Principal 2-Blocks and Sylow 2-Subgroups

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Abstract. Let G be a finite group with Sylow 2-subgroup $P \leqslant G$. Navarro–Tiep–Vallejo have conjectured that the principal 2-block of $N_G(P)$ contains exactly one irreducible Brauer character if and only if all odd-degree ordinary irreducible characters in the principal 2-block of G are fixed by a certain Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$. Recent work of Navarro–Vallejo has reduced this conjecture to a problem about finite simple groups. We show that their conjecture holds for all finite simple groups, thus establishing the conjecture for all finite groups.

1. **Introduction**

1.1. Let G be a finite group, $\ell > 0$ a prime, and let $\mathrm{Irr}_{\ell'}(G) \subseteq \mathrm{Irr}(G)$ be the ordinary irreducible characters of G whose degrees are coprime to ℓ. The McKay conjecture proposes that there is a bijection between the sets $\mathrm{Irr}_{\ell'}(G)$ and $\mathrm{Irr}_{\ell'}(N_G(P))$ where $N_G(P)$ is the normaliser of a Sylow ℓ-subgroup $P \in \mathrm{Syl}_\ell(G)$. In [Nav04] Navarro proposed a striking generalisation of this conjecture, which we refer to as the Galois-McKay conjecture. This conjecture states that there exists a bijection $\mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N_G(P))$ which is compatible with the action of certain Galois automorphisms.

1.2. Currently very little is known about the validity of the Galois-McKay conjecture. However, there is a known consequence of this conjecture which is much more tractable than the conjecture itself, see [Nav04, 5.2]. From now until the end of this article we denote by $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ the unique element of the Galois group of the maximal abelian extension $\mathbb{Q} \subseteq \mathbb{Q}^{\mathrm{ab}}$ that fixes 2-roots of unity and squares odd roots of unity.

Conjecture 1.3 (Navarro). *Assume G is a finite group and $P \in \mathrm{Syl}_2(G)$. We have $N_G(P) = P$ if and only if all odd-degree irreducible characters of G are σ-fixed.*

1.4. The first author has reduced Conjecture 1.3 to showing that each finite simple group is SN2S-Good, in the sense of [SF16, Definition 1]. Moreover, the combined efforts of [SF16; SFT18; SF17] complete the programme of showing each finite simple group is SN2S-Good, thus establishing Conjecture 1.3. Recently Navarro–Tiep–Vallejo [NTV18] have considered an analogue of Conjecture 1.3 which involves the principal ℓ-block. They show their analogue holds when ℓ is odd but, again, the $\ell = 2$ case seems to be harder. Their conjecture in the case $\ell = 2$, which is the focus of this article, is as follows.

Conjecture 1.5 (Navarro–Tiep–Vallejo). *Assume G is a finite group and $P \in \mathrm{Syl}_2(G)$. The principal 2-block of $N_G(P)$ contains only one irreducible Brauer character if and only if every odd-degree character in the principal 2-block of G is σ-fixed.*

1.6. In this form Conjecture 1.5 and Conjecture 1.3 do not appear to be related. However, it is a result of Brauer that the principal ℓ-block of a finite group has only one irreducible Brauer character if and only if the group has a normal ℓ-complement, see [Nav98, Corollary 6.13]. In fact, it is shown in [NV17, 6.7] that Conjecture 1.3 is a consequence of Conjecture 1.5 and [NV17, Theorem C]. In [NV17, Theorem B] Navarro–Vallejo have shown that to establish Conjecture 1.5 for all finite groups it is enough to establish the conjecture when G is an almost simple group whose socle is a finite non-abelian simple group of 2-power index. Using this approach we are able to establish the validity of Conjecture 1.5.

Theorem 1.7. *If G is an almost simple group whose socle is non-abelian and has 2-power index, then Conjecture 1.5 holds for G. In particular, Conjecture 1.5 holds for all finite groups.*

1.8. Now let S be a finite non-abelian simple group and $S \leqslant A \leqslant \mathrm{Aut}(S)$ an almost simple group with A/S a 2-group. As one might expect, checking that A satisfies Conjecture 1.5 is closely related to checking that S is SN2S-Good. Hence, we first consider when our previous work [SF16; SFT18; SF17] establishes that Conjecture 1.5 holds for A. This turns out to be the case unless S is one of the following groups: a group of Lie type defined in characteristic 2, $\mathsf{A}_{n-1}^{\pm}(q)$, $\mathsf{E}_6^{\pm}(q)$, ${}^2\mathsf{G}_2(q)$, J_1, J_2, J_3, Suz, HN. Therefore, these are the cases that we must consider here.

1.9. The layout of this paper is as follows. In Sections 2 and 3 we recall some general statements about normalisers of Sylow 2-subgroups and characters of principal 2-blocks. This allows us to establish when Conjecture 1.5 is a consequence of being SN2S-Good, as mentioned in 1.8, see Proposition 3.9. The sporadic groups mentioned in 1.8 and ${}^2\mathsf{G}_2(q)$ are treated in Section 4. In Section 5 we introduce finite reductive groups and give a criterion for an irreducible character of a finite reductive group to be σ-fixed. Using this and results of [SFT18; NT15] we treat the remaining exceptions from 1.8 in Sections 6 to 8.

2. Normalisers of Sylow 2-Subgroups

2.1. Assume S is a finite group with a trivial centre and let $S \leqslant A \leqslant \mathrm{Aut}(S)$ be an extension of S whose quotient A/S is a 2-group. If $Q \in \mathrm{Syl}_2(A)$ then $SQ/S \in \mathrm{Syl}_2(A/S)$ so we must have $SQ/S = A/S$, i.e., $A = SQ$. The intersection $P = S \cap Q \in \mathrm{Syl}_2(S)$ is a Sylow 2-subgroup of S, which is normal in Q. We wish to record some elementary lemmas regarding the relationship between $N_A(Q)$ and $N_S(P)$.

Lemma 2.2. *If $N_S(P) = P \times V$ has a normal 2-complement V, then $N_A(Q) = Q \times C_V(Q)$ has a normal 2-complement $C_V(Q)$. In particular, if S has a self-normalising Sylow 2-subgroup, then so does A.*

Proof. By assumption we have $N_S(P) = P \times V$ with $V \leqslant N_S(P)$ a $2'$-group. An easy calculation shows that $N_A(Q) = N_S(Q)Q$. Moreover, as $N_S(Q) \leqslant N_S(P)$ we have $N_S(Q) = P \times N_V(Q)$ so $N_A(Q) = Q \rtimes N_V(Q)$. Note that $C_S(P) = Z(P) \times V$ and as Q normalises both S and P it must normalise V. Hence Q normalises $N_V(Q)$ so $N_A(Q) = Q \times C_V(Q)$ as desired. ■

Lemma 2.3. *Assume $C_S(P) = Z(P)$ and $N_A(Q)$ has a normal 2-complement. Then $N_A(Q) = Q$.*

Proof. Assume $N_A(Q) = Q \times V$ has a normal 2-complement V. Any element $v \in V$ has odd order and centralises $P \leqslant Q$. However, as A/S is a 2-group we must have $v \in C_S(P) = Z(P) \leqslant P$ so $v = 1$. ■

3. Passing From Almost Simple to Simple Groups

If G is a finite group then we denote by $B_0(G)$ the principal 2-block of G. Moreover, we denote by $\mathrm{Irr}(B_0(G))$ the ordinary irreducible characters of G contained in the block and by $\mathrm{Irr}_{2'}(B_0(G)) = \mathrm{Irr}(B_0(G)) \cap \mathrm{Irr}_{2'}(G)$ those that have odd degree.

