A Note on Semisymplectic Actions of Lie Groups

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Abstract. A semisymplectic action of a Lie groups on a symplectic manifold is one where each element of the group acts either symplectically or antisymplectically. We find conditions that a semisymplectic action descends to an action on the symplectic reduced spaces. We consider a few examples, and in particular apply these ideas to reduction of N-body systems with Galilean invariance.

Sur les actions semisymplectiques des groupes de Lie

Résumé. Une action semisymplectique d'un groupe de Lie sur une variété symplectique est une action dont les éléments du groupe agissent de façon symplectique ou antisymplectique. Pour un sous-groupe distingué K de $\widehat{G}$ qui agit de façon hamiltonienne, on trouve des conditions pour qu' un sous-groupe de $\widehat{G}/K$ agisse de façon semisymplectique sur un espace réduit $\mathcal{P}_\mu$ par K. On applique ce résultat surtout aux systèmes à N-corps de symétrie Galiléenne.

Version française abrégée

Soit $\widehat{G}$ un groupe de Lie qui agit proprement sur une variété symplectique $(\mathcal{P},\omega)$ de façon que

$$g^*\omega = \chi(g)\omega,$$

où χ est un homomorphisme surjectif $\chi : \widehat{G} \to \mathbf{Z}_2 = \{\pm 1\}$, appellé le *caractère temporel* par Souriau [4]. De telles actions sont dites *semisymplectiques*. On note $G = \ker\chi$. L'action de G est donc symplectique.

La question de base qu'on se pose ici est la suivante : si on considère l'espace réduit $\mathcal{P}_\mu$ pour $\mu \in \mathfrak{g}^*$, alors sous quelles conditions $\widehat{G}/G$ agit-il sur $\mathcal{P}_\mu$? Plus généralement, soit K un sous-groupe distingué de $\widehat{G}$, contenu dans G dont l'action sur $\mathcal{P}$ est hamiltonienne : il existe une application moment $\Phi : \mathcal{P} \to \mathfrak{k}^*$, où $\mathfrak{k}$ est l'algèbre de Lie de K, et $\mathfrak{k}^*$ son dual. Alors quel sous-groupe de $\widehat{\Gamma} := \widehat{G}/K$ agit sur $\mathcal{P}_\mu$, pour $\mu \in \mathfrak{k}^*$?

Le groupe $\widehat{G}$ agit de façon naturelle sur $\mathfrak{k}$ et $\mathfrak{k}^*$ par des actions qui étendent les actions adjointes et coadjointes de K. On les appelle actions adjointes et coadjointes, notées $\mathrm{Ad}_g : \mathfrak{k} \to \mathfrak{k}$ et $\mathrm{Coad}_g = \mathrm{Ad}^*_{g^{-1}} : \mathfrak{k}^* \to \mathfrak{k}^*$. Muni du caractère temporel χ, définissons maintenant une nouvelle action de $\widehat{G}$ sur $\mathfrak{k}^*$ nommée *action coadjointe tordue par* χ:

$$\mathrm{Coad}^\chi_g\,\mu = \chi(g)\,\mathrm{Coad}_g\,\mu.$$

Par les mêmes méthodes que celles utilisées par Souriau [4, (11.17)], on démontre que l'application

$$\theta : G \to \mathfrak{k}^*, \quad \theta(g) = \Phi(g \cdot x) - \mathrm{Coad}_g^\chi \, \Phi(x),$$

est un cocycle bien-défini. Il s'en suit que l'application moment est équivariante par rapport à l'action donnée sur $\mathcal{P}$ et l'action sur $\mathfrak{k}^*$ donnée par

$$(g, \mu) \mapsto \mathrm{Coad}_g^\chi \, \mu + \theta(g).$$

Pour $\mu \in \mathfrak{k}^*$, on note par $\mathcal{O}_\mu$ son orbite sous l'action coadjointe de K modifiée par le cocycle θ. On supposera que $\mathcal{O}_\mu$ est une sous-variété localement fermée. Alors l'espace réduit $\mathcal{P}_\mu := \Phi^{-1}(\mathcal{O}_\mu)/K$ par l'action de K a une décomposition en stratification symplectique unique [3, 1].

