The Evidential Reasoning Approach for Multi-attribute Decision Analysis Under Both Fuzzy and Interval Uncertainty[1]

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Abstract

Many multiple attribute decision analysis (MADA) problems are characterised by both quantitative and qualitative attributes with various types of uncertainties. Incompleteness (or ignorance) and vagueness (or fuzziness) are among the most common uncertainties in decision analysis. The evidential reasoning (ER) and the interval grade evidential reasoning (IER) approaches have been developed in recent years to support the solution of MADA problems with interval uncertainties and local ignorance in decision analysis. In this paper, the ER approach is enhanced to deal with both interval uncertainty and fuzzy beliefs in assessing alternatives on an attribute. In this newly developed FIER approach, local ignorance and grade fuzziness are modelled under the integrated framework of a distributed fuzzy belief structure, leading to a fuzzy belief decision matrix. A numerical example is provided to illustrate the detailed implementation process of the FIER approach and its validity and applicability.

Keywords: Multiple attribute decision analysis; Uncertainty modelling; The evidential reasoning approach; Utility; Fuzzy sets; Fuzzy

1. Introduction

Many real world multiple attribute decision analysis (MADA) problems are characterised with both quantitative and qualitative attributes. In many circumstances, the attributes, especially qualitative ones, could only be properly assessed using human judgments, which are subjective in nature and are inevitably associated with uncertainties caused due to either or both of the following two phenomenon: 1) Human being's inability to provide complete judgments, or the lack of information, which is referred to as "ignorance" (incompleteness); 2) The vagueness of meanings about attributes and their assessments, which is referred to as "fuzziness" (vagueness).

For decades, many MADA methods have been developed, such as the well-know Analytical

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Hierarchy Process (AHP) (Saaty, 1988) and Multiple Attribute Utility Theory (Keeney and Raiffa, 1993; Belton and Stewart, 2002, etc.) as well as their extensions (Arbel and Vargas 1992, 1993; Salo and Hämäläinen 1992; Islame et al. 1997, etc.). In those methods, MADA problems are modelled using decision matrices, in which an alternative is assessed on each attribute by either a single real number or an interval value. Unfortunately, in many decision situations using a single number or interval to represent a judgement proves to be difficult and may be unacceptable. Information may be lost or distorted in the process of pre-aggregating different types of information, such as a subjective judgement, a probability distribution, or an incomplete piece of information.

Concerning the fuzziness of MADA problems, a large amount of fuzzy MADA methods have been proposed in the literature (Bellman and Zadeh, 1970; Yager, 1977, 1978, 1981; Laarhoven and Pedrycz, 1983; Dong, Shah and Wong, 1985; Tseng and Klein, 1992, etc.). Nevertheless, these pure fuzzy MADA approaches are essentially based on traditional evaluation methods and are unable to handle probabilistic uncertainties such as ignorance.

Different from the traditional MADA methods, the Evidential Reasoning (ER) approach (Yang and Singh, 1994; Yang and Sen, 1994; Yang 2001; Yang and Xu 2002a, 2002b), which is the combination of the D-S theory (Dempster, 1967; Shafer, 1976) with a distributed modelling framework, shed a new line to modelling complex MADA problems. The ER approach uses a distributed modelling framework, in which each attribute is accessed using a set of collectively exhaustive and mutually exclusive assessment grades. Probabilistic uncertainty including local and global ignorance is characterized by a belief structure in the ER approach, which can both model precise data and capture various types of uncertainties such as probabilities and vagueness in subjective judgments. Along with the application of ER modelling, experiences show that decision maker may not always be confident enough to provide subjective assessments to individual grades only, but at times wishes to be able to assess beliefs to sub-sets of adjacent grades. Such ignorance is referred to as local ignorance or interval uncertainty. It is to deal with the local ignorance that the interval grade ER (IER) approach is proposed (Xu et al 2005). Another extension to the basic ER approach is to take account of vagueness or fuzzy uncertainty, i.e. the assessment grades are no longer clearly distinctive crisp sets, but are defined as dependent fuzzy sets. In other words, the intersection of two fuzzy sets may not be empty. Yang et al 2006 proposed the fuzzy ER approach.

The aim of this paper is to integrate the main features of the above two approaches, and develop a general ER modelling framework and an attribute aggregation process, referred to as the fuzzy IER (FIER) algorithm, in order to deal with both fuzzy and interval grade assessments in MADA and provide a more powerful means to support the solution of complex MADA problems.

2. The FIER approach for MADA under fuzzy uncertainty

2.1. The new FIER distributed modelling framework using the fuzzy belief structure

Suppose a MADA problem has M alternatives $a_l, l = 1,\ldots,M$, one upper level attribute, referred to as general attribute, and L lower level attributes $e_i, i = 1,\ldots,L$, called basic attributes. The relative weights of the L basic attributes are denoted by $W = (w_1,\ldots,w_L)$, which are known and satisfy the conditions $0 \le w_i \le 1$ and $\sum_{i=1}^{L} w_i = 1$.

