
Induced Characters of Type D Weyl Groups and the Littlewood–Richardson Rule

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Abstract. For any ordinary irreducible character of a maximal reflection subgroup of type $D_a D_b$ of a type D Weyl group we give an explicit decomposition of the induced character in terms of Littlewood–Richardson coefficients.

1. Introduction

1.1. Let W be a finite Weyl group and $H \leq W$ a reflection subgroup. A natural question that occurs in the representation theory of W is to describe the decomposition of the induced character $\text{Ind}_H^W(\chi)$ for any (ordinary) irreducible character χ of H . By a general process one can reduce this to the case where W is irreducible and H is maximal. The classical situation occurs when W is the symmetric group \mathfrak{S}_n then H is a Young subgroup $\mathfrak{S}_a \mathfrak{S}_b$ with $a + b = n$. In this case the decomposition is given by the classical Littlewood–Richardson coefficients which are combinatorially computable using the Littlewood–Richardson rule. For exceptional type Weyl groups one can simply tackle this using the CHEVIE package [Gec+96].

1.2. The remaining cases occur when W is of type B_n or D_n . In the first case we have H is either of type D_n or $B_a B_b$ with $a + b = n$ (see [TF05, Lemma 5]). Both of these cases are treated, for instance, in the book of Geck and Pfeiffer (see [GP00, 5.6.1, 6.1.3]). Now if W is of type D_n then we have H is either of type A_{n-1} , D_{n-1} or $D_a D_b$ with $2 \leq a, b \leq n - 2$ and $a + b = n$ (see [TF05, Corollary 1]). For the so-called non-degenerate irreducible characters of W the multiplicities are easily deduced from the type B case. However the multiplicities for degenerate characters requires some work. It is the purpose of this note to consider the case where H is of type D_{n-1} or $D_a D_b$. Here we are able to get a complete description of the multiplicities in terms of Littlewood–Richardson coefficients. We suspect that this result is well known to the experts but we failed to find it in the literature. When H is of type A_{n-1} then certain special cases are considered in [GP00, 5.6.3] and [Gec15, 3.7] but a combinatorial formula in the general case is still to be obtained.

1.3. We briefly point out why such a result is relevant in a wider context and also the motivation for the author. Let \mathbf{G} be a connected reductive algebraic group over an algebraically closed field of characteristic p and $G = \mathbf{G}^F$ the fixed points under a Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$. In the representation theory of G the representations

of Weyl groups occur frequently and in many varied ways. In [Lus86] Lusztig showed that the restriction of a character sheaf of \mathbf{G} to the unipotent variety could be explicitly decomposed in terms of certain intersection cohomology complexes, which are defined on the closure of a unipotent conjugacy class. The coefficients in this decomposition correspond precisely to the coefficients in the decomposition of an induced character from a subgroup of a relative Weyl group of \mathbf{G} , which is a reflection subgroup when $Z(\mathbf{G})$ is connected. Recently the author has extended the work of Lusztig to obtain an explicit formula for the value of the characteristic function of a character sheaf at a unipotent element (see [Tay14]). In this formula such multiplicities also occur, as well as their F -coset analogues.

1.4. If \mathbf{G} is an adjoint simple group of type B_n then under certain mild restrictions Lusztig was able to prove, for unipotently supported character sheaves, his conjecture relating characteristic functions of character sheaves to the irreducible characters of G (see [Lus86, 5.3(c)] and [Lus84, 13.7]). A key role in the proof of Lusztig's result was the type B analogue of Proposition 2.7, as given in [GP00, 6.1.3]. It is the authors hope to extend Lusztig's result to the case where \mathbf{G} is of type D_n , where one needs the information contained in Proposition 2.7.

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2. The Result

2.1. Notation. Let G be a finite Weyl group. We assume that $\mathbb{Q} \subseteq \mathbb{K}$ is a fixed field extension, then the group algebra $\mathbb{K}G$ is semisimple; we denote by $\text{Irr}(G)$ the irreducible characters of G over \mathbb{K} . If $\chi, \psi \in \text{Irr}(G)$ are irreducible characters then we set

$$\langle \chi, \psi \rangle_G = \dim \text{Hom}_{\mathbb{K}G}(M_\chi, M_\psi),$$

where M_χ (resp. M_ψ) is a $\mathbb{K}G$ -module affording χ (resp. ψ). We extend this linearly to a bilinear map $\langle -, - \rangle_G : \mathbb{Z} \text{Irr}(G) \times \mathbb{Z} \text{Irr}(G) \rightarrow \mathbb{Z}$. Given any element $g \in G$ we will denote by $\text{Cl}_G(g)$ (resp. $C_G(g)$) the conjugacy class (resp. centraliser) of g . If G has reflection representation $G \rightarrow \text{GL}(V)$ then, for any subgroup $H \leq G$, we denote by j_H^G the Lusztig–Macdonald–Spaltenstein induction with respect to V as defined in [GP00, §5.2].