3.1. The main result of [NV17] states that Conjecture 1.5 holds for all finite groups if it holds for all almost simple groups A whose quotient A/S by its non-abelian socle S is a 2-group. In this section we develop several lemmas which allow us to deduce, in certain scenarios, that Conjecture 1.5 holds for A if it holds for S. Lemma 2.2 already goes in this direction given the following result of Brauer, see [Nav98, Corollary 6.13].

Lemma 3.2 (Brauer). *Let G be a finite group and $P \in \mathrm{Syl}_2(G)$. Then the principal block $B_0(G)$ contains only one irreducible Brauer character if and only if $N_G(P)$ has a normal 2-complement.*

3.3. Let G be a finite group. We now turn our attention to the statement that every character $\chi \in \mathrm{Irr}_{2'}(B_0(G))$ is σ-fixed. For a group X acting on the irreducible characters $\mathrm{Irr}(G)$ we write $\mathrm{Irr}_{2'}(G)_X$ for the members of $\mathrm{Irr}_{2'}(G)$ that are invariant under X.

Lemma 3.4. *Assume G is a finite group with normal subgroup $N \lhd G$ whose quotient G/N is a 2-group. Then given any odd-degree character $\chi \in \mathrm{Irr}_{2'}(G)$, the restriction $\mathrm{Res}_N^G(\chi) \in \mathrm{Irr}_{2'}(N)_{G/N}$ is irreducible. Furthermore, $\chi \in \mathrm{Irr}_{2'}(B_0(G))$ if and only if $\mathrm{Res}_N^G(\chi) \in \mathrm{Irr}_{2'}(B_0(N))_{G/N}$ and χ is σ-fixed if and only if $\mathrm{Res}_N^G(\chi)$ is σ-fixed.*

Proof. Let $\chi \in \mathrm{Irr}_{2'}(G)$ and let $\varphi \in \mathrm{Irr}(N)$ satisfy $\langle \mathrm{Res}_N^G(\chi), \varphi \rangle \neq 0$. By Clifford theory, $\chi(1)/\varphi(1)$ divides the index $[G : N]$ which is a 2-power. Since $\chi(1)$ is odd it follows that $\chi(1) = \varphi(1)$ and $\mathrm{Res}_N^G(\chi) = \varphi$ is irreducible and invariant under G/N. The second statement follows from the observation that the principal block $B_0(G)$ is the only block of G that covers $B_0(N)$, which is a consequence of [Nav98, Theorem 8.11] since the trivial character 1_G is the unique irreducible Brauer character lying over 1_N. The last statement follows from [SF16, Lemma 3.4]. ■

Corollary 3.5. *Let $A = SQ$ be an almost simple group with socle S and $Q \in \mathrm{Syl}_2(A)$. Assume $N_A(Q)$ has a normal 2-complement. If every Q-invariant $\chi \in \mathrm{Irr}_{2'}(B_0(S))$ is σ-fixed, then A satisfies Conjecture 1.5.*

Proof. This is an immediate consequence of Lemmas 3.2 and 3.4. ■

Corollary 3.6. *Let S be a finite simple group such that $N_S(P)$ has a normal 2-complement for some $P \in \mathrm{Syl}_2(S)$. Assume every $\chi \in \mathrm{Irr}_{2'}(B_0(S))$ is σ-fixed, i.e., S satisfies Conjecture 1.5. Then any almost simple group $S \leqslant A \leqslant \mathrm{Aut}(S)$ with A/S a 2-group satisfies Conjecture 1.5.*

Proof. If $Q \in \mathrm{Syl}_2(A)$ then by Lemma 2.2 $N_A(Q)$ has a normal 2-complement because $N_S(P)$ does. Moreover, by Lemma 3.4 we have every member of $\mathrm{Irr}_{2'}(B_0(A))$ is fixed by σ because every member of $\mathrm{Irr}_{2'}(B_0(S))$ is. ■

3.7. It is known that all finite simple groups are SN2S-Good, see [SF16; SFT18; SF17]. We would like to use these previous results to deduce that if a finite simple group S is SN2S-Good, then any almost simple group A with socle S and quotient A/S a 2-group satisfies Conjecture 1.5. Unfortunately there are exceptions to this, but we deal with all such cases in the following sections. Before proceeding we will need the following lemma.

Lemma 3.8. *Assume G is a finite group and $N \lhd G$ is a normal subgroup. If we identify $\mathrm{Irr}(G/N)$ with a subset of $\mathrm{Irr}(G)$ then we have $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$. Moreover, if N is a 2-group and $G/C_G(N)$ is a 2-group then we have*

$$\mathrm{Irr}(B_0(G/N)) = \{\chi \in \mathrm{Irr}(B_0(G)) \mid N \leqslant \mathrm{Ker}(\chi)\}.$$

Proof. This is just [Nav98, 7.6] together with the observation that the trivial character of G/N lifts to the trivial character of G. ■

Proposition 3.9. *Let S be a non-abelian finite simple group and let $S \leqslant A \leqslant \mathrm{Aut}(S)$ be an almost simple group with A/S a 2-group. If S is SN2S-Good, in the sense of [SF16, Definition 1], then A satisfies Conjecture 1.5 unless S is one of the following finite simple groups:*

- ${}^2\mathsf{G}_2(q)$, J_1, J_2, J_3, Suz, HN,
- *a simple group of Lie type defined in characteristic 2 whose quasi-split maximal torus is nontrivial,*
- $\mathsf{A}^{\pm}_{n-1}(q)$ *with q odd,*
- $\mathsf{E}^{\pm}_6(q)$ *with q odd.*

Proof. First suppose that S has a self-normalising Sylow 2-subgroup. Then A has a self-normalising Sylow 2-subgroup by Lemma 2.2. Hence, by Corollary 3.5, we have A satisfies Conjecture 1.5 if every $\chi \in \mathrm{Irr}_{2'}(B_0(S))$ is σ-fixed. However, if S is SN2S-Good then all odd-degree irreducible characters of S are σ-fixed.

If S does not have a self-normalising Sylow 2-subgroup and is not in the stated list of exceptions then, by [Kon05], S must be $\mathrm{PSp}_{2m}(q)$ with $q \equiv \pm 3 \pmod 8$ and $m \geq 2$.

