On Lusztig's Conjecture for Character Sheaves of Classical-Type Groups

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- Algebraic Closures: $\mathbb{K} = \overline{\mathbb{F}}_p$ ($p > 2$ a prime) and $\overline{\mathbb{Q}}_\ell$ ($\ell \neq p > 0$ a prime).
- $\mathbf{G}$ a connected reductive algebraic group over $\mathbb{K}$ such that $Z(\mathbf{G})$ is connected and $\mathbf{G}/Z(\mathbf{G})$ is simple of type B_n, C_n or D_n.

B_n	C_n	D_n
$\mathrm{SO}_{2n+1}(\mathbb{K})$	$\mathrm{CSp}_{2n}(\mathbb{K})$	$\mathrm{CO}_{2n}(\mathbb{K})^\circ$

- $F : \mathbf{G} \to \mathbf{G}$ is a split Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure $\mathbf{G}^F$.

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Goal

Let $\mathrm{Irr}(\mathbf{G}^F) = \{\chi_1, \dots, \chi_r\}$ be the irreducible $\overline{\mathbb{Q}}_\ell$-characters of $\mathbf{G}^F$ and let $\{g_1, \dots, g_r\}$ be conjugacy class representatives of $\mathbf{G}^F$. Give a method to explicitly compute the character table

$$(\chi_i(g_j))_{1 \leqslant i,j \leqslant r}.$$

Recall we have the $\overline{\mathbb{Q}}_\ell$-vector space of all class functions

$$\mathrm{Cent}(\mathbf{G}^F) = \{f : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell \mid f(xgx^{-1}) = f(g) \text{ for all } x, g \in \mathbf{G}^F\}.$$

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Possible Solution

Find a basis $\mathcal{A}$ of $\mathrm{Cent}(\mathbf{G}^F)$ such that:

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A Computable Basis

Take $\mathcal{A} = \{f_1, \dots, f_r\}$ to be the characteristic functions of the conjugacy classes of $\mathbf{G}^F$. In other words for each $1 \leqslant i \leqslant r$ we define

$$f_i(g) = \begin{cases} |C_{\mathbf{G}^F}(g)| & \text{if } xgx^{-1} = g_i \text{ for some } x \in G \\ 0 & \text{otherwise.} \end{cases}$$

- Lusztig (1984) formally introduces the notion of an almost character.
- Defined on a case by case basis.
- Explicit linear combination of irreducible characters of $\mathbf{G}^F$.
- Coefficients given by a Fourier transform matrix (classical types) or non-abelian analogue (exceptional types).

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Theorem (Lusztig, 1984)

A subset $\mathcal{B}$ of the set of almost characters is an orthonormal basis of $\mathrm{Cent}(\mathbf{G}^F)$. Furthermore the change of basis matrix from $\mathcal{B}$ to $\mathrm{Irr}(\mathbf{G}^F)$ is explicitly known.

Fix an F-stable maximal torus $\mathbf{T}_0 \leqslant \mathbf{G}$ and let $W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ be the Weyl group of $\mathbf{G}$. For each $w \in W$ define an F-stable maximal torus

$$\mathbf{T}_w := g\mathbf{T}_0 g^{-1} \qquad \text{some } g \in \mathbf{G} \text{ with } g^{-1}F(g) \mapsto w \in W.$$

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By Deligne–Lusztig (1976) we have for each $w \in W$ a virtual character

$$R_{\mathbf{T}_w}^{\mathbf{G}}(1) \in \mathbb{Z}\operatorname{Irr}(G).$$

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Example of an Almost Character

Let $\rho \in \mathrm{Irr}(W)$ then we define

$$R_\rho = \frac{1}{|W|}\sum_{w \in W} \rho(w) R_{\mathbf{T}_w}^{\mathbf{G}}(1) \in \mathbb{Q}\,\mathrm{Irr}(G).$$

We call R_ρ a uniform unipotent almost character.

$\mathscr{D}\mathbf{G}$:= the bounded derived category of $\overline{\mathbb{Q}}_\ell$-constructible sheaves on $\mathbf{G}$

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- An object $A \in \mathscr{D}\mathbf{G}$ is a complex

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

  of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbf{G}$ such that for each $i \in \mathbb{Z}$ the cohomology sheaf $\mathscr{H}^i(A)$ is constructible.
- In particular, for each $x \in \mathbf{G}$, the stalk $\mathscr{H}^i_x(A)$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Furthermore we have $\mathscr{H}^i_x(A) \neq 0$ for only finitely many $i \in \mathbb{Z}$.

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Definition

A character sheaf of $\mathbf{G}$ is a $\mathbf{G}$-equivariant simple object in $\mathscr{M}\mathbf{G}$. We denote by $\widehat{\mathbf{G}}$ the set of character sheaves of $\mathbf{G}$.

The Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defines a functor

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and ϕ_A induces an automorphism $\phi_A : \mathscr{H}_x^i(A) \to \mathscr{H}_x^i(A)$. We define the characteristic function of A to be $\chi_{A,\phi_A} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ given by

$$\chi_{A,\phi_A}(g) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(\phi_A, \mathscr{H}_g^i(A)).$$

Theorem (Lusztig, 1986)

There exists a family of isomorphisms $\{\phi_A : F^*A \to A \mid A \in \widehat{\mathbf{G}}^F\}$, *(unique up to multiplication by roots of unity), such that*

$$\mathcal{A} = \{\chi_{A,\phi_A} \mid A \in \widehat{\mathbf{G}}^F\}$$

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Conjecture (Lusztig, 1985)

There is an explicit ordering of the bases $\mathcal{A}$ and $\mathcal{B}$ so that the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$ is diagonal and the non-zero entries are roots of unity.

