PERTURBING A SYMMETRIC RESONANCE: THE MAGNETIC SPHERICAL PENDULUM

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We consider symmetric 2 degree of freedom Hamiltonian systems which are resonant because of the symmetry, such as the spherical pendulum. Perturbing such a system by breaking the symmetry (e.g. adding a magnetic term) creates bifurcations in the geometry of the families of periodic orbits, and the aim of this paper is to study this geometry. For brevity, we consider only the cases of symmetry breaking $\mathbf{O}(2) \to \mathbf{SO}(2)$ and $\mathbf{D}_4 \to \mathbf{Z}_4$. We follow the standard approach of Lyapounov-Schmidt reduction and normal form theory together with singularity theory to justify the truncation to normal form. However, the details of the singularity theory are new.

Introduction

The spherical pendulum is a 2-degree of freedom Hamiltonian system with $\mathbf{O}(2)$ symmetry. The stable equilibrium has double imaginary eigenvalues, say $\pm i$, due to this symmetry. At the linear level, every initial condition gives rise to a periodic orbit of period 2π, and symmetry methods[6] can be used to predict which of these persist in the full nonlinear system. One finds that there are two rotational modes and infinitely many planar modes (where the pendulum swings in a vertical plane), all of period approximately 2π. The global period-energy diagram for these modes is shown in Figure 1(a) below.

Now suppose we break the reflexional symmetry while preserving the rotational symmetry. This is realized if, for example, the pendulum carries an electric charge, and moves in the presence of a rotationally symmetric magnetic field. This perturbed system has only $\mathbf{SO}(2)$ symmetry, and one consequence is that the eigenvalues split: $\pm(1 \pm \lambda)i$, say, where λ is proportional to the strength of the magnetic field. Indeed, the magnetic field will favour one direction of rotation over the other. The $\mathbf{SO}(2)$ symmetry forces the two families of rotating solutions to persist. For small values of λ these will be the only nonlinear normal modes, and the infinite family of planar modes will no longer exist as modes (i.e. in families containing the equilibrium). On the other hand, a nondegeneracy argument applied to any given energy level shows that for sufficiently small values of λ these 'planar periodic orbits' will persist — albeit slightly perturbed and no longer planar. The study shows that this family pulls away from the equilibrium point, while clinging to the slower of the rotational modes (see Figure 1(b) and Figure 2).

The purpose of this short paper is to describe the geometry of these families of periodic orbits as the symmetry breaking parameter λ is varied. The technique used allows us to describe the periods of these periodic orbits. We also treat the case where the symmetry breaking is from $\mathbf{D}_4$ to $\mathbf{Z}_4$, where the question of stability of the periodic orbits is of interest as well. However the more general $\mathbf{D}_n$ to $\mathbf{Z}_n$ requires more sophisticated results from singularity theory and will be left to a separate publication.

In more detail, Section 1 describes the invariant Hamiltonians we will be considering, Section 2 consists of the calculations for the fourth order normal form and ends with a statement of the main theorem (Theorem 2.3) relating the modes of the full system to those of the truncated system. This relies on singularity theoretic methods, described in Section 3. Section 4 describes the stabilities of the periodic orbits, and finally in Section 5 we do the normal form computation for the spherical pendulum.

J.J. Duistermaat[4] was the first to study detuning resonances using singularity theory, although the details of the singularity theory are different, and he did not consider the 1:1 resonance because in generic systems (with no symmetry) it is of codimension 3. All other resonances are of codimension 1, including the 1:-1 resonance.

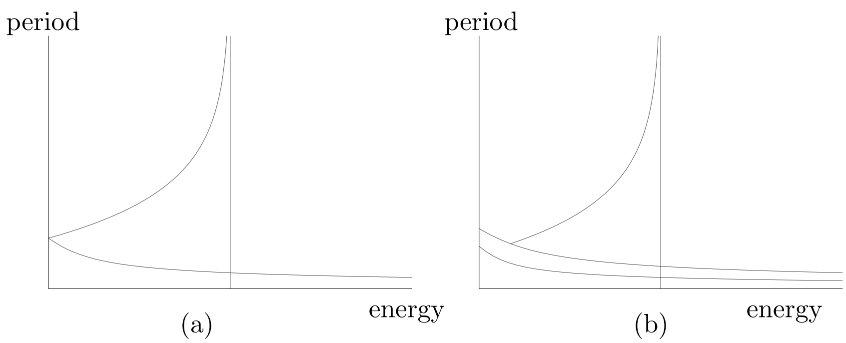

Figure 1. The period-energy diagram for (a) the spherical pendulum and (b) a magnetic perturbation. The lower curves are for the circular mode, and the upper curve is for the 'plane pendulum' mode and its perturbation. The vertical axis corresponds to the energy of the stable equilibrium, while the vertical asymptote represents the energy of the unstable equilibrium.