Suppose M alternatives are all assessed using a pre-defined set H. In basic ER methodology, set H is defined as a union of N assessment grades $H_p, p = 1,\ldots,N$, which are mutually exclusive and collectively exhaustive for the assessment of all attributes, and the whole set H_{1N} as follows:

$$H \square \{H_1, H_2, \ldots, H_N, H_{1N}\} \tag{1.}$$

According to Xu et al (2005), in the IER (Interval grade ER) methodology, the performances of alternatives can be assessed to an individual grade or a grade interval, the complete set of all individual grades and grade intervals, for assessing each attribute can be represented by

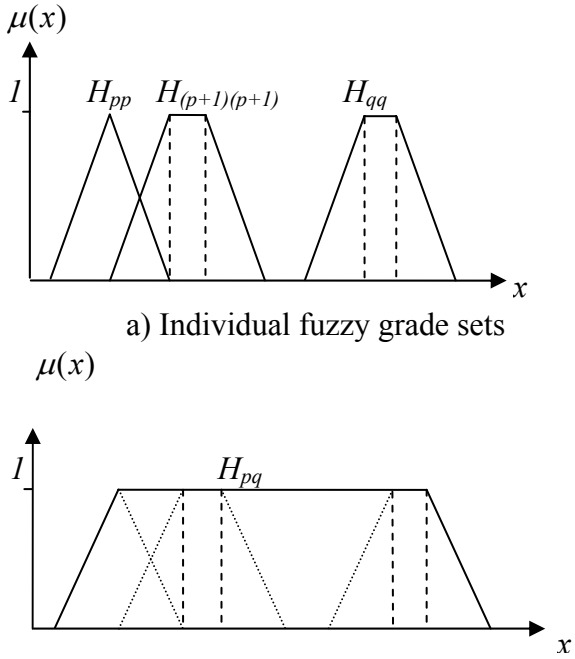
$$H \square \begin{Bmatrix} H_{11} & H_{12} & \cdots & H_{1(N-1)} & H_{1N} \\ & H_{22} & \cdots & H_{2(N-1)} & H_{2N} \\ & & \ddots & \vdots & \vdots \\ & & & H_{(N-1)(N-1)} & H_{(N-1)N} \\ & & & & H_{NN} \end{Bmatrix}, \tag{2.}$$

where H_{pp} ($p = 1,...,N$) in formula (2) denotes an individual grade. H_{pq} ($p = 1,...,N$, $q = p+1,...,N$) denotes the interval grade which is the union of individual grades $H_{pp}, H_{(p+1)(p+1)}, \ldots, H_{qq}$.

In the basic ER as well as the IER approach introduced above, all individual and interval assessment grades are assumed to be crisp and independent of each other. However, there are occasions where an assessment grade may represent a vague concept or standard and there may be no clear cut between the meanings of two adjacent grades. In this paper, we will drop the above assumption and allow grades to be vague and adjacent grades to be dependent. To simplify the discussion and without loss of generality, fuzzy sets will be used to characterise vague assessment grades and it is assumed that only two adjacent fuzzy grades have the overlap of meanings. This represents the most common features of fuzzy uncertainty in decision analysis.

In order to generalize the $H = \{H_{pq}, p = 1,...,N, q = p,...,N\}$ to fuzzy sets, we assume that a general set of fuzzy individual assessment grades $\{H_{pp}\}, p = 1,...,N$ are dependent on each other, which may be assumed to be either triangular or trapezoidal fuzzy sets or their combinations for

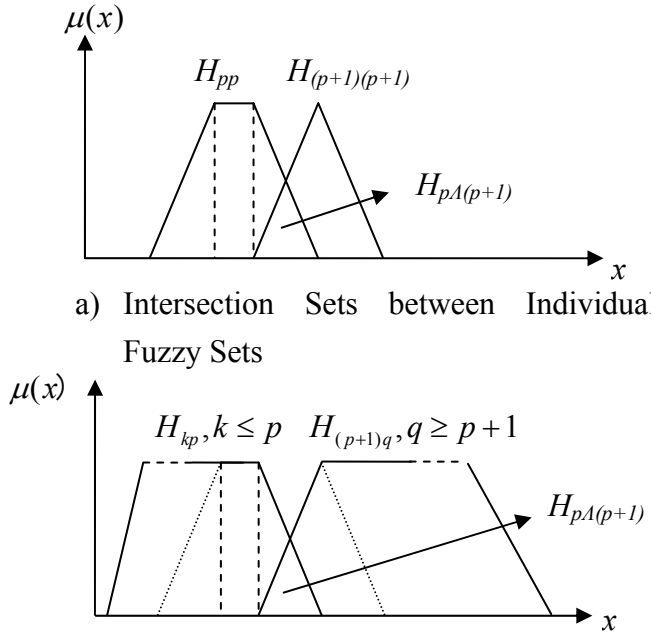
simplifying the discussion and without loss of generality. Assuming that only two adjacent fuzzy individual assessment grades may intersect, we denote by $H_{p\Lambda(p+1)}$ ($p = 1,...,N-1$) the fuzzy intersection subset of the two adjacent fuzzy individual assessment grades H_{pp} and $H_{(p+1)(p+1)}$ (Fig. 1).



a) Individual fuzzy grade sets

b) Interval fuzzy grade sets

Fig 1. Fuzzy sets definitions



a) Intersection Sets between Individual Fuzzy Sets

b) Intersection Sets between Interval or Interval Fuzzy Sets

Fig. 2. Intersections between fuzzy assessment grades

Furthermore, we define the sets $H_{pq}, p = 1,...,N, q = p+1,...,N$ as trapezoidal fuzzy sets which include individual grades H_{pp}, $H_{(p+1)(p+1)}, \ldots, H_{qq}$. If these individual assessment grades are triangular or trapezoidal fuzzy sets, every interval grade will be a trapezoidal fuzzy set (Fig. 1b). And we also define $H_{p\Lambda(p+1)}$ as the fuzzy intersection subset of the two adjacent fuzzy interval assessment grades H_{kp} and $H_{(p+1)q}$, where $k \le p$, $q \ge p+1$ (Fig. 2).

