Caustics in time reversible Hamiltonian systems

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Abstract

We consider the projection to configuration space of invariant tori in a time reversible Hamiltonian system at a point of zero momentum. At such points the projection has rank zero and the resulting caustic has a corner. We use caustic equivalence of Lagrangian mappings to find a normal form for such a corner in 3 degrees of freedom.

1 Introduction

Invariant tori arise in many Hamiltonian systems. For example, the closure of any bounded trajectory in a completely integrable system is a torus, and if a completely integrable system is slightly perturbed then by KAM theory many of these invariant tori persist, [2]. Invariant tori also abound near stable equilibrium points and stable periodic orbits. There is considerable interest in the geometry of the projections of invariant tori into configuration space, particularly if the system is a classical Hamiltonian system (i.e., of the form 'kinetic + potential'), see for example, [1], [7] and [10].

This paper carries further some work of J.B. Delos [7] in which he explains why projections of invariant tori in classical Hamiltonian systems with 2 degrees of freedom can have corners. At first sight, from a singularity theoretic point of view, one would expect the projection of an invariant 2-torus on to the configuration plane to have only fold and cusp singularities. However, Delos shows that in classical Hamiltonian systems if the torus contains a point $(q,p) = (q,0)$ (where q and p are conjugate position and momentum) then near such a point the projection $(q,p) \mapsto q$ of the torus has a 'folded handkerchief' singularity, that is, in suitable local coordinates (x,y) about this point, the projection takes the form $(x,y) \mapsto (x^2,y^2)$. The singularities of the projection of tori at points where $p \neq 0$ are of the expected type: folds or cusps. See Figure 1 for two typical trajectories in 2 degrees of freedom, one with these corners and one without. Further pictures of such projections can be found in the references cited above.

It turns out that the crucial property of a Hamiltonian system which causes corners to appear generically is time-reversibility; it is not strictly necessary for the system to be in the classical 'kinetic + potential' form, though because kinetic energy is even in momentum any classical system is time reversible. (In fact, I do not know of an interesting Hamiltonian system which fulfills the requirements of this paper but is not of the classical form.)

In this paper I consider invariant tori, or possibly more generally, invariant Lagrangian submanifolds, in time reversible systems, but now in 3 degrees of freedom. The main result is that the analogue of corners in 3 degrees of freedom is not just the corner of a cube (as Delos conjectured) but is more subtle, see Figure 2. The technique required in 3 degrees of freedom is also more subtle. In 2 degrees of freedom, the results use the standard theory about Lagrangian singularities via $\mathcal{R}^+$-equivalence of generating families, adapted to allow for the time-reversal symmetry; however in 3 degrees of freedom, the generating families have infinite $\mathcal{R}^+$-codimension, so it is necessary to weaken the equivalence relation to 'caustic equivalence', which involves J. Damon's theory of $\mathcal{K}_V$-equivalence.

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2 Invariant tori in reversible systems

We consider time reversible systems on a phase space T^*Q which is the cotangent bundle of an n-dimensional configuration space Q. We denote the natural projection by $\pi : T^*Q \to Q$. The time-reversal involution $\tau : T^*Q \to T^*Q$ is given by $\tau(q,p) = (q,-p)$, where $q \in Q$ and $p \in T^*_qQ$. We assume that the Hamiltonian $H : T^*Q \to \mathbb{R}$ is invariant under τ for then the associated Hamiltonian vector field X is reversed: $\tau_* X(q,p) = -X(q,-p)$. This implies that if $t \mapsto \gamma(t)$ is an integral curve of X then so is $t \mapsto \tau \circ \gamma(-t)$. Note that for $p = 0$, we have $\tau_* X(q,0) = -X(q,0)$, so $X(q,0)$ is 'vertical' for all $q \in Q$. It follows from these observations that if $\gamma(0) = (q,0)$ then the closure of the set $\{\gamma(t) \mid t \in \mathbb{R}\}$ is invariant under the involution τ, in which case we say it is *time reversible*.

We are interested in the case that the integral curve γ is dense in a Lagrangian subanifold which we will denote by L (for example, γ is a quasiperiodic orbit whose closure is a torus of dimension n). If L meets the fixed point set of τ — the zero-section of T^*Q, which, abusing notation, we will denote by Q — then L is τ-invariant: it is a time reversible Lagrangian submanifold.