On pose $\Gamma = G/K$ et $\widehat{\Gamma} = \widehat{G}/K$. Alors $\widehat{\Gamma}$ agit de façon naturelle sur l'ensemble des orbites $\mathcal{O}_\mu$ ($\mu \in \mathfrak{k}^*$), et aussi sur $\mathcal{P}/K$. Définissons

$$\widehat{\Gamma}_\mu := \{\gamma \in \widehat{\Gamma} \mid \gamma \cdot \mathcal{O}_\mu = \mathcal{O}_\mu\},$$

et $\chi_\mu : \widehat{\Gamma}_\mu \to \mathbf{Z}_2$ l'homomorphisme induit de χ par restriction et projection.

Théorème 1 *$\widehat{\Gamma}_\mu$ agit de façon semisymplectique sur $\mathcal{P}_\mu$, avec caractère temporel égal à χ_μ. Par ailleurs, pour deux éléments $\mu, \mu' \in \mathfrak{k}^*$ équivalents sous l'action de $\widehat{G}$, les espaces réduits $\mathcal{P}_\mu$ et $\mathcal{P}_{\mu'}$ sont isomorphes ainsi que les actions de Γ_μ et $\Gamma_{\mu'}$.*

Repère lié au centre de masse Il a souvent été remarqué que dans certains systèmes à N-corps dans $\mathbf{R}^n$ le passage au repère lié au centre de masse est une forme de réduction symplectique par rapport au groupe de translations. Les méthodes dévéloppées ici nous permettent de montrer que ces systèmes réduits ont toujours une symétrie de reversibilité. Nous utilisons une action du groupe Galiléen reversible sur l'espace des phases étendu $\mathcal{P} \times \mathbf{R}$. Si $K = \mathbf{R}^n$ est le sous-groupe des translations, alors pour $\mu \in (\mathbf{R}^n)^*$ on trouve $\widehat{\Gamma}_\mu \simeq \mathbf{O}(n) \times \mathbf{Z}_2$, et tous les espaces réduits $\mathcal{P}_\mu$ munis de leurs actions de $\widehat{\Gamma}_\mu$ sont isomorphes. Cette approche est différente de celle proposée par Souriau [4].

Let $\widehat{G}$ be a Lie group acting properly on a symplectic manifold $(\mathcal{P}, \omega)$ in such a way that for each $g \in \widehat{G}$,

$$g^*\omega \ = \ \chi(g)\omega,$$

where $\chi : \widehat{G} \to \mathbf{Z}_2 = \{\pm 1\}$ is a surjective homomorphism, called a *temporal character* by Souriau [4]. Such actions are said to be *semisymplectic*. The subgroup $G = \ker \chi$ acts symplectically on $\mathcal{P}$.

The principal question we address is: does the action of $\widehat{G}/G$ descend to an antisymplectic involution on $\mathcal{P}_\mu$, where $\mathcal{P}_\mu$ is the reduced space corresponding to $\mu \in \mathfrak{g}^*$? The answer depends on the group, and in general on the value of μ. In fact we answer a slightly more general question, where the reduction is not necessarily by all of G but by some subgroup K of G that is normal in $\widehat{G}$. This is the first step in the 'reduction by stages' procedure described in [2].

In the final section of this note we show how the general ideas developed here underlie the common observation that for systems which are Galilean invariant the equations of motion obtained by reduction to centre of mass coordinates are independent of the velocity at which the centre of mass is moving.

General results Let K be a subgroup of G which is normal in $\widehat{G}$ (written $K \lhd \widehat{G}$), and for which the action on $\mathcal{P}$ is Hamiltonian. This means that there is a momentum map $\Phi : \mathcal{P} \to \mathfrak{k}^*$ where $\mathfrak{k}^*$ is the dual of the Lie algebra $\mathfrak{k}$ of K.

