Decomposition Matrices of Unipotent Blocks

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(joint work with Olivier Brunat and Olivier Dudas)

Assume G is a finite group and $\mathrm{Irr}(G)$ is the set of complex-valued irreducible characters of G. Fix a prime $\ell > 0$ and let $\mathrm{IBr}(G)$ be the ℓ-modular Brauer characters of G, which are functions $G_{\ell'} \to \mathbb{C}$ where $G_{\ell'} \subseteq G$ is the set of elements whose order is coprime to ℓ.

If $f : G \to \mathbb{C}$ is a function then we denote by $f^0 := f|_{G_{\ell'}}$ the restriction of f to the ℓ'-elements of G. It is well known that if $\chi \in \mathrm{Irr}(G)$ then there exist integers $d_{\chi,\varphi} \geqslant 0$ such that

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi,\varphi}\varphi.$$

The resulting matrix $(d_{\chi,\varphi})$ is the (ℓ-)decomposition matrix of G. Obtaining information about this matrix is a central problem in the representation theory of finite groups and calculating exactly the entries $d_{\chi,\varphi}$ is an extremely challenging problem in general.

We will consider the case where $G = \mathbf{G}(k)$ is a finite reductive group and $\ell \neq p := \mathrm{char}(k)$, i.e., G is the group of k-points of a connected reductive algebraic group $\mathbf{G}$ defined over a finite field k. We will denote by $\bar{k}$ an algebraic closure of k. We then have a corresponding group $\mathbf{G}(\bar{k})$ of $\bar{k}$-points which contains G as a subgroup. We will let $\mathcal{C}_{\mathrm{u}}(\mathbf{G})$ denote the set of unipotent conjugacy classes of $\mathbf{G}(\bar{k})$.

After [7, 2, 6] we can associate to each irreducible character $\chi \in \mathrm{Irr}(G)$ a class $\mathcal{O}_\chi \in \mathcal{C}_{\mathrm{u}}(\mathbf{G})$, called the *unipotent support* of χ. It is a little delicate to define this class in general but if p is good for $\mathbf{G}$ and the centre $Z(\mathbf{G}(\bar{k}))$ is connected then it is shown in [8] that $\mathcal{O}_\chi$ is the unique unipotent class satisfying the following conditions:

- $\chi(u) \neq 0$ for some $u \in \mathcal{O}_\chi \cap G \neq \emptyset$
- if $v \in G$ is a unipotent element and $\chi(v) \neq 0$ then $v \in \overline{\mathcal{O}_\chi}$ (the Zariski closure of $\mathcal{O}_\chi$).

Example. If $1_G \in \mathrm{Irr}(G)$ is the trivial character then $\mathcal{O}_{1_G}$ is the class of regular unipotent elements and if $\mathrm{St}_G \in \mathrm{Irr}(G)$ is the Steinberg character then $\mathcal{O}_{\mathrm{St}_G}$ is the trivial unipotent class.

For finite reductive groups one has an important set of characters $\mathcal{E}(G,1) \subseteq \mathrm{Irr}(G)$, defined using ℓ-adic cohomology, known as the set of *unipotent characters*. These characters are a generic model for all the irreducible characters of G. Using the unipotent support we obtain a partition of the unipotent characters

$$\mathcal{E}(G,1) = \bigsqcup_{\mathcal{O} \in \mathcal{C}_{\mathrm{u}}(\mathbf{G})} \mathcal{E}(G,1,\mathcal{O})$$

where $\mathcal{E}(G,1,\mathcal{O}) = \{\chi \in \mathcal{E}(G,1) \mid \mathcal{O}_\chi = \mathcal{O}\}$. Note this set might be empty in general and the non-empty such sets are known as *families* of unipotent characters.

Example. Assume $G = \mathrm{Sp}_4(k)$ then $\mathcal{C}_{\mathrm{u}}(\mathbf{G}) = \{\mathcal{O}_{(1^4)}, \mathcal{O}_{(2,1^2)}, \mathcal{O}_{(2^2)}, \mathcal{O}_{(4)}\}$ where each class is labelled by the sizes of the Jordan blocks in the Jordan normal form of an element under the natural representation $\mathrm{Sp}_4(\bar{k}) \to \mathrm{GL}_4(\bar{k})$. It is well known that $|\mathcal{E}(G,1)| = 6$ and the sizes of the corresponding sets $\mathcal{E}(G,1,\mathcal{O})$ are

$\mathcal{O}$	$\mathcal{O}_{(1^4)}$	$\mathcal{O}_{(2,1^2)}$	$\mathcal{O}_{(2^2)}$	$\mathcal{O}_{(4)}$
$\lvert\mathcal{E}(G,1,\mathcal{O})\rvert$	1	0	4	1

Here $\mathcal{E}(G,1,\mathcal{O}_{(1^4)}) = \{\mathrm{St}_G\}$ and $\mathcal{E}(G,1,\mathcal{O}_{(4)}) = \{1_G\}$.

