A Note on Skew Characters of Symmetric Groups

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Abstract. In previous work Regev used part of the representation theory of Lie superalgebras to compute the values of a character of the symmetric group whose decomposition into irreducible constituents is described by semistandard (k, ℓ)-tableaux. In this short note we give a new proof of Regev's result using skew characters.

1. **Introduction**

1.1. For any partition $\alpha \in \mathsf{P}(n)$ of an integer $n \geqslant 0$ we have a corresponding irreducible character χ_α of the symmetric group $\mathfrak{S}_n$; we assume this labelling is as in [JK81]. In [Reg13] Regev observed that the values of the character $\Gamma_n = \sum_{a=0}^{n} \chi_{(a,1^{n-a})}$ obtained by summing over all hook partitions were particularly simple. Specifically if $\nu \in \mathsf{P}_r(n)$ is a partition of length r then we have

$$\Gamma_n(\nu) = \begin{cases} 2^{r-1} & \text{if all parts of } \nu \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Here we write $\Gamma_n(\nu)$ for the value of Γ_n at an element of cycle type ν.

1.2. To prove this result Regev considered a more general but related problem which we now recall. For any integers $k, \ell \geqslant 0$ and any partition $\alpha \in \mathsf{P}(n)$ we denote by $s_{k,\ell}(\alpha)$ the number of all semistandard (k, ℓ)-tableaux of shape α, see 3.1 for the definition. Motivated by the representation theory of Lie superalgebras Regev considered the following character of $\mathfrak{S}_n$

$$\Lambda_n^{k,\ell} = \sum_{\alpha \in \mathsf{P}(n)} s_{k,\ell}(\alpha)\chi_\alpha.$$

The main result of [Reg13] is the following.

Theorem 1.3 (Regev). *If $\nu = (\nu_1, \dots, \nu_r) \in \mathsf{P}_r(n)$ is a partition of length r then*

$$\Lambda_n^{k,\ell}(\nu) = \prod_{i=1}^{r} (k + (-1)^{\nu_i - 1}\ell).$$

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Remark 1.4. Note that $s_{k,\ell}(\alpha) \neq 0$ if and only if α is contained in the (k,ℓ)-hook, as defined in [BR87, 2.3]. In particular, we have $\Lambda_n^{k,\ell}$ is the same as the character $\chi_{\varphi^*_{(k,\ell),n}}$ defined in [Reg13].

1.5. Using formulas for the coefficients $s_{k,\ell}(\alpha)$ obtained in [BR87] Regev deduces that $\Lambda_n^{1,1} = 2\Gamma_n$ from which the statement of 1.1 follows immediately. Although Theorem 1.3 is stated purely in terms of the representation theory of the symmetric group Regev's proof uses, in an essential way, results of Berele–Regev on representations of Lie superalgebras [BR87]. As is noted in the introduction to [Reg13] it is natural to ask whether there is a proof of this result which uses only techniques from the symmetric group. The purpose of this note is to provide such a proof. Our proof is based on a description of the character $\Lambda_n^{k,\ell}$ as a sum of skew characters[1]. With this we can use the Murnaghan–Nakayama formula to compute the values of $\Lambda_n^{k,\ell}$ and thus prove Theorem 1.3. As a closing remark we use our description in terms of skew characters to show that $\Lambda_n^{1,1} = 2\Gamma_n$ using the branching rule.

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2. Background on Skew Characters

2.1. Let $\mathbb{N} = \{1, 2, \dots\}$ be the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout we use the term *diagram* to mean a subset of $\mathbb{N}^2$. For any two diagrams $S, T \subseteq \mathbb{N}^2$ we write $S \equiv T$ if there exist integers $i, j \in \mathbb{Z}$ such that $T = \{(a+i, b+j) \mid (a,b) \in S\}$; we say such diagrams are *equivalent*. The notion of connected diagram and connected components of a diagram have their usual natural meanings, see [Mac95, I, §1] for details. A diagram T will be called a *horizontal line*, resp., *vertical line*, if for any $(i,j), (i',j') \in T$ we have $i = i'$, resp., $j = j'$.