1 The Hamiltonian and its symmetry

The symmetry-breaking scenario is modelled by a family H_λ of Γ-invariant Hamiltonians, such that H_0 is Γ_0-invariant. We will say that such a family is (Γ_0, Γ)-*invariant*. We consider two possible symmetry-breaking scenarios: $\mathbf{O}(2) \to \mathbf{SO}(2)$ and $\mathbf{D}_4 \to \mathbf{Z}_4$. Thus $(\Gamma_0, \Gamma) = (\mathbf{O}(2), \mathbf{SO}(2))$ or $(\mathbf{D}_4, \mathbf{Z}_4)$.

After a suitable choice of coordinates[7] on $\mathbf{R}^4 \simeq \mathbf{C}^2$, the Hamiltonian can be written,

$$H_\lambda(z_1, z_2) = N_1 + N_2 + \lambda(N_1 - N_2) + \alpha(N_1 + N_2)^2 + \beta N_1 N_2 + \gamma S + \tilde{H}_\lambda(z_1, z_2), \tag{1.1}$$

where

$$\begin{cases} N_j = |z_j|^2 \ \text{ for } j = 1, 2, \\ S = \Re(z_1^2 \overline{z}_2^2), \end{cases} \tag{1.2}$$

and the (Γ_0, Γ)-invariant $\tilde{H}_\lambda$ is of order at least 5 with respect to (z, λ), where λ is given weight 2 and z weight 1. Moreover, since S is not an $\mathbf{SO}(2)$-invariant, we suppose $\gamma = 0$ if $(\Gamma_0, \Gamma) = (\mathbf{O}(2), \mathbf{SO}(2))$. The form $\lambda(N_1 - N_2)$ of the detuning term of the Hamiltonian can be arranged by rescaling time as a function of λ if necessary.

The quadratic part $N_1 + N_2$ of H_0 determines the linear approximation to the Hamiltonian, which is in 1:1 resonance and so generates the 2π-periodic flow: $(z_1, z_2) \mapsto (e^{it} z_1, e^{it} z_2)$. The degree 4 part of the Hamiltonian (1.1) is

$$H_\lambda^{(4)} = N_1 + N_2 + \lambda(N_1 - N_2) + \alpha(N_1 + N_2)^2 + \beta N_1 N_2 + \gamma S, \tag{1.3}$$

which is in normal form with respect to this quadratic part of H_0.

Remark 1.1 If we considered a general linear system that is only Γ-invariant then the symplectic Γ-representation is of type[6] 'complex+dual' and the passing 1:1 resonance is then[3,5] of codimension 1. It follows that essentially the same situation of 1:1 resonance of codimension 1 can be achieved without the symmetry-breaking scenario. In fact the results of this paper on the geometry of the nonlinear normal modes continue to hold for a family of Γ-invariant systems ($\Gamma = \mathbf{SO}(2)$ or $\mathbf{Z}_4$) as the family passes through a 1:1 resonance. The only changes are that certain new terms would appear, and the precise equations for the modes would therefore change.

2 Nonlinear normal modes

Let $H_\lambda(z)$ be a Hamiltonian depending on a parameter λ, such that the origin is a nondegenerate equilibrium for all λ.

