Algebraic Groups 2015 — Exam

Start Time: 09:00 on Monday 1st June 2015.
End Time: 17:00 on Friday 12th June 2015.

Guidance:

- Answer as many questions as possible.
- You may use any result stated in the course as part of your answers, including those given without proof. However, you should indicate in your solution when you use such a result.
- The statement of any question on this exam may be used as part of the solution to any other question.

Notation: The following notation and assumptions will be in force throughout the questions.

- K will denote an algebraically closed field whose characteristic will be denote by $\mathrm{char}(K)$. Unless stated otherwise this field is chosen arbitrarily, without any further assumptions.
- Any variety will be assumed to be a variety over K. In particular, $\mathbb{A}^n$ will denote affine n-space K^n and $\mathrm{Mat}_n(K)$ will denote the space of all square $n \times n$ matrices whose entries are contained in K, which may be identified with $\mathbb{A}^{n^2}$.
- For any variety X we denote by $K[X]$ the affine algebra of X. If $x \in X$ then we denote by $T_x(X) = \mathrm{Der}_K(K[X], K_x)$ the tangent space of X at x. If $X \subseteq \mathbb{A}^n$ is an algebraic set, then we denote by

$$\mathfrak{T}_x(X) = \{v \in \mathbb{A}^n \mid d_x(f)(v) = 0 \text{ for all } f \in \mathbf{I}(X)\} \subseteq \mathbb{A}^n$$

  the heuristic tangent space of X at x.
- If G is a group then we will denote by $e \in G$ the identity element. In particular, we denote by $T_e(G)$ the tangent space at the identity of an algebraic group G.
- For any $n \geqslant 1$ we define

$$Q_n = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \in \mathrm{Mat}_n(K) \qquad \text{and} \qquad J_{2n} = \begin{bmatrix} 0 & Q_n \\ -Q_n & 0 \end{bmatrix} \in \mathrm{Mat}_{2n}(K).$$

  We then have the classical groups

$$\begin{aligned} \mathrm{GL}_n(K) &= \{A \in \mathrm{Mat}_n(K) \mid \det(A) \neq 0\}, \\ \mathrm{SL}_n(K) &= \{A \in \mathrm{Mat}_n(K) \mid \det(A) = 1\}, \\ \mathrm{Sp}_{2n}(K) &= \{A \in \mathrm{GL}_n(K) \mid A^T J_{2n} A = J_{2n}\}, \end{aligned}$$

  and assuming that $\mathrm{char}(K) \neq 2$ we then also have the special orthogonal group

$$\mathrm{SO}_n(K) = \{A \in \mathrm{SL}_n(K) \mid A^T Q_n A = Q_n\}.$$

Question 1.

(a) Let R be a commutative ring with 1. Show that every prime ideal of R is a radical ideal.

(b) Let $f \in K[X_1, \dots, X_n]$ be an irreducible polynomial. Show that $\mathbf{I}(\mathbf{V}(f)) = \langle f \rangle$ and deduce that $\mathbf{V}(f)$ is irreducible.

(c) Assuming that $\det \in K[X_{ij} \mid 1 \leqslant i, j \leqslant n]$ is an irreducible polynomial show that $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ are connected.

Question 2. Let G be a connected algebraic group and N a finite normal subgroup. Show that N lies in the center of G.

Question 3. Let $G = \mathrm{GL}_n(K)$.

(a) Show that the determinant function $\det : G \to K$ is regular.

(b) Consider the determinant as a function $\det : \mathrm{Mat}_n(K) \to K$. Compute the differential $d_e(\det) : \mathrm{Mat}_n(K) \to K$ at the identity using the heuristic approach.

Question 4. Show that $\mathrm{SL}_2(K)$ and $\mathrm{Sp}_2(K)$ are isomorphic as algebraic groups.

Question 5. For both abelian connected 1-dimensional algebraic groups $\mathbb{G}_a$, $\mathbb{G}_m$ determine:

(a) the automorphism group of the variety,

(b) the automorphism group of the algebraic group and

(c) the module of differentials, giving a generator.

Moreover, are the Lie algebras $\mathrm{Lie}(\mathbb{G}_a)$ and $\mathrm{Lie}(\mathbb{G}_m)$ isomorphic? Justify your answer.

Question 6. Show that any unipotent element of an affine algebraic group has:

(a) finite order if $\mathrm{char}(K) > 0$,

(b) infinite order if $\mathrm{char}(K) = 0$.

[Hint: any unipotent matrix in $U_n(K) \leqslant \mathrm{GL}_n(K)$ is of the form $e + n$ for some nilpotent matrix n.]

Question 7. Let $G = \mathrm{SL}_n(K)$ and let $D \leqslant G$ be the subgroup of diagonal matrices.

(a) Show that the tangent space of D at the identity is

$$\mathfrak{T}_e(D) = \{(a_{ij}) \in \mathrm{Mat}_n(K) \mid a_{ij} = 0 \text{ if } i \neq j \text{ and } \textstyle\sum_{i=1}^n a_{ii} = 0\}.$$

(b) Let U, resp., U^-, be the subgroup of uni-upper, resp., uni-lower, triangular matrices. Let $\varphi : U^- \times D \times U \to G$ be the product map defined by $\varphi(v, t, u) = vtu$. Show that the differential $d_e\varphi$ is injective. [You may use the fact that $d_e\mu(x, y) = x + y$ for any $(x, y) \in \mathfrak{T}_e(G) \oplus \mathfrak{T}_e(G)$, where $\mu : G \times G \to G$ is the product map given by $\mu(g, h) = gh$.]

