPSRCA is not the best, but RPSRCA has the highest average classification accuracy on all seven datasets (86.86%), which shows the efficiency of the proposed RPSRCA. Moreover, it is obvious that, on all seven datasets, the standard deviation of RPSRCA is

[1] http://archive.ics.uci.edu/ml/datasets

Algorithm 1: RPSR-Based Classification Algorithm (RPSRCA).

Input: Training set $\mathcal{D}^{train}$, Test set $\mathcal{D}^{test}$ (each sample has K attributes).
Hyperparameter: Fusion order $\mathscr{F}$ (default value $= [1, 2, \ldots, K]$), Reliability vector $\mathbf{r}$ (default value $= [r_v \,|r_v = 1 - \frac{0.5(v-1)}{K-1}; v = 1, 2, \ldots, K]$, descending order).
Output: Predicted labels for all test samples $PredLabel$.
/* Initialization */
Initialize the list for predicted label: $PredLabel \leftarrow [\,]$.
Normalize $\mathcal{D}^{train}$ and $\mathcal{D}^{test}$.
  /* RPS generation method */
for $j = 1$ to K **do**
  Based on the j-th attribute of the samples in $\mathcal{D}^{train}$, establish the j-th GDM $\mathbf{f}_j(x)$ according to (10) to (13).
end for
for all test samples $\mathbf{x} \in \mathcal{D}^{test}$ **do**
  for $j = 1$ to K **do**
    Based on GDM $\mathbf{f}_j(x)$, calculate the MV $\mathbf{f}_j$ by (14).
    Perform weight analysis and obtain the weight vector $\mathbf{w}_j$ by (15) to (20).
    On the basis of $\mathbf{w}_j$, construct the weighted RPS $RPS_j^{\mathbf{w}}$ by (21) to (22).
  end for
  /* RPSR rule of combination */
  Rank $RPS_j^{\mathbf{w}}|_{j=1}^{K}$ according to the fusion order $\mathscr{F}$, and then obtain the ranked weighed RPS sources $RPS_{j_v}^{\mathbf{w}}|_{v=1}^{K}$.
  for $v = 1$ to K **do**
    Based on $RPS_{j_v}^{\mathbf{w}}$ and the v-th reliability factor r_v of reliability vector $\mathbf{r}$, produce the weighted RPS with reliability $RPS_{j_v}^{\mathbf{w},\mathbf{r}}$ according to (26).
  end for
  According to (27) to (28), combine the K weighted RPS with reliability $RPS_{j_v}^{\mathbf{w},\mathbf{r}}|_{v=1}^{K}$, and then acquire the fused RPS source $\widehat{RPS}$.
  /* Ordered probability transformation */
  Convert $\widehat{RPS}$ into probability P_{OPT} by (29).
  /* Classification */
  Predict the label of the test sample: $\tilde{\theta} \leftarrow \arg\max_{\theta \in \Theta} P_{OPT}(\theta)$.
  Append $\tilde{\theta}$ to $PredLabel$.
end for
return $PredLabel$

Algorithm 2: Hyperparameter Tuning Method for Reliability Vector and Fusion Order.