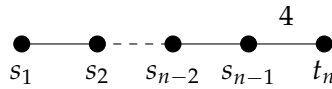
By a *partition* we will mean a finite (possibly empty) weakly decreasing sequence of strictly positive integers $\alpha = (\alpha_1, \dots, \alpha_k)$. For any such partition α we set

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = () \\ \alpha_1 + \dots + \alpha_k & \text{if } \alpha \neq (). \end{cases}$$

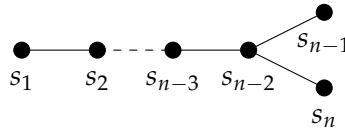
We say α is a *partition of* $n \geq 0$ if $|\alpha| = n$. By a *bipartition of* n we will refer to an

ordered pair of partitions $(\alpha; \beta)$ such that $|\alpha| + |\beta| = n$. For any $n \geq 0$ we will denote by $\mathcal{P}(n)$ (resp. $\mathcal{BP}(n)$) the set of partitions (resp. bipartitions) of n . We will also denote by $\mathcal{X}(n)$ the set of all pairs of integers (a, b) such that $a, b > 0$ and $a + b = n$. If n is not a positive integer then we set $\mathcal{P}(n) = \mathcal{BP}(n) = \mathcal{X}(n) = \emptyset$. Given any partition $\alpha = (\alpha_1, \dots, \alpha_k)$ we write $\ell(\alpha) = k$ for the length of the partition, which is simply 0 if α is empty. If $\beta = (\beta_1, \dots, \beta_\ell)$ is a second partition then we write $\alpha \cup \beta \in \mathcal{P}(|\alpha| + |\beta|)$ for the partition obtained by reordering the sequence $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell)$. Note we clearly have $\ell(\alpha \cup \beta) = \ell(\alpha) + \ell(\beta)$.

2.2. For any $n \geq 0$ let \tilde{W}_n be a Coxeter group of type B_n with generators $\{s_1, \dots, s_{n-1}, t_n\}$ and corresponding Coxeter diagram



For $0 \leq n \leq 1$ we set $W_n \leq \tilde{W}_n$ to be the trivial subgroup. Assume now that $n > 1$ then setting $s_n = t_n s_{n-1} t_n$ we have the subgroup $W_n = \langle s_1, \dots, s_n \rangle \leq \tilde{W}_n$ is naturally a Coxeter group of type D_n with corresponding Coxeter diagram



Note that W_2 is of type $D_2 = A_1 A_1$ and W_3 is of type $D_3 = A_3$.

2.3. We assume now that $n \geq 1$. Recall that there is a bijection $\mathcal{BP}(n) \rightarrow \text{Irr}(\tilde{W}_n)$ which we denote by $(\alpha; \beta) \mapsto [\alpha; \beta]$. We fix this bijection to be the one defined in [GP00, 5.5.6]. The group $W_n \leq \tilde{W}_n$ is an index 2 normal subgroup. In particular, the restriction $\overline{[\alpha; \beta]} = \text{Res}_{W_n}^{\tilde{W}_n}([\alpha; \beta])$ of any irreducible character $[\alpha; \beta] \in \text{Irr}(\tilde{W}_n)$ to W_n is either irreducible or the sum of two irreducible characters. From [GP00, 5.6.1] we recall that

$$\overline{[\alpha; \beta]} = \overline{[\beta; \alpha]}$$

is an irreducible character of W_n if and only if $\alpha \neq \beta$. Assuming $\alpha \neq \beta$ then we call the irreducible character $\overline{[\alpha; \beta]} \in \text{Irr}(W_n)$ *non-degenerate*. Now assume n is even and $\alpha \in \mathcal{P}(n/2)$ is a partition. Then we have

$$\overline{[\alpha; \alpha]} = \overline{[\alpha; \alpha]}_+ + \overline{[\alpha; \alpha]}_- ,$$