Now, note that since $q \equiv \pm 3 \pmod 8$, q is an odd power of an odd prime so S has index at most 2 in A. Specifically, if $A \neq S$ then $A = \mathrm{InnDiag}(S)$. Furthermore, $\mathrm{InnDiag}(S)$ has a self-normalising Sylow 2-subgroup by [SF17, Lemma 3.17]. By [MS16, Lemma 7.5] and [CE04, Lemma 21.14], we see that $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}_{2'}(B_0(G))$, where $G = \mathrm{Sp}_{2n}(q)$ is the Schur cover for S. Thus, by Lemma 3.8, we have $\mathrm{Irr}_{2'}(S) = \mathrm{Irr}_{2'}(B_0(S))$ since $S = G/Z(G)$ with $|Z(G)| = 2$. Using Corollary 3.5 we see that it suffices to show that there exists a character $\chi \in \mathrm{Irr}_{2'}(S)$ which is not σ-fixed but that every $\mathrm{InnDiag}(S)$-invariant character $\chi \in \mathrm{Irr}_{2'}(S)$ is σ-fixed. That is, it suffices to show that S is SN2S-Good. ■

4. The Groups ${}^2\mathsf{G}_2(q)$, J_1, J_2, J_3, Suz, HN

Proposition 4.1. *If S is one of the simple groups ${}^2\mathsf{G}_2(q)$, J_1, J_2, J_3, Suz, or HN and $S \leqslant A \leqslant \mathrm{Aut}(S)$ is an almost simple group with A/S a 2-group, then A satisfies Conjecture 1.5.*

Proof. If S is either ${}^2\mathsf{G}_2(q)$ or J_1 then S does not have a self-normalising Sylow 2-subgroup and the outer automorphism group has odd order or is trivial, respectively. In the other cases the outer automorphism group is order 2 but $\mathrm{Aut}(S)$ has a self-normalising Sylow 2-subgroup. Hence by Corollary 3.5, and [SF16, Theorems 4.2 and 4.3], it suffices to show that the odd degree characters illustrated in the proofs of [SF16, Theorems 4.2 and 4.3] that are not fixed by σ also lie in the principal block.

In the case of the Ree groups ${}^2\mathsf{G}_2(q)$ the character χ_4 in the notation of [Gec+96], mentioned in [SF16, Theorem 4.3] to not be σ-invariant, is the character ξ_7 in the notation of [LM80], which is shown there to be in the principal block.

From information in the GAP Character Table Library [Bre04] regarding $\mathsf{J}_1, \mathsf{J}_2, \mathsf{J}_3$, Suz, and HN in characteristic 2, we see that there are two characters of degree 77, 21, 85, 5005, and 133, respectively, which are interchanged by the action of σ and lie in the principal block. ■

5. Generalities on Reductive Groups

From now on we assume $p > 0$ is a fixed prime and $\mathbb{K} = \overline{\mathbb{F}}_p$ is an algebraic closure of the finite field $\mathbb{F}_p$ of cardinality p.

5.1. Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{K}$ and let $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism so that $G = \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite reductive group. Given such a pair $(\mathbf{G}, F)$ we will denote by $\mathcal{S}(\mathbf{G}, F)$ the set of all pairs $(\mathbf{T}, s)$ consisting of an F-stable maximal torus $\mathbf{T} \leqslant \mathbf{G}$ and a rational semisimple element $s \in \mathbf{T}^F$. We fix a regular embedding $\iota : \mathbf{G} \to \widetilde{\mathbf{G}}$, where $\widetilde{\mathbf{G}}$ is a connected reductive algebraic group over $\mathbb{K}$ with connected centre. The Frobenius endomorphism on $\widetilde{\mathbf{G}}$ will also be denoted by F and the group $\widetilde{G}$ denotes the finite group $\widetilde{\mathbf{G}}^F$.

5.2. Let $(\mathbf{G}^\star, F^\star)$ be a pair dual to $(\mathbf{G}, F)$ and similarly let $(\widetilde{\mathbf{G}}^\star, F^\star)$ be a pair dual to $(\widetilde{\mathbf{G}}, F)$. We will assume that $\iota^\star : \widetilde{\mathbf{G}}^\star \to \mathbf{G}^\star$ is a surjective homomorphism of algebraic groups dual to the regular embedding; note that $F^\star \circ \iota^\star = \iota^\star \circ F^\star$. To each pair $(\mathbf{T}^\star, s) \in \mathcal{S}(\mathbf{G}^\star, F^\star)$ we have a corresponding virtual character $R^{\mathbf{G}}_{\mathbf{T}^\star}(s)$ of $G = \mathbf{G}^F$. If $[s] \subseteq G^\star := \mathbf{G}^{\star F^\star}$ is a $\mathbf{G}^{\star F^\star}$-conjugacy class of semisimple elements then we have a corresponding (rational) Lusztig series $\mathcal{E}(G,[s]) \subseteq \mathrm{Irr}(G)$. These series form a partition

$$\mathrm{Irr}(G) = \bigsqcup_{[s]\subseteq G^\star} \mathcal{E}(G,[s])$$

of the irreducible characters. We will need the following well-known lemma concerning restriction of characters from $\widetilde{G}$ to G, see [Bon06, 11.7].

Lemma 5.3. *Assume $\chi \in \mathcal{E}(G,[s])$ is an irreducible character and $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ is an irreducible character covering χ. Then $\widetilde{\chi} \in \mathcal{E}(\widetilde{G},[\widetilde{s}])$ with $\widetilde{s} \in \widetilde{G}^\star$ satisfying $\iota^\star(\widetilde{s}) = s$.*

5.4. For the rest of this section we assume that $\gamma \in \mathrm{Gal}(\mathbb{Q}_{|\widetilde{G}|}/\mathbb{Q})$ where $\mathbb{Q}_{|\widetilde{G}|}$ is the field obtained from $\mathbb{Q}$ by adjoining a primitive $|\widetilde{G}|$th root of unity. Moreover, we assume that $\mathcal{E}(G,[s])$ is a γ-invariant Lusztig series and $\chi \in \mathcal{E}(G,[s])$ is an irreducible character. Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ be a character covering χ so that $\widetilde{\chi} \in \mathcal{E}(\widetilde{G},\tilde{s})$ with $\tilde{s} \in \widetilde{G}^\star$ satisfying $\iota^\star(\tilde{s}) = s$ by Lemma 5.3. The proof of [SFT18, 3.4] shows that $\mathcal{E}(\widetilde{G},[\tilde{s}])^\gamma = \mathcal{E}(\widetilde{G},[\tilde{t}])$ for some semisimple element $\tilde{t} \in \widetilde{G}^\star$. Now clearly $\widetilde{\chi}^\gamma \in \mathcal{E}(\widetilde{G},[\tilde{t}])$ and χ^γ is covered by $\widetilde{\chi}^\gamma$ so another application of Lemma 5.3 shows that $\iota^\star(\tilde{t}) = s$ because $\chi^\gamma \in \mathcal{E}(G,[s])$ by assumption.