Input: Training set $\mathcal{D}^{train}$ (each sample has K attributes).
Output: Optimal fusion order $\mathscr{F}^*$, Optimal reliability vector $\mathbf{r}^*$.
/* Initialization */
Initialize the past order list and the next order list for $\mathscr{F}$:
$Past \leftarrow [\,], Next \leftarrow [1, 2, \ldots, K]$.
Initialize the reliability vector $\mathbf{r}$ with coefficient $\eta = 0$:
$\mathbf{r} \leftarrow [r_v \,|r_v = 1 - \frac{\eta(v-1)}{K-1}; v = 1, 2, \ldots, K]$.
/* Hyperparameter tuning for fusion order */
Invoke Algorithm 3 $RecurFO(Past, Next)$, and then obtain the optimal fusion order $\mathscr{F}^*$.
/* Hyperparameter tuning for reliability vector */
Set the accuracy list: $AccList_1 \leftarrow [\,]$.
for $\eta \in \{0.25, 0.5, 0.75, 1\}$ **do**
  According to 10-fold cross validation, randomly split $\mathcal{D}^{train}$ into new training set $\mathcal{D}^{train'}$ and validation set $\mathcal{D}^{valid}$.
  Based on $\mathcal{D}^{train'}$ and $\mathcal{D}^{valid}$, randomly split obtain the accuracy Acc of Algorithm 1, where the reliability vector is calculated by η and the fusion order uses $\mathscr{F}^*$.
  Append Acc to $AccList_1$.
end for
Acquire the optimal coefficient:
$\eta^* \leftarrow \arg\max_\eta AccList_1$.
Get the optimal reliability vector $\mathbf{r}^*$ based on η^*.
return $\mathscr{F}^*$ and $\mathbf{r}^*$

much lower than other machine learning classifiers, especially on Iris dataset (0.13%), which proves the stability of RPSRCA for classification problems.

As shown in Table VII, compared with the evidential classifiers, the classification accuracy of RPSRCA is the highest on Wine (98.04%), Heart (85.70%), Ionosphere (87.97%), Sonar (75.39%), Australian (87.65%), and Diabetes (77.31%), and the standard deviation of RPSRCA is the lowest on Iris (0.13%), Ionosphere (0.51%), Australian (0.26%), and Diabetes (0.47%). On the Iris dataset, the accuracy of ER rule (96.06%) is higher than that of RPSRCA and other evidential classifiers. In addition, RPSRCA performs well on the Australian dataset (87.65%). RPSRCA is superior to the worst classifier (79.80%) by 7.85% and the second-best classifier (81.48%) by 6.17%. Furthermore, RPSRCA has the highest average classification accuracy (86.86%) in comparison of the evidential classifiers.

To summarize, the proposed RPSRCA outperforms the selected classifiers with the highest average classification accuracy (86.86%) and the lowest average standard deviation (0.64%), which demonstrates the efficiency and the stability of RPSRCA. The main reason is that RPSRCA is designed based on the three techniques of RPSR (RPSGM, RPSR rule, and OPT), which comprehensively considers concept of order in two aspects, namely the element order and the fusion order. Based on the RPSR framework, the proposed RPSRCA is able to handle the imprecision in the classification process and be less susceptible to samples on imprecise boundaries, thereby enhancing the stability of classification. Additionally, RPSRCA offers additional refinement of imprecise boundary by incorporating the concept of order, leading to further enhancements in classification performance and stability. Hence, RPSRCA can perform well in terms

Algorithm 3: Recursive Procedure for Searching Optimal Fusion Order $RecurFO(Past, Next)$.

```
Input: Past order list Past, Next order list Next.
Output: Optimal fusion order F*;
/* Base case */
if Next is emtpy then
  return Optimal fusion order F* ← Past
end if
  /* Recursive step */
Empty the accuracy list: AccList_2 ← [].
for all elements in the next order list cur ∈ Next do
  Remove cur from Next: Next' ← Next\cur.
  Concatenate the lists: F' ← [Past + cur + Next'].
  According to 10-fold cross validation, randomly
  split D^train into new training set D^train' and
  validation set D^valid.
  Based on D^train' and D^valid, obtain the accuracy
  Acc of Algorithm 1, where the reliability vector uses
  the default value and the fusion order takes F'.
  Append Acc to AccList_2.
end for
Acquire the current optimal element:
cur* ← arg max_{cur∈Next} AccList_2.
Update the past order list and the next order list:
Past* ← [Past + cur*], Next* ← Next\cur*.
return Invoke Algorithm 3 RecurFO(Past*, Next*)
```

of accuracy and stability, which further proves the efficiency of the proposed RPSR.