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There is an explicit ordering of the bases $\mathcal{A}$ and $\mathcal{B}$ so that the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$ is diagonal and the non-zero entries are scalars of absolute value 1.

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Remaining Problem

Compute explicitly the change of basis matrix C from $\mathcal{A}$ to $\mathcal{B}$.

Assume $\mathbf{P} \leqslant \mathbf{G}$ is a parabolic with Levi complement $\mathbf{L} \leqslant \mathbf{P}$. Lusztig has defined a map

$$A_0 \in \widehat{\mathbf{L}} \qquad \rightsquigarrow \qquad \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0) \in \mathscr{M}\mathbf{G}$$

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- for any $A \in \widehat{\mathbf{G}}$ there exists a Levi subgroup $\mathbf{L} \leqslant \mathbf{P}$ and a cuspidal character sheaf $A_0 \in \widehat{\mathbf{L}}$ such that $(A : \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)) \neq 0$. Furthermore the pair $(\mathbf{L}, A_0)$ is unique up to $\mathbf{G}$-conjugacy.

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Definition

We say $A \in \widehat{\mathbf{G}}$ is cuspidal if $(A : \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)) \neq 0$ implies $\mathbf{L} = \mathbf{P} = \mathbf{G}$.

Definition

We say $A \in \widehat{\mathbf{G}}$ is unipotently supported if $\mathscr{H}_u^i(A) \neq 0$ for some $i \in \mathbb{Z}$ and $u \in \mathbf{G}$ unipotent.

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Theorem (Lusztig)

If $A \in \widehat{\mathbf{G}}$ is cuspidal and unipotently supported then

$$A = \mathrm{IC}(\overline{\mathcal{O}_0 Z^\circ(\mathbf{G})}, \mathscr{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell)$$

where $\mathcal{O}_0 \subseteq \mathbf{G}$ is a unipotent conjugacy class and $\mathscr{E}_0$ is a cuspidal local system on $\mathcal{O}_0$.

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A is F-stable if and only if $F(\mathcal{O}_0) = \mathcal{O}_0$ and $F^*\mathscr{E}_0 \cong \mathscr{E}_0$. Choose

$$\varphi : F^*\mathscr{E}_0 \to \mathscr{E}_0 \qquad \rightsquigarrow \qquad \varphi_u : (\mathscr{E}_0)_u \to (\mathscr{E}_0)_u \quad \text{is} \quad q^* \,\mathrm{id}$$

for any split unipotent element $u \in \mathcal{O}_0^F$. Now $\varphi \boxtimes \mathrm{id} \rightsquigarrow \phi_A$.

Let $(\mathbf{L}_1, A_1), \ldots, (\mathbf{L}_k, A_k)$ be representatives for the conjugation action of $\mathbf{G}^F$ on

$$\{(\mathbf{L}, A) \mid \mathbf{L} \leqslant \mathbf{G} \text{ an } F\text{-stable Levi and } A \in \widehat{\mathbf{L}}^F \text{ cuspidal}\}.$$

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We get a block decomposition

$$\widehat{\mathbf{G}}_i^F = \{A \in \widehat{\mathbf{G}}^F \mid (A : \mathrm{ind}_{\mathbf{L}_i \subseteq \mathbf{P}_i}^{\mathbf{G}}(A_i)) \neq 0\}$$

$$\widehat{\mathbf{G}}^F = \bigsqcup_{i=1}^{k} \widehat{\mathbf{G}}_i^F \quad \rightsquigarrow \quad C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_k \end{bmatrix}$$

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Lemma (Shoji)

Let $\zeta_i \in \overline{\mathbb{Q}}_\ell$ be such that $\zeta_i \chi_{A_i,\phi_{A_i}}$ is an almost character of $\mathbf{L}_i^F$. The isomorphism ϕ_{A_i} induces an isomorphism ϕ_A for each $A \in \widehat{\mathbf{G}}_i^F$ such that C_i is $(-1)^{a_i}\zeta_i$ times the identity matrix for some $a_i \in \mathbb{Z}_{\geqslant 0}$.

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Theorem (T., 2013)

Assume $A \in \widehat{\mathbf{G}}^F$ *is a unipotently supported cuspidal character sheaf and* $q \equiv 1 \pmod 4$ *when* $\mathbf{G}$ *is of type* C_n *or* D_n *then* χ_{A,ϕ_A} *is the corresponding almost character of* G.

- When $\mathbf{G} = \mathrm{SO}_{2n+1}(\mathbb{K})$ due to Lusztig - "On the Character Values of Finite Chevalley Groups at Unipotent Elements" (1986).

By Kawanaka (1986) we have a map

$$u \in G \text{ unipotent} \qquad \rightsquigarrow \qquad \gamma_u \in \mathrm{Cent}(G)$$

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$$\chi \in \mathrm{Irr}(G) \qquad \rightsquigarrow \qquad \mathcal{O}_\chi \subseteq \mathbf{G} \text{ an } F\text{-stable unipotent class}$$

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Lemma (Lusztig, 1992)

Assume p and q are sufficiently large. For any $\chi \in \mathrm{Irr}(G)$ and any $u \in \mathcal{O}_\chi^F$ we have $|\langle D_G(\gamma_u), \chi\rangle|$ is a "small" integer. In other words it is bounded independently of q.

Question

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- Lusztig has expressed each GGGR as an explicit linear combination of characteristic functions of F-stable unipotently supported character sheaves.
- In particular we may translate the problem

$$\langle D_G(\gamma_u), \chi \rangle \qquad \rightsquigarrow \qquad \langle D_G(\chi_{A,\phi_A}), \chi \rangle$$

- The right hand side is known once Lusztig's conjecture is explicitly known.
- Then one only has to do the translation.