Finally, the generalized fuzzy assessment set H can be defined as follows.

$$H \square H_F = \{H_{pq}, p = 1,...,N, q = p,...,N\} \cup \{H_{p\Lambda(p+1)}, p = 1,...,N-1\}, \quad (3.)$$

where H_{pq} is a fuzzy set and $H_{p\Lambda(p+1)}$ is the intersection of two adjacent fuzzy sets H_{kp} and $H_{(p+1)q}$, where $k \le p$, $q \ge p+1$.

The assessment of an alternative on attribute a_l is then given by

$$S(a_l) = \{(C,\ \beta_i(C));\ C \in H_F,\ i = 1,...,L\}, \tag{4.}$$

where $\sum_{C \in H} \beta_i(C) = 1$ holds.

The mass functions are defined as follows:

$$m_i(C) = w_i \beta_i(C),\ \ i = 1,...,L,\ C \ne \Phi, C \in H_F \tag{5.}$$

$$m_i(\Phi) = 0, \tag{6.}$$

$$m_i(U) = 1 - w_i,\ \ i = 1,...,L \tag{7.}$$

where $m_i(U)$ in equation (7) is the remaining probability mass that is unassigned to any evaluation grades in set H_F after only attribute i has been taken into account. In other words, $m_i(U)$ represents the remaining role that other attributes can play in the assessment. $m_i(U)$ should eventually be assigned back to set H_F, in a way that is dependent upon the importance of other attributes.

3.2. The new FIER algorithm under both interval probabilistic and fuzzy uncertainties

Based on the fuzzy assessment set H_F, a FIER (Fuzzy Interval grade ER) recursive algorithm is developed as follows using the similar technique used in Yang and Singh 1994 and Yang, et al 2006 (the detailed proof is shown in the Appendix).

$$\tilde{m}_{I(1)}(H_{pq}) = m_1(H_{pq}),\ \ p = 1,...,N, q = p,...,N \tag{8.}$$

$$\tilde{m}_{I(1)}(H_{p\Lambda(p+1)}) = 0,\ p = 1,...,N-1 \tag{9.}$$

$$\tilde{m}_{I(1)}(U) = m_1(U) \tag{10.}$$

$$\begin{aligned}
\tilde{m}_{I(i+1)}(H_{pq}) = & -\tilde{m}_{I(i)}(H_{pq}) m_{i+1}(H_{pq}) \\
& + \sum_{k=1}^{p} \sum_{l=q}^{N} [\tilde{m}_{I(i)}(H_{kl}) m_{i+1}(H_{pq}) + \tilde{m}_{I(i)}(H_{pq}) m_{i+1}(H_{kl})] \\
& + \sum_{k=1}^{p-1} \sum_{l=q+1}^{N} [\tilde{m}_{I(i)}(H_{kq}) m_{i+1}(H_{pl}) + \tilde{m}_{I(i)}(H_{pl}) m_{i+1}(H_{kq})] \\
& + \tilde{m}_{I(i)}(U) m_{i+1}(H_{pq}) + \tilde{m}_{I(i)}(H_{pq}) m_{i+1}(U), \\
& p = 1,...,N, q = p,...,N
\end{aligned} \tag{11.}$$

$$\tilde{m}_{I(i+1)}(H_{p\wedge(p+1)}) = \sum_{k=1}^{p}\sum_{q=p+1}^{N}[\tilde{m}_{I(i)}(H_{kp})\cdot m_{i+1}(H_{(p+1)q}) + m_{i+1}(H_{kp})\cdot \tilde{m}_{I(i)}(H_{(p+1)q})]$$

$$+\sum_{k=1}^{p+1}\sum_{\substack{l=p\\ l\geq k}}^{N} m_{i+1}(H_{kl})\cdot \tilde{m}_{I(i)}(H_{p\wedge(p+1)}) + \tilde{m}_{I(i)}(H_{p\wedge(p+1)})m_{i+1}(U)$$

$$p = 1,...,N-1 \tag{12.}$$

$$\tilde{m}_{I(i+1)}(U) = \tilde{m}_{I(i)}(U)m_{i+1}(U) = \prod_{l=1}^{i+1} m_l(U) \tag{13.}$$

$$K = 1/[\sum_{p=1}^{N}\sum_{q=p}^{N} \tilde{m}_{I(L)}(H_{pq}) + \sum_{p=1}^{N-1}\mu_{p\wedge(p+1)}^{\max}\tilde{m}_{I(L)}(H_{p\wedge(p+1)}) + \tilde{m}_{I(L)}(U)] \tag{14.}$$

$$m_{I(L)}(H_{pq}) = K\cdot \tilde{m}_{I(L)}(H_{pq}),\ p = 1,...,N, q = p,...,N \tag{15.}$$

$$m_{I(L)}(\bar{H}_{p\wedge(p+1)}) = K\cdot \mu_{p\wedge(p+1)}^{\max}\cdot \tilde{m}_{I(L)}(H_{p\wedge(p+1)}),\ p = 1,...,N-1 \tag{16.}$$

$$m_{I(L)}(U) = K\cdot \tilde{m}_{I(L)}(U) \tag{17.}$$

After the L attributes have been combined one-by-one using the above FIER algorithm, the overall assessment of an alternative a_l can be obtained as:

$$\beta(H_{pq}) = \frac{m_{I(L)}(H_{pq})}{1-m_{I(L)}(U)},\ p = 1,...,N, q = p,...,N \tag{18.}$$

$$\beta(\bar{H}_{p\wedge(p+1)}) = \frac{m_{I(L)}(\bar{H}_{p\wedge(p+1)})}{1-m_{I(L)}(U)},\ p = 1,...,N-1 \tag{19.}$$