where $\overline{[\alpha; \alpha]}_\pm \in \text{Irr}(W_n)$. We say $\overline{[\alpha; \alpha]}_\pm$ is a *degenerate* character of W_n of type \pm . It will be convenient to also write $\overline{[\alpha; \beta]}_\pm$ even when $\alpha \neq \beta$; in this case the sign should simply be ignored. To conclude we have the irreducible characters of W_n are

$$\text{Irr}(W_n) = \{\overline{[\alpha; \beta]} \mid (\alpha; \beta) \in \mathcal{BP}(n) \text{ and } \alpha \neq \beta\} \cup \{\overline{[\alpha; \alpha]}_\pm \mid \alpha \in \mathcal{P}(n/2)\}.$$

For any $X \in \text{Irr}(W_n)$ we set

$$\varepsilon(X) = \begin{cases} 0 & \text{if } X \text{ is non-degenerate} \\ \pm 1 & \text{if } X \text{ is degenerate of type } \pm. \end{cases}$$

2.4. To make the above parameterisation of $\text{Irr}(W_n)$ concrete we must distinguish the irreducible characters $\overline{[\alpha; \alpha]}_{\pm}$ which we do following [GP00, 5.6.3]. Let

$$H_n^+ = \langle s_1, \dots, s_{n-2}, s_{n-1} \rangle \quad H_n^- = \langle s_1, \dots, s_{n-2}, s_n \rangle$$

then both of these are subgroups of W_n isomorphic to the symmetric group \mathfrak{S}_n on $\{1, \dots, n\}$. If n is even then they are not conjugate. Recall that we have a bijection $\mathcal{P}(n) \rightarrow \text{Irr}(\mathfrak{S}_n)$ defined as in [GP00, 5.4.7]; we denote this by $\alpha \mapsto [\alpha]$. We have a natural isomorphism $H_n^{\pm} \cong \mathfrak{S}_n$ given by $s_i \mapsto (i, i+1)$, for $1 \leq i \leq n-2$, and $s_{n-1}, s_n \mapsto (n-1, n)$. Under this isomorphism we will identify the set of characters $\text{Irr}(H_n^{\pm})$ with $\text{Irr}(\mathfrak{S}_n)$. Now for any $\alpha \in \mathcal{P}(n/2)$ we set

$$\overline{[\alpha; \alpha]}_+ = j_{H_n^+}^{W_n}([\alpha \cup \alpha]) \quad \overline{[\alpha; \alpha]}_- = j_{H_n^-}^{W_n}([\alpha \cup \alpha])$$

and this distinguishes the characters (c.f. 2.1).

2.5. Assume $n \geq 4$ then setting $s_0 = t_0 s_1 t_0 \in W_n$ with

$$t_0 = s_1 s_2 \cdots s_{n-1} t_n s_{n-1} \cdots s_2 s_1$$

we obtain the extended Coxeter diagrams



We set $\tilde{\mathcal{S}}_0 = \{t_0, s_1, \dots, s_{n-1}, t_n\}$ and $\mathcal{S}_0 = \{s_0, \dots, s_n\}$. Given $(a, b) \in \mathcal{X}(n)$ we have corresponding reflection subgroups $\tilde{W}_a \tilde{W}_b = \langle \tilde{\mathcal{S}}_0 \setminus \{s_a\} \rangle$ of \tilde{W}_n and

$$W_a W_b = \begin{cases} \langle \mathcal{S}_0 \setminus \{s_0, s_1\} \rangle & \text{if } a = 1 \\ \langle \mathcal{S}_0 \setminus \{s_{n-1}, s_n\} \rangle & \text{if } a = n-1 \\ \langle \mathcal{S}_0 \setminus \{s_a\} \rangle & \text{otherwise} \end{cases}$$

of W_n . Note that we clearly have $W_a W_b = \tilde{W}_a \tilde{W}_b \cap W_n$. Setting $H_a^{\pm} = W_a \cap H_n^{\pm}$ determines two copies of \mathfrak{S}_a inside W_a (similarly for W_b). With respect to these choices the degenerate characters of W_a and W_b are concretely labelled as in 2.4.