(c) Show that we have

$$\mathfrak{T}_e(G) = \{A \in \mathrm{Mat}_n(K) \mid \mathrm{tr}(A) = 0\},$$

where $\mathrm{tr} : \mathrm{Mat}_n(K) \to K$ is the trace.

(d) Deduce a formula for the dimension of G in terms of n.

Question 8. Let G be the symplectic group $\mathrm{Sp}_4(K)$. Give 6 generators of the vanishing ideal $\mathbf{I}(G) \subseteq K[X_{ij} \mid 1 \leqslant i, j \leqslant 4]$ and show that

$$\mathfrak{T}_e(G) = \{A \in \mathrm{Mat}_4(K) \mid J_4 A = -A^T J_4\}.$$

Deduce the dimension of G and describe, in short, why the Lie algebra of G is a subalgebra of the Lie algebra $\mathrm{Mat}_4(K)$ with the usual bracket $[A, B] = AB - BA$.

Question 9. Show that $\mathrm{SL}_2(K)$ is generated by the subgroups

$$U = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \middle| \, c \in K \right\} \qquad \text{and} \qquad U^- = \left\{ \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \middle| \, c \in K \right\}.$$

Question 10. Assume that $\mathrm{char}(K) \neq 2$ and let us fix an element $\omega \in K$ such that $\omega^2 = -2$.

(a) Show that $\mathfrak{T}_e(\mathrm{SL}_n(K)) \subseteq \mathrm{Mat}_n(K)$ is a $\mathrm{GL}_n(K)$-space, where the action is given by conjugation.

(b) Prove that there exists a unique homomorphism of algebraic groups $\varphi : \mathrm{SL}_2(K) \to \mathrm{SO}_3(K)$ such that

$$\varphi \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \omega c & c^2 \\ 0 & 1 & -\omega c \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \varphi \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\omega c & 1 & 0 \\ c^2 & \omega c & 1 \end{bmatrix}.$$

(c) Determine the image and kernel of φ. [You may assume here that $\mathrm{SO}_3(K)$ is connected.]

[Hint: (b). For the existence, find an appropriate basis for the action in (a).]

Question 11. Let X be an affine variety and for any $x \in X$ let $\mathfrak{m}_x \subseteq K[X]$ be the maximal ideal of functions vanishing at x. Any $D \in T_x(X)$ maps $\mathfrak{m}_x$ to 0, hence induces a linear map $\widetilde{D} : \mathfrak{m}_x/\mathfrak{m}_x^2 \to K$. Show that the map $D \mapsto \widetilde{D}$ defines a K-linear isomorphism $T_x(X) \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Question 12. Let k be any field and A a commutative k-algebra. Let $\mathcal{D} = \mathrm{Der}_k(A, A)$.

(a) Check that $\mathcal{D}$ is a Lie algebra with the Lie bracket given by $[D, D'] = D \circ D' - D' \circ D$ for all $D, D' \in \mathcal{D}$.

Suppose $\mathrm{char}(k) = p > 0$ and set $\mathrm{Ad}\, D = [D, -] : \mathcal{D} \to \mathcal{D}$ so that

$$(\mathrm{Ad}\, D)(D') = [D, D']$$

for all $D' \in \mathcal{D}$. Show that:

(b) $D^p \in \mathcal{D}$,

(c) $(\operatorname{Ad} D)^p = \operatorname{Ad} D^p$.

Question 13. Prove that we have an isomorphism $\operatorname{Aut}(\mathbb{D}_n) \cong \mathrm{GL}_n(\mathbb{Z})$ of abstract groups. [Hint: the automorphism group of $\mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$.]

Question 14. Denote by $K^* \subseteq \mathbb{A}^1$ the set $\mathbb{A}^1 - \{0\}$ and by X the disjoint union $\{0_1\} \sqcup K^* \sqcup \{0_2\}$, i.e., $0_1 \neq 0_2$ in X. We define two bijections $\phi_1, \phi_2 : \mathbb{A}^1 \to X$ by setting

$$\phi_i(x) = \begin{cases} x & \text{if } x \in K^* \\ 0_i & \text{if } x = 0. \end{cases}$$

We endow X with the topology generated by the images of the open sets in $\mathbb{A}^1$ through ϕ_1 and ϕ_2, i.e., the topology generated by the set $\{\phi_1(U), \phi_2(U) \mid U \subseteq \mathbb{A}^1 \text{ is an open set}\}$.

(a) Describe the open and closed sets of X.

(b) Argue that ϕ_1 and ϕ_2 are continuous maps $\mathbb{A}^1 \to X$ and that X is a quasi-compact topological space.

(c) Let $\mathcal{O}$ be the sheaf of regular functions of the affine variety $\mathbb{A}^1$. Show that the direct image sheaves $\mathcal{O}_1 = (\phi_1)_*(\mathcal{O})$ and $\mathcal{O}_2 = (\phi_2)_*(\mathcal{O})$ are equal. In other words, for any open set $U \subseteq X$ a K-valued function f on U lies in $\mathcal{O}_1$ if and only if it lies in $\mathcal{O}_2$. Conclude that X is a prevariety with this sheaf of functions.

(d) Show that X is not a variety.