2.2. For any $k \in \mathbb{N}_0$ we denote by C_k the set of all compositions $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_0^k$ of length k; we call α_i a *part* of α. For such a composition we denote by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_k$ and by α° the composition obtained from α by removing all parts equal to 0. If $n \in \mathbb{N}_0$ then we denote by $\mathsf{P}_k(n)$ the set of all $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathsf{C}_k$ such that $\alpha_1 \geqslant \cdots \geqslant \alpha_k > 0$ and $|\alpha| = n$, which are the partitions of n of length k. Moreover we denote by $\mathsf{P}(n)$ the set $\bigcup_{k\in\mathbb{N}} \mathsf{P}_k(n)$ of all partitions of n. To each partition $\alpha \in \mathsf{P}(n)$ we have a corresponding diagram $T_\alpha = \{(i,j) \mid 1 \leqslant j \leqslant \alpha_i\}$ called the *Young diagram* of α. For any two Young diagrams $T_\beta \subseteq T_\alpha$ the difference $T_\alpha \setminus T_\beta$ is called a *skew diagram*.

2.3. Let us denote by R the commutative unital graded ring $\bigoplus_{n\in\mathbb{N}_0} R_n$ where R_n is the $\mathbb{Z}$-module of all virtual characters of $\mathfrak{S}_n$. The multiplication $\widehat{\otimes}$ in this ring is given

[1] This idea was prompted by a recent question of Marcel Novaes on MathOverflow [Nov16], which is where we also first learned of Regev's work.

by tensor induction, i.e., we have

$$\chi \widehat{\otimes} \psi = \operatorname{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (\chi \otimes \psi)$$

for any $\chi \in R_m$ and $\psi \in R_n$. For any integer $a \in \mathbb{Z}$ we define an element $[a] \in R$ by setting

$$[a] = \begin{cases} \chi_{(a)} & \text{if } a \geqslant 0, \\ 0 & \text{if } a < 0. \end{cases}$$

Note that $[0] = \chi_{(0)}$ is the trivial character of $\mathfrak{S}_0 = \{1\}$, which is the unit in R. Now assume $\alpha \in \mathsf{P}(n)$ is a partition then the corresponding irreducible character χ_α of $\mathfrak{S}_n$ can be expressed via the following determinantal formula

$$\chi_\alpha = \det([\alpha_i - i + j]),$$

see [JK81, 2.3.15]. In such an expression we assume that i and j run over $1, \dots, k$ with k larger than the length of α and any part of α which is undefined is set to 0. If $S = T_\alpha \setminus T_\beta$ is a skew diagram with $|S| = n$ then following [JK81, 2.3.11] we define a corresponding *skew character* of $\mathfrak{S}_n$ by setting

$$\psi_S = \det([\alpha_i - \beta_j - i + j]).$$

The following are well known properties of skew characters which are easily deduced from the above determinantal formulas, see the argument in [Mac95, I, 5.7] for a proof of the first property.

Lemma 2.4. *Assume S and S' are skew diagrams then the following hold:*

(a) if $S_1, \dots, S_r$ are the connected components of S then $\psi_S = \psi_{S_1} \widehat{\otimes} \cdots \widehat{\otimes} \psi_{S_r}$,

(b) if $S \equiv S'$ then $\psi_S = \psi_{S'}$,

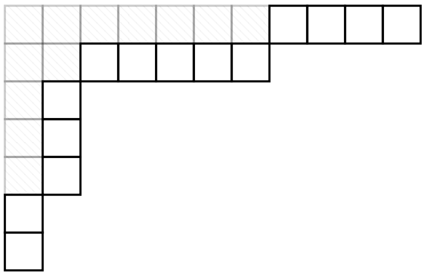
(c) if $S \equiv T_\alpha$ for some partition $\alpha \in \mathsf{P}(n)$ then $\psi_S = \chi_\alpha$.

