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Multiplicities of zero-schemes in quasihomogeneous corank-1 singularities $\mathbf{C}^n \to \mathbf{C}^n$

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Abstract

How many cusps does a swallowtail have,
After it becomes a stable map,
And how many swallowtails does a butterfly have,
After it . . . (with apologies to B. Dylan)

Introduction

Consider the map

$$\begin{array}{rccc} F: & \mathbf{C}^2 & \to & \mathbf{C}^2 \\ & (x,y) & \mapsto & (x, y^4+xy), \end{array}$$

(which is a section of the swallowtail singularity) and its perturbation

$$F_\varepsilon(x,y) = (x, y^4 + xy + \varepsilon y^2).$$

The singular set of F is given by $4y^3 + x = 0$, and the discriminant $\Delta(F)$ of F (the image of its singular set) is a curve with a singular point at the origin. The singular set of F_ε is also a smooth curve, but its image $\Delta(F_\varepsilon)$ is a curve with 2 cusps (A_2-points) and a double point (an $A_{(1,1)}$-point) — see Figure 1.

It turns out (and is well-known) that the number of cusps and double points is independent of the perturbation, provided the perturbation is a stable map. T. Fukuda and G. Ishikawa [3] show that the number of cusps is given by the dimension of a local

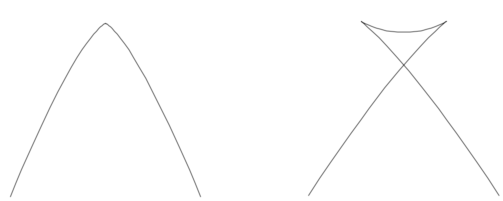

Figure 1: Discriminants of F and F_ε — the swallowtail

algebra associated to F, and independently J. Rieger [15] gives formulae for both the number of cusps and the number of double points in the case that F is of corank 1 — see also [16]. T. Gaffney and D. Mond [6] give formulae for both the number of cusps and the number of double points for a general $\mathcal{A}$-finitely-determined map-germ $\mathbf{C}^2 \to \mathbf{C}^2$.

In this paper, we consider the analogous problem for map-germs $F : \mathbf{C}^n \to \mathbf{C}^n$; that is, given such a map-germ, consider a perturbation which is stable, and ask how many occurrences of each isolated feature in $\Delta(F_\varepsilon)$ there are. The features are the *zero-schemes* of the title, and the numbers are the *multiplicities*. We are able to give answers in the case that F *is of corank 1*. In particular, if F is weighted homogeneous, then we give a closed formula (Theorem 1) for these numbers in terms of the weights and degrees of F. However, unlike Fukuda, Ishikawa and Rieger, we do not consider the case of real map-germs.

The final section 3 of the paper uses this result to give a formula for the multiplicities of the strata in the generalized swallowtail discriminant (Theorem 9).

A 3-dimensional example analogous to the swallowtail one above can be obtained by taking a section of the butterfly:

$$\begin{array}{rcl} F : \ \mathbf{C}^3 & \to & \mathbf{C}^3 \\ (x_1, x_2, y) & \mapsto & (x_1, x_2, y^5 + x_1 y^2 + x_2 y). \end{array}$$

Here the singular set is a smooth surface in $\mathbf{C}^3$, whose image $\Delta(F)$ is a surface with a cuspidal edge and a more degenerate point at the origin. A stable perturbation (or stabilization) F_ε can be given by

$$F_\varepsilon(x_1, x_2, y) = (x_1, x_2, y^5 + x_1 y^2 + x_2 y + \varepsilon y^3).$$

A schematic illustration of $\Delta(F_\varepsilon)$ is given in Figure 2. The interesting isolated features (zero-schemes) of $\Delta(F_\varepsilon)$ are the 2 swallowtail points (A_3-points), and the 2 points where a cuspidal edge passes through a smooth sheet ($A_{(2,1)}$-points). There could in principle be a further isolated feature, namely a triple point of $\Delta(F_\varepsilon)$ where three smooth sheets intersect ($A_{(1,1,1)}$-points), but such a singularity does not occur in this example. The purpose of this paper is to be able to predict these numbers from the form of F, without studying F_ε explicitly. For example, if y^5 were replaced by y^6 in the butterfly example above, then according to Theorem 1, any stabilization would have one $A_{(1,1,1)}$-point, six $A_{(2,1)}$-points and three A_3-points. See Example 2 below.

In general, let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a map-germ with a degenerate (non-stable) singularity, and let F_ε be a 1-parameter *stabilization* of F (that is, for $\varepsilon \neq 0$, the map F_ε is stable). We assume that F is of corank 1 (that is, dF_0 has rank $n-1$). If F is $\mathcal{A}$-finitely-determined, then the singularity of F at 0 splits up into a number of non-degenerate zero-dimensional stable singularities of F_ε, which we now describe.