Since K is normal, $\widehat{G}$ acts on it by conjugation and the linearisation of this action at the identity element defines an action of $\widehat{G}$ on $\mathfrak{k}$ which extends the adjoint action of K. The contragredient action of $\widehat{G}$ on $\mathfrak{k}^*$ similarly extends the coadjoint action of K. We continue to refer to these extended actions as adjoint and coadjoint actions, respectively, and denote them by $\mathrm{Ad}_g : \mathfrak{k} \to \mathfrak{k}$ and $\mathrm{Coad}_g = \mathrm{Ad}^*_{g^{-1}} : \mathfrak{k}^* \to \mathfrak{k}^*$.

Given the character $\chi : \widehat{G} \to \mathbf{Z}_2$ with kernel G, we define the *χ-twisted coadjoint action* of $\widehat{G}$ on $\mathfrak{k}^*$ by

$$\mathrm{Coad}^\chi_g\,\mu \;=\; \chi(g)\,\mathrm{Coad}_g\,\mu.$$

Exactly the same argument as used by Souriau [4, (11.17)], shows that the map

$$\theta \;:\; \widehat{G} \to \mathfrak{k}^*, \qquad \theta(g) \;=\; \Phi(g\cdot x) - \mathrm{Coad}^\chi_g\,\Phi(x),$$

is a well-defined cocycle on $\widehat{G}$ with values in $\mathfrak{k}^*$. The momentum map Φ is then equivariant with respect to the *θ-affine χ-twisted coadjoint action* of $\widehat{G}$ on $\mathfrak{k}^*$:

$$(g,\mu) \;\mapsto\; \mathrm{Coad}^{\chi,\theta}_g\,\mu \;=\; \mathrm{Coad}^\chi_g\,\mu + \theta(g). \tag{1}$$

If the group $\widehat{G}$ is semisimple or compact, then the momentum map can be chosen so that the associated cocycle θ vanishes.

For $\mu \in \mathfrak{k}^*$, denote by $\mathcal{O}_\mu$ the K-orbit of μ in $\mathfrak{k}^*$ under the restriction to K of the θ-affine χ-twisted coadjoint action. We suppose from now on that this orbit $\mathcal{O}_\mu$ is locally closed. Then the reduced space $\mathcal{P}_\mu = \Phi^{-1}(\mathcal{O}_\mu)/K$ has a unique decomposition as a symplectic stratified space [3, 1].

Let $\widehat{\Gamma} = \widehat{G}/K$ and $\Gamma = G/K$. The affine twisted coadjoint action of $\widehat{G}$ on $\mathfrak{k}^*$ induces an action of $\widehat{\Gamma}$ on the set of orbits $\mathcal{O}_\mu$ (that is, on $\mathfrak{k}^*/K$), and we define

$$\widehat{\Gamma}_\mu \;:=\; \{\, \gamma \in \widehat{\Gamma} \;\mid\; \gamma\cdot\mathcal{O}_\mu = \mathcal{O}_\mu \,\},$$

and $\chi_\mu : \widehat{\Gamma}_\mu \to \mathbf{Z}_2$ to be the homomorphism induced from χ by restriction and projection.

Note that $\widehat{\Gamma}_\mu$ is essentially independent of the choice of momentum map, for if $\Phi' = \Phi + \nu$ for some $\nu \in \mathfrak{k}^*$, then the associated cocycle is $\theta' = \theta - \delta\nu$, where $\delta\nu(g) = \mathrm{Coad}^\chi_g\,\nu - \nu$. The resulting K-orbits in $\mathfrak{k}^*$ are then related by $\mathcal{O}'_{\mu+\nu} = \mathcal{O}_\mu + \nu$, and $\widehat{\Gamma}'_{\mu+\nu} = \widehat{\Gamma}_\mu$. Thus $\widehat{\Gamma}_\mu$ and χ_μ depend on the action on $\mathcal{P}$ only through the cohomology class of the cocycle θ. In particular, if G is compact or semisimple then $\widehat{\Gamma}_\mu$ depends only on the Lie group structure and not on the given action on $\mathcal{P}$.