5.5. The kernel $\mathrm{Ker}(\iota^\star)$ is connected, see [Bon06, 2.5], so the Lang–Steinberg theorem shows that there exists an element $\tilde{z} \in \mathrm{Ker}(\iota^\star)^{F^\star}$ such that $\tilde{t} = \tilde{s}\tilde{z}$. By [Bon06, 2.6, 11.6] we have a bijection

$$\begin{aligned} \mathcal{E}(\widetilde{G},[\tilde{s}]) &\to \mathcal{E}(\widetilde{G},[\tilde{s}\tilde{z}]) \\ \widetilde{\chi} &\mapsto \widetilde{\chi} \otimes \theta_{\tilde{z}} \end{aligned}$$

where $\theta_{\tilde{z}} \in \mathrm{Irr}(\widetilde{G})$ is the lift of an irreducible character of the quotient $\widetilde{G}/G$. With this we can prove the following.

Proposition 5.6. *Let $\gamma \in \mathrm{Gal}(\mathbb{Q}_{|\widetilde{G}|}/\mathbb{Q})$ be a Galois automorphism and $\mathcal{E}(G,[s])$ a γ-invariant Lusztig series. Assume $\tilde{s} \in \widetilde{G}^\star$ is such that $\iota^\star(\tilde{s}) = s$ and $\widetilde{\chi} \in \mathcal{E}(\widetilde{G},[\tilde{s}])$ satisfies the following property:*

- ($\star$) *for any $\widetilde{\chi}' \in \mathcal{E}(\widetilde{G},[\tilde{s}])$ we have $\langle \widetilde{\chi}', R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})\rangle_{\widetilde{G}} = \langle \widetilde{\chi}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})\rangle_{\widetilde{G}}$ for all $(\widetilde{\mathbf{T}}^\star,\tilde{s}) \in \mathcal{S}(\widetilde{\mathbf{G}}^\star,F^\star)$ if and only if $\widetilde{\chi} = \widetilde{\chi}'$.*

Then if $\chi \in \mathcal{E}(G,s)$ is a constituent of $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$, so is χ^γ. In particular, if χ extends to $\widetilde{G}$ then $\chi^\gamma = \chi$.

Proof. The proof of [SFT18, 3.4] together with [Bon06, 11.5(b)] shows that, for some $\tilde{z} \in \mathrm{Ker}(\iota^\star)^{F^\star}$, we have

$$R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})^\gamma = R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s}\tilde{z}) = R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s}) \otimes \theta_{\tilde{z}} \tag{5.7}$$

for any $(\widetilde{\mathbf{T}}^\star,\tilde{s}) \in \mathcal{S}(\widetilde{\mathbf{G}}^\star,F^\star)$. Now, this implies that

$$\langle \widetilde{\chi}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})\rangle = \langle \widetilde{\chi}^\gamma, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})^\gamma\rangle = \langle \widetilde{\chi}^\gamma \otimes \theta_{\tilde{z}}^{-1}, R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{T}}^\star}(\tilde{s})\rangle$$

for any $(\widetilde{\mathbf{T}}^\star, \tilde{s}) \in \mathcal{S}(\widetilde{\mathbf{G}}^\star, F^\star)$. By assumption ($\star$) we thus have $\widetilde{\chi}^\gamma = \widetilde{\chi} \otimes \theta_{\tilde{z}}$. In particular, we must have $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}^\gamma) = \mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$, which implies χ and χ^γ are both constituents of $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi})$. ∎

Remark 5.8. It has been shown by Digne–Michel in [DM90, 6.3] that condition ($\star$) in Proposition 5.6 is satisfied in all but a few extreme cases. We also note that Proposition 5.6 is also true if γ is taken to be an automorphism of $\widetilde{G}$ stabilising G.

6. Characteristic 2

In this section we assume that $\mathbf{G}$ is simple and that $p = 2$.

6.1. Recall that $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism. In particular, there exists a unique *minimal* integer $d \geqslant 1$ for which F^d is a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-rational structure, where $q = 2^a \in \mathbb{N}$ for some $a \geqslant 1$. Recall that a maximal torus $\mathbf{T} \leqslant \mathbf{G}$ is said to be *quasi-split* if it is F-stable and contained in an F-stable Borel subgroup. Moreover, we say the Frobenius endomorphism F^d is *split* if for some maximal torus $\mathbf{T} \leqslant \mathbf{G}$ we have $F^d(t) = t^q$ for all $t \in \mathbf{T}$. We will need the following lemma.

Lemma 6.2. *Assume $\mathbf{T} \leqslant \mathbf{G}$ is a quasi-split maximal torus such that $\mathbf{T}^F = Z(\mathbf{G})^F$. Then we must have $q = 2$ and one of the following holds: F^d is split, in which case $\mathbf{T}^F = \{1\}$, or $\mathbf{G}^F = \mathrm{SU}_3(2)$, which is solvable.*

Proof. Let $\pi : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ be a simply connected cover of $\mathbf{G}$. We may lift F to a Steinberg endomorphism $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$, i.e., $F \circ \pi = \pi \circ F$. It follows from [DM91, 0.35] that $\pi(Z(\mathbf{G}_{\mathrm{sc}})) = Z(\mathbf{G})$ so $|Z(\mathbf{G})^F| \leqslant |Z(\mathbf{G}_{\mathrm{sc}})|$.

Using the formula in [Car93, 2.9] we can easily compute $|\mathbf{T}^F|$. If $q = 2$ and F^d is split then $\mathbf{T}^{F^d}$, hence also $Z(\mathbf{G})^F \leqslant \mathbf{T}^F$, is trivial. Assume conversely that $q \geqslant 2$ and if $q = 2$ then F^d is not split. Then by [Car93, 3.6.7] we have $\mathbf{T}^F \neq \{1\}$. We aim to show that $|Z(\mathbf{G}_{\mathrm{sc}})| < |\mathbf{T}^F|$ which implies that $\mathbf{T}^F \neq Z(\mathbf{G})^F$. If $\mathbf{G}$ is not of type A_{n-1} or E_6 then $Z(\mathbf{G}_{\mathrm{sc}})$ is trivial, see [MT11, Table 24.2]. If $\mathbf{G}$ is of type E_6 then $|Z(\mathbf{G}_{\mathrm{sc}})| = 3$ but $|\mathbf{T}^F| = (q-1)^6 \geqslant 3^6$ if G is of type $\mathsf{E}_6(q)$ and $|\mathbf{T}^F| = (q-1)^4(q+1)^2 \geqslant 3^2$ if G is of type $\mathsf{E}_6^-(q)$.