2.6. Assume $(a, b) \in \mathcal{X}(n)$ and $\alpha \in \mathcal{P}(a)$, $\beta \in \mathcal{P}(b)$ and $\gamma \in \mathcal{P}(n)$ are partitions. We define the corresponding Littlewood–Richardson coefficient to be

$$c_{\alpha\beta}^{\gamma} = \langle \text{Ind}_{\mathfrak{S}_a \mathfrak{S}_b}^{\mathfrak{S}_n}([\alpha] \boxtimes [\beta]), [\gamma] \rangle_{\mathfrak{S}_n}.$$

Here $\mathfrak{S}_a \mathfrak{S}_b$ denotes the Young subgroup of \mathfrak{S}_n preserving the sets $\{1, \dots, a\}$ and $\{a+1, \dots, n\}$. Furthermore, $[\alpha] \boxtimes [\beta]$ denotes the tensor product of the characters $[\alpha]$ and $[\beta]$ so that $([\alpha] \boxtimes [\beta])(wx) = [\alpha](w)[\beta](x)$ for all $wx \in \mathfrak{S}_a \mathfrak{S}_b$. Now let $\alpha = (\alpha_1; \alpha_2) \in \mathcal{BP}(a)$, $\beta = (\beta_1; \beta_2) \in \mathcal{BP}(b)$ and $\gamma = (\gamma_1; \gamma_2) \in \mathcal{BP}(n)$ be bipartitions. We will denote by \mathfrak{S}_α the symmetric group \mathfrak{S}_1 (resp. \mathfrak{S}_2) if $\alpha_1 = \alpha_2$ (resp. $\alpha_1 \neq \alpha_2$) and similarly for β . With this we define

$$a_{\alpha\beta}^\gamma = \sum_{\sigma \in \mathfrak{S}_\alpha} \sum_{\tau \in \mathfrak{S}_\beta} c_{\alpha_{\sigma(1)}\beta_{\tau(1)}}^{\gamma_1} c_{\alpha_{\sigma(2)}\beta_{\tau(2)}}^{\gamma_2}$$

where the sum is over all permutations such that $|\gamma_i| = |\alpha_{\sigma(i)}| + |\beta_{\tau(i)}|$ for $i \in \{1, 2\}$. If no such permutations exist then we have $a_{\alpha\beta}^\gamma = 0$. Note that this definition does not depend upon the ordering of the partitions in the bipartitions α , β or γ . With this notation in place we may now state our result.

Proposition 2.7. *Assume $n \geq 4$ and $(a, b) \in \mathcal{X}(n)$. Let $E = A \boxtimes B \in \text{Irr}(W_a W_b)$ be an irreducible character then we denote by $\alpha = (\alpha_1; \alpha_2) \in \mathcal{BP}(a)$ and $\beta = (\beta_1; \beta_2) \in \mathcal{BP}(b)$ bipartitions such that $A = \overline{[\alpha_1; \alpha_2]}_\pm$ and $B = \overline{[\beta_1; \beta_2]}_\pm$. Now let $X \in \text{Irr}(W_n)$ be an irreducible character and let $\gamma = (\gamma_1; \gamma_2) \in \mathcal{BP}(n)$ be a bipartition such that $X = \overline{[\gamma_1; \gamma_2]}_\pm$. Then*

$$\langle \text{Ind}_{W_a W_b}^{W_n}(E), X \rangle_{W_n} = \begin{cases} a_{\alpha\beta}^\gamma & \text{if } X \text{ is non-degenerate,} \\ \frac{1}{2}(a_{\alpha\beta}^\gamma + \varepsilon(A)\varepsilon(B)\varepsilon(X)c_{\alpha_1\beta_1}^{\gamma_1}) & \text{if } X \text{ is degenerate.} \end{cases}$$

Remark 2.8. We note that in the interesting case when $\varepsilon(A)\varepsilon(B)\varepsilon(X) \neq 0$, i.e. when $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$, then the result may be written as

$$\langle \text{Ind}_{W_a W_b}^{W_n}(E), X \rangle_{W_n} = \frac{1}{2} c_{\alpha_1\beta_1}^{\gamma_1} (c_{\alpha_1\beta_1}^{\gamma_1} + \varepsilon(A)\varepsilon(B)\varepsilon(X))$$

because $a_{\alpha\beta}^\gamma = (c_{\alpha_1\beta_1}^{\gamma_1})^2$.