Theorem 1 *$\widehat{\Gamma}_\mu$ acts semisymplectically on $\mathcal{P}_\mu$, with temporal character χ_μ. If $\mu' = \mathrm{Coad}^{\chi,\theta}_g\,\mu$ for some $g \in \widehat{G}$ then $\mathcal{P}_{\mu'}$ is symplectomorphic to $\mathcal{P}_\mu$, $\widehat{\Gamma}_{\mu'} = g\widehat{\Gamma}_\mu g^{-1}$ and the action of $\widehat{\Gamma}_{\mu'}$ on $\mathcal{P}_{\mu'}$ is symplectomorphic to that of $\widehat{\Gamma}_\mu$ on $\mathcal{P}_\mu$. Moreover, if H is a $\widehat{G}$-invariant Hamiltonian on P then the reduced Hamiltonian flow induced by H on $\mathcal{P}_{\mu'}$ is mapped by the symplectomorphism to that induced on $\mathcal{P}_\mu$.*

PROOF Since $K \lhd \widehat{G}$, it follows that $\widehat{\Gamma}$ acts on $\mathcal{P}/K$, and it is clear by construction that $\widehat{\Gamma}_\mu$ preserves $\mathcal{P}_\mu = \Phi^{-1}(\mathcal{O}_\mu)/K \subset \mathcal{P}/K$.

The symplectic forms on the symplectic strata in $\mathcal{P}_\mu$ depend uniquely on the symplectic form ω, and consequently any symmetry of ω that acts on $\mathcal{P}_\mu$ preserves the symplectic form on each stratum. Furthermore, replacing ω by $-\omega$ on $\mathcal{P}$ reverses the symplectic forms on the strata in $\mathcal{P}_\mu$, so that indeed $\widehat{\Gamma}_\mu$ acts semisymplectically on $\mathcal{P}_\mu$ with temporal character χ_μ. The rest of the proof is straightforward. □

Semidirect products It is interesting to consider the special case of semidirect products $\widehat{G} = \widehat{\Gamma} \ltimes K$ where $\widehat{\Gamma}$ is a compact subgroup of $\widehat{G}$ and $K \lhd G$. This includes many examples that arise in applications. Note that since in this case $\widehat{\Gamma}$ is a subgroup of $\widehat{G}$ it acts semisymplectically on $\mathcal{P}$. It also acts naturally on K and $\mathfrak{k}$ and by the χ-twisted coadjoint action on $\mathfrak{k}^*$. We denote elements of $\widehat{G} = \widehat{\Gamma} \ltimes K$ by (γ, k) where $\gamma \in \widehat{\Gamma}$, $k \in K$. Using multiplicative notation on $\widehat{\Gamma}$ and additive notation on K we write the product of two elements as $(\gamma_1, k_1)(\gamma_2, k_2) \;=\; (\gamma_1\gamma_2, k_1 + \gamma_1 k_2)$.

Proposition 1 *If $\widehat{G} = \widehat{\Gamma} \ltimes K$ and $\widehat{\Gamma}$ is compact then the momentum map $\Phi : \mathcal{P} \to \mathfrak{k}^*$ can be chosen to be $\widehat{\Gamma}$ equivariant with respect to the χ-twisted coadjoint action on $\mathfrak{k}^*$. The corresponding cocycle $\theta : \widehat{G} \to \mathfrak{k}^*$ satisfies:*

1. $\theta(\gamma, k) = \theta(1, k)$
2. $\theta(1, \gamma k) = \mathrm{Coad}^{\chi}_{\gamma}\, \theta(1, k)$.

In particular the restriction of θ to $\widehat{\Gamma}$ is identically zero and the restriction to K is equivariant with respect to the action of $\widehat{\Gamma}$ on K and its χ-twisted coadjoint action on $\mathfrak{k}^*$.

