An Axiom System for a Spatial Logic with Convexity

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Contents

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List of important predicates from Chapter 5

$hp[l]$	l is a half-plane
$\alpha[l_1, l_2]$	lines bounding half-planes l_1 and l_2 are coincident
$par[l_1, l_2]$	lines bounding half-planes l_1 and l_2 are parallel
$\Gamma[l_1, l_2, l_3]$	lines bounding half-planes l_1, l_2 and l_3 meet at a single point
$coor[l_1, l_2, l_3]$	lines bounding half-planes l_1, l_2 and l_3 form a coordinate frame
$\ll l_1, l_2 \gg$	lines bounding half-planes l_1 and l_2 form a general line pair
$\ll l_1, l_2 \gg \doteq \ll l_3, l_4 \gg$	lines bounding half-planes l_1, l_2, l_3 and l_4 form general line pairs that determine the same point
$add[l_1, l_2, l_3, a, b, c]$	$\overline{\mathbf{OA}} + \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ in reference to the coordinate frame formed by lines bounding half-planes l_1, l_2 and l_3
$mult[l_1, l_2, l_3, a, b, c]$	$\overline{\mathbf{OA}} \cdot \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ in reference to the coordinate frame formed by lines bounding half-planes l_1, l_2 and l_3
$power_n[l_1, l_2, l_3, a, b_n]$	$\overline{\mathbf{OA}}^n = \overline{\mathbf{OB}}$ in reference to the coordinate frame formed by lines bounding half-planes l_1, l_2 and l_3
$\tau^j_{(P,Q)}[l_1, l_2, l_3, m]$	m is fixed with respect to the coordinate frame formed by lines bounding half-planes l_1, l_2, l_3
$\beta[l, l_1, l_2, l_3]$	the point determined by $\ll l, l_2 \gg$ lies between the points determined by $\ll l, l_1 \gg$ and $\ll l, l_3 \gg$

See Chapter 5 for full explanation of the above.

Abstract

A spatial logic is any formal language with geometric interpretation. Research on region-based spatial logics, where variables are set to range over certain subsets of geometric space, have been investigated recently within the qualitative spatial reasoning paradigm in AI.
Building on the results from [Pra99] on spatial logics with convexity, we axiomatised the theory of $\langle ROQ(\mathbb{R}^2), conv, \leq\rangle$, where $ROQ(\mathbb{R}^2)$ is the set of regular open rational polygons of the real plane; $conv$ is the convexity property and $\leq$ is the inclusion relation. We proved soundness and completeness theorems. We also proved several expressiveness results. Additionally, we provide a historical and philosophical overview of the topic and present contemporary results relating to affine spatial logics.

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Keywords: spatial logic, convexity, axiomatisation.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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”It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way.”

Charles Dickens, *A Tale of Two Cities*

First, I wish to thank my abecedarians in Poland.

I wish to thank George Wilmers for being very understanding of my shortcomings and for teaching me the word *quixotic*. I wish to thank my supervisor, Ian Pratt-Hartmann for being strict and demanding with me. This thesis would not have happened if it was not for him. If I can ever consider myself a researcher, it is mainly thanks to you Ian.

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Last but not least, thank you Manchester, it has been an amazing adventure.

1 Introduction

Introduction This thesis concerns region-based spatial logic with convexity. What is spatial logic? Informally, spatial logic can be viewed as a formal language with geometrical interpretation, where variables range over geometrical entities and relation and function symbols are interpreted as geometrical relations and functions. In terms of geometry, it encompasses *inter alia* Euclidean geometry and topology. As an example consider a language with two primitive symbols, one denoting a ternary betweenness relation on points and the other denoting the relation ”the distance from point a to point b is the same as the distance from point c to point d”. This is one of the first spatial logics, investigated by Alfred Tarski (see [TG99]) and called by him *Elementary Geometry*.

Although the name may be an invention of the early twenty-first century, spatial logics have rich and diverse background. The origins of spatial logic can be traced to the early developments in formal geometry. The first, and still the best-known, formalisation of geometry was undertaken in the *Elements* by Euclid. It was the development of the tools of model theory and formal logic in the first half of 20^{th}century that allowed researchers to probe the inferential and expressive power of geometry. The novelty of this approach consists in changing the focus from geometry itself to the *language* that describes it. This allows one to describe several languages and compare them in terms of expressivity and tackle the problem of their computational complexity with mathematically precise tools.