To find the periodic orbits of the Hamiltonian system given by H_λ the standard technique[4,7] is to set up a variational problem on the loop space $C^1(S^1, \mathbf{C}^2)$, and apply Lyapounov-Schmidt reduction or the splitting lemma to reduce to a finite dimensional problem. This procedure introduces a new parameter, the period-shift τ, and the result is that in a neighbourhood of the equilibrium point the periodic orbits of H_λ of period $2\pi/(1+\tau)$ are the critical points of a $\Gamma \times S^1$-invariant function $h_{\lambda,\tau}$. Furthermore, the function $h_{\lambda,\tau}$ can be calculated to any finite order from the normal form of H_λ: if H_λ is in normal form to degree k with respect to the quadratic part $H_0^{(2)}$ then

$$h_{\lambda,\tau}(z) = H_\lambda - \tau H_0^{(2)} + O(k+1).$$

2.1 Study of the fourth-order normal form

For this subsection, denote the 4th order normal form $H_\lambda^{(4)}$ of (1.3) simply by H_λ. By the discussion above, the periodic orbits of H_λ of period $2\pi/(1+\tau)$ are given by the critical points of the $\Gamma \times S^1$-invariant function

$$h_{\lambda,\tau} := (\lambda-\tau)N_1 - (\lambda+\tau)N_2 + \alpha(N_1+N_2)^2 + \beta N_1 N_2 + \gamma S, \qquad (2.1)$$

where $\gamma = 0$ if $\Gamma = \mathbf{SO}(2)$. This is a weighted homogeneous function of degree 4, where the weights are defined by weight$(z_1, z_2, \tau, \lambda) = (1,1,2,2)$. In Section 3 we show that the full system is equivalent to this one in an appropriate sense, but in order to do so it will be necessary to assume the following nondegeneracy conditions:

$$\begin{cases} \alpha \neq 0,\ \beta \pm \gamma \neq 0,\ 4\alpha+\beta\pm\gamma \neq 0, \\ 2\alpha+\beta\pm\gamma \neq 0, \quad 4\alpha\beta+\beta^2-\gamma^2 \neq 0. \end{cases} \qquad (2.2)$$

Proposition 2.1 *Let $h_{\lambda,\tau}$ be given by (2.1) while satisfying the nondegeneracy conditions (2.2). For each value of λ, τ, $h_{\lambda,\tau}$ has the following critical points:*

group	*label*	*critical points*
Γ	(O)	$N_1 = N_2 = 0,\ (S=0)$
Γ	(A_1)	$N_2 = 0, N_1 = (\tau-\lambda)/2\alpha,\ (S=0)$
Γ	(A_2)	$N_1 = 0, N_2 = (\tau+\lambda)/2\alpha,\ (S=0)$
$\Gamma = \mathbf{SO}(2)$	(B)	$N_1 = \dfrac{\lambda}{\beta} + \dfrac{\tau}{(4\alpha+\beta)},\ N_2 = -\dfrac{\lambda}{\beta} + \dfrac{\tau}{(4\alpha+\beta)}$
$\Gamma = \mathbf{Z}_4$	$(B_\pm)$	$N_1 = \dfrac{\lambda}{\beta\pm\gamma} + \dfrac{\tau}{4\alpha+\beta\pm\gamma},\ N_2 = -\dfrac{\lambda}{\beta\pm\gamma} + \dfrac{\tau}{4\alpha+\beta\pm\gamma},$
		$S = \mp\dfrac{\lambda^2}{(\beta\pm\gamma)^2} \pm \dfrac{\tau^2}{(4\alpha+\beta\pm\gamma)^2}.$

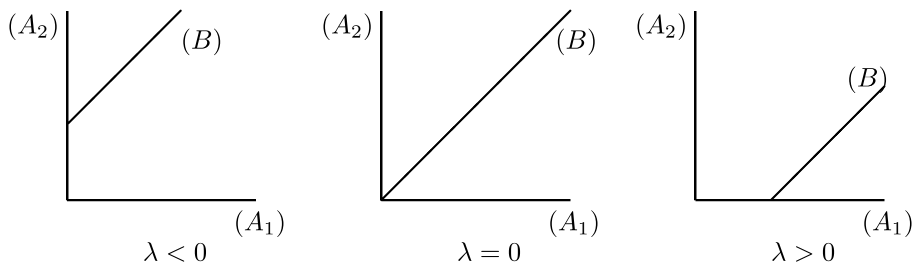

Figure 2. The NLNMs in N_1-N_2 space, for $\Gamma = \mathbf{SO}(2)$. Here we are assuming $\beta > 0$; if $\beta < 0$ the sign of λ should be reversed.