PROOF From any momentum map $\Phi : \mathcal{P} \to \mathfrak{k}^*$ for the action of K on $\mathcal{P}$ we can obtain a $\widehat{\Gamma}$-equivariant momentum map $\widetilde{\Phi}$ by averaging with respect to Haar measure on $\widehat{\Gamma}$:

$$\widetilde{\Phi}(x) \;=\; \int_{\widehat{\Gamma}} \mathrm{Coad}^{\chi}_{\gamma^{-1}}\, \Phi(\gamma x)\, d\gamma.$$

Let $\widetilde{\theta} : \widehat{G} \to \mathfrak{k}^*$ denote the corresponding cocycle. The $\widehat{\Gamma}$ equivariance of $\widetilde{\Phi}$ implies that $\widetilde{\theta}(\gamma, 0) = 0$ for all $\gamma \in \widehat{\Gamma}$. Hence $\widetilde{\theta}(\gamma, k) \;=\; \widetilde{\theta}((1, k)(\gamma, 0)) \;=\; \widetilde{\theta}(1, k) + \mathrm{Coad}_{(1,k)}\, \widetilde{\theta}(\gamma, 0) \;=\; \widetilde{\theta}(1, k)$ and

$$\widetilde{\theta}(1, \gamma k) = \widetilde{\theta}((\gamma, 0)(1, k)(\gamma^{-1}, 0)) = \widetilde{\theta}(\gamma, 0) + \mathrm{Coad}^{\chi}_{\gamma} \left(\widetilde{\theta}(1, k) + \widetilde{\theta}(\gamma^{-1}, 0)\right) = \mathrm{Coad}^{\chi}_{\gamma}\, \widetilde{\theta}(1, k).$$

□

It follows immediately from this proposition that for any $\mu \in \mathfrak{k}^*$ we have

$$\widehat{\Gamma}_\mu \;=\; \{\, \gamma \in \widehat{\Gamma} \;\mid\; \mathrm{Coad}^{\chi}_{\gamma}\, \mu = \mathrm{Coad}^{\theta}_{k}\, \mu \;\text{ for some } k \in K \,\}.$$

Direct products In a direct product $\widehat{G} = \widehat{\Gamma} \times K$ the group $\widehat{\Gamma}$ acts trivially on K. A particular case is where K acts on T^*X by cotangent lift, and $\Gamma = \mathbf{Z}_2^{\tau}$ acts by $\tau(q, p) = (q, -p)$. It follows that a $\widehat{\Gamma}$ equivariant cocycle $\theta : K \to \mathfrak{k}^*$ must take values in the fixed point set of the χ-twisted coadjoint action of $\widehat{\Gamma}$ on $\mathfrak{k}^*$. However the natural coadjoint action on $\mathfrak{k}^*$ is trivial and so the χ-twisted coadjoint action just multiplies the elements of $\mathfrak{k}^*$ by $\chi(\gamma)$ and the fixed point set consists only of the origin. Thus for any semisymplectic action of $\widehat{\Gamma} \times K$ the cocycle θ can be chosen to vanish. Moreover, if $\theta \equiv 0$, then

$$\widehat{\Gamma}_\mu \;=\; \begin{cases} \widehat{\Gamma} & \text{if } -\mu \in \mathcal{O}_\mu \\ \Gamma & \text{otherwise} \end{cases}$$

where $\mathcal{O}_\mu$ is now just the ordinary coadjoint orbit of K through μ.

If K is Abelian then the coadjoint orbits are just points and so $\widehat{\Gamma}_\mu = \widehat{\Gamma}$, and $\mathcal{P}_\mu$ has an antisymplectic symmetry, if and only if $\mu = 0$.

If K is compact the condition $-\mu \in \mathcal{O}_\mu$ can be determined from the Weyl group action on a Cartan subalgebra. Indeed, after identifying $\mathfrak{k}^*$ with $\mathfrak{k}$, choose a Cartan subalgebra containing μ. Then $-\mu \in \mathcal{O}_\mu$ if and only if there is an element w of the Weyl group for which $w\mu = -\mu$. For example, if $K = \mathbf{SO}(n) (n \geq 3)$

then there is always such an element, while if $K = \mathbf{SU}(n) (n \geq 3)$ then only those μ that are *balanced* satisfy the condition (a vector μ is balanced if μ and $-\mu$ have the same elements).

