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Math 517B - Group Theory (Spring 2017) Exercise Sheet 4

Exercise 4.1. Show that if G is $\mathrm{GL}_n(\mathbb{K})$, with $\mathbb{K}$ an algebraically closed field, then the centraliser $C_G(g)$ is connected for any element $g \in G$.

Exercise 4.2. Show that if (G, F) is $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ then we have a bijection

$$\begin{aligned}\{F\text{-stable } G\text{-conjugacy classes}\} &\to \{G^F\text{-conjugacy classes}\}\\ \mathcal{O} &\mapsto \mathcal{O}^F = \mathcal{O} \cap G^F\end{aligned}$$

Exercise 4.3. Assume (G, F) is $\mathrm{GL}_2(q)$. Find an explicit element $g \in G$ such that

$$g^{-1}F(g) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus construct, explicitly, an element of G^F which is G-conjugate to the element $d(\lambda, \lambda^q)$ with $\lambda \in C_{q^2-1} \setminus C_{q-1}$.

Exercise 4.4. Let (G, F) be $\mathrm{SL}_2(q)$ or $\mathrm{PGL}_2(q)$. Find an element $x \in G$ whose G-conjugacy class is F-stable but whose centraliser $C_G(x)$ is disconnected.

Exercise 4.5. Assume (G, F) is $\mathrm{GL}_2(q)$. Recall that in class we started to determine the traces

$$\mathrm{Tr}((g, t), \overline{\mathbb{Q}}_\ell[G^F/U_0^F])$$

on the Harish-Chandra bimodule. Assume $t = d(\mu_1, \mu_2)$ with $\mu_1, \mu_2 \in C_{q-1}$. Show that these traces are given by the following table.

$d(\lambda, \lambda)$ $\lambda \in C_{q-1}$	$d(\lambda, \lambda)u$ $\lambda \in C_{q-1}$	$d(\lambda_1, \lambda_2)$ $\lambda_1 \neq \lambda_2 \in C_{q-1}$	$d(\lambda, \lambda^q)$ $\lambda \in C_{q^2-1} \setminus C_{q-1}$
$(q-1)(q^2-1)\delta_{\lambda=\mu_1^{-1}=\mu_2^{-1}}$	$(q-1)\delta_{\lambda=\mu_1^{-1}=\mu_2^{-1}}$	$(q-1)\delta_{\{\lambda_1,\lambda_2\},\{\mu_1^{-1},\mu_2^{-1}\}}$	0

Here we have

$$\delta_{\lambda=\mu_1^{-1}=\mu_2^{-1}} = \begin{cases} 1 & \text{if } \lambda = \mu_1^{-1} = \mu_2^{-1} \\ 0 & \text{otherwise.} \end{cases} \qquad \delta_{\{\lambda_1,\lambda_2\},\{\mu_1^{-1},\mu_2^{-1}\}} = \begin{cases} 1 & \text{if } \{\lambda_1, \lambda_2\} = \{\mu_1^{-1}, \mu_2^{-1}\} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.6. Assume (G, F) is $\mathrm{GL}_2(q)$. Show that if $\alpha \boxtimes \beta \in \mathrm{Irr}(T_0^F)$ is an irreducible character of $T_0^F \cong C_{q-1} \times C_{q-1}$ then the values of the Harish-Chandra induced character $R_{T_0}^G(\alpha \boxtimes \beta)$ are given by the following table.

	$d(\lambda,\lambda)$	$d(\lambda,\lambda)u$	$d(\lambda_1,\lambda_2)$	$d(\lambda,\lambda^q)$
	$\lambda \in C_{q-1}$	$\lambda \in C_{q-1}$	$\lambda_1 \neq \lambda_2 \in C_{q-1}$	$\lambda \in C_{q^2-1} \setminus C_{q-1}$
$R^G_{T_0}(\alpha \boxtimes \beta)$	$(q+1)\alpha(\lambda)\beta(\lambda)$	$\alpha(\lambda)\beta(\lambda)$	$\alpha(\lambda_1)\beta(\lambda_2)+\alpha(\lambda_2)\beta(\lambda_1)$	0

Conclude that the values of the Steinberg character are given by the following table

	$d(\lambda,\lambda)$	$d(\lambda,\lambda)u$	$d(\lambda_1,\lambda_2)$	$d(\lambda,\lambda^q)$
	$\lambda \in C_{q-1}$	$\lambda \in C_{q-1}$	$\lambda_1 \neq \lambda_2 \in C_{q-1}$	$\lambda \in C_{q^2-1} \setminus C_{q-1}$
St_{G^F}	q	1	-1	0