2.5. For any $k, \ell, n \in \mathbb{N}_0$ we denote by $\mathsf{B}_{k,\ell}(n) \subseteq \mathsf{C}_k \times \mathsf{C}_\ell$ the set of all pairs $(\lambda \mid \mu)$ of compositions such that $|\lambda| + |\mu| = n$; we call these *bicompositions* of n. Now for each bicomposition $(\lambda \mid \mu) \in \mathsf{B}_{k,\ell}(n)$ we denote by $S_{(\lambda\mid\mu)}$ some (any) skew diagram whose connected components $H_1, \dots, H_r, V_1, \dots, V_s \subseteq S_{(\lambda;\mu)}$ are such that H_i is a horizontal line, resp., V_j is a vertical line, and $(|H_1|, \dots, |H_r|) = \lambda^\circ$, resp., $(|V_1|, \dots, |V_s|) = \mu^\circ$. It is easy to see that such a diagram exists. We then get a corresponding character of $\mathfrak{S}_n$

$$\psi_{(\lambda\mid\mu)} := \psi_{S_{(\lambda\mid\mu)}} = \chi_{(\lambda_1)} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{(\lambda_k)} \widehat{\otimes} \chi_{(1^{\mu_1})} \widehat{\otimes} \cdots \widehat{\otimes} \chi_{(1^{\mu_\ell})}. \tag{2.6}$$

This character does not depend upon the choice of skew diagram $S_{(\lambda\mid\mu)}$ by (a) and (b) of Lemma 2.4 because any two skew diagrams with equivalent connected components yield the same character of $\mathfrak{S}_n$. The expression in terms of irreducible characters follows from (a) and (c) because each connected component of $S_{(\lambda|\mu)}$ is equivalent to a Young diagram.

Example 2.7. Consider the bicomposition $(4,0,5;2,3) \in \mathsf{B}_{3,2}(14)$ then an example of a corresponding skew diagram $S_{(4,0,5;2,3)}$ is given by $T_{(11,7,2,2,2,1,1)} \setminus T_{(7,2,1,1,1)}$.



We then have $\psi_{(4,0,5;2,3)} = \chi_{(4)} \widehat{\otimes} \chi_{(0)} \widehat{\otimes} \chi_{(5)} \widehat{\otimes} \chi_{(1^2)} \widehat{\otimes} \chi_{(1^3)} = \chi_{(4)} \widehat{\otimes} \chi_{(5)} \widehat{\otimes} \chi_{(1^2)} \widehat{\otimes} \chi_{(1^3)}$.

2.8. The main tool we will use to prove Theorem 1.3 is the Murnghan–Nakayama formula for skew characters, which provides a recursive method for computing the values of such characters. To state this result we need to recall some notions concerning hooks, for which we follow [JK81, 2.3]. Recall that for any partition $\alpha \in \mathsf{P}_r(n)$ we define the *rim* of the corresponding Young diagram T_α to be

$$\mathcal{R}(T_\alpha) = \bigcup_{i=1}^{r} \{(i,j) \mid \alpha_{i+1} \leqslant j \leqslant \alpha_i\}$$

where we set $\alpha_{r+1} = 0$. Now assume $S = T_\alpha \setminus T_\beta$ is a skew diagram then we define the rim of this skew diagram to be

$$\mathcal{R}(S) = \mathcal{R}(T_\alpha) \cap S.$$

A *rim hook* of S is a connected diagram $R \subseteq \mathcal{R}(S)$ such that for any $(k,\ell) \in R$ and any $(i,j) \in \mathcal{R}(S) \setminus R$ we have either $i < k$ or $j < \ell$. If we have $|R| = a$ then we will also say that R is an *a-rim hook* of S. Now given a rim hook $R \subseteq \mathcal{R}(S)$ we denote by $\max(R)$, resp., $\min(R)$, the maximal, resp., minimal, integer i such that $(i,j) \in R$ for some $j \in \mathbb{N}$. The difference $\ell(R) = \max(R) - \min(R) \geqslant 0$ is called the *leg length* of the rim hook.