On the modular side we have a corresponding subset $\mathcal{B}(G,1) \subseteq \mathrm{IBr}(G)$ of Brauer characters, which is the union of the *unipotent blocks* of G. This set is defined by a corresponding subset $\mathcal{E}_\ell(G,1) \subseteq \mathrm{Irr}(G)$ of irreducible characters, which contains the set of unipotent characters. This correspondence is such that if $\chi \in \mathcal{E}_\ell(G,1)$ and $\varphi \in \mathrm{IBr}(G)$ then $d_{\chi,\varphi} \neq 0$ implies $\varphi \in \mathcal{B}(G,1)$.

In what follows we will be interested in the following part of the decomposition matrix

$$D = (d_{\chi,\varphi} \mid \chi \in \mathcal{E}_\ell(G,1) \text{ and } \varphi \in \mathcal{B}(G,1)).$$

This matrix is, in general, not square as $|\mathcal{E}_\ell(G,1)| \geqslant |\mathcal{B}(G,1)|$. However, it has been shown by Geck–Hiß that under some mild assumptions on ℓ we have $|\mathcal{E}(G,1)| = |\mathcal{B}(G,1)|$, this holds for instance if ℓ is very good for $\mathbf{G}$. This is known to be false in general.

Let us recall that we have a natural partial order $\preceq$ on $\mathcal{C}_{\mathrm{u}}(\mathbf{G})$ defined by $\mathcal{O}' \preceq \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ (the Zariski closure). With this in hand we can state Geck's conjecture on the decomposition matrix of G. To avoid introducing more notation we will work with a stronger assumption on ℓ than is actually stated in the conjecture. We note that a weak version of this conjecture was first proposed in Geck's PhD Thesis [3]. It was then further strengthened by Geck–Hiß [5] and reached the form we state here in [4].

Geck's Unitriangularity Conjecture. Assume ℓ is a very good prime for $\mathbf{G}$. Let $\mathcal{S}_{\mathbf{G}} = \{\mathcal{O} \in \mathfrak{Cl}_{\mathrm{u}}(\mathbf{G}) \mid \mathcal{E}(G,1,\mathcal{O}) \neq \emptyset\} = \{\mathcal{O}_1, \dots, \mathcal{O}_r\}$ where $\mathcal{O}_r \leqslant \cdots \leqslant \mathcal{O}_1$ is a total order refining the partial order $\preceq$ on $\mathcal{S}_{\mathbf{G}}$. Then there is an ordering of the Brauer characters in $\mathcal{B}(G,1)$ such that

$$D = \begin{array}{cl} \left[\begin{array}{ccc} D_1 & 0 & 0 \\ \star & \ddots & 0 \\ \star & \star & D_r \\ \hline \star & \star & \star \end{array}\right] & \begin{array}{c} \mathcal{E}(G,1,\mathcal{O}_1) \\ \vdots \\ \mathcal{E}(G,1,\mathcal{O}_r) \\ \\ \end{array} \end{array}$$

where each D_i is the identity matrix with rows labelled by the irreducible characters in $\mathcal{E}(G,1,\mathcal{O}_i)$.

Example. The poset $(\mathcal{S}_{\mathbf{G}}, \preceq)$ contains a unique maximal element, namely the class $\mathcal{O}_{\mathrm{reg}} \in \mathcal{S}_{\mathbf{G}}$ of regular unipotent elements, because $\mathcal{E}(G,1,\mathcal{O}_{\mathrm{reg}}) = \{1_G\}$. In the statement of the conjecture $\mathcal{O}_1 = \mathcal{O}_{\mathrm{reg}}$ and thus we should have 1_G^0 is an irreducible Brauer character, which it certainly is.

Similarly, the poset $(\mathcal{S}_{\mathbf{G}}, \preceq)$ contains a unique minimal element, namely the trivial class $\mathcal{O}_{\mathrm{triv}} \in \mathcal{S}_{\mathbf{G}}$, because $\mathcal{E}(G, 1, \mathcal{O}_{\mathrm{triv}}) = \{\mathrm{St}_G\}$. In the statement of the conjecture $\mathcal{O}_r = \mathcal{O}_{\mathrm{triv}}$ and St_G^0 could potentially have many irreducible constituents.

Since its inception several people have worked towards obtaining a proof of this conjecture. The conjecture was shown to be true by Dipper when $G = \mathrm{GL}_n(k)$ and Geck when $G = \mathrm{GU}_n(k)$. A particularly notable milestone in the life of the conjecture was achieved by Gruber–Hiß who showed the conjecture was true when G is a classical group and ℓ is a so-called linear prime for G. Together with O. Brunat and O. Dudas we have established the following.

Theorem (Brunat–Dudas–T.). Assume p is good for $\mathbf{G}$ and ℓ is very good for $\mathbf{G}$. If G has no component of type E_8 and $q \equiv 1 \pmod 4$ if G has a component of type E_7 then Geck's Unitriangularity Conjecture holds.

We are optimistic that our methods will be able to treat the cases of E_7 and E_8 and thus we hope to establish Geck's conjecture for all finite reductive groups, with appropriate assumptions on p and ℓ. As mentioned above the assumption that ℓ is very good is stronger than the assumption imposed in the original statement of the conjecture. Our result can be established with an assumption on ℓ matching that made in [4]. In fact, after work of Denoncin [1], it seems likely that some version of the unitriangularity can be established assuming only that ℓ is a good prime for $\mathbf{G}$.

References

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