RELATIVE PERIODIC ORBITS OF SYMMETRIC LAGRANGIAN SYSTEMS

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We announce two topological results that may be used to estimate the number of relative periodic orbits of different homotopy classes that are possessed by a symmetric Lagrangian system. The results are illustrated by applications to systems on tori and to strong force N-centre problems.

1. Introduction

Periodic orbits of Lagrangian system have been extensively studied by applying variational methods to action functionals defined on loop spaces [10,2,12,1,13,14]. Symmetries of the Lagrangian are often used to prove the existence of periodic orbits with particular properties. Typically the strategy is to minimize an action functional defined on a space of paths γ in the configuration space $\mathcal{M}$ which satisfy the property

$$\gamma(t+T) = g.\gamma(t) \tag{1}$$

for a fixed 'relative period' T and some fixed diffeomorphism g of $\mathcal{M}$, often referred to as a 'phase', that leaves the Lagrangian invariant. If g has order

k then the corresponding trajectory is periodic with period kT. Recent very striking applications of this idea have been the proofs of the existence of 'choreographies' of N-body problems[7,8,9].

The method is still valid if g does not have finite order, in which case the trajectories are only 'periodic modulo the action of g'. Whether or not g has finite order we refer to paths in $\mathcal{M}$ satisfying (1) as 'relative loops' and to the corresponding solutions of Lagrangian system as 'relative periodic orbits'. The variational theory for action functionals on loop spaces extends in a straightfoward way to relative loop spaces. Our aim in this paper is to outline some results on the topology of relative loop spaces that give lower bounds for the number of critical points possessed by 'well-behaved' action functionals.

Let L be a Lagrangian on a configuration manifold $\mathcal{M}$ which is invariant under a diffeomorphism $g : \mathcal{M} \to \mathcal{M}$. Our aim is to provide lower bounds for the number of minimizers and/or other critical points of the action functional

$$\mathcal{A}[\gamma] = \int_0^T dt L(\gamma, \dot{\gamma}, t) \tag{2}$$

defined on $\Lambda^1_g(\mathcal{M})$, a Sobolev space of paths in $\mathcal{M}$ satisfying (1) defined in §2.

Remark 1.1. We may allow L to depend periodically on time, with the same period T as in (1). If it is independent of time then the action functional is invariant under the time-translation action on relative loop space given by:

$$\gamma(t) \to \gamma(t + \tau) \text{ with } \tau \in \mathbb{R}$$

If g has finite order then this factors through an action of the circle S^1. Time dependence of L breaks this symmetry.

A 'well-behaved' action functional will have at least one local minimum in each connected component of $\Lambda^1_g(\mathcal{M})$. If in addition all the critical points are non-degenerate then the Morse inequalities provide lower bounds for the total number of critical points in *each* connected component, provided we have information on the homology groups of the component[6]. Weaker estimates for functionals with critical points that are not necessarily non-degenerate can be obtained from (Lusternik-Schnirelman) category theory[6]. In this paper we announce two main results:

- The number of connected components of the space of relative loops is equal to the number of orbits of a 'g-twisted action' of the fundamental group $\pi_1(\mathcal{M})$ on itself (Theorem 2.1).
- If $\mathcal{M}$ is $K(\pi, 1)$ then the connected components of the relative loop spaces are also $K(\pi, 1)$'s, with fundamental groups isomorphic to the isotropy subgroups of the g-twisted action of $\pi_1(\mathcal{M})$ on itself (Theorem 2.2).

The second result enables us to compute the category or homology groups of components of relative loop spaces in particular examples. Proofs of these theorems and other results in this paper will appear elsewhere. In §3 we present some examples to illustrate how these results may be used to estimate the numbers of relative periodic orbits of Lagrangian systems.