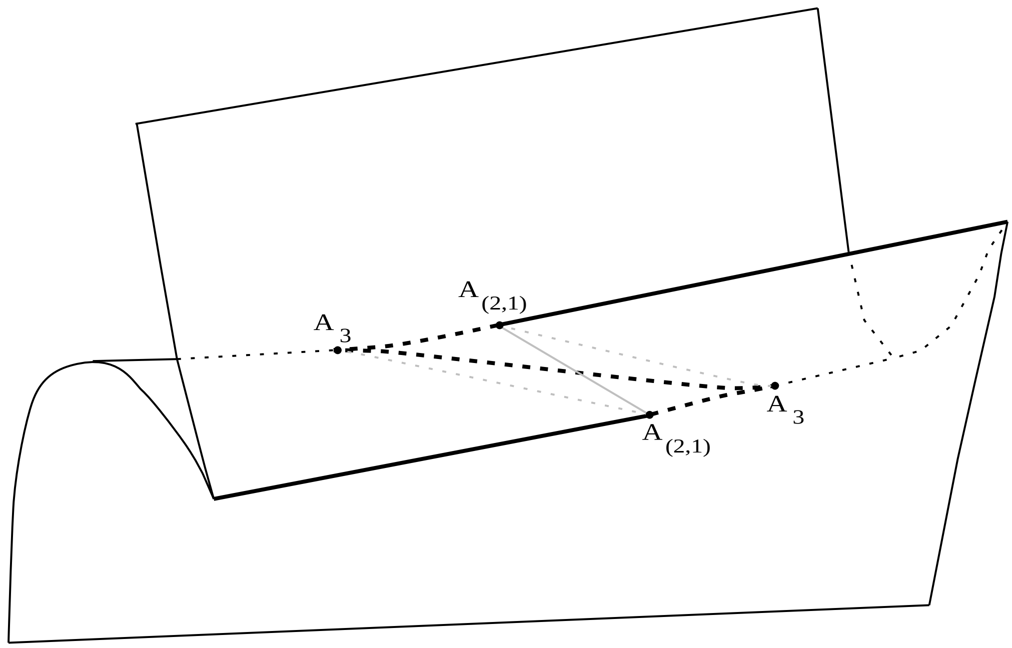

Figure 2: Discriminant of F_ε ($\varepsilon < 0$) — the butterfly
(thick lines are cuspidal edges, grey lines are self-intersections; broken lines are hidden)

A stable map-germ $G : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ has an A_k singularity ($k \leq n$) if it is left-right equivalent to the germ,

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^{k+1} + x_1 y^{k-1} + \cdots x_{k-1} y).$$

Moreover, any stable corank 1 map-germ is an A_k for some natural number k. As is easily seen from this normal form, the set of points in $\mathbf{C}^n$ where a stable map has an A_k singularity is a submanifold of codimension k (given by $x_1 = \cdots = x_{k-1} = y = 0$). The image of this set is then an immersed submanifold of codimension k. It turns out that a map with only corank 1 singularities is stable if and only if these submanifolds in the discriminant are in general position [11, (1.6)].

Definition Suppose the map $G : \mathbf{C}^n \to \mathbf{C}^n$ is stable (and defined on some open subset of $\mathbf{C}^n$). Let z be in the image of G, and put $S = G^{-1}(z) = \{s_1, \ldots, s_d\}$. Suppose G has an A_{r_j} singularity ($r_j \geq 0$) at s_j (for $j = 1, \ldots, d$). In the image, the corresponding submanifolds consisting of A_{r_j} singularities intersect at z, for $j = 1, \ldots, d$. Then z represents a *zero-scheme* if and only if this intersection is zero-dimensional. Since G is stable, these submanifolds are in general position so this occurs if and only if $r_1 + \cdots + r_d = n$. That is, after suppressing those r_j equal to zero, $\mathcal{P} = (r_1, \ldots, r_\ell)$ is a partition of n. We call such a multi-singularity an $A_{\mathcal{P}}$-singularity.

For example, in the case $n = 2$, the two possibilities of zero-schemes are a cusp, with $\mathcal{P} = (2)$, and a double-fold, with $\mathcal{P} = (1,1)$; for $n = 3$ the three possibilities are a swallowtail, with $\mathcal{P} = (3)$, a fold-cusp, with $\mathcal{P} = (2,1)$ and a triple fold, with $\mathcal{P} = (1,1,1)$ — as in the examples above.

The question we address is, given an $\mathcal{A}$-finite map-germ $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ (i.e. of finite $\mathcal{A}$-codimension or equivalently $\mathcal{A}$-finitely determined), and a partition $\mathcal{P}$ of n, how many $A_{\mathcal{P}}$ singularities are there in a stabilization of F, in a suitably small neighbourhood of 0? This number is independent of the particular stabilization chosen, and we denote it $\#A_{\mathcal{P}}(F)$ or simply $\#A_{\mathcal{P}}$.

We consider corank-1 map-germs from $X = (\mathbf{C}^n, 0)$ to $Y = (\mathbf{C}^n, 0)$. Choosing linearly adapted coordinates, we write

$$\begin{array}{rccc} F: & \mathbf{C}^{n-1} \times \mathbf{C} & \to & \mathbf{C}^{n-1} \times \mathbf{C} \\ & (x,y) & \mapsto & (x, f(x,y)). \end{array} \tag{1}$$

When F is weighted homogeneous, we put,

$$\begin{array}{rclrcl} w_0 & = & \mathrm{wt}(y), & w_i & = & \mathrm{wt}(x_i), \\ d & = & \mathrm{degree}(f), & w & = & \prod_{i=1}^{n-1} w_i. \end{array} \tag{2}$$

Let $\mathcal{P} = (r_1, \ldots, r_\ell)$ be a partition of n, with $r_1 \geq r_2 \geq \cdots \geq r_\ell \geq 1$, and call ℓ the *length* of $\mathcal{P}$. Define $N(\mathcal{P})$ to be the order of the subgroup of the permutation group S_ℓ which fixes $\mathcal{P}$. Here S_ℓ acts on $\mathbf{R}^\ell$ by permuting the coordinates. For example, for $\mathcal{P} = (4,4,2,2,2,1,1,1)$ we have $N(\mathcal{P}) = (2!)(3!)^2 = 72$.