Proof. For notational convenience we set $H = W_a W_b$, $\tilde{H} = \tilde{W}_a \tilde{W}_b$, $W = W_n$ and $\tilde{W} = \tilde{W}_n$. By the Clifford theoretic description of the irreducible characters of W and [GP00, 6.1.3] it is easy to deduce that

$$\langle \text{Ind}_H^{\tilde{W}}(E), [\gamma_1; \gamma_2] \rangle_{\tilde{W}} = \langle \text{Ind}_{\tilde{H}}^{\tilde{W}} \text{Ind}_H^{\tilde{H}}(E), [\gamma_1; \gamma_2] \rangle_{\tilde{W}} = a_{\alpha\beta}^\gamma. \quad (2.9)$$

Applying Frobenius reciprocity we obtain the multiplicity of any non-degenerate character in $\text{Ind}_H^W(E)$. Thus we are left with considering the case of degenerate characters.

Assume now that $X \in \text{Irr}(W)$ is degenerate so n is even. If B is non-degenerate then we have

$$s_n \text{Ind}_H^W(E) = \text{Ind}_H^W(s_n E) = \text{Ind}_H^W(E).$$

As conjugation by s_n permutes the degenerate characters $\overline{[\gamma_1; \gamma_2]}_\pm$ we get from (2.9) that

$$\langle \text{Ind}_H^W(E), X \rangle_W = \langle \text{Ind}_H^W(E), s_n X \rangle_W = \frac{1}{2} a_{\alpha\beta}^\gamma.$$

Similarly, if A is non-degenerate we may conjugate the induced character by the element

s_0 to deduce the same result. Hence we need only deal with the situation where both A and B are degenerate, which we now assume to be the case. In particular both a and b are even.

We denote by Π_X (resp. Δ_X) the character $\overline{[\gamma_1; \gamma_2]} = \overline{[\gamma_1; \gamma_2]}_+ + \overline{[\gamma_1; \gamma_2]}_-$ (resp. the difference character $\overline{[\gamma_1; \gamma_2]}_+ - \overline{[\gamma_1; \gamma_2]}_-$). Using the fact that any degenerate character $X \in \text{Irr}(W)$ may be written as $\frac{1}{2}(\Pi_X + \varepsilon(X)\Delta_X)$ we have

$$\langle \text{Ind}_H^W(E), X \rangle_W = \frac{1}{2} \langle \text{Ind}_H^W(E), \Pi_X \rangle_W + \frac{1}{2} \varepsilon(X) \langle \text{Ind}_H^W(E), \Delta_X \rangle_W.$$

Note that $\langle \text{Ind}_H^W(E), \Pi_X \rangle_W = a_{\alpha\beta}^\gamma$ by (2.9) so we have only to compute the multiplicity $\langle \text{Ind}_H^W(E), \Delta_X \rangle_W$. As A and B are both degenerate we can apply this approach to the character E to obtain

$$\begin{aligned} \text{Ind}_H^W(E) = \frac{1}{4} \Big[& \text{Ind}_H^W(\Pi_A \boxtimes \Pi_B) + \varepsilon(B) \text{Ind}_H^W(\Pi_A \boxtimes \Delta_B) \\ & + \varepsilon(A) \text{Ind}_H^W(\Delta_A \boxtimes \Pi_B) + \varepsilon(A)\varepsilon(B) \text{Ind}_H^W(\Delta_A \boxtimes \Delta_B) \Big]. \end{aligned}$$

We now observe that the character Π_A (resp. Π_B) is invariant under conjugation by s_0 (resp. s_n). Hence, applying the previous argument we see that the degenerate characters $\overline{[\gamma_1; \gamma_2]}_\pm$ occur with the same multiplicity in the first three induced characters. This implies that their inner product with the difference character Δ_X is 0. In particular

$$\langle \text{Ind}_H^W(E), \Delta_X \rangle_W = \frac{1}{4} \varepsilon(A)\varepsilon(B) \langle \text{Ind}_H^W(\Delta_A \boxtimes \Delta_B), \Delta_X \rangle_W.$$

For notational convenience we will set $\Theta = \text{Ind}_H^W(\Delta_A \boxtimes \Delta_B)$.

We now wish to compute the multiplicity on the right hand side but to do this we will need some information about conjugacy classes and the difference character Δ_X . For any partition $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}(n/2)$ we will denote by $w_{2\pi}^+ \in H_n^+ \leq W$ an element of cycle type $2\pi = (2\pi_1, \dots, 2\pi_k)$. This definition makes sense because of the isomorphism chosen in 2.4. We also denote by $w_{2\pi}^-$ the conjugate $s_n w_{2\pi}^+ s_n \in H_n^- \leq W$ and by $w_\pi \in \mathfrak{S}_{n/2}$ an element of cycle type π .