Of course, one must impose $N_1, N_2 \geq 0$ on these equations, so some critical points only exist for certain values of λ, τ. Translating these critical points of $h_{\lambda,\tau}$ into periodic orbits of H_λ gives the following.

Corollary 2.2 *Let H_λ be given by (1.3) while satisfying the nondegeneracy conditions (2.2). For each value of λ, H_λ has the following periodic orbits, with period $2\pi/(1+\tau)$,*

group	*mode*	*location of mode*	*period shift*
Γ	(A_1)	$N_2 = 0, N_1 > 0$	$\tau = \lambda + 2\alpha N_1$
Γ	(A_2)	$N_1 = 0, N_2 > 0$ $(S = 0)$	$\tau = -\lambda + 2\alpha N_2$
$\Gamma = \mathbf{SO}(2)$	(B)	$N_1 = N_2 + \dfrac{2\lambda}{\beta}$	$\tau = \frac{1}{2}(4\alpha + \beta)(N_1 + N_2)$
$\Gamma = \mathbf{Z}_4$	$(B_\pm)$	$N_1 = N_2 + \dfrac{2\lambda}{\beta \pm \gamma}$ $S = \pm N_1 N_2.$	$\tau = \frac{1}{2}(4\alpha + \beta \pm \gamma)(N_1 + N_2)$

The families (A_1) and (A_2) are of symmetry type $\widetilde{\mathbf{SO}(2)}$ and $\widetilde{\mathbf{SO}(2)}'$ respectively when $\Gamma = \mathbf{SO}(2)$ and of symmetry type $\widetilde{\mathbf{ZZ}_4}$ and $\widetilde{\mathbf{ZZ}_4}'$ when $\Gamma = \mathbf{Z}_4$. For $\lambda = 0$, the family (B) is of type $\mathbf{Z}_2^\kappa$, while the $(B_\pm)$ are of type $\mathbf{Z}_2^\kappa$ $\mathbf{Z}_2^{\kappa'}$ respectively. For $\lambda \neq 0$ these last symmetry types disappear, so the modes have generic symmetry.

Proof (of Proposition 2.1) The critical points can be found by requiring that the partial derivatives of $h_{\lambda,\tau}$ with respect to $z_1, \overline{z}_1, z_2, \overline{z}_2$ vanish. Another approach is to find the vector fields on the orbit space that are tangent

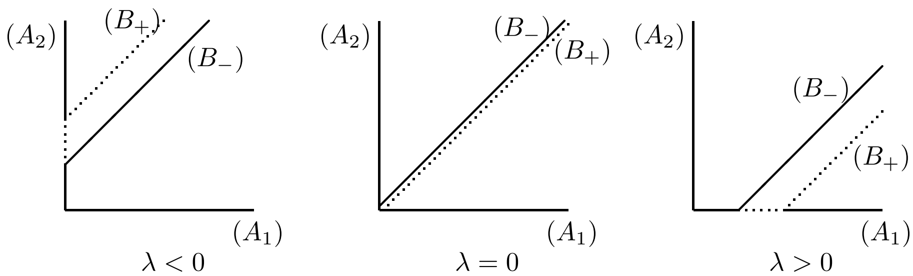

Figure 3. The NLNMs in N_1-N_2 space, for $\Gamma = \mathbf{Z}_4$, and $|\beta| > |\gamma|$. Here we are assuming $\beta > \gamma > 0$; if we change the sign of γ then (B_+) and (B_-) should be interchanged, while if we change the sign of both β and γ the sign of λ should be reversed. Note also that (B_+) and (B_-) are separated by having opposite signs of S. The dotted lines refer to unstable periodic orbits, the continuous lines to stable ones, see §4.

to the stratification by orbit type. For $G = \mathbf{SO}(2) \times S^1$ this module has just 2 generators $N_j\partial/\partial N_j$ $(j = 1, 2)$, and the G-orbits of critical points are found by differentiating $h_{\lambda,\tau}$ along these two vector fields. The equations are simply

$$\begin{aligned} N_1(\lambda - \tau + 2\alpha(N_1 + N_2) + \beta N_2) &= 0, \\ N_2(-\lambda - \tau + 2\alpha(N_1 + N_2) + \beta N_1) &= 0. \end{aligned}$$

This gives all 4 solutions $(O), (A_1), (A_2)$ and (B).