Theorem 1 *Let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ be a corank-1 weighted-homogeneous $\mathcal{A}$-finite map-germ, with weights and degrees as above. For any stabilization of F, and any partition $\mathcal{P}$ of n,*

$$\#A_{\mathcal{P}}(F) = \frac{w_0^{n-1}}{N(\mathcal{P})w} \prod_{j=1}^{n+\ell-1} \left(\frac{d}{w_0} - j \right),$$

where ℓ is the length of $\mathcal{P}$, and $N(\mathcal{P})$ is defined above.

Example 2 Let $F : \mathbf{C}^3 \to \mathbf{C}^3$ be defined by

$$F(x_1, x_2, y) = (x_1, x_2, y^6 + x_1 y^2 + x_2 y).$$

This map is weighted homogeneous, with weights and degrees given by $(w_1, w_2, w_0) = (4, 5, 1)$ and $d = 6$, so that $\frac{d}{w_0} = 6$, and $w = w_1 w_2 = 20$.

As already described above, the three types of zero-schemes that occur stably in dimension 3 are given by the partitions $\mathcal{P} = (3)$ (a swallowtail point), $\mathcal{P} = (2,1)$ (a cusp-fold point) and $\mathcal{P} = (1,1,1)$ (a triple fold point). The number of each of these occurring in a stabilization of F can be found from the formula of Theorem 1:

$$\begin{aligned}
\#A_{(3)} &= \frac{1}{1\times 20}(6-1)(6-2)(6-3) = 3\\
\#A_{(2,1)} &= \frac{1}{1\times 20}(6-1)\cdots(6-4) = 6\\
\#A_{(1,1,1)} &= \frac{1}{6\times 20}(6-1)\cdots(6-5) = 1,
\end{aligned}$$

as claimed earlier.

If the map-germ F is not weighted homogeneous, but is still $\mathcal{A}$-finite, then the multiplicities $\#A_{\mathcal{P}}$ can be computed as the dimensions of certain local algebras, see Corollary 5 and Example 8 below.

1 The $A_{\mathcal{P}}$ schemes

Associated to $X = \mathbf{C}^{n-1}\times\mathbf{C}$ and a partition $\mathcal{P}$ of n we will be considering various spaces. In particular,

$$\begin{aligned}
X_\ell &= \mathbf{C}^{n-1}\times\mathbf{C}^{\ell},\\
X^\ell &= \mathbf{C}^{n-1}\times\mathbf{C}^{\ell+n},
\end{aligned}$$

where $\ell = \mathrm{length}(\mathcal{P})$. The first of these spaces is used in this section, while the second is used in §2. We will also be considering a versal deformation $\widetilde{F}$ of F, with base $\mathbf{C}^d$, and then we denote $\widetilde{X}_\ell = \mathbf{C}^d\times X_\ell$, and similarly $\widetilde{X}^\ell = \mathbf{C}^d\times X^\ell$.

Let $\widetilde{F}:\widetilde{X}\to\widetilde{Y}$ be an $\mathcal{A}_e$-versal unfolding of F (with base $\mathbf{C}^d$), so that

$$\widetilde{F}(u,x,y) = (u,x,\widetilde{f}(x,y,u)) = (u,\widetilde{F}_u(x,y)).$$

Any stabilization F_ε of F can be induced from the versal deformation $\widetilde{F}$, so from now on we consider only this versal deformation.

For each partition $\mathcal{P} = (r_1,\dots,r_\ell)$ of n we consider (following ideas of Gaffney [5]) the subscheme $\widetilde{V}(\mathcal{P})$ of $\widetilde{X}_\ell := \mathbf{C}^d\times\mathbf{C}^{n-1}\times\mathbf{C}^\ell$, defined by

$$\widetilde{V}(\mathcal{P}) := \mathrm{clos}\left\{(u,x,y_1,\dots,y_\ell)\in\widetilde{X}_\ell \;\middle|\; \begin{array}{l}\bullet\ y_i\neq y_j,\\ \bullet\ \widetilde{F}(u,x,y_i)=\widetilde{F}(u,x,y_j),\text{ and}\\ \bullet\ \widetilde{F}_u\text{ has a singularity of type }A_{r_j}\\ \qquad\text{at }(u,x,y_j)\end{array}\right\},$$

where 'clos' means the analytic closure in $\widetilde{X}_\ell$.

Let $\pi = \pi_{\mathcal{P}} : \widetilde{V}(\mathcal{P}) \to \mathbf{C}^d$ be the restriction to $\widetilde{V}(\mathcal{P})$ of the Cartesian projection $\widetilde{X}_\ell \to \mathbf{C}^d$. For generic $u \in \mathbf{C}^d$, the fibre $\pi^{-1}(u)$ consists of those 'multi-points' (also known as 'sets') where $\widetilde{F}_u$ has an $A_{\mathcal{P}}$ multi-germ. We are thus interested in the degree of $\pi_{\mathcal{P}}$.

