Character Sheaves and GGGRs

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- $\mathbf{G}$ a connected reductive algebraic group defined over $\overline{\mathbb{F}_p}$.
- $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism defining an $\mathbb{F}_q$-rational structure $\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$.
- Fix a prime $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}_\ell}$. Interested in

$$\mathrm{Irr}(\mathbf{G}^F) \subset \mathrm{Cent}(\mathbf{G}^F) = \{f : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell \mid f(xgx^{-1}) = f(x)\}$$

Problem

Given $g \in \mathbf{G}^F$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ describe $\chi(g)$.

Two main cases to consider:

- $g \in \mathbf{G}^F_{\mathrm{ss}} = \{x \in \mathbf{G}^F \mid p \nmid \mathrm{o}(x)\}$
- $g \in \mathbf{G}^F_{\mathrm{uni}} = \{x \in \mathbf{G}^F \mid \mathrm{o}(x) = p^a\}$

For any F-stable maximal torus $\mathbf{T} \leqslant \mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ we have a virtual character

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F).$$

Theorem (Deligne–Lusztig, 1976)

For any $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ and $s \in \mathbf{G}_{\mathrm{ss}}^F$ we have

$$\chi(s) = \sum_{(\mathbf{T},\theta)/\sim} \langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi\rangle R_{\mathbf{T}}^{\mathbf{G}}(\theta)(s)$$

and

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(s) = \frac{1}{|C_{\mathbf{G}}^{\circ}(s)^F|} \sum_{\substack{x\in\mathbf{G}^F \\ x^{-1}sx\in\mathbf{T}^F}} \theta(x^{-1}sx).$$

$\mathscr{D}\mathbf{G}$:= the bounded derived category of $\overline{\mathbb{Q}}_\ell$-constructible sheaves on $\mathbf{G}$

$\mathscr{M}\mathbf{G}$:= the category of $\overline{\mathbb{Q}}_\ell$-perverse sheaves on $\mathbf{G}$

- Can think of an object $A \in \mathscr{D}\mathbf{G}$ as a bounded "complex"

$$\cdots \longrightarrow A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

  of $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathbf{G}$ such that for each $i \in \mathbb{Z}$ the cohomology sheaf $\mathscr{H}^i(A)$ is constructible.
- In particular, for each $x \in \mathbf{G}$, the stalk $\mathscr{H}_x^i(A)$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Furthermore we have $\mathscr{H}_x^i(A) \neq 0$ for only finitely many $i \in \mathbb{Z}$.

Definition

A character sheaf of $\mathbf{G}$ is a $\mathbf{G}$-equivariant simple object in $\mathscr{M}\mathbf{G}$. We denote by $\widehat{\mathbf{G}}$ the set of character sheaves of $\mathbf{G}$.

The Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ induces a functor

$$F^* : \mathscr{D}\mathbf{G} \to \mathscr{D}\mathbf{G}$$

which preserves $\widehat{\mathbf{G}}$. We say $A \in \mathscr{D}\mathbf{G}$ is *F-stable* if there exists an isomorphism

$$\phi_A : F^*A \to A \in \mathscr{D}\mathbf{G}.$$

We denote by $\widehat{\mathbf{G}}^F \subseteq \widehat{\mathbf{G}}$ the subset of F-stable character sheaves.

Definition

Assume now that $A \in \widehat{\mathbf{G}}^F$. For each $x \in \mathbf{G}^F$ and $i \in \mathbb{Z}$ we have

$$\mathscr{H}^i_x(F^*A) = \mathscr{H}^i_{F(x)}(A) = \mathscr{H}^i_x(A)$$

and ϕ_A induces an automorphism $\phi_A : \mathscr{H}^i_x(A) \to \mathscr{H}^i_x(A)$. We define the *characteristic function of A* to be $\chi_{A,\phi_A} : \mathbf{G}^F \to \overline{\mathbb{Q}}_\ell$ given by

$$\chi_{A,\phi_A}(g) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(\phi_A, \mathscr{H}^i_g(A)).$$

Theorem (Lusztig, 1986, 2012)

There exists a family of isomorphisms $\{\phi_A : F^*A \to A \mid A \in \widehat{\mathbf{G}}^F\}$ *(unique up to multiplication by roots of unity) such that*

$$\{\chi_{A,\phi_A} \mid A \in \widehat{\mathbf{G}}^F\}$$

is an orthonormal basis for $\mathrm{Cent}(\mathbf{G}^F)$.

Definition

We say $A \in \widehat{\mathbf{G}}$ is unipotently supported if $\mathscr{H}_u^i(A) \neq 0$ for some $i \in \mathbb{Z}$ and $u \in \mathbf{G}_{\mathrm{uni}}$.

