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Math 517B - Group Theory (Spring 2017) Exercise Sheet 1

Throughout K denotes an algebraically closed field.

Exercise 1.1. Let G be a linear algebraic group. Show that if $U \leqslant G$ is an abelian subgroup then $\overline{U} \leqslant G$ is also abelian.

Exercise 1.2. Let $f \in K[X_1, \dots, X_n]$ be an irreducible polynomial. Show that $\mathrm{I}(\mathrm{V}(\{f\})) = \langle f \rangle$ and deduce that $\mathrm{V}(\{f\})$ is irreducible. Conclude that $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ are connected.

Exercise 1.3. Let $\phi : X \to Y$ be a morphism between affine varieties. Show that if ϕ is injective then $\dim(X) \leqslant \dim(Y)$.

Exercise 1.4. Let $\mathscr{U}_n(K) \subseteq \mathrm{Mat}_n(K)$ denote the subset of unipotent matrices. Show that the following hold:

(a) $\mathscr{U}_n(K)$ is a Zariski closed subset of $\mathrm{Mat}_n(K)$,

(b) $\mathscr{U}_n(K)$ is irreducible.

Let $J_n \in \mathscr{U}_n(K)$ be a Jordan block of size $n \times n$ with every diagonal entry equal to 1. Furthermore, let $X \subseteq \mathscr{U}_n(K)$ be the set of all matrices whose Jordan normal form is J_n. We call the elements of X *regular unipotent*. Show that the following hold:

(c) X is a Zariski open subset of $\mathscr{U}_n(K)$,

(d) $\dim \mathscr{U}_n(K) = n(n-1)$.

(Hint: (b). Let $\mathrm{U}_n(K) \leqslant \mathrm{GL}_n(K)$ be the subgroup of uni-upper triangular matrices. Consider the map $\mathrm{GL}_n(K) \times \mathrm{U}_n(K) \to \mathrm{Mat}_n(K); (P, A) \mapsto P^{-1}AP$. (c). One possibility is to write the condition $A \in X$ as a condition on the rank of A. (d). Consider the map $\mathrm{GL}_n(K) \to \mathrm{Mat}_n(K); P \mapsto P^{-1}J_nP$.)

Exercise 1.5. Let G be an affine algebraic group over K. Assume $U, V \subseteq G$ are closed subgroups such that $G = UV$ and $U \cap V = \{1\}$. Show that:

(a) For every element $g \in G$ the map $U \times V \to G; (u, v) \to ugv$ is injective. In particular, the map $U \times V \to G; (u, v) \to ugv$ is injective.

(b) If G is connected then so are U and V.

(c) More generally $G^\circ = U^\circ V^\circ$.

(Hint: (b). The group $U^\circ \times V^\circ$ acts on G via $(u, v) \cdot g = ugv^{-1}$.)

Exercise 1.6. Assume X is a G-variety and G acts with finitely many orbits. Show that if $Y \subseteq X$ is an irreducible closed subset of X then there exists a unique G-orbit $\mathcal{O} \subseteq X$ such that $Y \cap \mathcal{O}$ is an open dense subset of Y.

Exercise 1.7. Assume G is an affine algebraic group over K.

(a) Let U, $V \leqslant G$ be closed subgroups of G with $V \leqslant N_G(U)$ or $U \leqslant N_G(V)$. Show that the subgroup $\langle U, V\rangle = UV \leqslant G$ is also closed and $\dim\langle U, V\rangle \leqslant \dim U + \dim V$.

(b) Show that if K is uncountable and $\dim G > 0$ then G is also uncountable.

(c) Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

in $\mathrm{SL}_2(\mathbb{Z})$. Show that A and B have finite order but the product AB has infinite order. Construct an example showing that the conclusion of (a) need not hold if one drops the assumption that either $V \leqslant N_G(U)$ or $U \leqslant N_G(V)$.

(Hint: (c). It is sufficient to show that $\mathrm{SL}_2(\mathbb{Z}) = \langle A, B\rangle$ but this is not necessary.)

Exercise 1.8. Assume T is a torus and let $N \leqslant T$ be a closed subgroup. Assuming that T/N has the structure of a linear algebraic group then show T/N is a torus.

Exercise 1.9. Let $G = \mathbb{D}_n$ and assume $n \geqslant 1$. Show that the following hold:

(a) $\bigcap_{\chi \in X(G)} \mathrm{Ker}(\chi) = 1$,

(b) if $\psi : X(G) \to X(G)$ is a $\mathbb{Z}$-module automorphism then there exists a K-algebra automorphism $\tilde{\psi} : K[G] \to K[G]$ such that $\tilde{\psi}|_{X(G)} = \psi$,

(c) there is an isomorphism $\mathrm{Aut}(G) \cong \mathrm{GL}_n(\mathbb{Z})$ where $\mathrm{Aut}(G)$ denotes the automorphism group of G as an algebraic group.

Exercise 1.10. Assume that $K = \overline{\mathbb{F}}_p$. If G is a linear algebraic group over K show that the following hold:

(a) every element of G has finite order,

(b) $g \in G$ is semisimple if and only if the order of g is coprime to p,

(c) $g \in G$ is unipotent if and only if the order of g is a power of p.

Using this, prove the Jordan decomposition in this case. Furthermore, show that if $\phi : G \to H$ is a homomorphism of linear algebraic groups then $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.

Exercise 1.11. Assume $p \in \mathbb{A}^n$ and consider the differentiation operator $d_p : K[X_1, \dots, X_n] \to K[X_1, \dots, X_n]$. If $f, g \in K[X_1, \dots, X_n]$ check that

$$d_p(f + g) = d_p(f) + d_p(g) \qquad \text{and} \qquad d_p(fg) = d_p(f)g(p) + f(p)d_p(g).$$

Using this show that if $X \subseteq \mathbb{A}^n$ is a closed subset and $f_1, \dots, f_r \in I(X)$ is a generating set of the vanishing ideal then for any $p \in X$ we have

$$T_p(X) = \{v \in \mathbb{A}^n \mid d_p(f_i)(v) = 0 \text{ for all } 1 \leqslant i \leqslant r\}.$$

Exercise 1.12. Recall that the determinant polynomial det $\in K[X_{ij} \mid 1 \leqslant i, j, \leqslant n]$ is given by

$$\det = \sum_{\rho \in \mathfrak{S}_n} \operatorname{sgn}(\rho) X_{1\rho(1)} \cdots X_{n\rho(n)}.$$

Show that

$$d_1(\det) = \sum_{i=1}^{n} X_{ii},$$

which is the trace polynomial.

Exercise 1.13. Recall that for an integer $n \geqslant 1$ we define matrices

$$J_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \qquad \text{and} \qquad \Omega_{2n} = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}.$$

Moreover we then define the symplectic group

$$G = \{A \in \mathrm{GL}_{2n}(K) \mid A^T \Omega_{2n} A = \Omega_{2n}\}.$$

Show that we have

$$T_1(G) \subseteq \{A \in \mathrm{Mat}_{2n}(K) \mid A^T \Omega_{2n} + \Omega_{2n} A = 0\}.$$

Assuming equality holds conclude that G acts on $T_1(G)$ by conjugation.