Proposition 1 *If $q \in L$ and $L \pitchfork_q Q$, then τ acts as $-I$ in a neighbourhood of q in L, and since $\pi(z) = \pi(\tau(z))$ we have that $\pi_{|L}(-z) = \pi_{|L}(z)$ for z in this neighbourhood. That is, $\pi_{|L}$ is an even map.* □

We now show that if L is a time reversible torus which is transverse to (and meets) Q then $L \cap Q$ consists of precisely 2^n points, where $n = \dim Q$.

Proposition 2 *Let L be a torus of dimension n, and $\tau : L \to L$ an involution with only isolated fixed points. If τ has one fixed point, then it has precisely 2^n fixed points.*

PROOF: This follows from the Lefschetz fixed point theorem in an appropriate form. However, since in this case the argument is particularly simple and there is not a good reference, I will outline a proof from first principles. Let N be the number of fixed points of τ. The involution τ generates the group $\mathbb{Z}_2$. Consider the quotient space $L/\mathbb{Z}_2$. This has N isolated singular points (where it is locally a cone on a projective space), and it can be triangulated so that each singular point is at a vertex (simplex of dimension 0). Now lift this triangulation up to L, and let C_k be the $\mathbb{Q}$-vector space generated by the simplices of dimension k. Note that τ induces an action on each C_k by permutation matrices (there are no -1 enties by construction). Thus, the trace of τ on C_k, which we denote by $\mathrm{Tr}(\tau;C_k)$, is equal to the number of simplexes of dimension k which are left fixed by τ. Since all N fixed points are isolated,

$$\mathrm{Tr}(\tau;C_k) = \begin{cases} 0 & \text{if } k > 0, \\ N & \text{if } k = 0. \end{cases}$$

Thus, $\sum_{k=o}^{n}(-1)^k \mathrm{Tr}(\tau;C_k) = N$. Now, since τ is a chain map on the chain complex,

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0,$$

an easy argument shows that $\sum_{k=o}^{n}(-1)^k \mathrm{Tr}(\tau;C_k) = \sum_{k=o}^{n}(-1)^k \mathrm{Tr}(\tau;H_k)$, where $H_k = H_k(C.) = H_k(L,\mathbb{Q})$. It remains therefore, to calculate $\mathrm{Tr}(\tau;H_k)$.

Denote the map on H_k induced by τ by τ_k. Since τ is an involution, so are the τ_k. Moreover, since L is a torus, $H_k(L,\mathbb{Q})$ is the k-th exterior power of $H_1(L,\mathbb{Q})$ and τ_k is the k-th exterior power of τ_1. Since τ_1 is an involution, there is a basis of $H_1(L,\mathbb{Q})$ with respect to which it is diagonal with ± 1's down the diagonal. Let $\tau_1 = I_r \oplus -I_s$. It is not hard to show that

$$\mathrm{Tr}(\tau_k) = \sum_{l=0}^{k} (-1)^{n-l} \binom{r}{k-l} \binom{s}{l}.$$

Multiplying this by t^k and summing over k gives,

$$\sum_{k=0}^{n} \mathrm{Tr}(\tau_k) t^k = (-1)^n (1+t)^r (1-t)^s.$$

To obtain the alternating sum put $t = -1$ and multiply by $(-1)^n$, so

$$L(\tau) = \begin{cases} 0 & \text{if } r > 0, \\ 2^n & \text{if } r = 0. \end{cases}$$

Thus $N = 0$ or 2^n for any involution with isolated fixed points. □

3 Lagrangian maps and generating families

Let $L \subset T^*Q$ be a Lagrangian submanifold. Since $\pi : T^*Q \to Q$ is a Lagrangian fibration its restriction $\pi_{|L}$ to L is by definition a Lagrangian map. (From now on we will denote $\pi_{|L}$ simply by π.) To study the local geometry of such maps we use generating families. Here we give a very brief outline of the theory of generating families as developed by V.I. Arnold and V.M. Zakalyukin. The details can be found in [3]. Recall that the *caustic* of a Lagrangian map is the set of its singular values.