2. The topology of relative loop spaces

The space of continuous relative loops:

$$\Lambda_g(\mathcal{M}) \doteq \{\gamma \in C^0(\mathbb{R}, \mathcal{M}) : \gamma(t+T) = g.\gamma(t)\}$$

is a (not necessarily complete) metric space with distance:

$$d_0(\gamma_1, \gamma_2) = \sup_{t \in [0,T]} dist_{\mathcal{M}}(\gamma_1(t), \gamma_2(t))$$

where $dist_{\mathcal{M}}(.,.)$ is a distance in $\mathcal{M}$. To employ variational methods we introduce $\Lambda^1_g(\mathcal{M})$, the space of curves γ such that in any local chart $\phi : U \to \mathbb{R}^d$ of $\mathcal{M}$ we have $\phi \circ \gamma \in H^1(\gamma^{-1}(U), \mathbb{R}^d)$. See [11] for a similar construction. The space $\Lambda^1_g(\mathcal{M})$ is a metric space with metric

$$d_1(\gamma_1, \gamma_2) = d_0(\gamma_1, \gamma_2) + \sqrt{|E(\gamma_1) - E(\gamma_2)|}, \quad E(\gamma) = \frac{1}{2}\int_0^T dt \|\dot{\gamma}\|^2_{\mathcal{M}} \quad (3)$$

and we have:

Proposition 2.1. *The natural embedding of $\Lambda^1_g(\mathcal{M})$ in $\Lambda^0_g(\mathcal{M})$ is a homotopy equivalence.*

It is also not difficult to show that $\Lambda_g(\mathcal{M})$ is homeomorphic to

$$\Lambda^g(\mathcal{M}) \doteq \{\gamma \in C^0([0,T], \mathcal{M}) : \gamma(T) = g.\gamma(0)\}.$$

To describe the algebraic topology of $\Lambda^1_g(\mathcal{M})$ it is therefore sufficient to describe that of $\Lambda^g(\mathcal{M})$. This is the topic of the following subsections.

2.1. *Main theorems*

In what follows for notational simplicity we denote a path and its homotopy class by the same symbol and use $*$ to denote both concatenation of paths and the induced operations on homotopy classes.

Assume that $\mathcal{M}$ is connected. Choose a base point $m \in \mathcal{M}$ and let

$$\Lambda_m^g(\mathcal{M}) \doteq \{\gamma \in \Lambda^g(\mathcal{M}) : \gamma(0) = m\},$$

the space of continuous paths from m to gm. Let $\Lambda_m(\mathcal{M}) \doteq \Lambda_m^{\mathrm{id}}(\mathcal{M})$ denote the space of continuous loops based at m. Note that the space of connected components of $\Lambda_m(\mathcal{M})$ is the fundamental group of $\mathcal{M}$: $\pi_0(\Lambda_m(\mathcal{M})) = \pi_1(\mathcal{M}, m)$.

Fix a particular path $\omega \in \Lambda_m^g(\mathcal{M})$. The map $\Phi_\omega(\gamma) = \omega^{-1} * \gamma$ is a bijection

$$\Phi_\omega : \pi_0(\Lambda_m^g(\mathcal{M})) \to \pi_0(\Lambda_m(\mathcal{M})) = \pi_1(\mathcal{M}, m)$$

where ω^{-1} is the path obtained by traversing ω 'backwards'. This bijection depends (only) on the homotopy class of ω in $\Lambda_m^g(\mathcal{M})$.

For any $\alpha \in \Lambda_m(\mathcal{M})$ let $g.\alpha$ be the loop in $\Lambda_{gm}(\mathcal{M})$ obtained by applying the diffeomorphism g to α and define an automorphism of $\pi_1(\mathcal{M}, m)$ by:

$$\alpha \mapsto \alpha_g = \omega^{-1} * g.\alpha * \omega.$$

Again this depends (only) on the homotopy class of ω in $\Lambda_m^g(\mathcal{M})$. Now define the *g-twisted action* of $\pi_1(\mathcal{M}, m)$ on itself by

$$\alpha \cdot \beta = \alpha_g * \beta * \alpha^{-1} \qquad \alpha, \beta \in \pi_1(\mathcal{M}, m). \tag{4}$$

The number of connected components of relative loop space is given by the following result.