In the case of A_{n-1} we have $|Z(\mathbf{G}_{\mathrm{sc}})| \leqslant n$ and if G is of type $\mathsf{A}_{n-1}(q)$ then $|\mathbf{T}^F| = (q-1)^{n-1} \geqslant 3^{n-1} > n$. Assume now that G is of type ${}^2\mathsf{A}_{n-1}(q)$. Let us write $n-1$ as $2k+\delta$, where $k = \lfloor (n-1)/2 \rfloor$ and $\delta \in \{0,1\}$. We then have $|\mathbf{T}^F| = (q-1)^{k+\delta}(q+1)^k \geqslant 3^k$ and $|Z(\mathbf{G}_{\mathrm{sc}})| \leqslant 2k+2 < 3^k$ whenever $k > 1$. Thus we need only consider the cases ${}^2\mathsf{A}_3$ and ${}^2\mathsf{A}_2$. If $\mathbf{G}$ is of type ${}^2\mathsf{A}_3$ then $Z(\mathbf{G}_{\mathrm{sc}})$ is trivial so we're done. If $\mathbf{G}$ is of type ${}^2\mathsf{A}_2$ then $Z(\mathbf{G})$ is trivial unless $\mathbf{G} = \mathbf{G}_{\mathrm{sc}}$. If $q \geq 4$, the above yields that $|Z(\mathbf{G}_{\mathrm{sc}})| < |\mathbf{T}^F|$, so we are left with the case $\mathbf{G}^F = \mathrm{SU}_3(2)$, in which case $|\mathbf{T}^F| = |Z(\mathbf{G}^F)| = 3$. ∎

6.3. We assume $\mathbf{T}_0 \leqslant \mathbf{B}_0 \leqslant \mathbf{G}$ are an F-stable maximal torus and Borel subgroup of $\mathbf{G}$, respectively. We set $\mathbf{U}_0 = R_u(\mathbf{B}_0)$ to be the unipotent radical of the Borel. If Φ

are the roots of $\mathbf{G}$ with respect to $\mathbf{T}_0$ then for each $\alpha \in \Phi$ we fix a closed embedding $x_\alpha : \mathbb{K}^+ \to \mathbf{G}$ such that

$$tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$$

for all $t \in \mathbf{T}_0$ and $c \in \mathbb{K}^+$. If $\Delta \subseteq \Phi$ are the simple roots determined by $\mathbf{B}_0$ then the set $\{x_\alpha(c) \mid \pm\alpha \in \Delta, c \in \mathbb{K}^+\}$ generates $\mathbf{G}$ by [Hum75, 27.5, Theorem].

6.4. Recall that we have a standard Frobenius endomorphism $F_2 : \mathbb{K} \to \mathbb{K}$ given by $F_2(c) = c^2$. There then exists a split Frobenius endomorphism $F_2 : \mathbf{G} \to \mathbf{G}$ such that $F_2 \circ x_\alpha = x_\alpha \circ F_2$ for all $\alpha \in \Phi$ and $F_2(t) = t^2$ for all $t \in \mathbf{T}_0$. Note the groups $\mathbf{U}_0$ and $\mathbf{B}_0$ are F_2-stable. More generally, if $r = 2^b$ with $b \geqslant 0$ we denote by F_r the b-fold composition $F_2 \circ \cdots \circ F_2$; we call this a *field automorphism* of $\mathbf{G}$. If $r = 1$ then this is the identity and if $r \geqslant 2$ then this is a split Frobenius endomorphism.

6.5. We define a *graph automorphism* of $\mathbf{G}$ to be a bijective morphism $\tau : \mathbf{G} \to \mathbf{G}$ such that $\mathbf{T}_0$ and $\mathbf{B}_0$ are τ-stable and $\tau \circ x_\alpha = x_{\rho(\alpha)} \circ F_2^{\varepsilon(\alpha)}$ for some bijection $\rho : \Delta \to \Delta$ and function $\varepsilon : \Delta \to \{0, 1\}$. Here we have $\varepsilon(\alpha) = 0$ unless $\mathbf{G}$ is of type B_2 or F_4 and α is a short root. Note that any graph automorphism commutes with F_2. After possibly composing with an inner automorphism of $\mathbf{G}$ we may, and will, assume that our Steinberg endomorphism F is of the form $F = F_r \circ \tau = \tau \circ F_r$ for some (possibly trivial) graph automorphism τ and some $r = 2^b$ with $b \geqslant 0$. Note $\mathbf{T}_0$, $\mathbf{U}_0$, and $\mathbf{B}_0$ are F-stable and we set $T_0 = \mathbf{T}_0^F$, $U_0 = \mathbf{U}_0^F$, and $B_0 = \mathbf{B}_0^F$.

6.6. Let us denote by $\Gamma(\mathbf{G}) \leqslant \mathrm{Aut}(\mathbf{G})$ the subgroup generated by field and graph automorphisms, so that $\Gamma(\mathbf{G}) = \langle F_2, \tau_0, \tau_1 \rangle$ for some (possibly trivial) graph automorphisms $\tau_0, \tau_1 \in \mathrm{Aut}(\mathbf{G})$ satisfying $\tau_0^2 \in \{1, F_2\}$ and $\tau_0^3 = 1$. We have a natural surjective map $C_{\mathrm{Aut}(\mathbf{G})}(F) \to \mathrm{Aut}(G)$, where $G = \mathbf{G}^F$, given by restriction and we denote by $\Gamma(G)$ the image of $C_{\Gamma(\mathbf{G})}(F)$ under this map. If $\mathbf{G}$ is simply connected and G is perfect then the quotient $S = G/Z$, where $Z := Z(G) = Z(\mathbf{G})^F$, is a finite simple group of Lie type in characteristic 2, see [MT11, 24.13, 24.14]. Moreover, by [GLS98, 2.5.1, 2.5.12(a), 2.5.14] we have

$$\mathrm{Aut}(S) \cong \mathrm{Aut}(G) \cong \widetilde{G}/Z(\widetilde{G}) \rtimes \Gamma(G).$$

In what follows we will denote by $\widetilde{S}$ the group $\widetilde{G}/Z(\widetilde{G})$ and we will also identify S with a subgroup of $\widetilde{S}$. We will need the following consequence of Lemma 6.2.

