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Math 517B - Group Theory (Spring 2017) Exercise Sheet 2

Throughout K denotes an algebraically closed field and $n \geqslant 1$ is a positive integer.

Recall that for any integer $N \geqslant 1$ we denote by $\mathrm{Mat}_N(K)$ the set of all $N \times N$ matrices. As a K-vector space $\mathrm{Mat}_N(K)$ has dimension N^2, with a basis given by the elementary matrices E_{ij} with $1 \leqslant i, j \leqslant N$. By E_{ij} we mean the matrix whose only non-zero entry, which is equal to 1, is contained in the ith row and jth column. Note that the multiplication of elementary matrices is given by the relation

$$E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$$

where δ_{jk} is the Kronecker delta. We can also think of $\mathrm{Mat}_N(K)$ as a Lie algebra endowed with the standard Lie bracket, in this case we denote $\mathrm{Mat}_N(K)$ by $\mathfrak{gl}_N(K)$.

Throughout these exercises we will assume that $G = \mathrm{Sp}_{2n}(K)$ is the finite symplectic group. You will find it particularly convenient to index the entries of a matrix $A \in \mathrm{Mat}_{2n}(K)$ as follows

$$A = \begin{bmatrix} a_{n,n} & \cdots & a_{n,1} & a_{n,-1} & \cdots & a_{n,-n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{1,n} & \cdots & a_{1,1} & a_{1,-1} & \cdots & a_{1,-n} \\ a_{-1,n} & \cdots & a_{-1,1} & a_{-1,-1} & \cdots & a_{-1,-n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{-n,n} & \cdots & a_{-n,1} & a_{-n,-1} & \cdots & a_{-n,-n} \end{bmatrix}.$$

Recall that we have

$$\mathrm{Sp}_{2n}(K) = \{A \in \mathrm{GL}_{2n}(K) \mid A^T\Omega_{2n}A = \Omega_{2n}\}$$

where

$$\Omega_{2n} = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \qquad \text{and} \qquad J_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Moreover, we will assume that the Lie algebra of G is given by

$$\mathfrak{g} = \{A \in \mathfrak{gl}_{2n}(K) \mid A^T\Omega_{2n} + \Omega_{2n}A = 0\}.$$

The fact that $\mathfrak{g}$ contains the Lie algebra of G was Exercise 1.13.

Exercise 2.1. Check that $J_n^2 = I_n$ and $\Omega_{2n}^2 = I_{2n}$, where $I_N \in \mathfrak{gl}_N(K)$ is the identity matrix. Show that we have

$$E_{i,j}\Omega_{2n} = \mathrm{sgn}(j)E_{i,-j} \qquad \text{and} \qquad \Omega_{2n}E_{i,j} = -\,\mathrm{sgn}(i)E_{-i,j}.$$

Here $\mathrm{sgn}(x)$ denotes the sign of an integer $x \in \mathbb{Z}$.

Exercise 2.2. Show that the intersection $G \cap D_{2n}(K)$ is given by

$$T_0 := \{\mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}) \mid t_i \in \mathbb{G}_m\}.$$

where we have

$$\mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}) = \sum_{i=-n}^{n} t_i^{\mathrm{sgn}(i)} E_{i,i}$$

Prove that this is a maximal torus of G and show that $N_G(T_0) \leqslant N_{\mathrm{GL}_{2n}(K)}(D_{2n}(K))$. [Hint: any maximal torus of G is contained in a maximal torus of $\mathrm{GL}_{2n}(K)$. Now consider centralisers of elements contained in T_0.]