Now, according to [GP00, 10.4.9] and [Gec15, 3.5], we have for all $w \in W$ that

$$\Delta_X(w) = \begin{cases} \pm(-1)^{n/2} 2^{\ell(\pi)} [\gamma_1](w_\pi) & \text{if } w \in \text{Cl}_W(w_{2\pi}^\pm) \text{ for some } \pi \in \mathcal{P}(n/2) \\ 0 & \text{otherwise.} \end{cases}$$

Here $[\gamma_1]$ denotes an irreducible character of the symmetric group $\mathfrak{S}_{n/2}$. Note that the statement in [GP00, 10.4.10] identifying the characters in [GP00, 10.4.6] with those in [GP00, 5.6.3] is not correct; the correct statement is given in [Gec15, 3.5]. We will also need the following identities concerning centraliser orders which are easily obtained by direct computation

$$|C_W(w_{2\pi}^\pm)| = |C_{\tilde{W}}(w_{2\pi}^\pm)| = 2^{\ell(\pi)} |C_{H_n^\pm}(w_{2\pi}^\pm)| = 2^{2\ell(\pi)} |C_{\mathfrak{S}_{n/2}}(w_\pi)|.$$

To compute the multiplicity $\langle \Theta, \Delta_X \rangle_W$ we will need to determine the value of the induced character at $w_{2\pi}^\pm$, for which we use the explicit induction formula given in [Isa06, pg. 64]. For this we need to determine the orbits of H acting on $H \cap \text{Cl}_W(w_{2\pi}^\pm)$ by conjugation. With this in mind we define

$$\mathcal{B}_\pi = \{(\delta, \epsilon) \in \mathcal{P}(a/2) \times \mathcal{P}(b/2) \mid \delta \cup \epsilon = \pi\},$$

(c.f. 2.1). We then have

$$\begin{cases} \{w_{2\delta}^+ w_{2\epsilon}^+, w_{2\delta}^- w_{2\epsilon}^- \mid (\delta, \epsilon) \in \mathcal{B}_\pi\} & \text{for } w_{2\pi}^+ \\ \{w_{2\delta}^+ w_{2\epsilon}^-, w_{2\delta}^- w_{2\epsilon}^+ \mid (\delta, \epsilon) \in \mathcal{B}_\pi\} & \text{for } w_{2\pi}^- \end{cases}$$

is a complete set of representatives for the desired orbits where

$$w_{2\delta}^- = s_0 w_{2\delta}^+ s_0 \in H_a^- \quad \text{and} \quad w_{2\epsilon}^- = s_n w_{2\epsilon}^+ s_n \in H_b^-.$$

In addition, $\{w_\delta w_\epsilon \mid (\delta, \epsilon) \in \mathcal{B}_\pi\} \subseteq \mathfrak{S}_{a/2} \mathfrak{S}_{b/2}$ is a complete set of representatives for the orbits of $\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}$ acting on $\mathfrak{S}_{a/2} \mathfrak{S}_{b/2} \cap \text{Cl}_{\mathfrak{S}_{n/2}}(w_\pi)$ by conjugation. Now applying the explicit induction formula we obtain

$$\begin{aligned} \Theta(w_{2\pi}^+) &= \sum_{(\delta, \epsilon) \in \mathcal{B}_\pi} \frac{|C_W(w_{2\pi}^+)|}{|C_H(w_{2\delta}^+ w_{2\epsilon}^+)|} \Delta_A(w_{2\delta}^+) \Delta_B(w_{2\epsilon}^+) \\ &\quad + \sum_{(\delta, \epsilon) \in \mathcal{B}_\pi} \frac{|C_W(w_{2\pi}^+)|}{|C_H(w_{2\delta}^- w_{2\epsilon}^-)|} \Delta_A(w_{2\delta}^-) \Delta_B(w_{2\epsilon}^-) \\ &= (-1)^{n/2} 2^{\ell(\pi)+1} \sum_{(\delta, \epsilon) \in \mathcal{B}_\pi} \frac{|C_{\mathfrak{S}_{n/2}}(w_\pi)|}{|C_{\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}}(w_\delta w_\epsilon)|} [\alpha_1](w_\delta) [\beta_1](w_\epsilon), \\ &= (-1)^{n/2} 2^{\ell(\pi)+1} \text{Ind}_{\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}}^{\mathfrak{S}_{n/2}} ([\alpha_1] \boxtimes [\beta_1])(w_\pi). \end{aligned}$$