VI. Conclusion

In artificial intelligence, evidence theory serves as an uncertainty reasoning method for processing data characterized by incompleteness and uncertainty. Recently, random permutation set (RPS) theory has emerged as an orderable extension of evidence theory and has attracted considerable interest. However, it faces two main challenges: the lack of a generation method for the element order in permutation mass function (PMF), and the absence of an effective strategy to determine the fusion order for permutation orthogonal sum (POS). To address the two issues, this paper proposes an RPS theory-based reasoning model, called random permutation set reasoning (RPSR). The main contributions are summarized as follows:

i) RPSR is proposed, including RPS generation method (RPSGM), RPSR rule of combination, and ordered probability transformation (OPT), which are illustrated by several numerical examples.

ii) RPSGM can construct RPS based on Gaussian discriminant model and weight analysis; RPSR rule incorporates POS with reliability vector, which can combine RPS sources with reliability in fusion order; OPT is used to convert RPS into a probability distribution for the final decision.

iii) An RPSR-based classification algorithm (RPSRCA) is proposed for classification problems and evaluated through a series of experiments. The experimental results show that RPSRCA is more efficient and stable than other classifiers.

In future work, we aim to apply RPSR into other fields, such as decision making and fault diagnosis. Besides, we intend to study the properties of the proposed RPSR rule, especially the relationship of Dempster's rule, ER rule, and RPSR rule. Moreover, low-complexity reasoning methods and efficient conflict management mechanisms of RPSR are also worth studying.

Acknowledgment

The authors greatly appreciate the reviewers' suggestions and the editor's encouragement.

References

[1] G. E. Box and G. C. Tiao, *Bayesian Inference in Statistical Analysis*. Hoboken, NJ, USA: John Wiley & Sons, 2011.
[2] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.
[3] D. Dubois and H. Prade, "Possibility theory," in *Granular, Fuzzy, and Soft Computing*. Berlin, Germany: Springer, 2023, pp. 859–876.
[4] A. P. Dempster, "Upper and lower probabilities induced by a multivalued mapping," *Ann. Math. Statist.*, vol. 38, no. 2, pp. 325–339, 1967.
[5] G. Shafer, *A Mathematical Theory of Evidence*, vol. 1. Princeton, NJ, USA: Princeton Univ. Press, 1976.
[6] F. Cuzzolin, *The Geometry of Uncertainty*. Berlin, Germany: Springer, 2021.
[7] Y. Deng, "Uncertainty measure in evidence theory," *Sci. China Inf. Sci.*, vol. 63, no. 11, pp. 1–19, 2020.
[8] Y. Deng, "Information volume of mass function," *Int. J. Comput. Commun. Control*, vol. 15, no. 6, 2020, Art. no. 3983.
[9] X. Deng and W. Jiang, "Quantum representation of basic probability assignments based on mixed quantum states," in *Proc. IEEE 24th Int. Conf. Inf. Fusion*, 2021, pp. 1–6.
[10] F. Xiao, "Generalization of dempster–shafer theory: A complex mass function," *Appl. Intell.*, vol. 50, no. 10, pp. 3266–3275, 2020.
[11] X. Gao, L. Pan, and Y. Deng, "Quantum pythagorean fuzzy evidence theory: A negation of quantum mass function view," *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 5, pp. 1313–1327, May 2022.
[12] Y. Deng, "Random permutation set," *Int. J. Comput. Commun. Control*, vol. 17, no. 1, 2022, Art. no. 4542.
[13] F. Xiao, "EFMCDM: Evidential fuzzy multicriteria decision making based on belief entropy," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 7, pp. 1477–1491, Jul. 2020.
[14] F. Xiao and W. Pedrycz, "Negation of the quantum mass function for multisource quantum information fusion with its application to pattern classification," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 45, no. 2, pp. 2054–2070, Feb. 2023.
[15] Z. Liu, X. Zhang, J. Niu, and J. Dezert, "Combination of classifiers with different frames of discernment based on belief functions," *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 7, pp. 1764–1774, Jul. 2021.
[16] N. Balakrishnan, F. Buono, and M. Longobardi, "A unified formulation of entropy and its application," *Physica A, Stat. Mechanics Appl.*, vol. 596, 2022, Art. no. 127214.
[17] H. Cui, L. Zhou, Y. Li, and B. Kang, "Belief entropy-of-entropy and its application in the cardiac interbeat interval time series analysis," *Chaos, Solitons Fractals*, vol. 155, 2022, Art. no. 111736.
[18] C. Qiang, Y. Deng, and K. H. Cheong, "Information fractal dimension of mass function," *Fractals*, vol. 30, 2022, Art. no. 2250110.
[19] Q. Zhou et al., "Information dimension of galton board," *Fractals*, vol. 30, no. 04, pp. 1–11, 2022.
[20] F. Xiao, J. Wen, and W. Pedrycz, "Generalized divergence-based decision making method with an application to pattern classification," *IEEE Trans. Knowl. Data Eng.*, vol. 35, no. 7, pp. 6941–6956, Jul. 2023.
[21] R. A. Hummel and M. S. Landy, "A statistical viewpoint on the theory of evidence," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 10, no. 2, pp. 235–247, Mar. 1988.
[22] Z.-W. Zhang, Z. Liu, Z. Ma, Y. Zhang, and H. Wang, "A new belief-based incomplete pattern unsupervised classification method," *IEEE Trans. Knowl. Data Eng.*, vol. 34, no. 11, pp. 5084–5097, Nov. 2022.