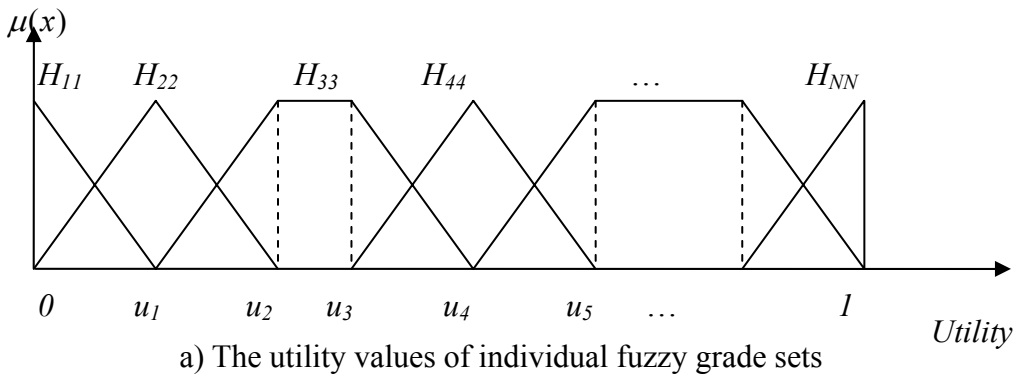
3. Fuzzy expected utilities for characterising alternatives

Utility is one of the most important concepts in decision analysis. In fuzzy MADA, however, utilities corresponding to fuzzy assessment grades can no longer be represented by singleton numerical values because the evaluation grades are fuzzy sets. In general, a fuzzy grade utility should have the same form as its corresponding fuzzy assessment grade. For example, if a fuzzy assessment grade is a triangular fuzzy number, its corresponding fuzzy grade utility should also be a triangular fuzzy number. In the FIER methodology according to the definitions of fuzzy grades in section 2.1, the utility values of an interval fuzzy grade can be calculated from the utility values of the correspondent fuzzy individual grades as shown in Fig. 3.

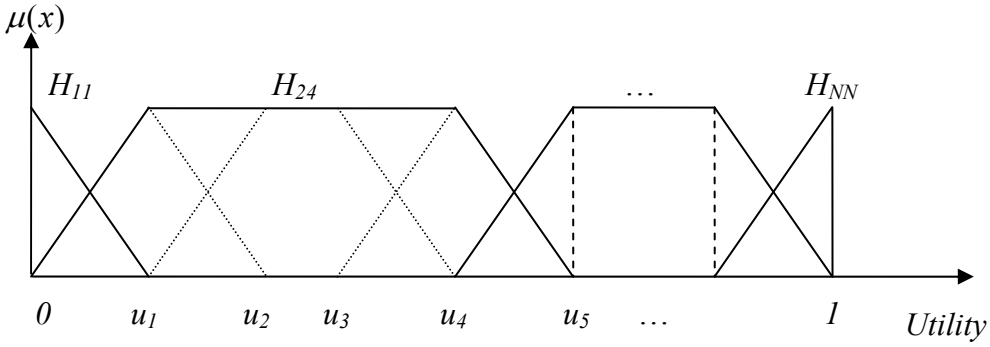
According to the basic ER methodology, the fuzzy expected utility of an aggregated assessment $S(y(a_l))$ for alternative a_l is defined as follows:

$$u(S(y(a_l))) = \sum_{p=1}^{N}\sum_{q=p}^{N}\beta(H_{pq})u(H_{pq}) + \sum_{p=1}^{N-1}\beta(\bar{H}_{p\wedge(p+1)})u(\bar{H}_{p\wedge(p+1)}) \tag{20.}$$

where $u(H_{pq})$ is the fuzzy grade utility of the assessment grade H_{pq}, and $u(\bar{H}_{p\Lambda(p+1)})$ is the fuzzy grade utility of the intersection fuzzy grade set $\bar{H}_{p\Lambda(p+1)}$. Without loss of generality, suppose $u(H_{pp})$ is the utility value of the grade H_{pp} with $u(H_{(p+1)(p+1)}) \geq u(H_{pp})$ as it is assumed that the grade $H_{(p+1)(p+1)}$ is preferred to H_{pp}. Suppose H_{11} is the least preferred fuzzy assessment grade, which has the lowest fuzzy grade utility, and H_{NN} is the most preferred fuzzy assessment grade, which has the highest fuzzy grade utility. Suppose $u(H_{pq})$ can take the lower bound value, the upper bound value and the two most possible values as $u_{\min}(H_{pq}), u_{\max}(H_{pq})$, $u_{MP1}(H_{pq})$ and $u_{MP2}(H_{pq})$ ($u_{MP1}(H_{pq}) \leq u_{MP2}(H_{pq})$) respectively if all grade sets are triangular or trapezoidal fuzzy sets. It is straightforward that the following equations hold according to the relationships of individual and interval grade sets:



a) The utility values of individual fuzzy grade sets



b) The utility values of Interval fuzzy grade sets

Fig 3 the utility of fuzzy grades

$$u_{\min}(H_{pq}) = u_{\min}(H_{pp}) \tag{21.}$$

$$u_{\max}(H_{pq}) = u_{\max}(H_{qq}) \tag{22.}$$

$$u_{MP1}(H_{pq}) = u_{MP1}(H_{pp}) \tag{23.}$$

$$u_{MP2}(H_{pq}) = u_{MP2}(H_{qq}) \tag{24.}$$

where in equation (21), the belief degree $\beta(H_{pq})$ could be assigned to the best grade in the interval grade H_{pq}, which is H_{qq}, and also can be assigned to the worst grade H_{pp} as shown in equation (22).

