Instructor: Jay Taylor

Math 517B - Group Theory (Spring 2017) Exercise Sheet 3

Assume G is an abstract group with an automorphism $F : G \to G$. We will assume that $\mathcal{O}$ is a G-set with a map $F' : \mathcal{O} \to \mathcal{O}$ such that

(F1) $F'(g \cdot x) = F(g) \cdot F'(x)$.

At this point we do not assume that $\mathcal{O}$ is transitive. For each element $x \in \mathcal{O}$ we denote by $[x]_G = \{g \cdot x \mid g \in G\}$ the corresponding G-orbit. Note the condition (F1) ensures that $G^F = \{g \in G \mid F(g) = g\}$ acts on the set $\mathcal{O}^{F'} = \{x \in \mathcal{O} \mid F'(x) = x\}$.

Exercise 3.1. For any element $x \in \mathcal{O}$ set $T_G(x, F') = \{g \in G \mid g \cdot F'(x) = x\}$. Show that the following hold:

(a) if $g \in T_G(x, F')$ then $T_G(x, F') = \mathrm{Stab}_G(x)g$, i.e., $T_G(x, F')$ is a coset of the stabiliser,

(b) if $g \in T_G(x, F')$ and $h \in \mathrm{Stab}_G(x)$ then $h \star g = hgF(h)^{-1} \in T_G(x, F')$.

Conclude that $\star$ defines an action of $\mathrm{Stab}_G(x)$ on $T_G(x, F')$, which we call F-conjugation. The orbits of this action are called F-conjugacy classes and the set of all such orbits is denoted by $H^1(F, T_G(x, F'))$.

Exercise 3.2. Let us denote by $\mathscr{L} : G \to G$ the map defined by $\mathscr{L}(g) = g^{-1}F(g)$. Show that for any $x \in \mathcal{O}$ the following are equivalent:

(a) $F'([x]_G) = [x]_G$,

(b) $T_G(x, F') \neq \emptyset$.

Furthermore, show that the following are equivalent:

(c) $[x]_G \cap \mathcal{O}^{F'} \neq \emptyset$,

(d) $T_G(x, F') \cap \mathscr{L}(G) \neq \emptyset$.

[Remark: this gives a motivation for why one might consider the Lang–Steinberg map $\mathscr{L}$.]

Exercise 3.3. Show that the map $\mathscr{L} : G \to G$ is never surjective if G is finite. [Hint: consider G/G^F.]

Exercise 3.4. Assume the G-action on $\mathcal{O}$ is transitive and fix an element $x_0 \in \mathcal{O}$. Assume $x_1, x_2 \in \mathcal{O}^{F'}$ are F'-fixed elements and let $g_1, g_2 \in G$ be such that $x_i = g_i \cdot x_0$. Show that the following are equivalent:

(a) x_1 and x_2 are in the same G^F-orbit,

(b) $\mathscr{L}(g_1)$ and $\mathscr{L}(g_2)$ are in the same $\mathrm{Stab}_G(x_0)$-orbit.

Conclude that the map $\mathcal{O}^{F'} \to T_G(x_0, F')$ defined by $g \cdot x_0 \to \mathscr{L}(g)$ induces a well defined bijection $\mathcal{O}^{F'}/G^F \to H^1(F, T_G(x, F'))$ between the G^F-orbits and the F-conjugacy classes.

Exercise 3.5. Assume $x_1, x_2 \in \mathcal{O}$ are in the same G-orbit, i.e., there exists an element $g \in G$ such that $x_2 = g \cdot x_1$. We denote by $\phi : G \to G$ the map defined by $\phi(h) = ghF(g^{-1})$. Show that the following hold:

(a) ϕ restricts to a bijective map $\phi : T_G(x_1, F') \to T_G(x_2, F')$,

(b) $\phi(h \star e) = {}^g h \star \phi(e)$ for any $h \in \mathrm{Stab}_G(x_1)$ and $e \in T_G(x_1, F')$.

Conclude that ϕ defines a bijective map $\phi : H^1(F, T_G(x_1, F')) \to H^1(F, T_G(x_2, F'))$.