Let $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ be a function germ, and let $F : (\mathbb{R}^n \times \mathbb{R}^a, 0) \to \mathbb{R}$ be a deformation of f. Denote $F(x,u)$ by $f_u(x)$, so $f_0 = f$. Let

$$C(F) = \{(x,u) \,|\, d(f_u)(x) = 0\}$$

be the set of critical points in the family F. We assume from now on that $C(F)$ is a submanifold (germ) of $\mathbb{R}^n \times \mathbb{R}^a$, in which case the projection $\pi_F : C(F) \to \mathbb{R}^a$ is a Lagrangian map (germ). The family F is said to be a *generating family* for π_F, and we will call f the *organizing centre* of F (and of π_F). The set of singular points of the map π_F are precisely the points (x,u) for which f_u

has a degenerate critical point at x, and thus the caustic of the Lagrangian map π_F is precisely the discriminant of the generating family F.

Given any Lagrangian map germ $\pi : (L,0) \to (Q,0)$ there is a family F as above with $\pi_F \sim \pi$ (where $\sim$ is Lagrangian equivalence). Of course, $a = \dim L = \dim Q$. Furthermore one can take $n = \dim\ker d\pi_0$ (necessarily, $n \geq \dim\ker d\pi_0$, and one can reduce to $n = \dim\ker d\pi_0$ by a splitting lemma argument). Two Lagrangian map germs are Lagrangian equivalent if and only if their generating families are $\mathcal{R}^+$-equivalent: there are diffeomorphism germs $\phi : (\mathbb{R}^a,0) \to (\mathbb{R}^a,0)$, and $\Phi : (\mathbb{R}^n \times \mathbb{R}^a,0) \to (\mathbb{R}^n \times \mathbb{R}^a,0)$, related by

$$\Phi(x,u) = (\psi(x,u),\phi(u)),$$

for some map germ ψ, and a function germ α on $(\mathbb{R}^a,0)$ such that

$$F \circ \Phi(x,u) = G(x,u) + \alpha(u).$$

Note that the map $x \mapsto \psi(x,0)$ is a diffeomorphism germ, so the organizing centres of $\mathcal{R}^+$-equivalent families are themselves $\mathcal{R}$-equivalent. Finally, a Lagrangian map germ is Lagrangian stable if and only if any associated generating family is an $\mathcal{R}^+$-versal deformation of its organizing centre.

In our application, the Lagrangian map germs in question are invariant under a $\mathbb{Z}_2$ action: $\pi(x) = \pi(-x)$. The only difference this makes to the discussion above is that the generating family $F(x,u)$ is odd in x, i.e. $F(-x,u) = -F(x,u)$. Furthermore, the Lagrangian equivalence respects the $\mathbb{Z}_2$ action if and only if the $\mathcal{R}^+$-equivalence between generating families does, i.e. $\psi(-x,u) = -\psi(x,u)$. We will call this $\mathcal{R}^+_{\mathbb{Z}_2}$-equivalence, even though the '+' is redundant as α must be 0.

Denote by $\mathcal{E}_n$ the ring of smooth function germs $(\mathbb{R}^n,0) \to \mathbb{R}$, by $\mathcal{E}_n^+$ the subring of those invariant under the action of $\mathbb{Z}_2$ (acting by $x \mapsto -x$), and denote by $\mathcal{E}_n^-$ the $\mathcal{E}_n^+$-module of odd function germs (i.e. $f(x) = -f(-x)$). Denote by m_n and m_n^+ the ideals in $\mathcal{E}_n$ and $\mathcal{E}_n^+$ respectively of germs vanishing at 0. For $f \in \mathcal{E}_n^-$ let $J_+(f)$ be the ideal in $\mathcal{E}_n^+$ generated by the partial derivatives of f. Define

$$J_-(f) = J_+(f).\mathcal{E}_n^-.$$

The $\mathcal{R}_{\mathbb{Z}_2}$-codimension is defined to be

$$\operatorname{cod}(f) = \dim_{\mathbb{R}}(\mathcal{E}_n^-/J_-(f)).$$

Applying the usual arguments of singularity theory, adapted to the world of odd functions, one obtains,