Exercise 2.3. Show that the Weyl group $W_G(T_0) = N_G(T_0)/T_0$ is isomorphic to the subgroup of the symmetric group $\mathfrak{S}_X$ on $X = \{-n, \ldots, -1, 1, \ldots, n\}$ which preserves the pairs $\{(i, -i) \mid 1 \leqslant i \leqslant n\}$. Furthermore, show that there is no complement of T_0 in $N_G(T_0)$, i.e., there exists no subgroup $N \leqslant N_G(T_0)$ such that $N_G(T_0) = N \ltimes T_0$. [Remark: A typical way to describe the isomorphism type of $W_G(T_0)$ is as the wreath product $C_2 \wr \mathfrak{S}_n$, where $C_2 = \{\pm 1\}$ is the cyclic group of order 2. Specifically $C_2 \wr \mathfrak{S}_n$ is a semidirect product $(C_2 \times \cdots \times C_2) \rtimes \mathfrak{S}_n$ of n copies of C_2 with $\mathfrak{S}_n$ acting by place permutation.]

Exercise 2.4. Let $t = \mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1})$. For any $-n \leqslant i \leqslant n$ let $e_i \in X(T_0)$ denote the character given by

$$e_i(t) = \begin{cases} t_i & \text{if } i > 0 \\ t_{-i} & \text{if } i < 0. \end{cases}$$

Note this means that $e_i = e_{-i}$. Show that $\{e_1, \ldots, e_n\}$ is a basis of $X(T_0)$. Moreover, check that for any $-n \leqslant i, j \leqslant n$ we have

$$tE_{i,j}t^{-1} = \alpha_{i,j}(t)E_{i,j}$$

where $\alpha_{i,j} \in X(T_0)$ is defined by $\alpha_{i,j} = \mathrm{sgn}(i)e_i - \mathrm{sgn}(j)e_j$.

Exercise 2.5. Show that the roots of G with respect to T_0 are given by

$$\Phi = \{\alpha_{i,j} \mid -n \leqslant i \neq j \leqslant n\}.$$

Determine the cardinality of Φ and compute $\dim \mathfrak{g}$. [Hint: The torus T_0 acts on more than one elementary matrix via the character $\alpha_{i,j}$. Consider linear combinations of these elementary matrices.]

Exercise 2.6. For each $\alpha \in \Phi$ construct a corresponding closed embedding $x_\alpha : \mathbb{G}_a \to G$ as in the lectures. By choosing appropriately normalised x_α's describe the homomorphisms $\varphi_\alpha : \mathrm{SL}_2(K) \to G$. What is the kernel of φ_α? Using this information describe completely the root datum of G with respect to T_0. Using this compute $|X(T_0)/\mathbb{Z}\Phi|$ and $|\check{X}(T_0)/\mathbb{Z}\check{\Phi}|$.

Exercise 2.7. If $B = B_{2n}(K) \cap G$ show that

$$B = \left\{ \begin{bmatrix} A & AJ_nS \\ 0 & J_nA^{-T}J_n \end{bmatrix} \middle| A \in B_n(K), S \in \mathrm{Mat}_n(K) \text{ such that } S = S^T \right\}.$$

Prove that this is a Borel subgroup of G containing T_0. Describe the corresponding simple and positive roots $\Delta \subseteq \Phi^+ \subseteq \Phi$.

Exercise 2.8. Let $V = K^{2n}$ then we have a natural representation $\rho : G \to \mathrm{GL}(V)$. Show that the map $(-,-) : V \times V \to K$ defined by $(v, w) = v^T \Omega_{2n} w$ is an alternating bilinear form on V. Note that $\rho(G)$ is precisely the stabiliser of $(-,-)$, i.e., for any $A \in G$ we have $(Av, Aw) = (v, w)$. Recall that a subspace $W \subseteq V$ is called *totally isotropic* if $(v, w) = 0$ for all $v, w \in W$. Show that each standard parabolic subgroup $\rho(P_I)$, with $I \subseteq \Delta$, is the stabiliser of a flag

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_r \subset V$$

where each V_i is a totally isotropic subspace of V. Conclude that the parabolic subgroups of G are precisely the stabilisers of such flags. [Hint: Pick an appropriate basis for V and construct the V_i out of this basis.]

Exercise 2.9. Maintaining the notation of the previous exercise show that for each standard Levi subgroup L_I, with $I \subseteq \Delta$, there exists a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$ such that $\rho(L_I) = \rho(G) \cap (\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_r))$. Describe the isomorphism type of L_I.