[23] J.-B. Yang and M. G. Singh, "An evidential reasoning approach for multiple-attribute decision making with uncertainty," *IEEE Trans. Syst., Man, Cybern.*, vol. 24, no. 1, pp. 1–18, Jan. 1994.
[24] J.-B. Yang and P. Sen, "A general multi-level evaluation process for hybrid MADM with uncertainty," *IEEE Trans. Syst., Man, Cybern.*, vol. 24, no. 10, pp. 1458–1473, Oct. 1994.
[25] J.-B. Yang and D.-L. Xu, "Evidential reasoning rule for evidence combination," *Artif. Intell.*, vol. 205, pp. 1–29, 2013.
[26] J.-B. Yang and D.-L. Xu, "On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty," *IEEE Trans. Syst., Man, Cybern.-Part A, Syst. Hum.*, vol. 32, no. 3, pp. 289–304, May 2002.
[27] S. Chen, Y. Wang, H. Shi, M. Zhang, and Y. Lin, "Evidential reasoning with discrete belief structures," *Inf. Fusion*, vol. 41, pp. 91–104, 2018.
[28] J. Wang, Z.-J. Zhou, C.-H. Hu, X.-X. Han, S.-W. Tang, and P.-Y. Ning, "An evidential reasoning rule considering parameter uncertainty," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 58, no. 2, pp. 1391–1404, Apr. 2022.
[29] X. Xu, J. Zheng, J.-B. Yang, D.-L. Xu, and Y.-W. Chen, "Data classification using evidence reasoning rule," *Knowl.-Based Syst.*, vol. 116, pp. 144–151, 2017.
[30] X. Xu et al., "Data classification based on attribute vectorization and evidence fusion," *Appl. Soft Comput.*, vol. 121, 2022, Art. no. 108712.
[31] X. Xu, D. Zhang, Y. Bai, L. Chang, and J. Li, "Evidence reasoning rule-based classifier with uncertainty quantification," *Inf. Sci.*, vol. 516, pp. 192–204, 2020.
[32] J.-B. Yang, Y.-M. Wang, D.-L. Xu, and K.-S. Chin, "The evidential reasoning approach for mada under both probabilistic and fuzzy uncertainties," *Eur. J. Oper. Res.*, vol. 171, no. 1, pp. 309–343, 2006.
[33] C. Fu, C. Xu, M. Xue, W. Liu, and S. Yang, "Data-driven decision making based on evidential reasoning approach and machine learning algorithms," *Appl. Soft Comput.*, vol. 110, 2021, Art. no. 107622.
[34] C. Fu, B. Hou, W. Chang, N. Feng, and S. Yang, "Comparison of evidential reasoning algorithm with linear combination in decision making," *Int. J. Fuzzy Syst.*, vol. 22, no. 2, pp. 686–711, 2020.
[35] P.-Y. Ning, Z.-J. Zhou, Y. Cao, S.-W. Tang, and J. Wang, "A concurrent fault diagnosis model via the evidential reasoning rule," *IEEE Trans. Instrum. Meas.*, vol. 71, 2022, Art. no. 3500916.
[36] J. Wang, Z.-J. Zhou, C.-H. Hu, S.-W. Tang, and Y. Cao, "A new evidential reasoning rule with continuous probability distribution of reliability," *IEEE Trans. Cybern.*, vol. 52, no. 8, pp. 8088–8100, Aug. 2022.
[37] L. Chang, C. Fu, W. Zhu, and W. Liu, "Belief rule mining using the evidential reasoning rule for medical diagnosis," *Int. J. Approx. Reasoning*, vol. 130, pp. 273–291, 2021.
[38] Z.-J. Zhou, G.-Y. Hu, C.-H. Hu, C.-L. Wen, and L.-L. Chang, "A survey of belief rule-base expert system," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 51, no. 8, pp. 4944–4958, Aug. 2021.
[39] L. Chang, C. Fu, Z. Wu, and W. Liu, "A data-driven method using BRB with data reliability and expert knowledge for complex systems modeling," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 52, no. 11, pp. 6729–6743, Nov. 2022.
[40] Y. Yang, D.-L. Xu, J.-B. Yang, and Y.-W. Chen, "An evidential reasoning-based decision support system for handling customer complaints in mobile telecommunications," *Knowl.-Based Syst.*, vol. 162, pp. 202–210, 2018.
[41] Z.-J. Zhou, Z.-C. Feng, C.-H. Hu, G.-Y. Hu, W. He, and X.-X. Han, "Aeronautical relay health state assessment model based on belief rule base with attribute reliability," *Knowl.-Based Syst.*, vol. 197, 2020, Art. no. 105869.
[42] S.-W. Tang, Z.-J. Zhou, C.-H. Hu, F.-J. Zhao, and Y. Cao, "A new evidential reasoning rule-based safety assessment method with sensor reliability for complex systems," *IEEE Trans. Cybern.*, vol. 52, no. 5, pp. 4027–4038, May 2022.
[43] R. R. Yager, "On the dempster-shafer framework and new combination rules," *Inf. Sci.*, vol. 41, no. 2, pp. 93–137, 1987.
[44] Z. Sunberg and J. Rogers, "A belief function distance metric for orderable sets," *Inf. Fusion*, vol. 14, no. 4, pp. 361–373, 2013.
[45] C. Cheng and F. Xiao, "A distance for belief functions of orderable set," *Pattern Recognit. Lett.*, vol. 145, pp. 165–170, 2021.