For a final direct product example consider the case $\widehat{G} = \mathbf{Z}_2 \times K$ with $K = \mathbf{SE}(2) = \mathbf{SO}(2) \ltimes \mathbf{R}^2$. The coadjoint orbit through $\mu = (\ell, \mathbf{p}) \in \mathfrak{so}(2)^* \oplus (\mathbf{R}^2)^*$ is a single point if $\mathbf{p} = 0$ and a cylinder of radius $||\mathbf{p}||$ otherwise. In the first case $-\mu \in \mathcal{O}_\mu$ if and only if $\mu = 0$, while in the second case $-\mu$ always lies in $\mathcal{O}_\mu$. Thus $\mathcal{P}_\mu$ inherits antisymplectic symmetries if and only if either the linear momentum is nonzero or if both the 'linear momentum' $\mathbf{p}$ and 'angular momentum' ℓ are zero.

Semidirect products with $\mathbf{K} = \mathbf{R}^n$ For a second set of examples we consider cases with $\widehat{G} = \widehat{\Gamma} \ltimes \mathbf{R}^n$. The restriction of the cocycle θ to $K = \mathbf{R}^n$ is *symplectic* and so can be identified with a skew-symmetric bilinear form on $\mathbf{R}^n$. The cocycle is $\widehat{\Gamma}$ equivariant if and only if this skew-symmetric form is $\widehat{\Gamma}$ semi-invariant:

$$\theta(\gamma k, \gamma \xi) \;=\; \chi(\gamma)\, \theta(k, \xi) \qquad \forall \gamma \in \widehat{\Gamma},\ k \in K = \mathbf{R}^n,\ \xi \in \mathfrak{k} = \mathbf{R}^n.$$

The equivariance of the cocycle implies that its image $\theta(K)$ is a $\widehat{\Gamma}$-invariant subspace of $(\mathbf{R}^n)^*$. Moreover $\mathrm{Coad}_k^\theta \mu = \mu + \theta(k)$ and so $\mathfrak{k}^*/K$ is the quotient space $(\mathbf{R}^n)^*/\theta(K)$. The groups $\widehat{\Gamma}_\mu$ are just the isotropy subgroups of the action of $\widehat{\Gamma}$ on this space induced from the χ-twisted coadjoint action on $(\mathbf{R}^n)^*$.

Note that if the χ-twisted coadjoint action of $\widehat{\Gamma}$ on $(\mathbf{R}^n)^*$ is an irreducible representation then either $\theta(K) = \{0\}$, and $\theta \equiv 0$, or $\theta(K) = (\mathbf{R}^n)^*$. In the latter case we always have $\widehat{\Gamma}_\mu = \widehat{\Gamma}$, while in the former case we have to calculate the isotropy subgroups of the χ-twisted coadjoint action of $\widehat{\Gamma}$ on the whole of $(\mathbf{R}^n)^*$.

As examples we consider two cases with $G = \mathbf{SE}(n)$ and $K = \mathbf{R}^n$. For the first case $\widehat{G} = \mathbf{Z}_2 \times \mathbf{SE}(n) = \widehat{\Gamma} \ltimes K$ with $\widehat{\Gamma} = \mathbf{Z}_2 \times \mathbf{SO}(n)$. When $n = 2$ or 3 this is a symmetry group for many mechanical systems. Reduction by $\mathbf{R}^n$ typically corresponds to choosing a coordinate frame which moves with the centre of mass of the body. Since $\mathbf{Z}_2$ acts trivially on K and $\mathfrak{k}$, the $\widehat{\Gamma}$ equivariance of θ implies that θ must be identically zero. The groups $\widehat{\Gamma}_\mu$ are therefore the isotropy subgroups for the χ-twisted coadjoint action of $\mathbf{Z}_2 \times \mathbf{SO}(n)$ on $(\mathbf{R}^n)^*$. Since the $\mathbf{SO}(n)$ action is isomorphic to its natural action on $\mathbf{R}^n$ and $\mathbf{Z}_2$ acts by $-I$ we have