Lemma 6.7. *We have $U_0 \leqslant G$ is a Sylow 2-subgroup, $N_G(U_0) = B_0 = U_0 \rtimes T_0$, and $C_G(U_0) = Z(U_0)Z$. Moreover, if G is perfect, then $N_G(U_0)$ has no normal 2-complement unless $q = 2$ and F^d is split, in which case $N_G(U_0) = U_0$.*

Proof. The statements about U_0, $N_G(U_0)$, and $C_G(U_0)$ are well known, see [MT11, 24.11] and [CE04, 2.31]. It follows that $N_G(U_0)$ has a normal 2-complement only when $T_0 = Z$ so the last statement follows from Lemma 6.2. ■

6.8. As U_0 is a Sylow 2-subgroup of G and Z is a $2'$-group we have $P = U_0Z/Z$ is a Sylow 2-subgroup of $S = G/Z$. The quotient $\widetilde{G}/G \cong \widetilde{S}/S$ has odd order so P is a Sylow 2-subgroup of $\widetilde{S}$. Our explicit choice of P is $\Gamma(G)$-invariant so if $\Gamma_2(G) \leqslant \Gamma(G)$ is a Sylow 2-subgroup of $\Gamma(G)$, then $P \rtimes \Gamma_2(G)$ is a Sylow 2-subgroup of $\widetilde{S} \rtimes \Gamma(G)$. Now, assume $S \leqslant A \leqslant \widetilde{S} \rtimes \Gamma(G)$ is an almost simple group whose quotient A/S is a 2-group. Up to conjugacy we can then assume that $A = S \rtimes Q_0$ with $Q_0 \leqslant \Gamma_2(G)$ a 2-subgroup. The group $Q := P \rtimes Q_0 \leqslant S \rtimes Q_0$ is a Sylow 2-subgroup of A and $P = S \cap Q$.

Lemma 6.9. *Assume $\mathbf{G}$ is simply connected and G is perfect so that $S = G/Z$ is a finite non-abelian simple group. If $A = S \rtimes Q_0$ is an almost simple group with $Q_0 \leqslant \Gamma(G)$ a 2-group, then $Q := P \rtimes Q_0 \in \mathrm{Syl}_2(A)$ and we have $N_A(Q) = Q$ if and only if $F_2 \in Q_0$. If $N_A(Q) \neq Q$, then $N_A(Q)$ has no normal 2-complement.*

Remark 6.10. Note that if $q = 2$ and F^d is split, then trivially we have $F_2 \in Q_0$ for any subgroup $Q_0 \leqslant \Gamma_2(G)$ because F_2 is the identity. Hence, if $F = F_2$ then we have $N_A(Q) = Q$ for any almost simple group $A = S \rtimes Q_0$, which follows from Lemma 6.7.

Proof (of Lemma 6.9). By [NTT07, Lemma 2.1] we have $N_A(Q)/Q \cong C_{N_G(U_0)/U_0Z}(Q_0)$. As we assume $Q_0 \leqslant \Gamma_2(G)$ we have the natural map $T_0/Z \to N_G(U_0)/U_0Z$ is a Q_0-equivariant isomorphism, hence $C_{N_G(U_0)/U_0Z}(Q_0) \cong C_{T_0/Z}(Q_0)$. Therefore, we have $N_A(Q) = Q$ is self-normalising if and only if $C_{T_0/Z}(Q_0) = \{1\}$. Furthermore, by Lemma 2.3 if $C_{T_0/Z}(Q_0) \neq \{1\}$ then $N_A(Q)$ has no normal 2-complement.

One direction of the statement is clear. If $tZ \in T_0/Z$ is such that $tZ = F_2(tZ) = t^2Z$ then clearly $t \in Z$ so $(T_0/Z)^{F_2} = \{1\}$. Hence, if $F_2 \in Q_0$ then $C_{T_0/Z}(Q_0) = \{1\}$.

We now prove the other direction, i.e., we aim to prove that $C_{T_0/Z}(Q_0) \neq \{1\}$ assuming $F_2 \notin Q_0$. We will denote by $\tilde{Q}_0 \leqslant C_{\Gamma(\mathbf{G})}(F)$ the preimage of $Q_0 \leqslant \Gamma(G)$ under the restriction map $C_{\Gamma(\mathbf{G})}(F) \to \Gamma(G)$. If $\mathbf{Z} = Z(\mathbf{G})$ then we have $C_{T_0/Z}(Q_0) \cong C_{\mathbf{T}_0/\mathbf{Z}}(\tilde{Q}_0)$ because $T_0/Z \cong (\mathbf{T}_0/\mathbf{Z})^F$. As $F_2 \notin Q_0$ we have $F_2 \notin \tilde{Q}_0$ so it suffices to show that $C_{\mathbf{T}_0/\mathbf{Z}}(\tilde{Q}_0) \neq \{1\}$.

If $\tilde{Q}_0 \leqslant K$ then clearly $C_{\mathbf{T}_0/\mathbf{Z}}(K) \leqslant C_{\mathbf{T}_0/\mathbf{Z}}(\tilde{Q}_0)$ so it suffices to show that $C_{\mathbf{T}_0/\mathbf{Z}}(K) \neq \{1\}$ for some subgroup $K \leqslant \Gamma(\mathbf{G})$ containing $\tilde{Q}_0$. Note that for any element $\gamma \in \Gamma(\mathbf{G})$ we have $Z(\mathbf{G}^\gamma) = Z(\mathbf{G})^\gamma$. If γ is a Steinberg endomorphism then this is just [MT11, 24.13] but an identical argument treats the general case. In particular, for any $\gamma \in \Gamma(\mathbf{G})$ we have an inclusion map

$$\mathbf{T}_0^\gamma/\mathbf{Z}^\gamma \to (\mathbf{T}_0/\mathbf{Z})^\gamma = C_{\mathbf{T}_0/\mathbf{Z}}(\gamma). \tag{6.11}$$

Assume $\mathbf{G}$ is of type A_1, B_n or C_n ($n > 2$), E_7, E_8, or G_2, then $\tilde{Q}_0 = \langle \phi \rangle$ where $\phi = F_2^m$ for some $m > 1$. By Lemma 6.2 we have $\mathbf{T}_0^\phi/\mathbf{Z}^\phi \neq \{1\}$ so we're done by (6.11) because $C_{\mathbf{T}_0/\mathbf{Z}}(\tilde{Q}_0) = C_{\mathbf{T}_0/\mathbf{Z}}(\phi)$. If $\mathbf{G}$ is of type B_2 or F_4 then $\tilde{Q}_0 = \langle \phi \rangle$ where $\phi = \tau_0^m$ for some $m > 2$. The exact same argument treats this case.

Finally, assume $\mathbf{G}$ is of type A_n ($n \geqslant 2$), D_n ($n \geqslant 4$) or E_6 then, as Q_0 is a 2-group, we have $\tilde{Q}_0 \leqslant \langle F_2, \psi \rangle$ where $\psi \neq 1$ is exactly one of τ_0 or τ_1. We have $\langle F_2, \psi \rangle = \langle F_2 \rangle \times \langle \psi \rangle$ and a simple application of Goursat's Lemma shows that, up to conjugacy in $\Gamma(\mathbf{G})$, we have $\tilde{Q}_0 \leqslant \langle F_2^m, \psi \rangle$ for some $m > 1$ or $\tilde{Q}_0 \leqslant \langle F_2^m \psi \rangle$ for some $m \geqslant 1$. As $F_2^m \psi$ is a Steinberg endomorphism the case where $\tilde{Q}_0 \leqslant \langle F_2^m \psi \rangle$ can be treated as above.