Remark 2.9. Note that if S is a skew diagram with connected components $S_1,\dots,S_r$ then we have $\mathcal{R}(S) = \mathcal{R}(S_1) \cup \cdots \cup \mathcal{R}(S_r)$ and moreover $R \subseteq \mathcal{R}(S)$ is a rim hook if and only if $R \subseteq \mathcal{R}(S_i)$ is a rim hook for some connected component.

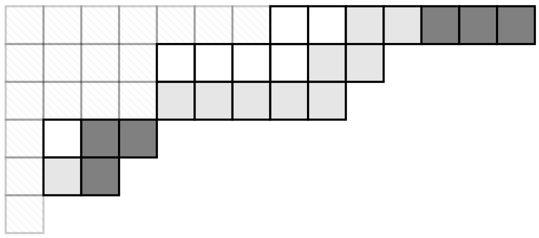
2.10. Note that a rim hook has the property that the difference $S \setminus R$ is again a skew diagram. Indeed, we have $R \subseteq \mathcal{R}(S) \subseteq \mathcal{R}(T_\alpha)$ is a rim hook of T_α, in the usual sense, and it is easily checked that $T_\alpha \setminus R$ is the Young diagram of a partition. Moreover, by assumption, we have $T_\beta \cap R = \varnothing$ and so clearly we have $T_\beta \subseteq T_\alpha \setminus R$ and $S \setminus R = (T_\alpha \setminus R) \setminus T_\beta$. With these ideas in place we may now state the Murnaghan–Nakayama formula for skew characters.

Theorem 2.11 ([JK81, 2.4.15]). *Assume $\nu \in \mathsf{P}_r(n)$ is a partition of n and let $a \in \mathbb{N}$ be a part of ν then we denote by $\hat{\nu} \in \mathsf{P}_{r-1}(n-a)$ the partition obtained by removing a. If S is a skew diagram with $|S| = n$ then*

$$\psi_S(\nu) = \sum_{R \subseteq \mathcal{R}(S)} (-1)^{\ell(R)} \psi_{S \setminus R}(\hat{\nu})$$

where the sum is taken over all a-rim hooks of S.

Example 2.12. Consider the following skew diagram $S = T_{(14,10,9,4,3,1)} \setminus T_{(7,4,4,1,1,1)}$.



The skew diagram S has four 3-rim hooks, namely

$$R_1 = \{(1,12),(1,13),(1,14)\} \qquad R_2 = \{(3,9),(2,9),(2,10)\}$$
$$R_3 = \{(3,7),(3,8),(3,9)\} \qquad R_4 = \{(5,3),(4,3),(4,4)\}.$$

Here we have highlighted the rim hooks R_1 and R_4 in dark grey and the remaining nodes of the rim $\mathcal{R}(S)$ in light grey. Now let $\nu = (10,4,4,3,2) \in \mathsf{P}_5(23)$ and $\hat{\nu} = (10,4,4,2) \in \mathsf{P}_4(20)$, which is obtained from ν by removing the part 3, then applying the Murnaghan–Nakayama formula we see that

$$\psi_S(\nu) = \psi_{S\setminus R_1}(\hat{\nu}) - \psi_{S\setminus R_2}(\hat{\nu}) + \psi_{S\setminus R_3}(\hat{\nu}) - \psi_{S\setminus R_4}(\hat{\nu})$$