Assume $\mathbf{P} \leqslant \mathbf{G}$ is a parabolic with Levi complement $\mathbf{L} \leqslant \mathbf{P}$. Lusztig has defined a map

$$A_0 \in \widehat{\mathbf{L}} \qquad \rightsquigarrow \qquad \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0) \in \mathscr{M}\mathbf{G}$$

called induction. The complex $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)$ satisfies the following properties:

- $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0) = A_0$ if $\mathbf{L} = \mathbf{P} = \mathbf{G}$.
- $\mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)$ is semisimple and all indecomposable summands are character sheaves.
- for any $A \in \widehat{\mathbf{G}}$ there exists a Levi subgroup $\mathbf{L} \leqslant \mathbf{P}$ and a cuspidal character sheaf $A_0 \in \widehat{\mathbf{L}}$ such that $(A : \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)) \neq 0$. Furthermore the pair $(\mathbf{L}, A_0)$ is unique up to $\mathbf{G}$-conjugacy.

Definition

We say $A \in \widehat{\mathbf{G}}$ is cuspidal if $(A : \mathrm{ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(A_0)) \neq 0$ implies $\mathbf{L} = \mathbf{P} = \mathbf{G}$.

Theorem (Lusztig)

If $A_0 \in \widehat{\mathbf{L}}$ *is cuspidal and unipotently supported then*

$$A_0 = \mathrm{IC}(\overline{\mathcal{O}_0} Z^\circ(\mathbf{L}), \mathscr{E}_0 \boxtimes \mathscr{L})[\dim \mathcal{O}_0 + \dim Z^\circ(\mathbf{L})]$$

where:

- $\mathcal{O}_0 \subseteq \mathbf{G}$ *is a unipotent conjugacy class,*
- $\mathscr{E}_0$ *is an* $\mathbf{L}$*-equivariant cuspidal local system on* $\mathcal{O}_0$*,*
- $\mathscr{L}$ *is a tame local system on* $Z^\circ(\mathbf{L})$*.*

Furthermore, the quotient group $W_{\mathbf{G}}(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ *is a Weyl group and*

$$\mathrm{End}_{\mathscr{D}\mathbf{G}}(\mathrm{ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(A_0)) \cong \overline{\mathbb{Q}}_\ell W_{\mathbf{G}}(\mathbf{L}, \mathscr{L})$$

In particular, we have a bijection

$$\{A \in \widehat{\mathbf{G}} \mid (A : \mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)) \neq 0\} \longleftrightarrow \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}, \mathscr{L}))$$

Denote by $\mathcal{N}_{\mathbf{G}}$ the set of all pairs $\iota = (\mathcal{O}_\iota, \mathscr{E}_\iota)$ where:

- $\mathcal{O}_\iota \subset \mathbf{G}$ is a unipotent class,
- $\mathscr{E}_\iota$ is a $\mathbf{G}$-equivariant local system on $\mathcal{O}_\iota$.

Theorem (Lusztig, 1984)

Denote by $\nu \in \mathcal{N}_{\mathbf{L}}$ the cuspidal pair $(\mathcal{O}_0, \mathscr{E}_0)$ and assume that $\mathscr{L} = \overline{\mathbb{Q}}_\ell$. Then there is a subset $\mathscr{I}(\mathbf{L}, \nu) \subseteq \mathcal{N}_{\mathbf{G}}$ and a natural bijection

$$\begin{aligned} \mathscr{I}(\mathbf{L}, \nu) &\to \{A \in \widehat{\mathbf{G}} \mid (A : \mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)) \neq 0\} \\ \iota &\mapsto K_\iota. \end{aligned}$$

Hence also a bijection

$$\begin{aligned} \mathscr{I}(\mathbf{L}, \nu) &\to \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L})) \\ \iota &\mapsto E_\iota. \end{aligned}$$

Let $A \in \widehat{\mathbf{G}}^F$ be an F-stable summand of $\mathrm{ind}_{\mathbf{L}}^{\mathbf{G}}(A_0)$ then we can assume:

$$F(\mathbf{L}) = \mathbf{L} \qquad F(\mathcal{O}_0) = \mathcal{O}_0 \qquad F^*\mathscr{E}_0 \cong \mathscr{E}_0 \qquad F^*\mathscr{L} \cong \mathscr{L}.$$

In particular we have:

- F induces an automorphism of $W_{\mathbf{G}}(\mathbf{L})$ and $W_{\mathbf{G}}(\mathbf{L}, \mathscr{L})$,
- If A is parameterised by $E \in \mathrm{Irr}(W_{\mathbf{G}}(\mathbf{L}, \mathscr{L}))$ then this is fixed by F.