$$\widehat{\Gamma}_\mu \;=\; \begin{cases} \mathbf{Z}_2 \times \mathbf{SO}(n) & \text{if } \mu = 0 \\ \widetilde{\mathbf{O}}(n-1) & \text{if } \mu \neq 0 \end{cases}$$

where $\widetilde{\mathbf{O}}(n-1)$ is generated by the $\mathbf{SO}(n-1)$ subgroup of $\mathbf{SO}(n)$ that fixes μ, together with the antisymplectic generator of $\mathbf{Z}_2$ composed with an element of $\mathbf{SO}(n)$ that maps μ to $-\mu$. Thus all the reduced phase spaces $\mathcal{P}_\mu$ inherit antisymplectic symmetries. Note that when n is even the antisymplectic symmetries are generated by one that commutes with the symplectic symmetries. However this is not true when n is odd.

For the second case we consider $\widehat{G} = \mathbf{E}(n) = \widehat{\Gamma} \ltimes K$ with $\widehat{\Gamma} = \mathbf{O}(n)$. The action of $\mathbf{O}(n)$ on $(\mathbf{R}^n)^*$ is obtained by taking the natural action and then composing the orientation reversing operators by $-I$. This action is irreducible and so $\theta : \mathbf{R}^n \to (\mathbf{R}^n)^*$ is either identically zero or surjective. For $n > 2$ there are no $\mathbf{SO}(n)$ invariant skew-symmetric forms on $\mathbf{R}^n$ and so only the case $\theta \equiv 0$ can occur. For $\theta \equiv 0$ we have

$$\widehat{\Gamma}_\mu \;\cong\; \begin{cases} \mathbf{O}(n) & \text{if } \mu = 0 \\ \mathbf{Z}_2 \times \mathbf{SO}(n-1) & \text{if } \mu \neq 0 \end{cases}$$

where $\mathbf{SO}(n-1)$ is the subgroup of $\mathbf{SO}(n)$ that fixes μ and $\mathbf{Z}_2$ is generated by reflection in the hyperplane perpendicular to μ. Again all the reduced phase spaces $\mathcal{P}_\mu$ inherit antisymplectic symmetries. However in this case the antisymplectic symmetries are always generated by one that commutes with the symplectic symmetries.

For $\widehat{G} = \mathbf{E}(2) = \widehat{\Gamma} \ltimes K$ with $\widehat{\Gamma} = \mathbf{O}(2)$ and $K = \mathbf{R}^2$ a nontrivial θ is possible. Indeed this occurs for the natural semisymplectic action of $\mathbf{E}(2)$ on $\mathbf{R}^2$ and in applications to systems of point vortices on the plane. When θ is nontrivial the $\widehat{\Gamma}$ equivariance implies that it is necessarily surjective and so in this case $\widehat{\Gamma}_\mu$ is always equal to $\widehat{\Gamma} = \mathbf{O}(2)$.

Reduction to centre of mass coordinates In the previous section we noted that reducing a system on $\mathcal{P}$ with $\widehat{G} = \mathbf{Z}_2 \times \mathbf{SE}(n)$ symmetry by $K = \mathbf{R}^n$ results in a system on $\mathcal{P}_\mu$ with symmetry group $\widehat{\Gamma}_\mu = \mathbf{Z}_2 \times \mathbf{SO}(n)$ if $\mu = 0$ and $\widehat{\Gamma}_\mu = \widetilde{\mathbf{O}}(n-1)$ if $\mu \neq 0$. However it is a common observation that in many systems with Euclidean invariance reduction to centre of mass coordinates results in systems which are isomorphic to each other for all μ, including $\mu = 0$. Here we show that this follows from the ideas discussed above when the system is invariant under an action of the Galilean group. This approach is different to the one given by Souriau [4]