Theorem 2.1. *The map Φ_ω induces a bijection*

$$\pi_0(\Lambda^g(\mathcal{M})) \cong \overline{\pi_1(\mathcal{M}, m)}^g,$$

where $\overline{\pi_1(\mathcal{M}, m)}^g$ is the set of orbits of the g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself.

Remark 2.1. If g is homotopic to the identity then $\Lambda^g(\mathcal{M})$ is homotopy equivalent to the loop space $\Lambda(\mathcal{M}) \doteq \Lambda^{\mathrm{id}}(\mathcal{M})$ and the g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself is just conjugation. We therefore recover the well known result that the connected components of the loop space map bijectively to the conjugacy classes of the fundamental group.

Remark 2.2. The g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself induces an affine action of $H_1(\mathcal{M})$ on itself:

$$< \alpha > \cdot < \beta > = \ g. < \alpha > - < \alpha > + < \beta >$$

where $< . >$ denote the homology class and $g. < \alpha >$ denotes the natural action of g on $H_1(\mathcal{M})$. When $\pi_1(\mathcal{M}, m)$ is abelian this is the same as the action of $\pi_1(\mathcal{M}, m)$ on itself. More generally it is easier to calculate than the $\pi_1(\mathcal{M}, m)$ action and in typical applications may be used to describe relative periodic orbits in terms of winding numbers.

We now describe the topology of the connected components of a relative loop space in the special case that $\mathcal{M}$ is a $K(\pi, 1)$. This means that all its homotopy groups except the fundamental group are trivial. Examples of $K(\pi, 1)$'s include tori, the plane $\mathbb{R}^2$ with N points removed, and the configuration spaces of planar N-body problems.

Theorem 2.2. *Assume $\mathcal{M}$ is a $K(\pi, 1)$. Then for any $\gamma \in \Lambda^g_m(\mathcal{M})$ the connected component of $\Lambda^g(\mathcal{M})$ containing γ, denoted $\Lambda^g_\gamma(\mathcal{M})$, is also a $K(\pi, 1)$ with*

$$\pi_1(\Lambda^g_\gamma(\mathcal{M})) \ \cong \ Z^g_{\pi_1(\mathcal{M})}(\Phi_\omega(\gamma))$$

where

$$Z^g_{\pi_1(\mathcal{M})}(\Phi_\omega(\gamma)) \doteq \{\alpha \in \pi_1(\mathcal{M}) : \quad \alpha_g * \Phi_\omega(\gamma) * \alpha^{-1} = \Phi_\omega(\gamma)\}$$

i.e. the isotropy subgroup (or centralizer) at $\Phi_\omega(\gamma)$ of the g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself.

We note that all $K(\pi, 1)$'s with isomorphic fundamental groups are homotopy equivalent to each other[4,16], and so this result determines the homotopy types of connected components of relative loop spaces. The homology groups can be computed algebraically as the homology groups of the fundamental group[5].

2.2. *A simple example*

Let $\mathcal{M} = T^1$, the circle, and consider first the loop space $\Lambda(T^1)$. The 'g-twisted action' of $\pi_1(T^1)$ on itself is just conjugation, and since $\pi_1(T^1) \cong \mathbb{Z}$ is abelian this is trivial. So $\pi_0(\Lambda(M)) \cong \mathbb{Z}$, the homotopy classes of loops being specified precisely by their winding numbers. Since T^1 is a $K(\pi, 1)$, Theorem 2.2 says that each component of loop space is also a $K(\pi, 1)$ with fundamental group isomorphic to $\mathbb{Z}$, and therefore has the homotopy type of a circle.