Proposition

Assume we fix an isomorphism $\varphi_0 : F^*\mathscr{E}_0 \to \mathscr{E}_0$ and an extension $\widetilde{E}$ of E to $W_{\mathbf{G}}(\mathbf{L}, \mathscr{L}) \rtimes \langle F \rangle$ (similarly an extension $\widetilde{E}_\iota$ of E_ι). Then this induces isomorphisms

$$\phi_A : F^*A \to A \qquad \phi_\iota : F^*K_\iota \to K_\iota$$

Theorem (T., 2014)

$$\chi_{A,\phi_A}|_{\mathbf{G}^F_{\text{uni}}} = \sum_{\iota \in \mathscr{I}(\mathbf{L},\nu)^F} \langle \widetilde{E}_\iota, \operatorname{Ind}_{W_{\mathbf{G}}(\mathbf{L},\mathscr{L}).F}^{W_{\mathbf{G}}(\mathbf{L}).F}(\widetilde{E}) \rangle_{W_{\mathbf{G}}(\mathbf{L}).F} \cdot \chi_{K_\iota,\phi_\iota}$$

Theorem (Lusztig, T.)

Let $a_\iota = -\dim \mathcal{O}_\iota - \dim Z^\circ(\mathbf{L})$ *then we have*

$$\chi_{K_\iota,\phi_\iota} = (-1)^{a_\iota} q^{(\dim \mathbf{G} + a_\iota)/2} P_{\iota',\iota} Y_{\iota'}$$

Theorem (Bonnafé, Shoji, Waldspurger)

Assume p *is good for* $\mathbf{G}$ *and one of the following holds:*

- $Z(\mathbf{G})$ *is connected and* $\mathbf{G}/Z(\mathbf{G})$ *is simple,*
- $\mathbf{G}$ *is* $\mathrm{SL}_n(\overline{\mathbb{F}_p})$, $\mathrm{Sp}_{2n}(\overline{\mathbb{F}_p})$ *or* $\mathrm{SO}_n(\overline{\mathbb{F}_p})$.

Then the functions $Y_{\iota'}$ *are explicitly computable.*

Assume now that p is good for $\mathbf{G}$. By Kawanaka (1986) we have a map

$$u \in \mathbf{G}^F_{\mathrm{uni}} \qquad \rightsquigarrow \qquad \gamma_u \in \mathrm{Cent}(G)$$

where γ_u is the character of a generalised Gelfand–Graev representation. These satisfy the following properties:

- γ_u is obtained by inducing a linear character from a p-subgroup of $\mathbf{G}^F$,
- $\gamma_u = \gamma_v$ if $xux^{-1} = v$ for some $x \in \mathbf{G}^F$,
- γ_1 is the regular character and γ_u is a Gelfand–Graev character when u is a regular element.

Problem

Describe the multiplicities $\langle \gamma_u, \chi \rangle$ for all $\chi \in \mathrm{Irr}(\mathbf{G}^F)$.

Consider $\mathbf{G}^F = \mathrm{GL}_n(q)$ and $\mathbf{B}$ the upper triangular matrices then

$$\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}(1_{\mathbf{B}^F}) = \sum_{\rho \in \mathrm{Irr}(\mathfrak{S}_n)} \rho(1)\chi_\rho$$

and

$$\mathcal{E}(\mathbf{G}^F, 1) = \{\chi_\lambda \mid \lambda \vdash n\}$$

is the set of unipotent characters.

Theorem (Kawanaka)

$$\langle \Gamma_\mu, \chi_\lambda \rangle = \begin{cases} 1 & \textit{if } \lambda^* = \mu \\ 0 & \textit{if } \lambda^* \lhd \mu \end{cases}$$

Example

$\chi_{(n)} = 1_{\mathrm{GL}_n(q)}$ occurs in the regular representation with multiplicity 1 and in no other GGGR.

If p and q are sufficiently large then Lusztig has given an explicit decomposition

$$\gamma_u \qquad \rightsquigarrow \qquad \{\chi_{K_\iota,\phi_\iota} \mid \iota \in \mathcal{N}_{\mathbf{G}}^F\}$$

and has conjectured an explicit decomposition

$$\chi \in \mathrm{Irr}(\mathbf{G}^F) \qquad \rightsquigarrow \qquad \{\chi_{A,\phi_A} \mid A \in \widehat{\mathbf{G}}^F\}$$

If we solve this conjecture then the multiplicity $\langle \gamma_u, \chi \rangle$ can be reduced to the multiplicities

$$\langle \chi_{A,\phi_A}, \chi_{K_\iota,\phi_\iota} \rangle$$

and these are given by our main theorem!