Proposition 3 (i) *Let $f \in \mathcal{E}_n^-$ and let k be an odd integer. If*

$$m_n^{k+2} \cap \mathcal{E}_n^- \subset m_n^+ J_-(f)$$

then f is k-$\mathcal{R}_{\mathbb{Z}_2}$-determined (in the space $\mathcal{E}_n^-$).
(ii) *The deformation $F : (\mathbb{R}^n \times \mathbb{R}^a,0) \to \mathbb{R}$ of f is $\mathcal{R}^+_{\mathbb{Z}_2}$-versal if and only if $\mathbb{R}\{\dot{F}_1,\ldots,\dot{F}_a\}$ spans $\mathcal{E}_n^-/J_-(f)$, where $\dot{F}_i = \partial F/\partial u_i(x,0)$.* □

EXAMPLE. Let $f = x^5 + y^5$. Then $J_-(f) = \mathcal{E}_2^-.\{x^4,y^4\}$, so

$$m_2^+ J_-(f) = \mathcal{E}_2^-.\{x^6,x^5y,x^4y^2,x^2y^4,xy^5,y^6\} = m_2^6 \cap \mathcal{E}_2^-.$$

Thus f is 5-determined (w.r.t. $\mathcal{R}_{\mathbb{Z}_2}$-equivalence).

4 Two degrees of freedom

This section essentially reproduces the result of J.B. Delos [7] using generating families; the main purpose is to illustrate the technical difficulties which arise in 3 degrees of freedom.

We have the following set up: $L \subset T^*Q$ is a Lagrangian submanifold invariant under the involution $\tau : T^*Q \to T^*Q$ (defined above) with the point $q = (q,0) \in L$, and $L \pitchfork_q Q$. The projection $\pi : L \to Q$ is $\mathbb{Z}_2$-invariant, and so has rank 0 at q. By section 2, there is an odd generating family $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ whose associated Lagrangian projection $\pi_F : C(F) \to \mathbb{R}^2$ is $\mathbb{Z}_2$-Lagrangian equivalent to π. The organizing centre f of F is also odd.

Proposition 4 *Any $\mathbb{Z}_2$-stable Lagrangian map germ $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ is Lagrangian equivalent to one of the following two germs:*

$$\begin{aligned} \pi_+(x,y) &= (x^2, y^2), \\ \pi_-(x,y) &= (x^2 - y^2, 2xy). \end{aligned}$$

PROOF: As π is $\mathbb{Z}_2$-stable, F must be an odd $\mathbb{Z}_2$-versal family, so the organizing centre f must have $\mathbb{Z}_2$-codimension at most 2. Up to $\mathcal{R}_{\mathbb{Z}_2}$-equivalence, the only odd functions with this property are

$$\begin{aligned} f_+(x,y) &= \tfrac{1}{3}(x^3 + y^3), \\ f_-(x,y) &= \tfrac{1}{3}x^3 - xy^2. \end{aligned}$$

This follows from the usual approach to classifying critical points (see e.g. [4]), using the determinacy estimate in Proposition 3. A $\mathbb{Z}_2$-versal deformation of $f_\pm$ is given by

$$F_\pm(x,y,u,v) = f_\pm(x,y) - ux - vy.$$

The result follows. □

Remark 5 Of these two Lagrangian $\mathbb{Z}_2$-stable singularities, only π_+ can occur in the projection of invariant Lagrangian submanifolds in classical Hamiltonian systems. The reason is as follows. Let $H(q,p) = K(p) + V(q)$ be the Hamiltonian, where K is the kinetic energy, which is a positive definite homogeneous quadratic function in p, and V is the potential energy. The invariant submanifold L lies on an energy level $H = E$, for some $E \in \mathbb{R}$, and suppose that $q = (q,0) \in L$. Since K is positive definite, the image of the projection of the submanifold must lie in the 'Hill's region' $\{q \in Q \mid V(q) \leq E\}$. Moreover, q is a regular point of the boundary of this Hill's region as otherwise $(q,0)$ would be an equilibrium point of the system and there would not be an invariant Lagrangian submanifold through it. Thus the image of the torus lies on one side of a smooth curve through q. The map π_- does not have this property, only the map π_+ does. This leaves open the question of whether a π_- singularity can occur stably in a nonclassical system.

It is natural to ask what degeneracies of projections of time reversible Lagrangian submanifolds might occur, after all in 2 degrees of freedom one might expect to have a 2 parameter family of invariant tori. However, such a degeneracy would require there to be a π_- singularity, either in a transition from a π_+ to a π_- or as a coalescing of a π_+ and a π_-. In either case there is the probably non-physical π_-, so we do not pursue this. The sort of degeneracy that is more likely to occur is that the Lagrangian submanifold (torus) itself becomes singular, for example along an invariant submanifold of lower dimension such as a periodic orbit.