[46] C. Bandt and B. Pompe, "Permutation entropy: A natural complexity measure for time series," *Phys. Rev. Lett.*, vol. 88, no. 17, 2002, Art. no. 174102.
[47] L. Chen and Y. Deng, "Entropy of random permutation set," *Commun. Stat. Theory Methods*, vol. 53, no. 11, pp. 4127–4146, 2024.
[48] J. Deng and Y. Deng, "Maximum entropy of random permutation set," *Soft Comput.*, vol. 26, no. 21, pp. 11 265–11 275, 2022.
[49] F. Smarandache, J. Dezert, and J.-M. Tacnet, "Fusion of sources of evidence with different importances and reliabilities," in *Proc. IEEE 13th Int. Conf. Inf. Fusion*, 2010, pp. 1–8.
[50] C. Fu, M. Xue, W. Chang, D. Xu, and S. Yang, "An evidential reasoning approach based on risk attitude and criterion reliability," *Knowl.-Based Syst.*, vol. 199, 2020, Art. no. 105947.
[51] H. Tabia, M. Daoudi, J.-P. Vandeborre, and O. Colot, "A new 3D-matching method of nonrigid and partially similar models using curve analysis," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 33, no. 4, pp. 852–858, Apr. 2011.
[52] Z. Liu, L. Huang, K. Zhou, and T. Denoeux, "Combination of transferable classification with multisource domain adaptation based on evidential reasoning," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 5, pp. 2015–2029, May 2021.
[53] P. Smets, "The combination of evidence in the transferable belief model," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 12, no. 5, pp. 447–458, May 1990.
[54] M. Zhou, Y.-W. Chen, X.-B. Liu, B.-Y. Cheng, and J.-B. Yang, "Weight assignment method for multiple attribute decision making with dissimilarity and conflict of belief distributions," *Comput. Ind. Eng.*, vol. 147, 2020, Art. no. 106648.
[55] Z. Feng, Z.-J. Zhou, C. Hu, L. Chang, G. Hu, and F. Zhao, "A new belief rule base model with attribute reliability," *IEEE Trans. Fuzzy Syst.*, vol. 27, no. 5, pp. 903–916, May 2019.
[56] L.-H. Yang, F.-F. Ye, J. Liu, Y.-M. Wang, and H. Hu, "An improved fuzzy rule-based system using evidential reasoning and subtractive clustering for environmental investment prediction," *Fuzzy Sets Syst.*, vol. 421, pp. 44–61, 2021.
[57] H. Wang, Y.-P. Fang, and E. Zio, "Resilience-oriented optimal post-disruption reconfiguration for coupled traffic-power systems," *Rel. Eng. Syst. Saf.*, vol. 222, 2022, Art. no. 108408.
[58] Z. Wen, Z. Liu, S. Zhang, and Q. Pan, "Rotation awareness based self-supervised learning for SAR target recognition with limited training samples," *IEEE Trans. Image Process.*, vol. 30, pp. 7266–7279, 2021.
[59] F. Xiao, "CED: A distance for complex mass functions," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 4, pp. 1525–1535, Apr. 2021, doi: 10.1109/TNNLS.2020.2984918.
[60] F. Xiao, "Multi-sensor data fusion based on the belief divergence measure of evidences and the belief entropy," *Inf. Fusion*, vol. 46, pp. 23–32, 2019.
[61] H. Liao, Z. Ren, and R. Fang, "A deng-entropy-based evidential reasoning approach for multi-expert multi-criterion decision-making with uncertainty," *Int. J. Comput. Intell. Syst.*, vol. 13, no. 1, pp. 1281–1294, 2020.
[62] Z. Han, C. Zhang, H. Fu, and J. T. Zhou, "Trusted multi-view classification with dynamic evidential fusion," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 45, no. 2, pp. 2551–2566, Feb. 2023.
[63] S. Sachan, F. Almaghrabi, J.-B. Yang, and D.-L. Xu, "Evidential reasoning for preprocessing uncertain categorical data for trustworthy decisions: An application on healthcare and finance," *Expert Syst. Appl.*, vol. 185, 2021, Art. no. 115597.
[64] L.-L. Chang, Z.-J. Zhou, Y. You, L.-H. Yang, and Z.-G. Zhou, "Belief rule based expert system for classification problems with new rule activation and weight calculation procedures," *Inf. Sci.*, vol. 336, pp. 75–91, 2016.
[65] C. K. Murphy, "Combining belief functions when evidence conflicts," *Decis. Support Syst.*, vol. 29, no. 1, pp. 1–9, 2000.
[66] Y. Deng, W. Shi, Z. Zhu, and Q. Liu, "Combining belief functions based on distance of evidence," *Decis. Support Syst.*, vol. 38, no. 3, pp. 489–493, 2004.
[67] P. Xu, Y. Deng, X. Su, and S. Mahadevan, "A new method to determine basic probability assignment from training data," *Knowl.-Based Syst.*, vol. 46, pp. 69–80, 2013.
[68] P. Smets, "Decision making in the TBM: The necessity of the pignistic transformation," *Int. J. Approx. Reasoning*, vol. 38, no. 2, pp. 133–147, 2005.