Proposition 3 *If $\mathcal{P} = (r_1, \ldots, r_\ell)$ is a partition of n, then*

$$\#A_{\mathcal{P}} = \frac{1}{N(\mathcal{P})}\text{degree}(\pi(\mathcal{P})).$$

PROOF Let $\mathbf{y} = (y_1, \ldots, y_\ell) \in \widetilde{V}(\mathcal{P})$ and $\sigma \in S_\ell$. We have

$$\mathbf{y}_\sigma := (y_{\sigma(1)}, \ldots, y_{\sigma(\ell)}) \in \widetilde{V}(\mathcal{P})$$

if and only if $r_{\sigma(j)} = r_j$ for each $j = 1, \ldots, \ell$. There are $N(\mathcal{P})$ such σ. The points $\mathbf{y}$ and $\mathbf{y}_\sigma$ are distinct, but the corresponding sets $\{y_1, \ldots, y_\ell\}$ are the same, and it is the sets that are counted in $\#A_{\mathcal{P}}$. □

Let $\widetilde{\mathcal{I}}(\mathcal{P})$ be the ideal in $\mathcal{O}_{\widetilde{X}_\ell}$ defining $\widetilde{V}(\mathcal{P})$, and put

$$\mathcal{I}(\mathcal{P}) = (\widetilde{\mathcal{I}}(\mathcal{P}) + \langle u_1, \ldots, u_d \rangle) / \langle u_1, \ldots, u_d \rangle \subset \mathcal{O}_{X_\ell},$$

corresponding to the intersection of $\widetilde{V}(\mathcal{P})$ with $\{0\} \times X_\ell = X_\ell$. The main theorem follows from the remaining two propositions of this section.

It follows from the definition of $\widetilde{\mathcal{I}}(\mathcal{P})$, that *at generic points of* $\widetilde{V}(\mathcal{P})$ (i.e. where $y_i \neq y_j$),

$$\widetilde{\mathcal{I}}(\mathcal{P}) = \left\langle (\partial_y \tilde{f})_1, \ldots, (\partial_y^{r_1} \tilde{f})_1, \ldots, (\partial_y \tilde{f})_\ell, \ldots, (\partial_y^{r_\ell} \tilde{f})_\ell \right\rangle + \left\langle \tilde{f}_1 - \tilde{f}_2, \ldots, \tilde{f}_1 - \tilde{f}_\ell \right\rangle, \quad (3)$$

where $\tilde{f}_k$ denotes $\tilde{f}$ evaluated at (u, x, y_k), for $1 \leq k \leq \ell$, and $(\partial_y^i \tilde{f})_k$ denotes the i^{th} partial derivative of $\tilde{f}$ with respect to y at the point (u, x, y_k), for $1 \leq k \leq \ell$ and $1 \leq i \leq r_k$.

Proposition 4 *Suppose $\widetilde{V}(\mathcal{P})$ is non-empty. (a) $\widetilde{V}(\mathcal{P})$ is smooth of dimension d;*
(b) $\pi(\mathcal{P}) : \widetilde{V}(\mathcal{P}) \to \mathbf{C}^d$ is finite and $\pi^{-1}(\pi(0)) = \{0\}$;
(c) the degree of $\pi(\mathcal{P})$ coincides with $\dim_{\mathbf{C}} \mathcal{O}_{X_\ell}/\mathcal{I}(\mathcal{P})$.

It follows from this proposition that the ideal $\mathcal{I}(\mathcal{P})$ is a complete intersection.

PROOF (a) Since $\widetilde{F}$ is versal, it follows a fortiori that it is a stable map, and then part (a) follows immediately from [9, Proposition 2.13].

(b) The projection $\pi_{\mathcal{P}} : \widetilde{V}(\mathcal{P}) \to \mathbf{C}^d$ is a finite mapping. In fact, for a generic $u \in \mathbf{C}^d$, the fibre $\pi^{-1}(u)$ is finite and consists of those 'multi-points' where $\widetilde{F}_u$ has an $A_{\mathcal{P}}$ multi-germ. The germ $\widetilde{F}_0 = F$ is $\mathcal{A}$-finite. So, by the Mather-Gaffney geometric criterion ([4] or [17, Theorem 2.1]), it is stable away from zero. Thus, $\pi^{-1}(\pi(0)) = \{0\}$.

(c) Since $\widetilde{V}(\mathcal{P})$ is smooth and hence is Cohen-Macaulay at zero, the degree of $\pi_{\mathcal{P}}$ coincides with $\dim_{\mathbf{C}} \mathcal{O}_{X_\ell}/\mathcal{I}(\mathcal{P})$ [8, Prop. 5.12]. □

Note that combining Propositions 3 and 4(c) gives a method for computing the multiplicities even in the case that F is not weighted homogeneous, provided we can compute $\mathcal{I}(\mathcal{P})$:

Corollary 5

$$\#A_{\mathcal{P}} = \frac{1}{N(\mathcal{P})} \dim_{\mathbf{C}} \left(\frac{\mathcal{O}_{X_\ell}}{\mathcal{I}(\mathcal{P})} \right).$$

In Section 2 we show how to compute $\mathcal{I}(\mathcal{P})$ and we give an example of how this applies. We also prove the following, which combined with the corollary above, proves Theorem 1.

Proposition 6 *If F is weighted homogeneous, with weights and degree as in (2), then*

$$\dim_{\mathbf{C}} \left(\frac{\mathcal{O}_{X_\ell}}{\mathcal{I}(\mathcal{P})} \right) = \frac{1}{w_0^\ell w} \prod_{j=1}^{n+\ell-1} (d - j w_0).$$

2 Multiple point schemes

Nearby the $(A_{r_1} + \cdots + A_{r_\ell}) = A_{(r_1,\ldots,r_\ell)}$ multi-germs, there are points in the target with $(r_1+1)+(r_2+1)+\cdots+(r_\ell+1) = (n+\ell)$ preimages. We shall follow D. Mond [14] and define an $(n+\ell)$-tuple scheme in $X^\ell = \mathbf{C}^{n-1} \times \mathbf{C}^{n+\ell}$, which on the appropriate diagonal specializes to the ideal defining $A_{(r_1,\ldots,r_\ell)}$ multi-germs (Proposition 7 below).