because $\ell(R_1) = \ell(R_3) = 0$ and $\ell(R_2) = \ell(R_4) = 1$

3. **Proof of Theorem 1.3**

3.1. We will now prove Theorem 1.3 but before proceeding we recall some definitions from [BR87, 2.1]. Specifically, let $D = \{1 < \cdots < k < 1' < \cdots < \ell'\}$ be a totally ordered set. If $\alpha \in \mathsf{P}(n)$ is a partition and $(\lambda \mid \mu) \in \mathsf{B}_{k,\ell}(n)$ is a bicomposition then we say a function $f : T_\alpha \to D$ is a (k,ℓ)-tableau of shape α and weight $(\lambda \mid \mu)$ if $\lambda_i = |\{x \in T_\alpha \mid f(x) = i\}|$ for any $1 \leqslant i \leqslant k$ and $\mu_j = |\{x \in T_\alpha \mid f(x) = j'\}|$ for any $1 \leqslant j \leqslant \ell$. As in [BR87, 2.1] we say f is *semistandard* if $T_f = f^{-1}(\{1,\ldots,k\})$ is a Young tableau whose rows are weakly increasing and whose columns are strictly increasing and $T_\alpha \setminus T_f$ is a skew tableau whose columns are weakly increasing and whose rows are strictly increasing. If $s_{(\lambda|\mu)}(\alpha)$ is the number of semistandard (k,ℓ)-tableaux of shape α and weight $(\lambda \mid \mu)$ then $s_{k,\ell}(\alpha) := \sum_{(\lambda|\mu)} s_{(\lambda|\mu)}(\alpha)$ is the number of all semistandard (k,ℓ)-tableaux of shape α, where the sum runs over $\mathsf{B}_{k,\ell}(n)$.

Lemma 3.2. *For any $k, \ell, n \in \mathbb{N}_0$ we have*

$$\Lambda_n^{k,\ell} = \sum_{(\lambda|\mu)\in \mathsf{B}_{k,\ell}(n)} \psi_{(\lambda|\mu)}.$$

Proof. The decomposition of the character on the right hand side of (2.6) into irreducible constituents has been described in [BR87, Lemma 3.23]. Specifically we have

$$\psi_{(\lambda|\mu)} = \chi_{(\lambda_1)}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{(\lambda_k)}\widehat{\otimes}\chi_{(1^{\mu_1})}\widehat{\otimes}\cdots\widehat{\otimes}\chi_{(1^{\mu_\ell})} = \sum_{\alpha\in\mathsf{P}(n)} s_{(\lambda|\mu)}(\alpha)\chi_\alpha.$$

Note that when $\ell = 0$ this statement is just Young's rule and as in [BR87] the general case can be proved easily by induction on ℓ using the definition of (k,ℓ)-tableaux. With this we see that

$$\Lambda_n^{k,\ell} = \sum_{\alpha\in\mathsf{P}(n)} s_{k,\ell}(\alpha)\chi_\alpha = \sum_{\alpha\in\mathsf{P}(n)}\sum_{(\lambda|\mu)\in\mathsf{B}_{k,\ell}(n)} s_{(\lambda|\mu)}(\alpha)\chi_\alpha = \sum_{(\lambda|\mu)\in\mathsf{B}_{k,\ell}(n)} \psi_{(\lambda|\mu)}$$