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Figure 1: Two trajectories in a time reversible 2 degree of freedom Hamiltonian (the Henon-Heiles Hamiltonian). The figure on the left has 4 corners, while the one on the right has only folds and cusps: it is not a time reversible torus.

5 Three degrees of freedom

We begin by mimicking the constructions for 2 degrees of freedom. The Lagrangian projection will be defined by an odd generating family whose organizing centre f is a critical point of corank 3 (so without loss of generality, an odd function of 3 variables). Thus f is a homogeneous ternary cubic plus (possibly) higher order terms. After a suitable linear choice of coordinates, any real nondegenerate homogeneous ternary cubic can be written in the form,

$$f_c^{\pm}(x,y,z) = x^3 + 2cx^2z \pm xz^2 + y^2z. \tag{1}$$

Of these, only $f_{\pm 1}^{+}$ are degenerate. This expression is known as the Legendre normal form for a non-degenerate ternary cubic. (A homogeneous polynomial is said to be *non-degenerate* if the origin is the only critical point over $\mathbb{C}$.)

Proposition 6 *Suppose* $(c,\pm) \notin \{(1,+),(-1,+)\}$*, then* $f_c^{\pm}$ *is 3-*$\mathcal{R}_{\mathbb{Z}_2}$*-determined and has codimension 4. Moreover,*

$$\mathcal{F}_c^{\pm}(x,y,z,t,u,v,w) = x^3 + 2(c+t)x^2z \pm xz^2 + y^2z + ux + vy + wz \tag{2}$$

is an $\mathcal{R}_{\mathbb{Z}_2}^{+}$*-versal deformation of* $f_c^{\pm}$*.*

PROOF: A simple computation shows that

$$m_3^4 \subset m_3^2.J(f),$$

which, by intersecting with $\mathcal{E}_3^-$, implies

$$m_3^5 \cap \mathcal{E}_3^- \subset m_3^+.J_-(f).$$

Now apply Proposition 3. □

Thus we have proved that every odd corank 3 critical point has $\mathcal{R}_{\mathbb{Z}_2}$-codimension at least 4. It follows that there does not exist a $\mathbb{Z}_2$ stable Lagrange map germ $(\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$. In fact any $\mathbb{Z}_2$ invariant Lagrangian germ $(\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ has infinite codimension, and so is not finitely $\mathcal{R}_{\mathbb{Z}_2}^{+}$-determined. The infinite codimension comes from the modulus c that occurs in the organizing centre: one can show that in general, if f is a non-simple germ and F any non-versal deformation of f, then the associated Lagrange map π_F has infinite codimension. We therefore cannot hope to classify, or give a normal form for, generic $\mathbb{Z}_2$-invariant Lagrangian maps, at least not under smooth Lagrangian equivalence.

There are two possible approaches to circumventing this problem. One is to use topological Lagrangian equivalence, and the other is to use a weaker version of Lagrangian equivalence which S. Janeczko and M. Roberts call caustic equivalence in [8, 9]. We take the second approach.

Caustic equivalence is designed to ensure that the caustics of caustic equivalent Lagrangian maps are diffeomorphic. The definition is in terms of generating families using J. Damon's notion of $\mathcal{K}_V$-equivalence which we define first; for more details, see [6].

Definition: Let $g_1, g_2 : (\mathbb{R}^a, 0) \to (\mathbb{R}^b, 0)$ be two map germs, and $V \subset (\mathbb{R}^b, 0)$ a subvariety germ. We say g_1 and g_2 are $\mathcal{K}_V$-equivalent if there are diffeomorphisms H of $(\mathbb{R}^a \times \mathbb{R}^b, (0,0))$ and h of $(\mathbb{R}^a, 0)$ such that

- $H(u,v) = (h(u), \theta(u,v))$, for some map θ,
- $H(\mathbb{R}^a \times V) = \mathbb{R}^a \times V$,
- $H(u, g_1(u)) = (h(u), g_2 \circ h(u))$.

Remark In our application, V is not analytically trivial at the origin, i.e. every analytic flow preserving V fixes 0, which implies that $\theta(u,0) = 0$ and $\mathcal{K}_V$ is a geometric subgroup of $\mathcal{K}$, [6].