We take $\widehat{G} = \widehat{\Gamma} \ltimes K$ where $\widehat{\Gamma} = (\mathbf{Z}_2 \times \mathbf{O}(n)) \ltimes \mathbf{R}^n$ and $K = \mathbf{R}^n \times \mathbf{R}$. A general element of $\widehat{G}$ is represented by $g = (\tau, A, \mathbf{v}, \mathbf{b}, s)$, where $\tau \in \mathbf{Z}_2$, $A \in \mathbf{O}(n)$, $\mathbf{v}, \mathbf{b} \in \mathbf{R}^n$ and $s \in \mathbf{R}$, or by

$$g = \begin{pmatrix} A & \mathbf{v} & \mathbf{b} \\ 0 & \tau & s \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(n+2)$$

Multiplication in the group is induced from that in $\mathbf{GL}(n+2)$.

The character χ is given by the projection from $\widehat{G}$ to $\mathbf{Z}_2$ which we identify with $\{\pm 1\}$. For simplicity we assume that $\mathcal{P} = T^*\mathbf{R}^{nN}$ and define the action of $\widehat{G}$ on $\mathcal{P} \times \mathbf{R}$, the product of phase space with time, by:

$$(\tau, A, \mathbf{v}, \mathbf{b}, s).(q, p, t) \;=\; (A.q + \mathbf{v}t + \mathbf{b}, \tau(A.p + m.\mathbf{v}), \tau t + s) \tag{2}$$

where $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$ with $q_i, p_i \in \mathbf{R}^n$, and $A.q = (Aq_1, \ldots, Aq_N)$, $A.p = (Ap_1, \ldots, Ap_N)$ and $m.\mathbf{v} = (m_1\mathbf{v}, \ldots, m_N\mathbf{v})$ for some $m \in \mathbf{R}^N$. Thus $\mathcal{P}$ is the phase space and $\widehat{G}$ the symmetry group for a Galilean invariant system of N particles in $\mathbf{R}^n$ with masses $m_1, \ldots, m_N$. Let M be the total mass $M = \sum_{i=1}^N m_i$.

The action of $\widehat{G}$ is defined only on $\mathcal{P} \times \mathbf{R}$ and not directly on $\mathcal{P}$. However by trivially extending the canonical symplectic form on $\mathcal{P}$ to a degenerate form on $\mathcal{P} \times \mathbf{R}$ we make this into a *pre-symplectic* group action and can proceed as above. In particular a momentum map for the action of K on $\mathcal{P} \times \mathbf{R}$ is given by $\Phi(q,p,t) = (\sum_{i=1}^N p_i, 0) \in \mathbf{R}^n \oplus \mathbf{R}$. It is easy to see that the K-reduced phase space $(\mathcal{P} \times \mathbf{R})_{(\mu,0)}$ is isomorphic as a symplectic space to $\mathcal{P}_\mu$.

The momentum map Φ is equivariant with respect to the θ-affine, χ-twisted coadjoint action of $\widehat{G}$ on $\mathfrak{k}^* = (\mathbf{R}^n \oplus \mathbf{R})^*$ with $\theta(\tau, A, \mathbf{v}, \mathbf{b}, s) = \tau M \mathbf{v}$. This action is transitive on the image of Φ and so all the reduced phase spaces $(\mathcal{P} \times \mathbf{R})_{(\mu,0)} \cong \mathcal{P}_\mu$ are symplectomorphic to each other. Moreover the isotropy subgroups $\widehat{\Gamma}_\mu$ are all conjugate to $\widehat{\Gamma}_0 \cong \mathbf{Z}_2 \times \mathbf{O}(n)$ and their actions on $\mathcal{P}_\mu$ are symplectomorphic to the action of $\mathbf{Z}_2 \times \mathbf{O}(n)$ on $\mathcal{P}_0$. Finally we note that this implies that the reduced flows induced by a $\widehat{G}$-invariant Hamiltonian on the space $\mathcal{P}_\mu$ are all symplectomorphic to each other. Thus the dynamics are the same in any coordinate system moving with a constant velocity.

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