Jixiang Deng received the bachelor's degree from Northwestern Polytechnical University, Xi'an, China, in 2020. He is currently working toward the master's degree with the Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, China. His research interests include Dempster-Shafer evidence theory, fuzzy set theory, information fusion, and pattern recognition.

Yong Deng (Member, IEEE) received the PhD degree in precise instrumentation from Shanghai Jiao Tong University, Shanghai, China, in 2003. From 2005 to 2011, he was an associate professor with the Department of Instrument Science and Technology, Shanghai Jiao Tong University. From 2010, he was a professor with the School of Computer and Information Science, Southwest University, Chongqing, China. From 2012, he was a visiting professor in Vanderbilt University, Nashville, TN, USA. From 2016, he was also a professor with the School of Electronic and Information Engineering, Xi'an Jiaotong University, Xi'an, China. From 2017, he is the full professor of Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu, China. From 2017, he is the adjunct professor of Medical center, Vanderbilt University, Nashville, TN, USA. He has published more than 100 papers in referred journals. His research interests include Dempster-Shafer evidence theory, decision making, information fusion and complex system modeling. He severed as the program member of many conference such as International Conference on Belief Functions. He severed as many editorial board members such as Editorial Board member of *Applied Intelligence* and *Journal of Organizational and End User Computing*. He severed as many guest editor such as *International Journal of Approximate Reasoning*, *Mathematical Problems in Engineering and Sustainability*. He severed as the reviewer for more than 30 journals. He has received numerous honors and awards, including the Elsevier Highly Cited Scientist in China from 2014 to present.