Now we return to generating families. Let $F_1, F_2 : (\mathbb{R}^n \times \mathbb{R}^a, 0) \to \mathbb{R}$ be deformations of the function germ f. Let $\mathcal{F} : (\mathbb{R}^n \times \mathbb{R}^b, 0) \to \mathbb{R}$ be a versal deformation of f, and let $V \subset (\mathbb{R}^b, 0)$ be the discriminant of this deformation. Each F_i is induced from $\mathcal{F}$ by a map $g_i : \mathbb{R}^a \to \mathbb{R}^b$. Note that the caustic of π_{F_i} is the set $g_i^{-1}(V)$. We say F_1 and F_2 are *caustic equivalent* if the map germs g_1 and g_2 are $\mathcal{K}_V$-equivalent. (We are being sloppy: F_i is not necessarily induced from $\mathcal{F}$, but it is equivalent to a generating family which is induced from $\mathcal{F}$. Since equivalent generating families define equivalent Lagrange maps, this sloppiness is unimportant.)

Theorem 7 *Let $f_c^{\pm}$ be given by (1), and $\mathcal{F}_c^{\pm}$ its versal deformation given in (2). Suppose $F : (\mathbb{R}^3 \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ is a deformation of $f_c^{\pm}$, for some $(c, \pm)$ satisfying,*

$$(c, \pm) \notin \{(1,+), (-1,+), (\sqrt{3}/2, +), (-\sqrt{3}/2, +)\},$$

such that the map $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ inducing F from $\mathcal{F}$ is transverse to the t-axis, then F is caustic equivalent to the generating family,

$$F_c^{\pm}(x,y,z,p,q,r) = f_c^{\pm}(x,y,z) - px - qy - rz. \tag{3}$$

The proof of this result is deferred to the final section.

We now proceed by describing the caustics of the generic invariant Lagrange projections which are physically allowable in classical Hamiltonian systems, see Remark 5. Let $f_c^{\pm}$ be given by (1), with generic 3-parameter deformation, given by (3). The associated Lagrangian map germ is given by,

$$\pi_c^{\pm}(x,y,z) = (3x^2 + 4cxz \pm z^2, 2yz, 2cx^2 \pm 2xz).$$

Recall from Theorem 7 that the exceptional values of $(c, \pm)$ are given by $(c^2, \pm) = (1, +),\ (3/4, +)$.

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Figure 2: The caustic of π_c^+, for $|c| < 1$, $c^2 \neq 3/4$.

Figure 3: The simplest possible caustic of a time reversible torus.

Lemma 8 *If $|c| > 1$ or '$\pm = -$', then the image of $\pi_c^\pm$ cannot be contained on one side of a smooth surface.*

PROOF: This is a straightforward calculation. First note that the restriction of $\pi_c^\pm$ to the plane $y = 0$ maps to the plane $q = 0$. This restriction map is surjective in both the cases in the hypothesis. Moreover, the lines $(0, \pm y, 0)$ map to two line segments, one on each side of the p-r plane. The lemma follows. □

Thus we are left with $\pi_c = \pi_c^+$ for $|c| < 1$. The origin is a Σ^3-point of π_c. There are Σ^2 points of π_c near 0 if and only if $c^2 = 3/4$, which is excluded by hypothesis. (There are 3 real branches of Σ^2-points if $c = -\sqrt{3}/2$ and one if $c = \sqrt{3}/2$, in the first case these are all hyperbolic umbilics, while in the second they are elliptic umbilics.) In this range of values of $|c|$, there are 3 branches of $\Sigma^{1,1}$ points which give rise to 3 cuspidal edges on the caustic. Further calculations show that the caustic is as drawn in Figure 2. The complement of the caustic has three components. Each point of the inner component has 8 preimage points; each point of the middle component has 4 points in its preimage, while the outer component is not in the image of the projection. The modulus c can be interpreted as a measure of the relative volumes of these three regions.