Now consider $\Lambda^g(T^1)$ where $g : T^1 \to T^1$ is a reflection. Choose one of the two fixed points of the reflection to be the base point m. We may choose ω to be the trivial path from m to m. Then for each $\alpha \in \pi_1(T^1, m) \cong \mathbb{Z}$ we have $\alpha_g = -\alpha$ and so the 'g-twisted action' (4) is the translation

$$\alpha.\beta \ = \ \beta - 2\alpha. \tag{5}$$

This has two orbits, $\overline{\pi_1(T^1)}^g \cong \mathbb{Z}_2$, and the isotropy subgroups are trivial. It follows from Theorems 2.1 and 2.2 that the space of relative loops $\Lambda^g(T^1)$ has two components, each of which is contractible. We strongly recommend that the reader convinces him/herself that this is true by drawing some pictures!

We note that exactly the same calculations hold for $\mathcal{M} = \mathbb{C}\backslash\{0\}$, since this is homotopy equivalent to T^1. Generalisations of these calculations to both N-tori and $\mathbb{C}\backslash\{N \text{ points}\}$ will be given in the next section, along with applications to systems of coupled rotors and N-centre problems.

3. Applications

We first recall two general results on the existence of critical points.

Existence of minima We will say that a continuous function $\mathcal{A}$ on a metric space X is coercive if every sequence γ_n in X either has a convergent subsequence or a subsequence on which $\mathcal{A}[\gamma_n] \to +\infty$. Note that if $\mathcal{A}$ is coercive then the sublevel sets $X^c = \{x \in X : \ \mathcal{A}[x] \leq c\}$ are all sequentially compact and therefore complete. A coercive function that is bounded below necessarily attains its minimum in each connected component of X. If A is a smooth function on a Hilbert manifold (without boundary) then these minima are critical points of $\mathcal{A}$ [15].

Lower bounds from category theory A smooth function A on a Hilbert manifold X is said to satisfy the Palais-Smale condition if every sequence γ_n in X with $\mathcal{A}[\gamma_n]$ bounded and $D\mathcal{A}[\gamma_n] \to 0$ has a convergent subsequence. If $\mathcal{A}$ is bounded from below and satisfies the Palais-Smale condition, and all the sublevel sets X^c are complete, then the number of critical points of $\mathcal{A}$ is greater than or equal to the (Lusternik-Schnirelman) category of X [12]. Category is a homotopy invariant which is equal to 1 if the space is contractible, to 2 for an N-sphere and to $N+1$ for real projective N-space and for the N-torus T^N.

3.1. *Geodesic flows on T^N and coupled rotors*

Let $\mathcal{M} = T^N = \mathbb{R}^N/\Omega$ where $\Omega \subset \mathbb{R}^N$ is a lattice. Consider a Lagrangian system with kinetic energy given by a Riemannian metric on T^N. This metric lifts to an Ω-invariant metric on $\mathbb{R}^N$. Let $E(\mathbb{R}^N)$ and $E(T^N)$ denote the groups of isometries of $\mathbb{R}^N$ and T^N with respect to these metrics. The elements of $E(T^N)$ lift to elements of $E(\mathbb{R}^N)$ that commute with Ω and we have

$$E(T^N) \;\cong\; N_{E(\mathbb{R}^N)}(\Omega)/\Omega$$

where $N_{E(\mathbb{R}^N)}(\Omega)$ is the normalizer of Ω in $E(\mathbb{R}^N)$. Note that the action of g on T^N induces a linear map on $H_1(T^N) \cong \mathbb{Z}^N$ which we will denote by $\overline{g}$.

Proposition 3.1.

(1) If rank$(\overline{g} - id) = N$ in $H_1(T^N)$ then $\Lambda^1_g(T^N)$ has $|\det(\overline{g} - id)| < \infty$ connected components, each of which is contractible.

(2) If rank$(\overline{g} - id) = N - l$ with $0 < l \leq N$ then $\Lambda^1_g(T^N)$ has an infinite number of connected components, each of which has the homotopy type of T^l, and so has category $l + 1$.

Note that this generalises the calculations in §2.2.