For $G = \mathbf{Z}_4 \times S^1$, the vector fields on the quotient are more complicated: there are 7 generators in all, and for brevity we omit the details. □

2.2 The full system

Given a family of Hamiltonians H_λ with the usual invariance properties, the question remains of whether its nonlinear normal modes correspond in some way to those of the fourth-order normal form. The answer is given in the following theorem.

Theorem 2.3 *Let $H_\lambda(z_1, z_2)$ be a family of smooth (Γ_0, Γ)-invariant Hamiltonian systems on $\mathbf{C}^2 \simeq \mathbf{R}^4$, where $(\Gamma_0, \Gamma) = (\mathbf{O}(2), \mathbf{SO}(2))$ or $(\mathbf{D}_4, \mathbf{Z}_4)$. Suppose moreover that the weighted 4-jet of H is in the form (1.3) and satisfies the nondegeneracy conditions (2.2). Then there exists a family of germs of (Γ_0, Γ)-equivariant homeomorphisms $\psi_{\tau,\lambda} : \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ and a family of continuous functions $T_\lambda : \mathbf{R} \to \mathbf{R}$, such that the periodic orbits of period*

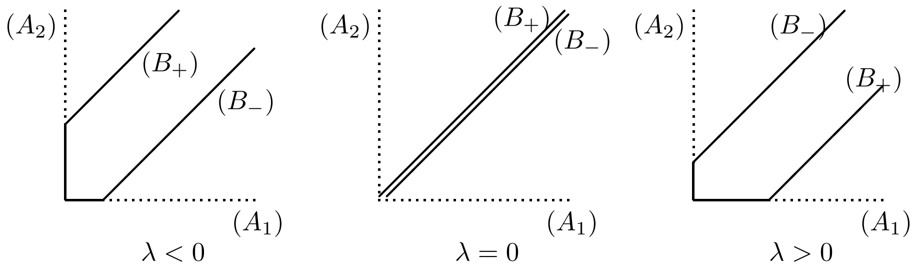

Figure 4. The NLNMs in N_1-N_2 space, for $\Gamma = \mathbf{Z}_4$, and $|\gamma| > |\beta|$. Here we are assuming $\gamma > \beta > 0$; if the signs of either or both are changed, then exactly the same modifications are applicable here as in Figure 3. Again, the dotted lines refer to unstable periodic orbits, the continuous lines to stable ones, see §4.

$2\pi/(1+\tau)$ *of the full system* H_λ *are identified under* $\psi_{\tau,\lambda}$ *with those periodic orbits of period* $2\pi/(1+T_\lambda(\tau))$ *of the fourth order normal form.*

The proof of this theorem relies on singularity theory methods. We present these first, before returning to the proof.

3 The singularity theory

For this section, it is necessary to complexify the functions $f_{\lambda,\tau}$, so rather than being functions on $\mathbf{R}^4$ (or $\mathbf{C}^2$) we now consider $f_{\lambda,\tau}(z)$ to be a family of holomorphic functions on $\mathbf{C}^4$, with complex parameters τ, λ. We write

$$F(z,\tau,\lambda) = (f_{\lambda,\tau}(z), \tau, \lambda).$$

We will say a family $f_{\lambda,\tau}$ of such functions is (G_0, G)-*invariant* if each function is G-invariant, and the functions $f_{0,\tau}$ are G_0-invariant. Similarly for (G_0, G)-equivariant maps. Here $G = \Gamma \times S^1$, and $G_0 = \Gamma_0 \times S^1$.

Definition 3.1 We say a family of (G_0, G)-invariant functions $f_{\lambda,\tau}(z)$ is *equivalent* to another family $f'_{\lambda,\tau}(z)$ if there exists a (G_0, G)-equivariant diffeomorphism $\Psi : \mathbf{C}^4 \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}^4 \times \mathbf{C} \times \mathbf{C}$ of the form

$$\Psi(z,\tau,\lambda) = (Z(z,\tau,\lambda), T(\tau,\lambda), \Lambda(\lambda))$$

and a diffeomorphism $\Phi : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C} \times \mathbf{C}$ of the same form $\Phi(y,\tau,\lambda) = (Y(y,\tau,\lambda), T(\tau,\lambda), \Lambda(\lambda))$, satisfying

$$F' = \Phi \circ F \circ \Psi.$$

This equivalence forms one of the geometric subgroups of $\mathcal{A}$ defined by Damon[2]. Furthermore, we say the families $f_{\lambda,\tau}$ and $f'_{\lambda,\tau}$ are C^0-*equivalent* if the maps Ψ and Φ are only homeomorphisms.

