Structure of Root Data and Smooth Regular Embeddings

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New Perspectives in Representation Theory of Finite Groups
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- $g \in G$ and $\langle g \rangle_{\widetilde{G}} \cap G = \langle g \rangle_G$ then

$$\chi_i(g) = \frac{\widetilde{\chi}(g)}{r}$$

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The representation theory of $\mathbf{G}^F$ is harder when $Z(\mathbf{G})$ is disconnected.

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- $\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times \cong \mathbf{T} \leqslant \mathbf{G}$ an F-stable maximal torus

$$\mathbf{G} \times_{Z(\mathbf{G})} \mathbf{T} = (\mathbf{G} \times \mathbf{T})/\{(z, z^{-1}) \mid z \in Z(\mathbf{G})\}$$

Example ($\mathbf{G} = \mathsf{SL}_n(\mathbb{F})$)

- if $p \nmid n$ then $\mathbf{G} \times_{Z(\mathbf{G})} \mathbf{T} \cong \mathsf{GL}_n(\mathbb{F})$
- if $n = p^k$ then $\mathbf{G} \times_{Z(\mathbf{G})} \mathbf{T} \cong \mathbf{G} \times \mathbf{T}$

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Regular Embedding (Lusztig '88)

A closed embedding $\iota : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ is a regular embedding if:

- $\widetilde{\mathbf{G}} = \iota(\mathbf{G})Z(\widetilde{\mathbf{G}})$ and $Z(\widetilde{\mathbf{G}})$ is connected
- $F : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$ is a Steinberg endomorphism and $\iota \circ F = F \circ \iota$.

We then have $\mathbf{G}^F \cong \iota(\mathbf{G})^F \lhd \widetilde{\mathbf{G}}^F$.

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Definition

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$$\begin{aligned} \mathbb{Z}\Phi &= \{a_1e_1 + \cdots + a_ne_n \mid a_1 + \cdots + a_n = 0\} \\ &= \mathrm{Span}_{\mathbb{Z}}\{e_1 - e_2, \ldots, e_{n-1} - e_n\} \end{aligned}$$

and $X(\mathbf{T})/\mathbb{Z}\Phi$ has no torsion as $X(\mathbf{T}) = \mathbb{Z}\Phi \oplus \mathbb{Z}e_n$.

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Example: $\mathsf{SL}_n(\mathbb{F}) \hookrightarrow \mathsf{GL}_n(\mathbb{F})$.

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If we have a regular embedding $\iota : \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ then

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- $\mathcal{T} = (T, \emptyset, \check{T}, \emptyset)$ and $h : T \twoheadrightarrow A$

$$X \oplus_{(A,f,h)} T = \{(x,t) \in X \oplus T \mid f(x) = h(t)\}.$$

and a surjective homomorphism

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Proposition (T.)

Assume $\mathbf{G}_n$ is one of

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There exists a smooth regular embedding $\mathbf{G}_n \hookrightarrow \widetilde{\mathbf{G}}_n$ such that each Levi subgroup of $\widetilde{\mathbf{G}}_n$ is isomorphic to

$$\mathsf{GL}_{n_1}(\mathbb{F}) \times \cdots \times \mathsf{GL}_{n_r}(\mathbb{F}) \times \widetilde{\mathbf{G}}_m$$

where $n = n_1 + \cdots + n_r + m$.

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where $\mathscr{R} = (X, \Phi, \check{X}, \check{\Phi})$ is semisimple, $\mathcal{T} = (T, \emptyset, \check{T}, \emptyset)$ is a torus, and $\Phi \subseteq K \subseteq X$ is a submodule.

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Remark

Recall that $\mathbf{G} = \mathbf{G}_{\mathrm{der}} Z^{\circ}(\mathbf{G})$. Assume $\mathcal{R}(\mathbf{G}) \in \mathscr{R}[\mathcal{R}, \mathcal{T}, K]$ then

$$\mathcal{R}(\mathbf{G}_{\mathrm{der}}) \cong \mathcal{R} \qquad \mathcal{R}(Z^{\circ}(\mathbf{G})) \cong \mathcal{T} \qquad X(\mathbf{G}_{\mathrm{der}} \cap Z^{\circ}(\mathbf{G})) \cong (X/K)_{p'}$$

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Theorem (T.)

The map $\mathsf{Aut}(A) \to \mathscr{R}$ defined by $\psi \mapsto \mathcal{R} \oplus_{(A,f,\psi\circ h)} \mathcal{T}$ induces a bijection

$$\mathsf{Aut}_{(\mathcal{R},f)}(A) \setminus \mathsf{Aut}(A) / \mathsf{Aut}_{(\mathcal{T},h)}(A) \to \mathscr{R}[\mathcal{R}, \mathcal{T}, K].$$

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Corollary (T.)

If A has s invariant factors and $\mathsf{rk}(T) \geqslant s + 1$ then $\mathscr{R}[\mathcal{R}, \mathcal{T}, K]$ has cardinality 1.

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- $\dim(Z^\circ(\mathbf{G})) = 2 \rightsquigarrow$ there's only one, e.g., $\mathsf{GL}_n(\mathbb{F}) \times (\mathbb{F}^\times)^k$.

Asai's Reduction Techniques

- Reduce proving a property (P) for $(\mathbf{G}, F)$ to the case where $\mathbf{G}_{\text{der}}$ is simple and simply connected. This assumes $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism.
- Get new proofs of Asai's results and extend them to show that they are compatible with Steinberg endomorphisms.