Conjugacy Classes and Complex Characters in Finite Reductive Groups

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Postgraduate Group Theory Conference
St. Andrews, May 2010

Let G be a connected reductive algebraic group which has an $\mathbb{F}_q$ rational structure, where $q = p^a$ for some prime $p > 0$. Let $F : G \to G$ be the associated Frobenius endomorphism. Then we have the fixed point group $G^F = \{x \in G \mid F(x) = x\} = G(q)$ is a finite reductive group.

Example

Let $G = \mathrm{GL}_n(\overline{\mathbb{F}_p})$. Then we have a standard Frobenius endomorphism given by $F_q(x_{ij}) = (x_{ij}^q)$. The fixed point group is the finite group $G^{F_q} = \mathrm{GL}_n(q)$.

Why study $\mathrm{GL}_n(q)$ in this way?
We can use powerful tools of algebraic geometry to obtain information about G and, using the Frobenius endomorphism, we can pass this information to G^F.

In any algebraic group we have a so called abstract Jordan-Chevalley decomposition of its elements. In other words for any $g \in G$ there exists unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, where g_s is diagonalisable and every eigenvalue of g_u is 1. We call g_s the semisimple part of g and g_u the unipotent part of g.

We get associated decompositions of elements in the finite group G^F. Semisimple and unipotent elements in G^F are characterised by their orders. We have $x \in G^F$ is semisimple if and only if x has order prime to p and x is unipotent if and only if x has order a power of p.

Example

Let $G = \mathrm{GL}_2(\overline{\mathbb{F}_5})$ and $G^F = \mathrm{GL}_2(5)$.

$$s = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad (s^4 = 1) \qquad\qquad u = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (u^5 = 1)$$

The Jordan decomposition shows us that, to determine conjugacy classes in G, it suffices to determine the semisimple and unipotent classes. This is mostly very well understood.

- Let $T \leqslant G$ be a maximal torus of G, (in the case of $\mathrm{GL}_n(\overline{\mathbb{F}_p})$ this can be chosen to be the subgroup of all diagonal matrices). Then semisimple classes of G are determined by the orbits of the Weyl group acting on T.
- Unipotent classes are more complicated. If p is a so called good prime for G then the unipotent classes were explicitly determined by Bala & Carter. If G is a classical group then the unipotent classes are parameterised by their Jordan normal forms.

Without too much extra hassle we can then use this to determine the classes of G^F.

Let H be a finite group and $\rho : H \to \mathrm{GL}_n(\mathbb{C})$ a complex representation of H. We then define the character of ρ to be the function $\chi_\rho : H \to \mathbb{C}$ defined by $\chi_\rho(h) = \mathrm{Trace}(\rho(h))$ for all $h \in H$. Recall that given ρ we can always construct a $\mathbb{C}H$-module $V_\rho = \mathbb{C}^n$ by defining the action of H to be $h \cdot v = \rho(h)v$ for all $v \in V$. We say χ_ρ is an irreducible character if V_ρ is a simple $\mathbb{C}H$ module.

Recollections

- A complex representation of H is uniquely determined by its character.
- The irreducible characters of H form a basis for the space of all complex class functions. In particular any character is an $\mathbb{N}$-linear combination of irreducibles.
- The number of irreducible characters is the same as the number of conjugacy classes in H.

In general given a finite group it is impossible to create a canonical bijection between irreducible characters and conjugacy classes. Can be done in special cases, (for example in the symmetric group using partitions). In 1980 Lusztig posed the following problem for a finite reductive group.

Problem

Let χ be a complex irreducible character of G^F. Show that there exists a unique F-stable unipotent class $\mathcal{O}$ in G which has the property that

$$\sum_{g\in\mathcal{O}^F}\chi(g)\neq 0$$

and $\mathcal{O}$ has maximal dimension amongst all classes with this property.

If such a unipotent class exists then we call this the unipotent support of χ.

If p is a good prime for G then it is known that unipotent supports always exist, (shown by Lusztig for p, q large and Geck for p a good prime). The solution to this problem is achieved by considering Kawanaka's theory of Generalised Gelfand-Graev Representations. Therefore we have a surjective map from irreducible characters to classes. What does this tell us?

Recall that if χ is a complex irreducible character of G^F then the degree of χ is a polynomial in q. In other words for some $n_\chi > 0$ and $a_\chi \geqslant 0$

$$n_\chi \cdot \chi(1) = q^{a_\chi} + \text{higher powers of } q.$$

The value a_χ was shown by Geck & Malle to be $\dim \mathfrak{B}_u$, where $\mathfrak{B}_u$ is the variety of Borel subgroups containing u and u is any element of the unipotent support of χ.

Let Φ be the map sending an irreducible character to its unipotent support. The following was stated by Lusztig in his book, (without proof).

Statement

Let p be a good prime for G and assume the centre of G is connected. Let $\mathcal{O}$ be an F-stable unipotent class of G and $u \in \mathcal{O}$. We write $A_G(u) = C_G(u)/C_G(u)^\circ$ for the component group of the centraliser. Then there exists an irreducible character χ such that $\Phi(\chi) = \mathcal{O}$ and $n_\chi = |A_G(u)|$.

A rigorous proof was later given independently by Hézard and Lusztig.

Problem

Show the above statement holds even when the centre of G is disconnected.

Why is this worthwhile? Statement was used by Hézard to prove a conjecture of Kawanaka, (for the case of G with connected centre), that any unipotently supported virtual character of G^F is a $\mathbb{Z}$-linear combination of Generalised Gelfand Graev Representations. Also managed to give a characterisation of GGGR's based on the degree of the character and certain vanishing properties on unipotent classes. Hope to show the results of Hézard hold for any finite reductive group defined over a good prime.

Recently Lusztig has shown a way to construct a map from conjugacy classes in the Weyl group to unipotent classes in an algebraic group. It would be interesting to see if there is any connection between the conjugacy classes in the Weyl group and complex irreducible characters sharing the same unipotent class in G.