Similarly, suppose $u(\bar{H}_{p\Lambda(p+1)})$ can take the lower bound value, the upper bound value and the two most possible values as $u_{\min}(\bar{H}_{p\Lambda(p+1)})$, $u_{\max}(\bar{H}_{p\Lambda(p+1)})$, $u_{MP1}(\bar{H}_{p\Lambda(p+1)})$ and $u_{MP2}(\bar{H}_{p\Lambda(p+1)})$ ($u_{MP1}(\bar{H}_{p\Lambda(p+1)}) = u_{MP2}(\bar{H}_{p\Lambda(p+1)})$) respectively, and the following equations hold:

$$u_{\min}(\bar{H}_{p\Lambda(p+1)}) = u_{\min}(H_{(p+1)(p+1)}) \tag{25.}$$

$$u_{\max}(\bar{H}_{p\Lambda(p+1)}) = u_{\max}(H_{pp}) \tag{26.}$$

Accordingly, the fuzzy expected utility $u(S(y(a_l)))$ is also a fuzzy number. From equations (20-26), the maximum utility value of alternative a_l could be calculated as:

$$u_{\max}(a_l) = \sum_{p=1}^{N}\sum_{q=p}^{N}\beta(H_{pq})u_{\max}(H_{qq}) + \sum_{p=1}^{N-1}\beta(\bar{H}_{p\Lambda(p+1)})u_{\max}(\bar{H}_{pp}) \tag{27.}$$

Similarly, in the worst case, if the uncertainty turned out to be against the assessed alternative, with the belief degree $\beta(H_{pq})$ being assigned to H_{pp} (the worst grade in the interval grade H_{pq}) and $\beta(\bar{H}_{p\Lambda(p+1)})$ assigned to H_{pp}, then the minimum utility value would be given by:

$$u_{\min}(a_l) = \sum_{p=1}^{N}\sum_{q=p}^{N}\beta(H_{pq})u_{\min}(H_{pp}) + \sum_{p=1}^{N-1}\beta(\bar{H}_{p\Lambda(p+1)})u_{\min}(\bar{H}_{(p+1)(p+1)}) \tag{28.}$$

We can also define the two most possible utilities and their average value as follows:

$$u_{MPV1}(a_l) = \sum_{p=1}^{N}\sum_{q=p}^{N}\beta(H_{pq})u_{MPV1}(H_{pp}) + \sum_{p=1}^{N-1}\beta(\bar{H}_{p\Lambda(p+1)})u_{MPV1}(\bar{H}_{p\Lambda(p+1)}) \tag{29.}$$

$$u_{MPV2}(a_l) = \sum_{p=1}^{N}\sum_{q=p}^{N}\beta(H_{pq})u_{MPV2}(H_{qq}) + \sum_{p=1}^{N-1}\beta(\bar{H}_{p\Lambda(p+1)})u_{MPV2}(\bar{H}_{p\Lambda(p+1)}) \tag{30.}$$

$$u_{AVG-MPV}(a_l) = \frac{u_{MP1}(a_l) + u_{MP2}(a_l)}{2} \quad (31.)$$

4. Application of the FIER approach to a new product screening problem

The company concerned is an electronic manufacturer, which manufactures a wide range of electronic entertainment products. Every year, the company identifies market requirements and comes up with a list of potential product development projects. Suppose there are three new computer game projects available: Motor Cycling, Sport Bass Fishin' and Play TV Baseball. However, at a preliminary design phase, the assessment of a project on multiple criteria is mainly based on experts' subject judgments. Experts' opinions may be expressed by belief degrees (or possibility measures) based on basic evaluation grades, i.e. {Bad, Poor, Average, Good, and Excellent}. As such, the basic evaluation grade set can be defined as a set H as follows:

$$H = \{H_{11},\ H_{22},\ H_{33},\ H_{44},\ H_{55}\} = \{\text{Bad, Poor, Average, Good, Excellent}\}$$

Due to the high level of uncertainty involved in this NPD problem, however, these evaluation grades may not be regarded as crisp sets. For example, it is difficult to separate the grade Bad from the grade Poor especially if evaluations need to be given between these two grades. Also it is not surprising that for some evaluations the experts prefer to give the belief degree measures on interval grades. For example, the TIMING for Sport Bass Fishin' is {(H34, 1.0)}, which means that means that 100% belief is given to interval grade H_{34}, i.e. the worst assessment for Sport Bass Fishin' on TIMING is Average and the highest is Good. However, the exact belief degree to each of the two grades is not known. In a similar way, the incomplete opinions of the experts in evaluating this NPD problem can be captured conveniently by the following fuzzy evaluation grades.

$$H_F = \{H_{pq}, p = 1,...,5, q = p,...,5\} \cup \{H_{p\wedge(p+1)}, p = 1,...,4\}$$

Based on the experts' opinions, we can approximate all the five individual assessment grades by either triangular or trapezoidal fuzzy numbers as shown in table 3, and the maximum degree of membership for every fuzzy intersection set is 0.5.