We are going to do the computations on the quotient space, so the following statement will be for functions on the complex analytic space $X = \mathbf{C}^4/G^{\mathbf{C}}$, where $G^{\mathbf{C}}$ is the complexification of G. In fact, X is just the space whose ring of functions is the ring of invariants, considered as complex functions rather than real. The module Θ_X of vector fields on X tangent to the stratification by orbit type consists of the image of the equivariant vector fields on $\mathbf{C}^4$.

Theorem 3.2 (Geometric criteria) *Let $f_{\lambda,\tau}$ be a family of (G_0, G)-invariant functions, which satisfies*

- *$f_{0,0}$ has an isolated critical point at 0 in the orbit space X;*
- *for each $\tau_0 \neq 0$ the critical points of f_{0,τ_0} are nondegenerate, and the critical values are distinct unless the points are equivalent under G_0; and*
- *for each $\lambda_0 \neq 0$, the family $f_{\lambda_0,\tau}$ is stable: every degenerate critical point and every multiple critical value is versally deformed by varying τ;*

then the family is finitely determined with respect to the equivalence defined above.

A function f on X has a critical point at x if $J_X(f) \subset \mathcal{M}_x$ — the maximal ideal of functions vanishing at x, where $J_X(f)$ is the Jacobian ideal generated by the derivatives of f along the vector fields in Θ_X. The critical point is nondegenerate if, at x, $J_X(f) = \mathcal{M}_x$. By Nakayama's lemma this is equivalent to the easier to verify property $J_X(f) + \mathcal{M}_x^2 = \mathcal{M}_x$.

That a degenerate critical point is to be versally deformed has the usual algebraic definition that the 'extended right tangent space' $J_X(f)$ plus the unfolding velocities generate the whole ring of germs. Finally if two nondegenerate critical points have a coincident critical value, then this is versally deformed if the values separate at non-zero speed as the parameter is varied. One can weaken the criteria above, but it is not needed for this paper. The theorem is proved using sheaf-theoretic methods, and will be written in a more general context elsewhere. Similar results are explained in the survey paper of C.T.C. Wall[10], and equivariant versions in a paper of R.M. Roberts[9].

Corollary 3.3 *If $f_{\lambda,\tau}(z)$ is a family of (G_0, G)-invariant functions satisfying the geometric criteria for finite determinacy, and which is weighted homogeneous of degree d, then any $f'_{\lambda,\tau}$ whose d-jet coincides with f is C^0-equivalent to f.*

This follows from the finite determinacy of $f_{\lambda,\tau}$ by a theorem of J. Damon[2] on finite determinacy for geometric subgroups of $\mathcal{A}$.

3.1 Proof of Theorem 2.3

To prove the theorem, we need to verify that $h_{\lambda,\tau}$ satisfies the geometric criteria for finite determinacy given in Theorem 3.2. The result then follows from Corollary 3.3 above, since a homeomorphism takes critical points to critical points, as singular fibres are topologically inequivalent to regular (smooth) fibres.

Let $h_{\lambda,\tau}$ be the family of functions defined in (2.1), and satisfying (2.2). The real critical points of $h_{\lambda,\tau}$ are given in Proposition 2.1. However, to use the geometric criterion, it is necessary to complexify the functions $h_{\lambda,\tau}$ to a family of functions on $\mathbf{C}^4$ and include the complex critical points too. It turns out that one introduces no new branches of critical points, and the complex critical points of $h_{\lambda,\tau}$ are given by the formulae in Proposition 2.1, except that the invariants N_1, N_2, S and T as well as the parameters λ, τ can now take complex values.