Jian-Bo Yang received the BEng. and MEng. degrees in system control engineering from North Western Polytechnic University, Xi'an, China, in 1981 and 1984, respectively, and the PhD degree in decision sciences and system engineering from Shanghai Jiao Tong University, Shanghai, China, in 1987. He was a faculty member of the University of Manchester Institute of Science and Technology, Manchester, U.K., in 1990 and from 1998 to 2004, the University of Birmingham, Birmingham, U.K., from 1995 to 1997, the University of Newcastle-upon-Tyne, Newcastle upon Tyne, U.K., from 1991 to 1995, and Shanghai Jiao Tong University from 1987 to 1989. He is a professor of Decision and System Sciences and the director of the Decision and Cognitive Sciences Research Center with the Alliance Manchester Business School, University of Manchester, Manchester. In the past three decades, he has been conducting research in multiple criteria decision analysis under uncertainty, multiple objective optimization, intelligent decision support systems, hybrid quantitative and qualitative decision modeling using techniques from operational research, artificial intelligence, and systems engineering, and dynamic system modeling, simulation, and control for engineering and management systems. His current applied research is in design decision making, risk modeling and analysis, production planning and scheduling, quality modeling and evaluation, supply chain modeling and supplier assessment, and evaluation of products, systems, projects, and policies. His research has been supported by EPSRC, DEFRA and SERC, HKRGC in U.K. and in Hong Kong, NSFC in China, and EC in Europe and industry. He has published 3 books, more than 140 papers in international journals/book chapters, and a similar number of papers in conferences and developed several software packages in these areas, including the windows-based intelligent decision system via evidential reasoning approach.