As usual, given a corank-1 map-germ $F : \mathbf{C}^n \to \mathbf{C}^n$ we choose linearly adapted coordinates on $\mathbf{C}^n$ so that $F(x,y) = (x, f(x,y))$ as in (1). Having chosen such coordinates on $\mathbf{C}^n$, we denote the coordinates of X^ℓ by

$$(x, \mathbf{y}) = (x, y_1^0, \ldots, y_1^{r_1}, y_2^0, \ldots, y_2^{r_2}, \ldots, y_\ell^0, \ldots, y_\ell^{r_\ell}).$$

We define an ideal $\mathcal{J}(f, \mathcal{P}) \subset \mathcal{O}_{X^\ell}$ by

$$\mathcal{J}(f, \mathcal{P}) = \langle h_i \mid i = 1, \ldots, n+\ell-1 \rangle,$$

with

$$h_i = V^{-1}. \begin{vmatrix} 1 & y_1^0 & \cdots & (y_1^0)^{i-1} & f_1^0 & (y_1^0)^{i+1} & \cdots & (y_1^0)^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_1^{r_1} & \cdots & (y_1^{r_1})^{i-1} & f_1^{r_1} & (y_1^{r_1})^{i+1} & \cdots & (y_1^{r_1})^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_\ell^0 & \cdots & (y_\ell^0)^{i-1} & f_\ell^0 & (y_\ell^0)^{i+1} & \cdots & (y_\ell^0)^{n+l-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_\ell^{r_\ell} & \cdots & (y_\ell^{r_\ell})^{i-1} & f_\ell^{r_\ell} & (y_\ell^{r_\ell})^{i+1} & \cdots & (y_\ell^{r_\ell})^{n+l-1} \end{vmatrix},$$

where $V = V(y_1^0, \cdots, y_1^{r_1}, \cdots, y_\ell^0, \cdots, y_\ell^{r_\ell})$ is the Vandermonde determinant and

$$f_k^i = f(x, y_k^i).$$

It follows from Cramer's rule that the ideal $\mathcal{J}(f, \mathcal{P})$ defines the set of points in X^ℓ where all the f_k^i coincide [14]. (Note that in the h_i some superscripts are indices, while others represent powers!)

For the versal deformation $\widetilde{F}$, one defines the ideal $\mathcal{J}(\tilde{f}, \mathcal{P})$ in $\mathcal{O}_{\tilde{X}^\ell}$ in exactly the same way, with $\tilde{f}_k^i = \tilde{f}(u, x, y_k^i)$.

In X^ℓ there is a diagonal of particular interest, namely,

$$\Delta(\mathcal{P}) = \{(x, \mathbf{y}) \in X^\ell \mid y_k^i = y_k^j,\ \forall i, j = 1, \ldots, r_k,\ \forall k = 1, \ldots, \ell\},$$

which can be parametrized in the obvious way by $(x, y_1, \ldots, y_\ell)$:

$$(x, \mathbf{y}) = (x, y_1, \ldots, y_1, y_2, \ldots, y_2, \ldots, y_\ell, \ldots, y_\ell), \tag{4}$$

with y_i occurring $r_i + 1$ times. This corresponds to an embedding j_ℓ of X_ℓ into X^ℓ. Of course, there is a similar embedding of $\widetilde{X}_\ell$ in $\widetilde{X}^\ell$. A generic point of $\Delta(\mathcal{P})$ is one of the form (4) with $y_i \neq y_j$, for $i \neq j$. We often simply write Δ in place of $\Delta(\mathcal{P})$.

Let $\mathcal{I}_{\Delta(\mathcal{P})}$ be the ideal defining $\Delta(\mathcal{P})$, that is

$$\mathcal{I}_{\Delta(\mathcal{P})} = \left\langle y_k^i - y_k^0, \mid i = 1, \ldots, r_k,\ k = 1, \ldots, \ell \right\rangle,$$

and let $\mathcal{J}_\Delta(f, \mathcal{P})$ be the $\mathcal{O}_{X^\ell}$ ideal defined by

$$\mathcal{J}_\Delta(f, \mathcal{P}) = \mathcal{J}(f, \mathcal{P}) + \mathcal{I}_{\Delta(\mathcal{P})}.$$

It was shown in [9] that at a generic point of $V(\mathcal{J}_\Delta(f, \mathcal{P}))$, f has a singularity of type A_{r_j} at (x, y_j), and $f(x, y_1) = \ldots = f(x, y_l)$ (see proof of Proposition 7(c) below).

Proposition 7 *(a) The ideal $\mathcal{J}(\tilde{f}, \mathcal{P})$ is reduced, and the multiple point variety $V(\mathcal{J}(\tilde{f}, \mathcal{P})) \subset \widetilde{X}^\ell$ is smooth of dimension $d + n$ (or is empty);*
(b) $\mathcal{J}_\Delta(f, \mathcal{P})$ is a complete intersection singularity;
(c) Let $j_\ell : X_\ell \hookrightarrow X^\ell$ be the embedding with image $\Delta(\mathcal{P})$ given in (4). Then the surjection $j_\ell^* : \mathcal{O}_{X^\ell} \to \mathcal{O}_{X_\ell}$ satisfies $j_\ell^*(\mathcal{J}_\Delta(f, \mathcal{P})) = \mathcal{I}(\mathcal{P})$ and consequently induces an isomorphism

$$j_\ell^* : \frac{\mathcal{O}_{X^\ell}}{\mathcal{J}_\Delta(f, \mathcal{P})} \xrightarrow{\simeq} \frac{\mathcal{O}_{X_\ell}}{\mathcal{I}(\mathcal{P})}.$$