Proof. The g-twisted action of $\pi_1(T^N) \cong H_1(T^N) \cong \mathbb{Z}^N$ on itself is given by

$$\alpha \cdot \gamma \;=\; \gamma + (\overline{g} - \mathrm{id}).\alpha.$$

It follows that $Z_{\pi_1(\mathcal{M})}(\gamma) = \ker(\overline{g} - \mathrm{id})$ and so is isomorphic to $\mathbb{Z}^l$ where l is the corank of $\overline{g} - \mathrm{id}$. Theorem 2.2 implies that the connected components of $\Lambda^1_g(T^N)$ are either contractible ($l = 0$) or have the homotopy type of T^l.

If rank$(\overline{g} - \mathrm{id}) < N$ then the orbits of the g-twisted action lie in proper affine subspaces of $\mathbb{Z}^N$, so there must be an infinite number of them. If rank$(\overline{g} - \mathrm{id}) = N$ then the number of orbits is equal to the area of a fundamental domain of the lattice $(g - \mathrm{id})\Omega$ (regarding g as an element of $E(\mathbb{R}^N)$) divided by the area of a fundamental domain of the lattice Ω. This is equal to $|\det(\overline{g} - \mathrm{id})|$. □

On T^N we consider Lagrangians of the form

$$L(x, \dot{x}, t) \;=\; \frac{1}{2}|\dot{x}|^2 - V(x, t)$$

where $x \in T^N$ and V is T-periodic in t. For $V \equiv 0$ this gives a geodesic flow on T^N, while more general systems may be interpreted as systems of coupled rotors or pendula. If V is invariant under $g \in E(T^N)$ then so is L and relative periodic orbits satisfying (1) are critical points of the action

$$\mathcal{A}[x] \;=\; \int_0^T dtL(x, \dot{x}, t)$$

on the relative loop space $\Lambda_g^1(T^N)$. Since T^N is compact we have:

(i) $\Lambda_g^1(T^N)$ is complete,
(ii) $\mathcal{A}$ is bounded from below,
(iii) The Palais-Smale condition holds for $\mathcal{A}$ in $\Lambda_g^1(\mathcal{M})$.

Thus the number of critical points of $\mathcal{A}$ in each connected component of $\Lambda_g^1(T^N)$ will be bounded below by the category of that component. So we can conclude:

- If $\text{rank}(\overline{g} - \text{id}) = N$ then there are $|\det(\overline{g} - \text{id})|$ relative periodic orbits, lying in different homotopy classes in $\Lambda_g^1(\mathcal{M})$, which minimise the action functional.
- If $\text{rank}(\overline{g} - \text{id}) = N - l < N$ then there is an infinite number of minimising relative periodic orbits of different homotopy types. Moreover each homotopy class contains at least $l+1$ relative periodic orbits.

By Remark 1.1 in the autonomous case, when V is independent of t, the relative periodic orbits form S^1 orbits (provided g has finite order), and so there will be an infinite number in each homotopy class. In this case it would interesting to be able to estimate the number of S^1 orbits of critical points in each component using an equivariant category theory.

3.2. *Strong force N-centre problems*

For a second application, we consider the motion of a particle in the plane, identified with $\mathbb{C}$, which is attracted to each of N fixed points at $p_1, \ldots, p_N$ by a 'strong force' potential field. The Lagrangian on the configuration space $\mathcal{M} = \mathbb{C}\backslash\{p_1, \ldots, p_N\}$ is

$$L(z, \dot{z}, t) \;=\; \frac{1}{2}|\dot{z}|^2 - \sum_{j=1}^{N} V_j(z, t)$$

where each potential V_j is negative, T-periodic in t and satisfies

$$\lim_{|z|\to\infty} V_j(z,t) = 0 \qquad \text{uniformly in } t.$$

Moreover we assume that there exist neighbourhoods U_j of p_j such that V_j is smooth outside U_j and satisfies

$$-\frac{A_j}{|z-p_j|^a} \;\leq\; V_j(z,t) \;\leq\; -\frac{B_l}{|z-p_l|^a} \qquad a \geq 2$$

for some constants $A_j, B_j > 0$ when $z \in U_j$. The conditions on the potential imply:

(i) The action functional $\mathcal{A}$ is bounded from below,
(ii) $\mathcal{A} \to \infty$ on sequences of paths that 'converge' to paths containing collisions $z = p_j$ [1],
(iii) $\mathcal{A} \to \infty$ on sequences of homotopically non-trivial closed loops containing points for which the distances from the centres go to infinity.