Exercise 3.6. Let $\widetilde{\mathcal{O}}$ denote the set of pairs (x, h) such that $x \in \mathcal{O}$ and $h \in T_G(x, F') \neq \emptyset$. Show that we have a G-action on $\widetilde{\mathcal{O}}$ given by

$$g \cdot (x, h) = (g \cdot x, ghF(g^{-1})).$$

Furthemore, show that we have a bijection between the orbits of G acting on $\widetilde{\mathcal{O}}$ and the orbits of G^F acting on $\mathcal{O}^{F'}$. [Remark: We're not assuming here that the action is transitive.]

For the following exercise we need some notation. Assume $n \geqslant 1$ is an integer then we denote by $\mathcal{P}(n)$ the set of partitions of n, i.e., the set of all sequences $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = n$. For each integer $i \geqslant 1$ and partition $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}(n)$ we set

$$a_i(\lambda) = |\{1 \leqslant j \leqslant k \mid \lambda_j = i\}|.$$

We say i is a part of λ if $a_i(\lambda) > 0$ and we call $a_i(\lambda)$ the multiplicity. We will need the subset

$$\mathcal{P}_0(n) = \{\lambda \in \mathcal{P}(n) \mid a_i(\lambda) \text{ is even for all odd integers } i > 1\}.$$

In other words $\mathcal{P}_0(n)$ are those partitions where each odd part, different from 1, occurs an even number of times.

Exercise 3.7. Assume $n \geqslant 1$ is an integer and let G be the symmetric group $\mathfrak{S}_n$. If $\lambda \in \mathcal{P}(n)$ is a partition of n then we have a corresponding conjugacy class $\mathcal{C}_\lambda \subseteq G$ such that the elements in $\mathcal{C}_\lambda$ have cycle type λ. Moreover the map $\lambda \mapsto \mathcal{C}_\lambda$ gives a bijection between $\mathcal{P}(n)$ and the conjugacy classes of G. We consider the conjugacy class

$$\mathcal{O}_0 := \begin{cases} \mathcal{C}_{(2^k)} & \text{if } n = 2k \\ \mathcal{C}_{(2^k,1)} & \text{if } n = 2k+1. \end{cases}$$

Every element of $\mathcal{O}_0$ is thus a product of $k = \lfloor n/2 \rfloor$ disjoint transpositions. With this we take $F : G \to G$ to be the automorphism defined by $F(g) = w_0 g w_0^{-1}$ where $w_0 \in \mathcal{O}_0$ is a fixed class representative. We have G acts on itself by conjugation and (F1) is satisfied. Note, each conjugacy class of G is clearly F-stable. Show that the following hold.

(a) For any $w \in G$ we have $\mathscr{L}(g) \cap T_G(w, F) \neq \emptyset$ if and only if $\mathcal{O}_0 \cap C_G(w) \neq \emptyset$.

(b) For any integer $1 < m \leqslant n$ we have $\mathcal{C}^F_{(m,1^{n-m})} \neq \emptyset$ if and only if m is even.

(c) We have $\mathcal{C}^F_\lambda \neq \emptyset$ if and only if $\lambda \in \mathcal{P}_0(n)$.

(d) Show that if $\lambda \in \mathcal{P}_0(n)$ then $\mathcal{C}_\lambda^F$ is a single G^F-conjugacy class. Hence the map $\lambda \mapsto \mathcal{C}_\lambda^F$ defines a bijection between $\mathcal{P}_0(\lambda)$ and the set of G^F-conjugacy classes.

[Hint: (b). If $w \in \mathcal{C}_{(m,1^{n-m})}$ then $C_G(w) \cong C_m \times \mathfrak{S}_{n-m}$. (d). Show that if $\mathcal{O}_0 \cap C_G(w)$ is non-empty then it is a single $C_G(w)$-conjugacy class. Remark: the group G^F is isomorphic to the hyperoctohedral group, which is the Weyl group of type B_k where $k = \lfloor n/2 \rfloor$.]

From this point forward we assume G is a *connected* affine algebraic group and F is a Steinberg endomorphism.

We will also assume that the following condition holds

(F2) $\mathrm{Stab}_G(x) \leqslant G$ is a closed subgroup of G for any $x \in \mathcal{O}$.

In this case we want to improve the parameterisation given in 3.3 using the Lang–Steinberg theorem.

Exercise 3.8. Assume G acts transitively on $\mathcal{O}$ then $\mathcal{O}^{F'} \neq \emptyset$.