Now, let $K = \langle \phi, \psi \rangle$ where $\phi = F_2^m$ for some $m > 1$. As ψ is an automorphism of $\mathbf{G}$ as an algebraic group we have by [GLS98, 1.15.2(d)], see also [Spr09, 10.3.5], that the group $\bar{\mathbf{G}} = \mathbf{G}^{\psi} \leqslant \mathbf{G}$ is a simple algebraic group with maximal torus and Borel subgroup $\bar{\mathbf{T}} = \mathbf{T}_0^{\psi} \leqslant \bar{\mathbf{B}} = \mathbf{B}_0^{\psi}$. These subgroups are stable under F_2 so ϕ restricts to a split Frobenius endomorphism on these groups endowing $\bar{\mathbf{G}}$ with an $\mathbb{F}_{2^m}$-rational structure, see [Gec03, 4.1.5]. As $\bar{\mathbf{T}}$ is a quasi-split maximal torus of $\bar{\mathbf{G}}$ with respect to ϕ we have by Lemma 6.2 that $\bar{\mathbf{T}}^{\phi}/Z(\bar{\mathbf{G}})^{\phi} \neq \{1\}$ is non-trivial so $C_{\mathbf{T}_0/\mathbf{Z}}(K) = C_{\mathbf{T}_0/\mathbf{Z}}(\phi) \cap C_{\mathbf{T}_0/\mathbf{Z}}(\psi) \neq \{1\}$. ■

Proposition 6.12. *Assume* $\mathbf{G}$ *is simple and simply connected and* $p = 2$. *If* G *is perfect, so that* $S = G/Z$ *is a non-abelian simple group, and* $S \leqslant A \leqslant \mathrm{Aut}(S)$ *is an almost simple group with* A/S *a* 2*-group, then* A *satisfies Conjecture 1.5.*

Proof. The group S has a strongly split BN-pair, in the sense of [CE04, 2.20], and satisfies the hypothesis in [CE04, 6.14]. Indeed, G satisfies this hypothesis by [CE04, 6.15] hence so does S because it's a quotient of G by a $2'$-group. According to [CE04, 6.18], as $C_S(P) = Z(P)$, we have every 2-block of S is either the principal block or a block of defect zero. In particular, this implies that

$$\mathrm{Irr}_{2'}(S) = \mathrm{Irr}_{2'}(B_0(S)) \tag{6.13}$$

because any irreducible character of S with maximal defect is in the principal block.

We will write $A = S \rtimes Q_0$, with $Q_0 \leqslant \Gamma_2(G)$ a 2-group, and set $Q = P \rtimes Q_0$ a Sylow 2-subgroup of A as in 6.8. If $F_2 \in Q_0$ then we are in the case that $N_A(Q) = Q$ by Lemma 6.9. It follows from the proof of [SFT18, 5.8] that every Q_0-invariant member of $\mathrm{Irr}_{2'}(S)$ is fixed by σ, so A satisfies Conjecture 1.5 by Corollary 3.5. Now assume $F_2 \notin Q_0$ then $N_A(Q)$ has no normal 2-complement, c.f., Lemma 6.9. We must show that there exists an odd degree character $\chi \in \mathrm{Irr}_{2'}(B_0(A))$ which is not σ-fixed.

It suffices to find an A-invariant character $\chi \in \mathrm{Irr}_{2'}(S)$ which is not σ-fixed and extends to A. Indeed, if $\widetilde{\chi} \in \mathrm{Irr}_{2'}(A)$ is such an extension, then $\widetilde{\chi}$ is in the principal block of A and is not σ-fixed, see (6.13) and Lemma 3.4. If $q = 2$ and F is twisted, then $\Gamma(G)$ has cardinality two or three, so $A = S$ because $\widetilde{S}/S$ is odd and $F_2 \notin Q_0$. In this case the existence of an odd degree character of S which is not σ-fixed was shown in the closing paragraphs of the proofs of [SF16, 4.9, 4.11, 4.12, and 4.15].

Now we can assume $q > 2$. In the proof of [SFT18, 6.4] and [SF16, 4.12] it is shown that there exists an odd-degree character $\chi \in \mathrm{Irr}_{2'}(G)$ with the following properties: χ extends to $\widetilde{G}$, is Q_0-invariant, has Z in its kernel, and is not σ-fixed. As χ extends to $\widetilde{G}$, it follows from [Spä12, 3.4] that χ extends to its inertia group $G \rtimes \Gamma(G)_{\chi}$ in the semidirect product $G \rtimes \Gamma(G)$. Let $\widetilde{\chi} \in \mathrm{Irr}(G \rtimes \Gamma(G)_{\chi})$ be such an extension. Then clearly Z is in the kernel of $\widetilde{\chi}$ so we may view this as a character of $S \rtimes \Gamma(G)_{\chi}$. As $Q_0 \leqslant \Gamma(G)_{\chi}$ we have $\mathrm{Res}_A^{S \rtimes \Gamma(G)_{\chi}}(\widetilde{\chi})$ is an extension of χ so A satisfies Conjecture 1.5. ■

6.14. We end this section by making some remarks on the work in [SFT18, §5, §6] concerning groups in characteristic 2. For the following remarks we adopt the notation of [SFT18]. Assuming $\mathbf{G}$ is simply connected and G is perfect the statement of [SFT18, 5.7] is correct but there is insufficient detail in the proof. The statement we require (and use) is that $C_{N_G(P)/PZ}(Q) = \{1\}$ if and only if $F_2 \in \Gamma_Q(G)$ where $Z = Z(G)$. Arguing exactly as in the proof of [SFT18, 5.7] we have $C_{N_G(P)/PZ}(Q) \cong C_{T_0/Z}(\Gamma_Q(G))$. The proof of Lemma 6.9 now shows that $C_{T_0/Z}(\Gamma_Q(G)) = \{1\}$ if and only if $F_2 \in \Gamma_Q(G)$ giving the statement.

6.15. In the proof of [SFT18, 6.4] one has to be more careful. Writing $F = F_q \circ \tau$ one should really work with $\bar{q} = q^d$ instead of q, where d is the order of τ. Replacing q by $\bar{q}$ the argument given in [SFT18, 6.4] works for twisted groups but needs to be modified for split groups, specifically addressing the case that $\Gamma_Q(G)$ contains an automorphism of the form $F_2^s \circ \tau_0$ for some integer $s \geq 1$. Arguments analogous to those in [SFT18, 6.4], [SF16] work here, considering semisimple elements s of the form

$$t_{r,\gamma}(\eta) = \check{\alpha}(\eta) \cdot \gamma(\check{\alpha})(\eta^r) \cdots \gamma^{e-1}(\check{\alpha})(\eta^{r^{e-1}}),$$

for appropriate choices of $\eta \in \mathbb{K}^\times$ and integral powers r of 2. Here $\check{\Delta} = \{\check{\alpha}_1, \dots, \check{\alpha}_n\}$ are the simple coroots, $\gamma \in \Gamma(\mathbf{G})$ is an appropriate graph automorphism of order $e \geqslant 1$, and $\check{\alpha} \in \check{\Delta}$ is a fixed simple coroot whose γ-orbit has length e.