Table 3 Membership functions of the fuzzy assessment grades and their fuzzy utilities

Linguistic term	Worst (W)	Poor (P)	Average (A)	Good (G)	Excellent (E)
Membership functions of fuzzy grade utilities	(0, 0, 0.2)	(0, 0.2, 0.4)	(0.2, 0.4, 0.6,0.8)	(0.6, 0.8, 1)	(0.8, 1, 1)

By using our proposed FIER methodology, the aggregated performance distribution of all the three alternative projects can be calculated. The expected maximum and minimum utilities can also be calculated according to formulae (27)-(31), as shown in table 5. A final rank order can be obtained as follows. Sport Bass Fishin' is possibly better than Motor Cycling and Play TV Baseball

according to the average MPV values of all the three projects presented. However, it is obviously that Sport Bass Fishin' does not absolutely dominate the other two projects. This is because

Table 4 Belief Matrix of the Performance Assessment Problem

Criteria	Weights	Motor Cycling	Sport Bass Fishin'	Play TV Baseball
TIMING	0.1	{(H44, 1.0)}	{(H34, 1.0) }	{(H12, 1.0)}
PRICE	0.1	{(H11, 1.0)}	{(H44, 0.9), (H15, 0.1)}	{(H45, 0.9), (H15, 0.1)}
LOGISTICS	0.05	{(H44, 1.0)}	{(H45, 1.0)}	{(H45, 1.0)}
SALES	0.02	{(H33, 1.0)}	{(H33, 1.0)}	{(H22, 1.0)}
MFGTECH	0.02	{(H44, 1.0)}	{(H33, 0.6), (H44, 0.4)}	{(H22, 1.0)}
MFGCAP	0.02	{(H44, 1.0)}	{(H44, 1.0)}	{(H45, 1.0)}
SUPPLY	0.05	{(H34, 1.0)}	{(H34, 1.0)}	{(H45, 1.0)}
DESIGN	0.1	{(H11, 1.0)}	{(H44, 0.8), (H15, 0.2)}	{(H45, 0.8), (H15, 0.2)}
DIFFADV	0.08	{(H11, 1.0)}	{(H55, 1.0)}	{(H55, 1.0)}
PAYOFFS	0.08	{(H11, 1.0)}	{(H44, 0.8), (H15, 0.2)}	{(H45, 0.8), (H15, 0.2)}
LOSSES	0.08	{(H44, 1.0)}	{(H22, 0.9), (H15, 0.1)}	{(H23, 0.9), (H15, 0.1)}
R&DUNC	0.25	{(H34, 1.0)}	{(H34, 0.9), (H15, 0.1)}	{(H12, 0.8), (H15, 0.2)}
NONR&D	0.05	{(H44, 1.0)}	{(H33, 0.8), (H15, 0.2)}	{(H12, 0.8), (H15, 0.2)}

Table 5 Fuzzy expected utilities and ranking order of alternatives

	Fuzzy expected utility u				
	Lower bound $u_{\min}$	Most possible value u_{MP1} and u_{MP2}		Upper bound $u_{\max}$	Avg. of MPV $u_{AVG-MPV}$
Motor Cycling	0.2893	0.4253	0.5538	0.7517	0.4896
Sport Bass Fishin'	0.4231	0.5958	0.7360	0.8944	0.6659
Play TV Baseball	0.2831	0.3960	0.6458	0.7406	0.5209

$u_{\min}$ (Sport Bass Fishin') = 0.4231 < $u_{\max}$ (Motor Cycling) = 0.7517

< $u_{\max}$ (Play TV Baseball) = 0.7406

While in the sense of MPV dominance, we can obtain:

u_{MP1} (Sport Bass Fishin') = 0.5958 > u_{MP2} (Motor Cycling) = 0.5538

This means that Sport Bass Fishin' is preferred to Motor Cycling in the sense of MPV dominance, or

Sport Bass Fishin' $\succ_{MPV}$ Motor Cycling

While the relationships between Sport Bass Fishin' and Play TV Baseball and between Play TV Baseball and Motor Cycling are not clear even in the sense of the most possible value dominance due to the uncertainties in the initial assessment data. In order to generate clearer dominant relations, more information or more accurate evaluations are needed.

5. Concluding remarks

Incompleteness and fuzziness are among the most common uncertainties in complex MADA problems. The new development as reported in this paper further extends the capability of the ER approach to utilise information with both local ignorance or interval uncertainty and fuzzy linguistic evaluation grades. Expert judgements can be captured by our proposed FIER method in such a convenient way that the evaluations made by experts, which are incomplete and fuzzy in nature, do not need to be converted to some strictly defined formats that may inevitably lead to the loss of important information, as shown in some classical MADA methods. In this sense, our FIER method can be used to deal with various types of uncertainties to help the DMs in making more informative decisions.

Similar with the previous ER approach, this FIER method is aimed to generate the preference orders of alternatives without having to gather perfect or complete information as is often done in real life decision making. However, the results obtained using the new methods may be an incomplete preference order as well due to the incompleteness and fuzziness in initial data, as illustrated in the example. In such cases, more information may be needed to support specific decision making such as finding a single winner in a performance assessment problem. Further research is needed to investigate the process of information gathering for sensitivity analyse.

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Appendix. Proof of the fuzzy IER aggregation algorithm

In the derivation of the IER algorithms in Equations (6)–(8), it was assumed that the evaluation grades are independent of each other. Due to the dependency of the adjacent fuzzy assessment grades on each other as shown in Fig. 1, the IER algorithms can no longer be employed without modification to aggregate attributes assessed using such fuzzy grades. However, the evidence theory provides scope to cope with such fuzzy assessments. The ideas similar to those used to develop the non-fuzzy evidential reasoning algorithm (Yang and Singh, 1994; Yang and Sen, 1994; Yang and Xu 2002a) are used to deduce the following fuzzy evidential reasoning algorithm.