as desired. ■

Proof (of Theorem 1.3). Choose a part a of ν and let $\hat{\nu} \in \mathsf{P}_{r-1}(n-a)$ be the partition obtained by removing the part a from ν. If $\lambda \in \mathsf{C}_k$ is a composition such that $\lambda_i \geqslant a$ then we denote by $\lambda \downarrow_i a \in \mathsf{C}_k$ the composition obtained by replacing λ_i with $\lambda_i - a$. Similarly, for any composition $\lambda \in \mathsf{C}_k$ we denote by $\lambda \uparrow_i a \in \mathsf{C}_k$ the composition obtained by replacing λ_i with $\lambda_i + a$. Consider the skew diagram $S_{(\lambda|\mu)}$ with $(\lambda \mid \mu) \in \mathsf{B}_{k,\ell}(n)$ a bicomposition. If $R \subseteq \mathcal{R}(S_{(\lambda|\mu)})$ is an a-rim hook then, by Remark 2.9, we have $R \subseteq \mathcal{R}(V)$ for some connected component $V \subseteq S_{(\lambda|\mu)}$. By definition V is either a horizontal or vertical line. It is easy to see that such a diagram contains an a-rim hook if and only if $|V| \geqslant a$ and if such a rim hook exists then it is unique. Moreover we have $\ell(R) = 0$ if V is a horizontal line and $\ell(R) = a-1$ if V is a vertical line. Considering the definition of the character $\psi_{(\lambda|\mu)}$ and applying Theorem 2.11 we see that

$$\psi_{(\lambda|\mu)}(\nu) = \sum_{\lambda_i\geqslant a} \psi_{(\lambda\downarrow_i a|\mu)}(\hat{\nu}) + \sum_{\mu_j\geqslant a} (-1)^{a-1}\psi_{(\lambda|\mu\downarrow_j a)}(\hat{\nu})$$

where the first, resp., second, sum is over all $1 \leqslant i \leqslant k$, resp., $1 \leqslant j \leqslant \ell$, such that $\lambda_i \geqslant a$, resp., $\mu_j \geqslant a$. Now clearly every bicomposition $(\lambda' \mid \mu') \in \mathsf{B}_{k,\ell}(n-a)$ arises from exactly $k+\ell$ bicompositions $(\lambda \mid \mu) \in \mathsf{B}_{k,\ell}(n)$ via the process $\downarrow_i a$, specifically from the k bicompositions $(\lambda' \uparrow_i a \mid \mu')$ and the ℓ bicompositions $(\lambda' \mid \mu' \uparrow_j a)$ with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Putting things together we see that

$$\Lambda_n^{k,\ell}(\nu) = \sum_{(\lambda|\mu)\in\mathsf{B}_{k,\ell}(n)} \psi_{(\lambda|\mu)}(\nu) = (k + (-1)^{a-1}\ell)\Lambda_{n-a}^{k,\ell}(\hat{\nu}).$$

Here we have implicitly used Lemma 3.2. An easy induction argument completes the proof. ■

Remark 3.3. We now assume that $k = \ell = 1$ then the skew characters occurring in $\Lambda_n^{1,1}$

are of the form $\psi_{(a|n-a)}$ with $0 \leqslant a \leqslant n$. Applying Lemma 2.4 we see that $\psi_{(a|n-a)} = \psi_S$ where $S = T_\alpha \setminus T_\beta$ with $\alpha = (a+1, 1^{n-a})$ and $\beta = (1)$. By [JK81, 2.4.16] we have

$$\psi_{(a|n-a)} = \sum_{\gamma \in \mathsf{P}(n)} c^{\alpha}_{\beta\gamma} \chi_\gamma$$

where $c^{\alpha}_{\beta\gamma}$ is the usual Littlewood–Richardson coefficient. As $\beta = (1)$ the Littlewood–Richardson coefficient $c^{\alpha}_{\beta\gamma}$ is described by the branching rule, c.f., [JK81, 2.4.3]. Applying this rule we easily deduce that

$$\psi_{(a|n-a)} = \begin{cases} \chi_{(1^n)} & \text{if } a = 0, \\ \chi_{(a,1^{n-a})} + \chi_{(a+1,1^{n-a-1})} & \text{if } 0 < a < n, \\ \chi_{(n)} & \text{if } a = n. \end{cases}$$

Alternatively viewing $\psi_{(a|n-a)}$ as the character $\chi_{(a)} \widehat{\otimes} \chi_{(1^{n-a})}$ one could apply Pieri's rule to deduce the same result. This gives an alternative way to see that $\Lambda_n^{1,1} = 2\Gamma_n$.

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