For any $x \in \mathcal{O}$ we define $A_G(x, F') = \mathrm{Stab}_G^\circ(x) \setminus T_G(x, F')$ to be the set of cosets

$$\mathrm{Stab}_G^\circ(x)h \subseteq T_G(x, F')$$

of the connected component $\mathrm{Stab}_G^\circ(x)$ contained in $T_G(x, F')$. We have a natural surjective map $\overline{} : T_G(x, F') \to A_G(x, F')$ defined by $\overline{g} = \mathrm{Stab}_G^\circ(x)g$. In class we proved the following.

Lemma 3.9. *The action $\star$ of $\mathrm{Stab}_G(x)$ on $T_G(x, F')$ induces an action of $\mathrm{Stab}_G(x)$ on $A_G(x, F')$ which factors through the finite component group $A_G(x) := \mathrm{Stab}_G^\circ(x)/\mathrm{Stab}_G(x)$. We denote by $H^1(F, A_G(x, F'))$ the orbits of this $A_G(x)$-action which we again call F-conjugacy classes.*

Exercise 3.10. Show that the surjective map $\overline{} : T_G(x, F') \to A_G(x, F')$ induces a bijection

$$H^1(F, T_G(x, F')) \to H^1(F, A_G(x, F')).$$

Conclude that if G acts transitively on $\mathcal{O}$ and $\mathrm{Stab}_G(x)$ is connected for some (any) $x \in \mathcal{O}$ then G^F acts transitively on $\mathcal{O}^{F'}$. [Hint: The hard part is surjectivity. Assume $g_1, g_2 \in T_G(x, F')$ are such that

$$\mathrm{Stab}_G^\circ(x)g_2 = \mathrm{Stab}_G^\circ(x)gg_1F(g^{-1})$$

for some $g \in \mathrm{Stab}_G(x)$ then $g_2 = ghg_1F(g^{-1})$ for some $h \in \mathrm{Stab}_G^\circ(x)$. Check that the $\mathrm{Stab}_G(x_0)$ is stable under $F' = \iota_{g_1} \circ F$ so that F' is a Steinberg endomorphism of $\mathrm{Stab}_G^\circ(x_0)$. Apply the Lang–Steinberg theorem to F' in $\mathrm{Stab}_G^\circ(x)$.]

Exercise 3.11. Assume G is a connected affine algebraic group with Steinberg endomorphism $F : G \to G$. In class we showed that $F' = \iota_g \circ F$ is also a Steinberg endomorphism where $\iota_g : G \to G$ is the conjugation map defined by $\iota_g(x) = gxg^{-1}$. Show that the fixed point groups G^F and $G^{F'}$ are isomorphic. [Hint: construct an element $h \in G$ such that $\iota_h \circ F' = F \circ \iota_h$ then ι_h gives the desired isomorphism.]

Exercise 3.12. Assume $H, K \leqslant G$ are conjugate subgroups of G, i.e., $K = {}^gH$ for some element $g \in G$. Furthermore, assume K is F-stable then $n = \mathscr{L}(g)$ satisfies ${}^nF(H) = H$. Show that $\iota_g : G \to G$ restricts to an isomorphism $H \to K$ satisfying $\iota_g \circ F_n = F \circ \iota_g$. Conclude that ι_g restricts to an isomorphism $H^{F_n} \to K^F$. [Remark: we have $F_n : G \to G$ is a Steinberg endomorphism and $\iota_g : G^{F_n} \to G^F$ is an isomorphism mapping H^{F_n} onto K^F.]

Exercise 3.13. Assume $G = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $F : G \to G$ the standard Frobenius endomorphism defined by $F(x_{ij}) = (x_{ij}^q)$. Moreover, we assume $T_0 \leqslant G$ is the maximal torus of diagonal matrices and $W_G(T_0) = N_G(T_0)/T_0$. Show that F acts trivially on $W_G(T_0)$. Conclude that the G^F-conjugacy classes of F-stable maximal tori of G are parameterised by the conjugacy classes of $W_G(T_0) \cong \mathfrak{S}_n$. The conjugacy classes of $\mathfrak{S}_n$ are parameterised by cycle types and thus by partitions of n. If $T_\lambda \leqslant G$ is an F-stable maximal torus parameterised by an element in $W_G(T_0)$ of cycle type λ show that we have

$$T_\lambda^F \cong C_{q^{\lambda_1}-1} \times \cdots \times C_{q^{\lambda_k}-1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$ and C_m denotes a cyclic group of order m. [Hint: use 3.12 and reduce to the case of a single cycle, which we did in class.]