Items (ii) and (iii) imply that $\mathcal{A}$ is coercive (in the sense defined at the beginning of this section) on the non-trivial homotopy classes in $\Lambda^1(\mathcal{M})$. It is clearly not coercive on the trivial homotopy class. Coercivity on classes of relative loops will be discussed below. The action functional also satisfies the Palais-Smale condition [1].

3.2.1. *Periodic orbits*

We first describe the implications of the results of §2 for periodic orbits of this system. The configuration space $\mathcal{M}$ is a $K(\pi, 1)$ with fundamental group $\pi_1(\mathcal{M}) = F_N$, the free group on N generators. The connected components of $\Lambda^1(\mathcal{M})$ correspond bijectively to the conjugacy classes of F_N, which are easily described. The remarks above on coercivity and Theorem 2.1 imply that each non-null homotopy class contains at least one periodic orbit of the system. Note that $H_1(\mathcal{M}) = \mathbb{Z}^N$ and the homology class of a loop is determined precisely by the N winding numbers of the loop with respect to the N points p_j. However these winding numbers are crude invariants: each homology class contains an infinite number of homotopy classes.

The centralizer of a non-trivial element $\gamma \in F_N$, ie the set of elements that commute with it, consists only of 'roots' and powers of γ, and so is isomorphic to $\mathbb{Z}$. Theorem 2.2 therefore implies that each non-null homotopy class of $\Lambda^1(\mathcal{M})$ has the homotopy type of a circle and so its category equals 2. It follows that each of these classes must contain at least two periodic orbits.

3.2.2. *Relative periodic orbits*

Let g be a rotation or reflection of $\mathbb{C}$ that permutes the points p_j and assume that if two points lie on the same orbit of g then the corresponding potentials are identical. Then the Lagrangian L is g invariant and we can seek relative periodic orbits with relative period T and phase g.

Rather than attempting to describe the calculations in general, we consider the particular case of $N = 2$, $p_1 = 1$, $p_2 = -1$ and $g.z = \overline{z}$, ie two centres fixed by a reflection. We may choose $0 \in \mathbb{C}$ as the base point m. Then $\pi_1(\mathcal{M}, m)$ is the group freely generated by α_1 and α_2, where α_j is a loop that starts at m and winds once round p_j in a clockwise direction. The automorphism $\alpha \mapsto \alpha_g$ of $\pi_1(\mathcal{M}, m)$ is defined by its action on these generators, namely $\alpha_j \mapsto \alpha_j^{-1}$, and the g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself is defined by the action of these generators:

$$\alpha_j \cdot \beta \; = \; \alpha_j^{-1} \beta \alpha_j^{-1} \qquad \text{for all } \beta \in \pi_1(\mathcal{M}, m).$$

Note that α_j^k is equivalent under this action to either 1 (if k is even) or α_j (if k is odd). This is similar to the behaviour observed in the simple example in §2.2. Every other element lies in the orbit of an element of the form

$$\alpha_1^{r_1} \alpha_2^{s_1} \dots \alpha_1^{r_l} \alpha_2^{s_l} \tag{6}$$

where all the r_j and s_j are non-zero. There are an infinite number of such orbits, and so $\Lambda_g^1(\mathcal{M})$ has an infinite number of connected components.