A similar calculation for the case $w_{2\pi}^-$ shows that

$$\Theta(w_{2\pi}^\pm) = \pm (-1)^{n/2} 2^{\ell(\pi)+1} \text{Ind}_{\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}}^{\mathfrak{S}_{n/2}} ([\alpha_1] \boxtimes [\beta_1])(w_\pi).$$

We are now ready to compute the multiplicity $\langle \Theta, \Delta_X \rangle_W$ but we first observe that

$$\Theta(w_{2\pi}^+) \Delta_X(w_{2\pi}^+) = \Theta(w_{2\pi}^-) \Delta_X(w_{2\pi}^-)$$

for any partition $\pi \in \mathcal{P}(n/2)$, which follows immediately from the above formulas. With this we can now compute the desired multiplicity as follows

$$\begin{aligned} \langle \Theta, \Delta_X \rangle_W &= \sum_{\pi \in \mathcal{P}(n/2)} \frac{1}{|C_W(w_{2\pi}^+)|} \Theta(w_{2\pi}^+) \Delta_X(w_{2\pi}^+) \\ &\quad + \sum_{\pi \in \mathcal{P}(n/2)} \frac{1}{|C_W(w_{2\pi}^-)|} \Theta(w_{2\pi}^-) \Delta_X(w_{2\pi}^-) \end{aligned}$$

$$\begin{aligned}
&= 4 \sum_{\pi \in \mathcal{P}(n/2)} \frac{1}{|C_{\mathfrak{S}_{n/2}}(w_\pi)|} \text{Ind}_{\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}}^{\mathfrak{S}_{n/2}} ([\alpha_1] \boxtimes [\beta_1])(w_\pi) [\gamma_1](w_\pi) \\
&= 4 \langle \text{Ind}_{\mathfrak{S}_{a/2} \mathfrak{S}_{b/2}}^{\mathfrak{S}_{n/2}} ([\alpha_1] \boxtimes [\beta_1]), [\gamma_1] \rangle_{\mathfrak{S}_{n/2}} \\
&= 4c_{\alpha_1 \beta_1}^{\gamma_1}.
\end{aligned}$$

This completes the proof. ■

2.10. We now rephrase Proposition 2.7 when $(a, b) \in \{(1, n-1), (n-1, 1)\}$ to obtain the well-known type D branching rules. However first we require some notation. For any partition $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{P}(n)$ let $I(\alpha)$ be the set of all $d \in \{1, \dots, k\}$ such that $d = k$ or $d < k$ and $\alpha_d > \alpha_{d+1}$. For any $d \in I(\alpha)$ we denote by $\alpha^{(d)}$ the partition obtained from α by replacing α_d with $\alpha_d - 1$ and removing any zero entries. Furthermore, for any bipartition $\gamma = (\gamma_1; \gamma_2) \in \mathcal{BP}(n)$ we denote by $Z(\gamma) \subseteq \mathcal{BP}(n-1)$ the set

$$\{(\gamma_1^{(d)}; \gamma_2), (\gamma_2; \gamma_1^{(d)}) \mid d \in I(\gamma_1)\} \cup \{(\gamma_1; \gamma_2^{(d)}), (\gamma_2^{(d)}; \gamma_1) \mid d \in I(\gamma_2)\}.$$

The following is deduced immediately from the type A branching rules (see [GP00, 6.1.8]), Frobenius reciprocity and Proposition 2.7.

Corollary 2.11. *Assume $n \geq 4$ and let $H \leq W_n$ be the subgroup $W_1 W_{n-1}$ or $W_{n-1} W_1$ (c.f. 2.5). Then for any irreducible characters $B = [\beta_1; \beta_2]_{\pm} \in \text{Irr}(H)$ and $X = [\gamma_1; \gamma_2]_{\pm} \in \text{Irr}(W_n)$ we have*

$$\langle \text{Res}_H^{W_n}(X), B \rangle_H = \begin{cases} 0 & \text{if } \beta \notin Z(\gamma) \\ 1 & \text{if } \beta \in Z(\gamma) \end{cases}$$

where $\beta = (\beta_1; \beta_2) \in \mathcal{BP}(n-1)$ and $\gamma = (\gamma_1; \gamma_2) \in \mathcal{BP}(n)$.

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