Proof (a) The dimension is clear: for each value of (u, x, Y) in the target there are finitely many points (u, x, y) which map to this under $\widetilde{F}$. The smoothness is less obvious, but follows from [9].
(b) The ideals $\langle u_1, \ldots, u_d \rangle$ and $\mathcal{I}_\Delta$ have d and n generators respectively, and the

intersection of $V(\mathcal{J}(f,\mathcal{P}))$ with the diagonal $\Delta(\mathcal{P})$ is reduced to a single point (the origin) so that for dimensional reasons the ideal is a complete intersection.
(c) It is proved in [9, Lemma 2.7] that at generic points of $\Delta(\mathcal{P})$ one has,

$$\mathcal{J}_\Delta(f,\mathcal{P}) = \Big\langle (\partial_y f)_1, \ldots, (\partial_y^{r_1} f)_1, \ldots, (\partial_y f)_\ell, \ldots, (\partial_y^{r_\ell} f)_\ell \Big\rangle + \langle f(x,y_i) - f(x,y_1); 2 \leq i \leq \ell \rangle + \mathcal{I}_{\Delta(\mathcal{P})},$$

where the $(\partial_y^i f)_k$ are as in (3). It follows that generically $j_\ell^* \mathcal{J}_\Delta(f,\mathcal{P}) = \mathcal{I}(\mathcal{P})$. Part (c) then follows from the fact that two reduced complete intersection ideals that coincide generically are in fact the same. □

Proof of Proposition 6 According to Proposition 7(c) it is enough to compute $\dim(\mathcal{O}_{X^\ell}/\mathcal{J}_\Delta(f,\mathcal{P}))$, and if f is weighted homogeneous this last can be computed by Bezout's theorem [12] since $\mathcal{J}_\Delta(f,\mathcal{P})$ is a complete intersection.

The generators of $\mathcal{J}_\Delta(f,\mathcal{P})$ are the h_j and the $y_k^i - y_k^0$. For each $j = 1, \ldots, n+\ell-1$ one has

$$\text{degree}(h_j) = d - jw_0,$$

while the other generators have degree w_0. The product of all the degrees of the generators is therefore

$$\left(\prod_{j=1}^{n+\ell-1} (d - jw_0) \right) w_0^n.$$

Since $\mathcal{J}_\Delta(f,\mathcal{P})$ is a weighted homogeneous complete intersection (Proposition 7(b)), we can apply Bezout's theorem [12], whence its colength is

$$\frac{1}{w_0^{\ell+n} w} \left(\prod_{j=1}^{n+\ell-1} (d - jw_0) \right) w_0^n = \frac{1}{w_0^\ell w} \prod_{j=1}^{n+\ell-1} (d - jw_0),$$

as required. □

Example 8 Let $f : \mathbf{C}^3 \to \mathbf{C}^3$ be the non-weighted-homogeneous map-germ given by

$$f(x_1, x_2, y) = (x_1, x_2, y^5 + x_1 y + {x_2}^2 y^2 + x_2 y^3).$$

(this is denoted 5_2 in the classification in [10]: note that this is not equivalent to a weighted-homogeneous map since the discriminant Milnor number and the $\mathcal{A}_e$-codimension do not coincide [2]).

Using Maple (see the Appendix below for the programme) we computed the three ideals $\mathcal{I}(\mathcal{P})$ for the three possible partitions. First we computed $\mathcal{J}(f,\mathcal{P})$, then substituted $\mathcal{I}_\Delta$. By Proposition 7 this gives $\mathcal{I}(\mathcal{P})$, and one then deduces the multiplicity from Corollary 5. The results are

$$\begin{aligned}\mathcal{I}((2,1)) &= \Big\langle -3\,y_1^2y_2^2 - 2\,y_2^3y_1 + x_1,\ 3\,y_1^2y_2 + 6\,y_2^2y_1 + y_2^3 + x_2^2, \\ &\qquad -y_1^2 - 6\,y_1y_2 - 3\,y_2^2 + x_2,\ 2\,y_1 + 3\,y_2\Big\rangle \\ \mathcal{I}((3)) &= \Big\langle 15\,y_1^4 + x_1,\ -20\,y_1^3 + x_2^2,\ 10\,y_1^2 + x_2\Big\rangle \\ \mathcal{I}((1,1,1)) &= \langle 1\rangle\,.\end{aligned}$$

It follows that

$$\begin{aligned}\#A_{(2,1)} &= 3 \\ \#A_{(3)} &= 3 \\ \#A_{(1,1,1)} &= 0.\end{aligned}$$

Note that $\#A_{(3)}$ is given in [10], but the values of the other two invariants are new.

Applying Theorem 1 or the method above to the corank-1 simple germs classified by Marar and Tari [10] enables us to 'complete' their Table 1 by giving the new invariants $\#A_{(1,2)}$ and $\#A_{(1,1,1)}$. It turns out that these are all zero, except for $\#A_{(1,2)}(5_k)$ for $k = 1, 2, 3$. The results are:

$$\#A_{(1,2)}(5_1) = 2, \qquad \#A_{(1,2)}(5_2) \;=\; \#A_{(1,2)}(5_3) \;= 3.$$

In particular, all the simple germs $f : (\mathbf{C}^3, 0) \to (\mathbf{C}^3, 0)$ satisfy $\#A_{(1,1,1)}(f) = 0$.

3 Multiplicities of strata in generalized swallowtails

In this final section, we use Theorem 1 to give a simple formula for the local multiplicity of the closure of each stratum in the discriminant of an isolated A_k singularity.