Every relative loop γ defines a full loop $g.\gamma * \gamma$ by concantenating it with its image under the action of g on $\mathcal{M}$. The action functional will be coercive on the component of $\Lambda_g^1(\mathcal{M})$ containing γ if the action functional on the full loop space $\Lambda(\mathcal{M})$ is coercive on the component containing $g.\gamma * \gamma$. If γ is represented by (6) in $\pi_1(\mathcal{M}, m)$ then $g.\gamma * \gamma$ is represented by

$$\alpha_1^{-r_1} \alpha_2^{-s_1} \dots \alpha_1^{-r_l} \alpha_2^{-s_l} \alpha_1^{r_1} \alpha_2^{s_1} \dots \alpha_1^{r_l} \alpha_2^{s_l}$$

which is non-null if and only if γ does not lie in the orbit of one of the classes 1, α_1 or α_2. Hence every component of $\Lambda_g^1(\mathcal{M})$ except those corresponding to the orbits of 1, α_1 and α_2 will contain a relative periodic orbit of the symmetric strong force two centre problem. It is easily seen that the action functional is not coercive on the 1, α_1 and α_2 components.

The g-twisted action of $\pi_1(\mathcal{M}, m)$ on itself is free, and so all the isotropy subgroups are trivial and, by Theorem 2.2, every connected component of $\Lambda^1_g(\mathcal{M})$ is contractible. So a non-autonomous system need have only one relative periodic orbit in each component on which the action functional is coercive. However the following argument *suggests* that an autonomous system will have an infinite number of relative periodic orbits in each such component.

Assume that the action functional of the autonomous system is an S^1-invariant Morse function on $\Lambda^1_g(\mathcal{M})$. The critical points form S^1 orbits. Choose a component of $\Lambda^1_g(\mathcal{M})$. Let the number of critical orbits of the action functional in the chosen component with index i be n_i. Since the Poincaré polynomial of S^1 is $1 + t$ and that of the component is 1, the Morse inequalities for a function with non-degenerate critical manifolds are equivalent to the equation[3]:

$$\sum_{i=0}^{\infty} n_i t^i (1+t) - 1 \; = \; (1+t) \sum_{i=0}^{\infty} q_i t^i$$

for some $q_i \geq 0$. It follows that all the n_i for i even must be strictly positive. An alternative approach to proving this estimate for the number of critical points in each non-trivial component would be to use an S^1 equivariant category theory.

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References

1. A. Ambrosetti and V. Coti Zelati, *Periodic solutions for Lagrangian systems with singular potentials*, Birkhauser, 1994.
2. V. Benci, *Periodic solutions for Lagrangian systems on a compact manifold*, J.Diff.Eq. **63** (1986) 135-161.
3. R. Bott, *Lectures on Morse theory, old and new*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 331-358.
4. R. Bott and W. Tu, *Differential forms in algebraic topology*, Springer Verlag, 1995.
5. K.S. Brown, *Cohomology of groups*, Springer Verlag, 1982.
6. K. Chang, *Infinite dimensional Morse theory and multiple solution problems*, Birkhauser, 1991.

7. A. Chenciner and R. Montgomery, *A remarkable solution of the 3-body problem in the case of equal masses*, Ann. Math. **152** (2000) 881-901.
8. A. Chenciner, J. Gerver , R. Montgomery and C. Simó *Simple choreographic motions of N-bodies: A preliminary study*, pp. 287-308 in *Geometry, mechanics and dynamics* (eds: P. Newton, P.Holmes and A. Weinstein), Springer Verlag, 2002.
9. D.F. Ferrario and S. Terracini, *On the existence of collisionless equivariant minimizers for the classical N-body problem*, to appear in Invent. Math. 2003.
10. W. Gordon, *Conservative dynamical systems involving strong force*, Trans. Amer. Math. Soc. **204** (1975) 113-135.
11. J. Jost, *Riemannian geometry and geometric analysis*, Springer Verlag, 1995.
12. J. Mawhin and M. Willhelm, *Critical point theory and Hamiltonian systems*, Springer Verlag, 1990.
13. R. Montgomery, *The N-body problem, the braid group, and action-minimizing periodic solution*, Nonlinearity **11** (1998) 363-376.
14. L. Sbano, *The topology of the three-body problem with zero total angular momentum and the existence of periodic orbits*, Nonlinearity **11** (1998) 641-658.
15. M. Struwe, *Variational methods*, Springer Verlag, 1990.
16. G.W. Whitehead, *Elements of homotopy theory* (3rd edn), Springer Verlag, 1995.