7. Type A_{n-1}

In this section we assume $\mathbf{G} = \mathrm{SL}_n(\mathbb{K})$ and $p \neq 2$.

Lemma 7.1. *If $s \in G^\star$ is a 2-element, then $\chi^\sigma = \chi$ for all $\chi \in \mathcal{E}(G,[s]) \cap \mathrm{Irr}_{2'}(G)$ unless $G = \mathrm{SL}_2(q)$ and $q \equiv \pm 3 \pmod 8$. In this latter case if $\chi \in \mathcal{E}(G,[s]) \cap \mathrm{Irr}_{2'}(G)$ does not extend to $\widetilde{G}$, then it is not σ-fixed.*

Proof. As s is a 2-element we have $\mathcal{E}(G,s)$ is σ-invariant by [SFT18, 3.4]. Assume $\chi \in \mathcal{E}(G,[s])$ is an odd-degree irreducible character and let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ be a character covering χ. By [DM90, 6.3] the condition ($\star$) in Proposition 5.6 is satisfied so we have $\widetilde{\chi}$ covers χ^σ. Moreover, if χ extends to $\widetilde{G}$ then $\chi^\sigma = \chi$. It is shown in [SFT18, 10.2] that if χ does not extend to $\widetilde{G}$ then $n = 2^r$, for some $r \geqslant 1$, and $\mathrm{Res}^{\widetilde{G}}_G(\widetilde{\chi}) = \chi + {}^g\chi$ for some $g \in \widetilde{G}$. If $r > 1$ then one can show that χ and ${}^g\chi$ are σ-fixed using the argument of [SFT18, §10]. The case of $\mathrm{SL}_2(q)$ is easily checked using the character table given in [Bon11].

Proposition 7.2. *Assume $\mathbf{G} = \mathrm{SL}_n(\mathbb{K})$ and $p \neq 2$. If G is perfect, so that $S = G/Z(G)$ is a non-abelian simple group, and $S \leqslant A \leqslant \mathrm{Aut}(S)$ is an almost simple group with A/S a 2-group, then A satisfies Conjecture 1.5.*

Proof. Let us start with the case where $G = \mathrm{SL}_2(q)$ and $q \equiv \pm 3 \pmod 8$. If $P \leqslant S$ is a Sylow 2-subgroup then $N_S(P) \cong \mathrm{SL}_2(3)$ does not have a normal 2-complement. There are precisely 2 odd degree characters of G that do not extend to $\widetilde{G}$ and these are labelled $R'_\sigma(\theta_0)$ in [Bon11, Table 5.4]; they are both in the principal block of G by [Bon11, 7.1.1(e)]. As G is perfect and $R'_\sigma(\theta_0)$ has odd degree it must have $Z(G) \cong C_2$ in its kernel so $R'_\sigma(\theta_0) \in \mathrm{Irr}_{2'}(B_0(S))$ by Lemma 3.8. As these are not σ-fixed, cf., Lemma 7.1, we have Conjecture 1.5 holds for S.

Assume now that $A \neq S$. Recall that if $q = p^m$ then $\mathrm{Aut}(S) \cong \mathrm{PGL}_2(q) \rtimes C_m$, where the cyclic group acts via field automorphisms. As p is odd and $q \equiv \pm 3 \pmod 8$ we must have m is odd, so $A \cong \mathrm{PGL}_2(q)$. We thus have A has a self-normalising Sylow 2-subgroup, see [CF64, Lemma 3]. It follows easily from the character table of $\mathrm{GL}_2(q)$, see [DM91, §15.9], that the only odd degree characters of $\mathrm{PGL}_2(q)$ are the trivial and Steinberg characters, which are σ-fixed; thus Conjecture 1.5 holds for A.

We now consider the case where either $G \neq \mathrm{SL}_2(q)$ or $q \equiv \pm 1 \pmod 8$. As $\mathbf{G}$ is of type A and q is odd we have by [CE04, 21.14] that

$$\mathrm{Irr}(B_0(G)) = \bigcup_{s \in G^\star} \mathcal{E}(G,[s]) \tag{7.3}$$

where the sum is taken over all 2-elements in the dual group $G^\star$. Lemma 7.1 thus implies that every member of $\mathrm{Irr}_{2'}(B_0(G))$ is fixed by σ and by Lemma 3.8, the same is true of $\mathrm{Irr}_{2'}(B_0(S))$. According to [Kon05, §1, Corollary] the normaliser $N_S(P)$ has a normal 2-complement, so A satisfies Conjecture 1.5 by Corollary 3.6. ∎

8. Type E_6

In this section we assume $\mathbf{G}$ is simply connected of type E_6 and $p \neq 2$.

Proposition 8.1. *Assume* $\mathbf{G}$ *is simply connected of type* E_6 *and* $p \neq 2$. *If* G *is perfect, so that* $S = G/Z(G)$ *is a non-abelian simple group, and* $S \leqslant A \leqslant \mathrm{Aut}(S)$ *is an almost simple group with* A/S *a* 2*-group, then* A *satisfies Conjecture 1.5.*

Proof. Write $A = SQ$ where $Q \in \mathrm{Syl}_2(A)$. First, note that if $P \in \mathrm{Syl}_2(S)$, then $N_S(P)$ has a normal 2-complement, see [Kon05, §1, Corollary]. Thus it suffices by Corollary 3.6 to show that every member of $\mathrm{Irr}_{2'}(B_0(S))$ is σ-fixed.

Consider an adjoint quotient $\widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}_{\mathrm{ad}}$ of $\widetilde{\mathbf{G}}$. The kernel of this map is the (connected) centre of $\widetilde{\mathbf{G}}$, so by the Lang–Steinberg theorem we have $\widetilde{G}/Z(\widetilde{G}) \cong \widetilde{\mathbf{G}}_{\mathrm{ad}}^F$. Now let $\chi \in \mathrm{Irr}_{2'}(B_0(S))$ be non-unipotent. By [NT15, Lemma 4.13], together with the proceeding remark, χ extends to a character $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G}/Z(\widetilde{G}))$. By inflation, we may view χ as a character of G and $\widetilde{\chi}$ as a character of $\widetilde{G}$ extending χ. By Lemma 3.8, we have $\chi \in \mathrm{Irr}(B_0(G))$, so a result of Broué–Michel shows that $\chi \in \mathcal{E}(G,[s])$ with $s \in G^\star$ a 2-element, see [CE04, 9.12]. By [SFT18, 3.4] the series $\mathcal{E}(G,[s])$ is σ-stable and condition ($\star$) of Proposition 5.6 is satisfied by [DM90] because χ is not unipotent. Hence, as χ extends to $\widetilde{G}$, we have χ is σ-fixed by Proposition 5.6. ∎

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