We need to see which of these critical points are degenerate. The two cases $\Gamma = \mathbf{SO}(2)$ and $\Gamma = \mathbf{Z}_4$ are very similar, though the computations in the second case are somewhat longer. Consequently, we only outline the computations in the case of $\Gamma = \mathbf{SO}(2)$.

Let $\Gamma = \mathbf{SO}(2)$. The (generally) 4 critical points are given in Proposition 2.1, and the values of $h_{\lambda,\tau}$ at these points, are given by

$$\begin{array}{ll} (O)\ 0 & (A_1)\ -(\tau-\lambda)^2/(4\alpha) \\ (A_2)\ -(\tau+\lambda)^2/(4\alpha) & (B)\ \lambda^2/\beta - \tau^2/(4\alpha+\beta) \end{array} \tag{3.1}$$

These critical points can coincide (and so be degenerate) as follows:

$$\begin{array}{ll} (A_1) = (O) : \tau - \lambda = 0, & (A_2) = (O) : \tau + \lambda = 0, \\ (A_1) = (B) : \beta\tau - (4\alpha+\beta)\lambda = 0, & (A_2) = (B) : \beta\tau + (4\alpha+\beta)\lambda = 0 \end{array}$$

The nondegeneracy conditions (2.2) ensure that no two of these degeneracies occurs at the same point outside the origin.

The only possible double critical values coming from two distinct critical points are:

$$\begin{array}{lll} h_{\lambda,\tau}((B)) = h_{\lambda,\tau}((O)) & \text{when} & (4\alpha+\beta)\lambda^2 = \beta\tau^2. \\ h_{\lambda,\tau}((A_1)) = h_{\lambda,\tau}((A_2)) & \text{when} & \lambda\tau = 0. \end{array}$$

These degeneracies also do not occur for the same parameter values as any of the others, by the nondegeneracy condition (2.2). The fact that $h_{\lambda,\tau}((A_1)) = h_{\lambda,\tau}((A_2))$ when $\lambda = 0$ is due to the fact that that system is $\mathbf{O}(2)$-invariant, and is therefore not a degeneracy.

The bifurcation diagram $\mathcal{B}(h_{\lambda,\tau})$ thus consists of 7 lines through the origin in parameter space. At points of 4 of these, the critical points degenerate, while on the remaining 3, two critical points have coincident values. The nondegeneracy conditions ensure that these 7 lines are distinct.

It remains to verify the sufficient geometric criteria required by Theorem 3.2. This is just a series of algebraic computations. Firstly, when $\lambda = \tau = 0$ the origin is indeed the only critical point. Secondly, when $\lambda = 0$ and $\tau = \tau_0 \neq 0$, the function $f = h_{0,\tau_0}$ is Γ_0-stable, for by (3.1) the critical values are distinct (except for (A_1) and (A_2) which are Γ_0-equivalent as already pointed out), and nondegenerate. Finally, for each $\lambda_0 \neq 0$ computations show that the degeneracies are all versally deformed by varying τ. For example, when $\tau = 0$ then $h_{\lambda_0,0}(A_1) = h_{\lambda_0,0}(A_2)$. The difference between the critical values in general is $h_{\lambda_0,\tau}(A_2) - h_{\lambda_0,\tau}(A_1) = \tau\lambda_0/\alpha$, and since $\lambda_0 \neq 0$ this difference varies with non-zero speed with τ, as required.

4 Stabilities of the normal modes

We briefly describe the stabilities of the periodic orbits in the modes whose existence we have proved. Consider first the case $\Gamma = \mathbf{SO}(2)$. The Floquet operator of the rotating modes (A_j) $(j = 1, 2)$ have imaginary eigenvalues for reasons of symmetry (the representation of $\Sigma \simeq \mathbf{SO}(2)$ is cyclospectral[6]). For the other (B) mode the eigenvalues are zero, since the periodic orbits are not isolated in their energy level.

