Generalised Gelfand–Graev Characters in Small Characteristics

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Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\overline{\mathbb{F}_p}$ of the finite field $\mathbb{F}_p$ of characteristic $p > 0$. We will assume that p is a *good prime* for $\mathbf{G}$ (the assumption $p > 5$ is sufficient for all cases). Furthermore, let us assume that $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ defining an $\mathbb{F}_q$-rational structure so that the fixed point group $G = \mathbf{G}^F$ is a finite reductive group. We will denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G (for example over $\mathbb{C}$).

In [4, 5] Kawanaka has constructed for every unipotent element $u \in G$ a character Γ_u of G called a *generalised Gelfand–Graev character* (GGGC). The definition of Γ_u is somewhat delicate and depends upon the study of the unipotent conjugacy classes of $\mathbf{G}$. However, here are some of the important properties:

- $\Gamma_u = \Gamma_v$ if $u = gvg^{-1}$ for some $g \in G$,
- $\Gamma_u = \mathrm{Ind}_V^G(\varphi_u)$ where $V \leqslant G$ is a p-subgroup (depending upon the $\mathbf{G}$-conjugacy class containing u) and φ_u is a linear character of V,
- if $u = 1$ then $V = \{1\}$ is the trivial subgroup and Γ_1 is the character of the regular representation of G,
- if u is a regular unipotent element of $\mathbf{G}$ (i.e., $\dim C_{\mathbf{G}}(u)$ is minimal) then V is a Sylow p-subgroup of G and Γ_u is a Gelfand–Graev character (see [2, 14.29]).

In his study of GGGCs Kawanaka conjectured the following surprising relationship between the unipotent conjugacy classes of $\mathbf{G}$ and the irreducible characters of G (see [4, 3.3.3]).

Conjecture. For any irreducible character $\chi \in \mathrm{Irr}(G)$ there exists a unique F-stable unipotent class $\mathcal{O}_\chi^* \subseteq \mathbf{G}$, called the *wave front set* of χ, such that:

(WF1) $\langle \Gamma_u, \chi \rangle \neq 0$ for some $u \in {\mathcal{O}_\chi^*}^F$,
(WF2) $\langle \Gamma_u, \chi \rangle \neq 0$ implies $\dim \mathcal{O}_u < \dim \mathcal{O}_\chi^*$ or $u \in \mathcal{O}_\chi^*$,

where $\mathcal{O}_u$ is the $\mathbf{G}$-conjugacy class containing $u \in G$.

We point out here that Kawanaka's conjecture is related to, and inspired by, an earlier conjecture of Lusztig on the existence of a *unipotent support* for $\chi \in \mathrm{Irr}(G)$. See [8] or [9] for more details. Let us also note that as Γ_1 is the regular character of G there always exists a class satisfying (WF1). Hence, another way to state the conjecture is that there is a unique class of maximal dimension satisfying (WF1).

Recall that we have a partial ordering $\leqslant$ on the set of unipotent conjugacy classes of $\mathbf{G}$ defined by $\mathcal{O}' \leqslant \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ (the Zariski closure). It is well known that if $\mathcal{O}' \leqslant \mathcal{O}$ then we have $\dim \mathcal{O}' \leqslant \dim \mathcal{O}$ with equality if and only if $\mathcal{O}' = \mathcal{O}$. Hence one could ask whether the following stronger geometric condition holds

(WF2') $\langle \Gamma_u, \chi \rangle \neq 0$ implies $\mathcal{O}_u \leqslant \mathcal{O}_\chi^*$.

In [8] Lusztig showed that Kawanaka's conjecture is true under the assumption that p and q are sufficiently large. Furthermore, using Lusztig's work Achar and

Aubert showed that the geometric refinement (WF2') is satisfied in [1] (again under the assumption that p and q are sufficiently large). The following removes these restrictions and is proved in [9].

Theorem. Assume only that p is a good prime for $\mathbf{G}$ then the wave front set exists for every irreducible character $\chi \in \mathrm{Irr}(G)$ and the geometric condition (WF2') is satisfied.

Lusztig's proof of Kawanaka's conjecture is based on a formula which decomposes the GGGCs in terms of characteristic functions of intersection cohomology complexes on the closures of unipotent conjugacy classes with coefficients in various local systems. Our approach is to show that this formula holds whenever p is an *acceptable prime* for $\mathbf{G}$; this is the main result of [9]. The assumption that p is an acceptable prime comes from the work of Letellier [6], on which our result relies heavily. Although this is a stronger assumption than p good it gives us enough information to prove the theorem by using various reduction arguments.

We finish by noting the following application of our result. The GGGCs are projective characters in characteristic $\ell \neq p$ a prime different from p. By Brauer reciprocity the multiplicities of irreducible characters in GGGCs provides information on the ℓ-decomposition numbers. In particular, the existence of the wave-front set can be used to ensure that certain decomposition numbers will be zero. This information was recently used by Dudas and Malle in [3] as part of their determination of new decomposition matrices for unipotent blocks of exceptional-type groups.

References

[1] P. N. Achar and A.-M. Aubert, *Supports unipotents de faisceaux caractères*, J. Inst. Math. Jussieu **6** (2007), no. 2, 173–207.

[2] F. Digne and J. Michel, *Representations of finite groups of Lie type*, vol. 21, London Mathematical Society Student Texts, Cambridge: Cambridge University Press, 1991.

[3] O. Dudas and G. Malle, *Decomposition matrices for exceptional groups at $d = 4$*, preprint (Oct. 2014), arXiv:1410.3754 [math.RT].

[4] N. Kawanaka, *Generalized Gelfand–Graev representations and Ennola duality*, in: *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, vol. 6, Adv. Stud. Pure Math. Amsterdam: North-Holland, 1985, 175–206.

[5] N. Kawanaka, *Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field. I*, Invent. Math. **84** (1986), no. 3, 575–616.

[6] E. Letellier, *Fourier transforms of invariant functions on finite reductive Lie algebras*, vol. 1859, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2005.

[7] G. Lusztig, *Characters of reductive groups over a finite field*, vol. 107, Annals of Mathematics Studies, Princeton, NJ: Princeton University Press, 1984.

[8] G. Lusztig, *A unipotent support for irreducible representations*, Adv. Math. **94** (1992), no. 2, 139–179.

[9] J. Taylor, *Generalised Gelfand–Graev representations in small characteristics*, preprint (Aug. 2014), arXiv:1408.1643 [math.RT].