The First problem is that the non-fuzzy IER algorithm, which follows the basic ER combination rules, is given in recursive forms. In each recursive step, a normalization procedure is taken to ensure that the possibility mass assigned to an empty set is set to zero. However, with the normalization process the non-fuzzy IER can not be transformed directly to a fuzzy IER algorithm. This is because the basic ER and IER algorithms are derived from the D-S combination rules, in which the normalizations can be postponed to the end of the recursive algorithm. According to the Yen 1994, we can draw the following conclusion.

Theorem: In the ER and IER recursive combination rules, the normalization can be postponed to the end of the recursive algorithms without changing the results.

Another challenge is that the intersection $H_{p\Lambda(p+1)}$ of the two adjacent evaluation grades H_{kp} and $H_{(p+1)q}$, where $k \le p$, $q \ge p+1$, is not empty in general, as shown in Fig.1. As in Yang et al 2006, the fuzzy intersection subset $H_{p\Lambda(p+1)}$, whose maximum degree of membership is represented by $\mu^{\max}_{p\Lambda(p+1)}$ and is usually less than one, will be normalized as a fuzzy subset $\bar{H}_{p\Lambda(p+1)}$ with the maximum membership degree being one, as shown in Fig. 2, so that $\bar{H}_{p\Lambda(p+1)}$ can be measured as a formal fuzzy set therefore assessed in the same scale as the other defined fuzzy evaluation grades such as H_{pq} (Fig. 1).

From the combination rule of the D-S theory and the ER methodology, equation (12-14) can be obtained straightforward. Then we need to prove formula (15-18) in which the assessments for the first $I(i)$ attributes are combined with that for attribute $i+1$ to generate an assessment for the $I(i+1)$ attributes.

We can separate the whole sets into three categories. The first category includes the basic interval grade sets H_{pq} for $q \ge p$, the second is the fuzzy intersection sets $H_{p\Lambda(p+1)}$ for $p = 1,...,N-1$, and the third is the set U.

It can be shown that these sets have the following relations:

$$H_{kq} \bigcap H_{pl} = H_{pq}, \quad q \ge p, \quad k = 1,...,p, \quad l = q,...,N$$

$$H_{kp} \bigcap H_{(p+1)q} = H_{p\Lambda(p+1)}, \quad k = 1,...,p, \quad q = p+1,...,N, \; p = 1,...,N-1$$

$$H_{kl} \bigcap H_{p\Lambda p+1} = H_{p\Lambda(p+1)}, \quad k = 1,...,p+1, \quad l = p,...,N, \quad k \le l, \quad p = 1,...,N-1$$

$H_{p\Lambda(p+1)} \bigcap U = H_{p\Lambda(p+1)}, \quad p = 1,...,N-1$

$H_{pq} \bigcap U = H_{pq}, \quad q \geq p$

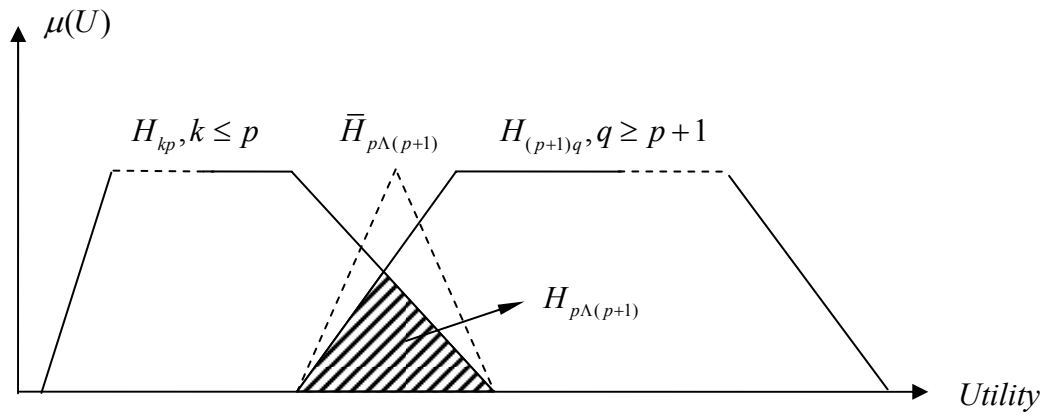

Fig. 6. Intersection of fuzzy sets

Regarding H_{pq}, only $H_{kq} \bigcap H_{pl}$ and $H_{pq} \bigcap U$ have the results of H_{pq}, although without any relation with the set $H_{p\Lambda(p+1)}$, so the combination results for H_{pq} are consistent with those of the basic IER algorithm, i.e.:

$$\begin{aligned}
\tilde{m}_{I(i+1)}(H_{pq}) &= -\tilde{m}_{I(i)}(H_{pq}) m_{i+1}(H_{pq}) \\
&+ \sum_{k=1}^{p} \sum_{l=q}^{N} [\tilde{m}_{I(i)}(H_{kl}) m_{i+1}(H_{pq}) + \tilde{m}_{I(i)}(H_{pq}) m_{i+1}(H_{kl})] \\
&+ \sum_{k=1}^{p-1} \sum_{l=q+1}^{N} [\tilde{m}_{I(i)}(H_{kq}) m_{i+1}(H_{pl}) + \tilde{m}_{I(i)}(H_{pl}) m_{i+1}(H_{kq})] \\
&+ \tilde{m}_{I(i)}(U) m_{i+1}(H_{pq}) + \tilde{m}_{I(i)}(H_{pq}) m_{i+1}(U),
\end{aligned}$$