Consider the stable A_k map $F : \mathbf{C}^k \to \mathbf{C}^k$,

$$F(x_1, \ldots, x_{k-1}, y) = (X_1, \ldots, X_{k-1}, Y) = (x_1, \ldots, x_{k-1}, y^{k+1} + x_1y^{k-1} + \cdots + x_{k-1}y).$$

This map is clearly weighted homogeneous, with weights $\mathrm{wt}(x_i) = \mathrm{wt}(X_i) = i + 1$, $\mathrm{wt}(y) = 1$ and $\mathrm{wt}(Y) = k + 1$. The discriminant $\Delta(F)$ is stratified by the various $A_{\mathcal{P}}$ multi-germs, where $\mathcal{P} = (r_1, \ldots, r_\ell)$ is a partition of any $n \le k + 1 - \ell$. Denote this stratum by $\Delta_{\mathcal{P}}$ and its closure by $Z_{\mathcal{P}}$. $Z_{\mathcal{P}}$ is an algebraic subvariety of $\mathbf{C}^k$ of dimension $D = k - n$.

Note that if $n > k + 1 - \ell$ then $\Delta_{\mathcal{P}}$ is empty, as observed by Goryunov [7, §4.3]. Indeed, close to $\Delta_{\mathcal{P}}$ there are points with at least $\sum_i (r_i + 1) = (n + \ell)$ preimages; however F has multiplicity $k + 1$ so that $n + \ell \le k + 1$ (Goryunov's $D(\mu_1, \ldots, \mu_k)$ corresponds to our $\Delta_{\mathcal{P}}$ for $\mathcal{P} = (\mu_1 + 1, \ldots, \mu_k + 1)$).

Theorem 9 *The multiplicity of $Z_{\mathcal{P}}$ at the origin is given by,*

$$\frac{1}{N(\mathcal{P})}(D+1)D(D-1)\dots(D-\ell+2),$$

where $D = \dim(Z_{\mathcal{P}})$ and $N(\mathcal{P})$ is defined in the introduction.

To prove this, we first need a lemma on the geometric structure of A_k discriminants.

Lemma 10 *Let $Z_{\mathcal{P}}$ be as above, and let (z_i) be any sequence of points in $Z_{\mathcal{P}}$ converging to 0. Then*

$$T_0 Z_{\mathcal{P}} := \lim_{i\to\infty} T_{z_i} Z_{\mathcal{P}} = \{(\mathbf{X}, Y) \mid X_{k-n+1} = X_{k-n+2} = \dots = X_{k-1} = Y = 0\}.$$

PROOF As is well-known and easy to see, the discriminant of F coincides with the discriminant of the orbit map $\sigma_0 : \mathbf{C}^k_s \to \mathbf{C}^k_t$ for the action of the permutation group S_{k+1}, where $\mathbf{C}^k_s$ is identified with the subspace of $\mathbf{C}^{k+1}$ the sum of whose coordinates vanishes, and S_{k+1} acts on $\mathbf{C}^{k+1}$ by permuting the coordinates. Consider the extension σ of σ_0 to $\mathbf{C}^{k+1}$ defined as usual by,

$$\begin{array}{rcl} \sigma : \mathbf{C}^{k+1} & \longrightarrow & \mathbf{C}^{k+1} \\ (y_1, \dots, y_{k+1}) & \mapsto & (\sum_i y_i,\ \sum_{i<j} y_i y_j, \dots, y_1 \dots y_{k+1}). \end{array}$$

Clearly, $\mathbf{C}^k_t$ is to be identified with the subspace of $\mathbf{C}^{k+1}$ with vanishing first coordinate. It will be more convenient for computations to change coordinates in the target of σ so that σ takes the form

$$\widetilde{\sigma}(y_1, \dots, y_{k+1}) = (\sum_i y_i,\ \sum_i y_i^2,\ \sum_i y_i^3, \dots, \sum_i y_i^{k+1}).$$

Note that the linear subspaces of the form $T_0 Z_{\mathcal{P}}$ are preserved by the differential at the origin of this change of coordinates; indeed this differential is a diagonal matrix.

Denote by $\widetilde{\Delta}$ the discriminant of $\widetilde{\sigma}$.

Given the partition $\mathcal{P} = (r_1, \dots, r_\ell)$ of n, the stratum $\widetilde{\Delta}_{\mathcal{P}}$ is the image under $\widetilde{\sigma}$ of $\Sigma_{\mathcal{P}} \subset \mathbf{C}^{k+1}$. Let $D + 1 = \dim(\widetilde{\Delta}_{\mathcal{P}})$ (so $D = \dim(Z_{\mathcal{P}})$ as in the theorem). It is convenient to extend $\mathcal{P}$ by $D + 1 - \ell$ zeros, so that $r_j = 0$ for $j = \ell + 1, \dots, D + 1$. The stratum $\Sigma_{\mathcal{P}} \subset \mathbf{C}^{k+1}$ is parametrized by

$$(y_1, \dots, y_{D+1}) \mapsto (y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_\ell, \dots, y_\ell, y_{\ell+1}, \dots, y_{D+1}),$$

where y_j occurs with multiplicity $r_j + 1$, and the y_j are distinct.