To return to time reversible invariant tori, by Proposition 2 such a torus would have 2^n corners (in n degrees of freedom). Thus the simplest possible caustic associated to a time reversible torus in 3 degrees of freedom should be as depicted in Figure 3. In general, of course, such a caustic could have in addition other stable Lagrangian singularities arising from singularities of the projection at points distinct from the fixed point set of τ. Such singularities do not involve any symmetry, so are just those found in the usual list of 3-dimensional Lagrangian singularities: the elliptic and hyperbolic umbilics, and the swallowtail, as well as the cuspidal edge, [3]. Indeed, if there is a 1-parameter family of time reversible invariant tori, and the parameter c describing the modulus associated to a corner passes through the value $\sqrt{3}/2$, three hyperbolic umbilics should appear (or disappear) on the caustic, and if it passes through $-\sqrt{3}/2$, an elliptic umbilic point should appear on the caustic inside the image of the torus.

To justify experimentally the results of this paper, consider the Hamiltonian,

$$H(p,q) = 0.5(p_1^2 + q_1^2) + 0.55(p_2^2 + q_2^2) + 0.625(p_3^2 + q_3^2) + q_1 q_2 q_3.$$

The associated Hamiltonian vector field, with initial condition $p = 0$, $q = (0.2, 0.2, 0.2)$, was integrated numerically using Kaos, the package developed by J. Guckenheimer and S. Kim. With $\{q_1 = 0\}$ as a Poincaré section, the trajectory is shown in Figure 4. It is clear that this figure is consistent with the picture of the caustic in Figure 3.

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Figure 4: A Poincaré section of a time reversible 3-torus. The right hand picture is a close-up of the top left corner of the left hand picture.

In a linear Hamiltonian system of the form,

$$H(p,q) = \sum \omega_i (p_i^2 + q_i^2),$$

the invariant tori are products of circles in the coordinate planes. Their projection to the configuration space would be a rectangular box with edges parallel to the coordinate axes, whose corners are caustics of the map π_c^+ for $c = -\sqrt{3}/2$. The box would be densely filled by a trajectory performing a 3-dimensional Lissajous figure.

CONJECTURE. Suppose $(0,0) \in T^*\mathbb{R}^3$ is an elliptic equilibrium point of a classical Hamiltonian system, with Hamiltonian $H(p,q) = K(p) + V(q)$, so $dH_{(0,0)} = 0$ and $d^2H_{(0,0)}$ is positive (or negative) definite. Suppose that there is a continuous family of invariant tori tending to $(0,0)$, call these $T_\varepsilon \subset \{H = \varepsilon\}$, the ε energy level. Each of the 8 corners of the caustic of T_ε is equivalent to π_c^+ for some $c \in [-1,1]$. Let C_ε be any one of these corners chosen to depend continuously on ε, and let c_ε be the associated value of c. I conjecture that,

$$\lim_{\varepsilon \to 0} c_\varepsilon = -\frac{\sqrt{3}}{2}.$$

6 Proof of Theorem 7

To prove Theorem 7, it is necessary to compute Θ_V, the module of smooth vector fields tangent to V, where V is the discriminant of the versal deformation $\mathcal{F}_c^\pm$ of $f_c^\pm$. The module Θ_V^{an} of analytic vector fields tangent to V can be computed with the aid of the software package Macaulay developed by D. Bayer and M. Stillman. The procedure is briefly as follows. First homogenize $\mathcal{F}_c^\pm$ with a new variable d (of weight 1) and compute the equation $\tilde{h}$ of the discriminant $\tilde{V}$ in c,d,u,v,w-space by elimination — the variable t can be ignored as it is equivalent to c. This homogenized base space of the versal deformation I denote by $\tilde{S}$. Use the resolution command to obtain a presentation of the matrix (row vector) $\partial\tilde{h}$ of partial derivatives of $\tilde{h}$:

$$\mathcal{O}_{\tilde{S}}^6 \xrightarrow{\lambda} \mathcal{O}_{\tilde{S}}^5 \xrightarrow{\partial\tilde{h}} \mathcal{O}_{\tilde{S}}.$$

The image of λ is the module consisting of vector fields which annihilate $\tilde{h}$. Because $\tilde{V}$ is homogeneous, the module $\Theta_{\tilde{V}}$ is generated by the annihilator of $\tilde{h}$ and the Euler field $\frac{\partial}{\partial c} + \frac{\partial}{\partial d} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$. To obtain V from $\tilde{V}$ we put $d = 1$, and for Θ_V, we intersect $\Theta_{\tilde{V}}$ with the submodule of vector fields with no $\frac{\partial}{\partial d}$-term. The result is that Θ_V is generated by six vector fields. In fact we only need the 1-jets of these vector fields at $(c_0,0,0,0)$, and only 4 of the generators have non-zero 1-jets. We record these 1-jets here.