The case $\Gamma = \mathbf{Z}_4$ is more interesting. The stabilities of the modes of the $\mathbf{D}_4$ system (at $\lambda = 0$) depend[8] on the values of β, γ. With λ included, the non-zero eigenvalues of the periodic orbits of the fourth-order normal form (1.3) are given as follows:

mode	eigenvalues	mode	eigenvalues
(A_1)	$\pm\left[\gamma^2 N_1^2 - (\beta N_1 - 2\lambda)^2\right]^{1/2}$	(B_+)	$\pm\left[-2\gamma N_1((\beta+\gamma)N_1 - 2\lambda)\right]^{1/2}$
(A_2)	$\pm\left[\gamma^2 N_2^2 - (\beta N_2 + 2\lambda)^2\right]^{1/2}$	(B_-)	$\pm\left[2\gamma N_1((\beta-\gamma)N_1 - 2\lambda)\right]^{1/2}$.

These eigenvalues are computed from the matrix $L = JD^2H$ evaluated at the periodic orbit in question. Here J is of course the usual skew-symmetric matrix arising in Hamilton's equations. The matrix L also has a double eigenvalue at zero corresponding to the directions along the orbit and transverse to the energy level.

To pass from the fourth-order normal form to the full system, one uses exactly the same perturbation argument as in Montaldi *et al*[8].

5 Normal form for the magnetic spherical pendulum

In this section, we translate the nondegeneracy condition (2.2) on the fourth order normal form into one on the coefficients in the magnetic spherical pendulum. Since λ only multiplies a detuning term, the only condition on the magnetic term is that the speed of the splitting should be non-zero. The conditions on α, β and γ refer merely to the unperturbed spherical pendulum.

Consider a spherical pendulum in 3-space, with the z-axis vertical, and the origin at the centre. The potential and kinetic energies, in terms of $q_1 = x$ and $q_2 = y$ and $z = -\sqrt{\ell^2 - q_1^2 - q_2^2}$, are

$$V(q_1, q_2) = mg(z + \ell), \qquad T = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2m\ell^2}(q_1 p_1 + q_2 p_2)^2,$$

where ℓ is the length of the pendulum, m the mass of the bob, and g the acceleration due to gravity. The p_j are the canonical momenta associated to the coordinates q_j. The Hamiltonian is $H = V + T$, which is of course $\mathbf{O}(2)$ invariant, and its Taylor expansion to degree 4 is,

$$H^{(4)}(q_1, q_2, p_1, p_2) = \frac{mg}{2\ell}(q_1^2 + q_2^2) + \frac{1}{2m}(p_1^2 + p_2^2) + \frac{mg}{8\ell^3}(q_1^2 + q_2^2)^2 + \frac{1}{2m\ell^2}(q_1 p_1 + q_2 p_2)^2.$$

Rescaling q and p canonically, rescaling time by $\sqrt{\ell/g}$ and introducing

$$z_1 = \tfrac{1}{2}((q_1 + ip_1) + i(q_2 + ip_2)), \quad z_2 = \tfrac{1}{2}((q_1 + ip_1) - i(q_2 + ip_2))$$

puts the quadratic part of the Hamiltonian in the form used in Section 1, and

$$H^{(4)} = (N_1 + N_2) + \frac{1}{8\mu}\left((N_1 + N_2)^2 - 6N_1N_2\right) + \frac{1}{8\mu}\left(2(N_1 + N_2)(z_1 z_2 + \overline{z}_1\overline{z}_2) + 5(z_1^2 z_2^2 + \overline{z}_1^2\overline{z}_2)\right)$$

where $\mu = m\sqrt{\ell^3 g}$. This expression is written as much as is possible in terms of the invariants N_1 and N_2. To obtain the normal form to fourth order, one averages over the S^1-action arising from the linear part. All the terms in the expression above that are not written entirely in terms of N_1 and N_2 are of average zero, so the normal form to degree 4 is

$$H^{(4)}(z_1, z_2) = (N_1 + N_2) + \frac{1}{8\mu}\left((N_1 + N_2)^2 - 6N_1N_2\right).$$

The coefficients in the fourth order normal form (1.3) are thus $\alpha = 1/8\mu$ and $\beta = -3/4\mu$. The nondegeneracy conditions (2.2) are therefore satisfied, so there is a homeomorphism mapping the nonlinear normal modes of the magnetic spherical pendulum to those of the fourth order normal form, which are given by Corollary 2.2, and the corresponding Figure 2. Moreover, since $\beta < 0$ an inspection of Corollary 2.2 shows that for all $\lambda \neq 0$, the family (B) bifurcates off the slower branch of rotating modes as claimed in the introduction.

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