Write $\widetilde{\sigma}_{\mathcal{P}}$ for the restriction of $\widetilde{\sigma}$ to $\Sigma_{\mathcal{P}}$. Using this parametrization of $\Sigma_{\mathcal{P}}$, $\widetilde{\sigma}_{\mathcal{P}}$ has the form,

$$\widetilde{\sigma}_{\mathcal{P}}(y_1, \dots, y_{D+1}) = (\sum_i (r_i + 1) y_i,\ \sum_i (r_i + 1) y_i^2, \dots, \sum_i (r_i + 1) y_i^{k+1}).$$

Thus, at a point $y \in \Sigma_{\mathcal{P}}$, the differential of $\widetilde{\sigma}_{\mathcal{P}}$ is

$$d\widetilde{\sigma}_{\mathcal{P}}(y) = \begin{bmatrix} r_1+1 & \cdots & r_{D+1}+1 \\ 2(r_1+1)y_1 & \cdots & 2(r_{D+1}+1)y_{D+1} \\ \vdots & & \vdots \\ (k+1)(r_1+1)y_1^k & \cdots & (k+1)(r_{D+1}+1)y_{D+1}^k \end{bmatrix}.$$

Notice that the top $(D+1) \times (D+1)$ minor is equal to

$$(D+1)! \left(\prod (r_i+1)\right) V(y_1, \dots, y_{D+1}),$$

where V is the Vandermonde determinant, which is non-vanishing on $\widetilde{\Delta}_{\mathcal{P}}$. Consequently, at points of $\widetilde{\Delta}_{\mathcal{P}}$, the tangent space to $\widetilde{\Delta}_{\mathcal{P}}$ projects isomorphically onto $\mathbf{C}^{D+1}$ (defined by the vanishing of the last $k-D$ coordinates).

Finally, since $\widetilde{\sigma}$ is weighted-homogeneous, and the last $k-D$ components are of strictly higher degree than the first $D+1$, it follows that in the limit as

$$(y_1, \dots, y_{D+1}) \to (0, \dots, 0),$$

the tangent space to $\widetilde{\Delta}_{\mathcal{P}}$ tends to $\mathbf{C}^{D+1}$. Intersecting source and target with $\mathbf{C}_s^k$ and $\mathbf{C}_t^k$ respectively shows that the same is true of the tangent space to $\Delta_{\mathcal{P}}$, as required. $\square$

Proof of Theorem 9 It follows from this lemma that the multiplicity at 0 of $Z_{\mathcal{P}}$ is given by the intersection multiplicity of $Z_{\mathcal{P}}$ with the n-dimensional subspace

$$\{(\mathbf{X}, Y) \mid X_1 = \cdots = X_{k-n} = 0\},$$

which is complementary to the unique limiting tangent space $T_0 Z_{\mathcal{P}}$, and it remains for us to compute this multiplicity.

To this end, consider the map $g : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$g(u_1, \dots, u_{n-1}, y) = (u_1, \dots, u_{n-1}, y^{k+1} + u_1 y^{n-1} + \cdots + u_{n-1} y),$$

which is induced from F by the immersion $\gamma : \mathbf{C}^n \to \mathbf{C}^k$,

$$\gamma(u_1, \dots, u_{n-1}, y) = (0, \dots, 0, u_1, \dots, u_{n-1}, y),$$

in the sense that $F \circ \gamma = \gamma \circ g$.

By the lemma, this inclusion is transverse to $\Delta(F)$ away from the origin, so that it is $\mathcal{K}_{\Delta(F)}$-finite, and consequently, g is $\mathcal{A}$-finite (Damon [1]). Moreover, a stabilization g_ε of g is obtained by perturbing the embedding γ to an embedding γ_ε transverse to $\Delta(F)$, and *a fortiori* transverse to $Z_{\mathcal{P}}$. If γ_ε is transverse to $Z_{\mathcal{P}}$, then $\text{image}(\gamma_\varepsilon) \cap Z_{\mathcal{P}} = \text{image}(\gamma_\varepsilon) \cap \Delta_{\mathcal{P}}$ is a finite set (for dimensional reasons).

The points of this intersection are precisely the image under γ_ε of the points in $\mathbf{C}^n$ (the image of g_ε) over which g_ε has an $A_{\mathcal{P}}$ singularity. Since g is weighted homogeneous, the number of such points is given by Theorem 1. A simple computation then proves Theorem 9. $\square$

Appendix: A Maple Programme

The Maple programme used for computing $\mathcal{I}(\mathcal{P})$ is short and simple, so can be included here. It runs (at least) on MapleV Release 4.

```
> restart;
> with(linalg);
```

Define function f, and partition $\mathcal{P}$:

```
> f := y^5 + x[1]*y + x[2]^2*y^2 + x[2]*y^3 ;
> P := [1,2];
```

Find dimension of space and length of partition and check that $\mathcal{P}$ is indeed a partition of n:

```
> n := nops(indets(f));
> ell := nops(P);
> if convert(P,`+`) <> n
>   then print(`ERROR, P should be a partition of n`)
> fi;
```

A trick to get indices for the multiple point scheme:

```
> Y := array(1..ell,0..max(op(P)));
> YY := [seq(seq(Y[i,j],j=0..P[i]),i=1..ell)];
> V:=factor(det(vandermonde(YY)));
```

Define the generators h_i of the multiple point scheme:

```
> h := proc(i::integer)
>   local W, j;
>   W := vandermonde(YY);
>   for j to nops(YY) do
>     W[j,i+1] := subs(y=YY[j], f)
>   od;
>   simplify(factor(det(W))/V)
> end;
```

The ideal $\mathcal{J}(f, \mathcal{P})$:

```
> J := [seq(h(i), i=1..n+ell-1)]:
```

Equations for the diagonal $\Delta(P)$:

```
> Delta := {seq( seq(Y[i,j]=y[i], j=0..P[i]), i=1..ell)};
```

Now compute $\mathcal{J}_\Delta$, restricted to Δ — in other words $\mathcal{I}(\mathcal{P})$:

```
> IP := subs(Delta, J);
```

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