Regarding $H_{p\Lambda(p+1)}$, we only need to consider the cases $H_{kp} \bigcap H_{(p+1)q} = H_{p\Lambda(p+1)}$, $H_{kl} \bigcap H_{p\Lambda p+1} = H_{p\Lambda(p+1)}$, $H_{p\Lambda(p+1)} \bigcap U = H_{p\Lambda(p+1)}$. So

$$\begin{aligned}
\tilde{m}_{I(i+1)}(H_{p\Lambda(p+1)}) &= \sum_{k=1}^{p} \tilde{m}_{I(i)}(H_{kp}) \cdot \sum_{q=p+1}^{N} m_{i+1}(H_{(p+1)q}) + \sum_{k=1}^{p} m_{i+1}(H_{k,p}) \cdot \sum_{q=p+1}^{N} \tilde{m}_{I(i)}(H_{(p+1)q}) \\
&+ \sum_{k=1}^{p+1} \sum_{\substack{l=p \\ l \geq k}}^{N} [m_{i+1}(H_{kl}) \cdot \tilde{m}_{I(i)}(H_{p\Lambda(p+1)}) + \tilde{m}_{I(i)}(H_{kl}) \cdot m_{i+1}(H_{p\Lambda(p+1)})] \\
&+ \tilde{m}_{I(i)}(H_{p\Lambda(p+1)}) m_{i+1}(U) + \tilde{m}_{I(i)}(U) m_{i+1}(H_{p\Lambda(p+1)}),
\end{aligned}$$

Because $m_{i+1}(H_{p\Lambda(p+1)}) = 0$, then

$$\tilde{m}_{I(i+1)}(H_{p\Lambda(p+1)}) = \sum_{k=1}^{p} \tilde{m}_{I(i)}(H_{kp}) \cdot \sum_{q=p+1}^{N} m_{i+1}(H_{(p+1)q}) + \sum_{k=1}^{p} m_{i+1}(H_{kp}) \cdot \sum_{q=p+1}^{N} \tilde{m}_{I(i)}(H_{(p+1)q})$$
$$+ \sum_{k=1}^{p+1} \sum_{\substack{l=p \\ l \geq k}}^{N} m_{i+1}(H_{kl}) \cdot \tilde{m}_{I(i)}(H_{p\Lambda(p+1)}) + \tilde{m}_{I(i)}(H_{p\Lambda(p+1)}) m_{i+1}(U)$$

It is straightforward to obtain

$$\tilde{m}_{I(i+1)}(U) = \tilde{m}_{I(i)}(U) m_{i+1}(U) = \prod_{l=1}^{i+1} m_l(U)$$

Finally, after all the L Attributes are combined, we have,

$$\tilde{m}(H_{pq}) = \tilde{m}_{I(L)}(H_{pq}),$$

$$\tilde{m}(U) = \tilde{m}_{I(L)}(U) \text{ and}$$

$$\tilde{m}(H_{p\Lambda(p+1)}) = \tilde{m}_{I(L)}(H_{p\Lambda(p+1)}).$$

Since the fuzzy subset $H_{p\Lambda(p+1)}$ is the intersection of the two fuzzy assessment grades H_{kp} and $H_{(p+1)q}$, its maximum degree of membership is normally not equal to unity. In order to capture the exact probability mass assigned to $H_{p\Lambda(p+1)}$, its membership function needs to be normalized. If this were not done, then probability mass assigned to $H_{p\Lambda(p+1)}$ would have nothing to do with its shape or height. In other words, as long as the two fuzzy assessment grades H_{kp} and $H_{(p+1)q}$ intersect, the probability mass assigned to $H_{p\Lambda(p+1)}$ would always be the same, no matter how large or small the intersection subset may be. So, it is necessary to normalise the membership function of $H_{p\Lambda(p+1)}$. After normalization, we have (Yang et al, 2006),

$$\tilde{m}(\bar{H}_{p\Lambda(p+1)}) = \mu_{p\Lambda(p+1)}^{\max} \cdot \tilde{m}(H_{p\Lambda(p+1)}) = \mu_{p\Lambda(p+1)}^{\max} \cdot \tilde{m}_{I(L)}(H_{p\Lambda(p+1)})$$

According to Yen 1990, the results should be normalized at the end:

$$K = 1/[\sum_{p=1}^{N} \sum_{q=p}^{N} \tilde{m}(H_{pq}) + \sum_{p=1}^{N-1} \tilde{m}(\bar{H}_{p\Lambda(p+1)}) + \tilde{m}(U)]$$
$$= 1/[\sum_{p=1}^{N} \sum_{q=p}^{N} \tilde{m}_{I(L)}(H_{pq}) + \sum_{p=1}^{N-1} \mu_{p\Lambda(p+1)}^{\max} \tilde{m}_{I(L)}(H_{p\Lambda(p+1)}) + \tilde{m}_{I(L)}(U)]$$

$$m(H_{pq}) = K \cdot \tilde{m}(H_{pq}) = K \cdot \tilde{m}_{I(L)}(H_{pq})$$

$$m(\bar{H}_{p\Lambda(p+1)}) = K \cdot \tilde{m}(\bar{H}_{p\Lambda(p+1)}) = K \cdot \mu_{p\Lambda(p+1)}^{\max} \cdot \tilde{m}_{I(L)}(H_{p\Lambda(p+1)})$$

$$m(U) = K \cdot \tilde{m}(U) = K \cdot \tilde{m}_{I(L)}(U)$$

Q.E.D.