$$j^1v_1 = \begin{pmatrix} 0 \\ u \\ v \\ w \end{pmatrix}, \qquad j^1v_2 = \begin{pmatrix} (4c_0^2 \mp 3)(u \mp c_0 w) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$j^1v_3 = \begin{pmatrix} (4c_0^2 \mp 3)(c_0^2 \mp 1)v \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad j^1v_4 = \begin{pmatrix} (4c_0^2 \mp 3)(c_0^2 \mp 1)w \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Remark These six vector fields generate the module Θ_V^{an} of analytic vector fields tangent to V (or rather the real part of the complex discriminant) but not necessarily the module Θ_V of smooth vector fields tangent to V. However, all we need is that these vector fields are contained in the module of smooth vector fields tangent to V.

PROOF OF THEOREM 7: Let S denote the germ of c,u,v,w-space at $((c_0,\pm),0,0,0)$, with $(c_0^2,\pm) \neq (1,+), (3/4,+)$. Let $V \subset S$ be the germ of the discriminant of $\mathcal{F}_c^{\pm}$ defined in (2). Suppose $g : (\mathbb{R}^3,0) \to S$ is transverse to the c-axis. Coordinates in $\mathbb{R}^3$ can be chosen so that g takes the form

$$(p,q,r) \mapsto (c_0 + h(p,q,r), p, q, r),$$

where $h \in m_3$. We wish to show that g is $\mathcal{K}_V$-equivalent to the map,

$$g_1(p,q,r) = (c_0,p,q,r).$$

We use the unipotency results of Bruce, du Plessis and Wall, [5]. Consider the submodule $\Theta_{V,1} \subset \Theta_V$ defined by,

$$\Theta_{V,1} = m_S^2\Theta_S \cap \Theta_V + \mathcal{E}_S.\{v_2, v_3, v_4\}.$$

The group this defines is a jet-unipotent subgroup of $\mathcal{K}$ in the sense of [5], since the 1-jet part of $\Theta_{V,1}$ consists only of strictly upper triangular matrices. Let $\mathcal{G}$ be the subgroup of $\mathcal{K}$ generated by $\Theta_{V,1}$ and $m_3^2\Theta_3$ (the latter being a submodule of the module of vector fields on the source). This is also jet-unipotent. With this $\mathcal{G}$ we use the notation of [5, Proposition (4.1)]. Let A be the module of smooth map germs $g : (\mathbb{R}^3,0) \to S$, with $g(0) = (c_0,0,0,0)$, and let

$$M = \{g \in A \,|\, g(p,q,r) = (h(p,q,r),0,0,0), h \in m_3\} + m_3^2.A.$$

First note that the map g_1 is 2-$\mathcal{K}_V$-determined by [6], since,

$$\begin{aligned} T\mathcal{K}_{V,e}.g_1 &= tg_1(\Theta_3) + g_1^*\Theta_V \\ &= (m_3,\, \mathcal{E}_3,\, \mathcal{E}_3,\, \mathcal{E}_3). \end{aligned}$$

Let $L = L(J^2\mathcal{G})$ and $m \in M$. Working modulo m_3^3 we have,

$$L.j^2(g_1+m) + m_3.M \quad = \quad t(g_1+m)(m_3^2\Theta_3) + (g_1+m)^*\Theta_{V,1} + m_3.M$$

$$
\begin{aligned}
&= m_3^2.\{(h_p,1,0,0),\,(h_q,0,1,0),\,(h_r,0,0,1)\} + \\
&\qquad (g_1+m)^*\{j^1v_2,\,j^1v_3,\,j^1v_4\} + m_3.M \\
&= m_3^2.\{(0,1,0,0),\,(0,0,1,0),\,(0,0,0,1)\} + \\
&\qquad (g_1+m)^*\{j^1v_2,\,j^1v_3,\,j^1v_4\} + m_3.M \\
&= M,
\end{aligned}
$$

provided $(4c_0^2 \mp 3)(c_0^2 \mp 1) \neq 0$. Thus, by Nakayama's lemma, $L.j^2(g_1+m) = M$ for all $m \in M$, and the result follows. □

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