Constructing a spatial logic If we were to custom-build a spatial logic, the first problem we are going to face is the choice of underlying geometric space. Many approaches have been studied, in most of them however either $\mathbb{R}^n$ for some n or some more general topological space is considered. The choice of $\mathbb{R}^3$ comes to mind first, due to its proximity to the space of our every-day experience. It proves a very hard object of study, hence often times $\mathbb{R}^2$ is studied in lieu of $\mathbb{R}^3$. Obviously, given its use throughout mathematics and computer science, there are perfectly good reasons to study $\mathbb{R}^2$ in its own right.

Having set on the underlying geometric space, say X, we are faced with another decision. Should the variables range over elements of X or some subset $S \subseteq 2^X$? In the first case we would be talking about *point-based* spatial logics, in the second about *region-based* spatial logics. There are, obviously, good reasons to study point-based spatial logics. After all, one can argue that, since points are "atoms" of most geometric spaces, it seems reasonable, to construct logical formalisms that mirror the granular nature of geometric spaces. Also, it would seem that our intuitions about the space we inhabit and which ultimately serves as a basis for any mathematical interpolation is inherently point-based. However, many a mathematician has pondered the "strange" status of points. For example, points in $\mathbb{R}^n$ have no dimension (or are of 0-dimension) and yet they serve as the building blocks for all other many-dimensional geometric entities. After all, in our day-to-day spatial reasoning tasks we do not rely on points as the building blocks of nature. It is rather regions that we reason with. Ultimately then, it might be the idiosyncrasies of Greek mathematical tradition, pinnacle of which was Euclid's *Elements*, enforced by centuries of repetition, that is to blame for our attachment to points.

The development of automated computing that has begun in the last century exposed yet more weaknesses of the point-based approach. Point-based spatial logics are in many cases computationally heavy. One can also argue that, since point-based approach is often connected with numerical, *quantitative* approach, data handling processes are much more error-prone. This led some researches to pursue alternative, region-based approach. This field of study has become known as *qualitative spatial reasoning* (QSR). Qualitative in this context means that all the primitive relations and functions are of non-numerical nature. The hope was above all that this will make spatial reasoning more tractable from the computational point of view. For example consider a language with a single relation symbol C understood as the contact relation. Intuitively two sets are in contact if their boundaries share at least one point. This spatial logic was investigated under many guises, most notably within the qualitative spatial reasoning paradigm.

As we saw, there are compelling reasons to choose region-based approach over the point-based one. The question arises now as to what *sort* of regions should we consider? We could obviously decide to consider all $S \subseteq 2^X$ for a given space X. Are there any reasons to consider a special class of regions rather than give them all an equal footing? One such reason is the admittedly vague notion of *well-behavedness*. In what follows we attempt to make this notion more precise. (More technical treatment of the topic is to be found in chapter 2 where all the terms used here are given proper mathematical definitions.)

First of all to smooth out the reasoning with regions, we would like to weed out as many "special cases" as possible. Assuming we are working with some topological space, this can be done by considering only *regular* subsets of that space as plausible region-candidates. This gets rid of many a "strange" set e.g. of fractal nature. In the next step we need to decide whether we consider our regions to contain their boundaries or not. In the first case we end up with regular *closed* sets and in the second case with regular *open* sets. From a formal point of view, this is not an essential choice. In the remainder we will consider mainly regular open variants (and everything we say can be applied *mutatis mutandis* to the regular closed case). However, in describing work done in the past we often use regular closed variant as well.

The class of all regular open subsets of some topological space is already a good choice for the well-behaved regions. Apart from what has been mentioned already, by a well-known result the elements of the class of regular open subsets of some topological space form a Boolean Algebra. That is, operations of sum, product and complement of regular open sets conform to the laws of Boolean Algebra. We can do better still. We can look inside this class for some better region-candidates.

Note that in other areas of computer science, the approximation of real life objects as polygons is nearly universal. The concept of polygons is widely used in computational geometry. It is also employed in many practical applications, like virtual reality, computer vision or virtual production.[1] We single

[1] We do not deal in detail with these approaches here. For more information please consult [PS85] for a gentle introduction to computational geometry and [BZ01] for computer vision and virtual manufacturing applications.

out two classes: (regular open) *polygons* and (regular open) *rational polygons*. The fact that it is countable, makes the second subclass especially interesting from the point of view of computer science applications.

The choice of geometric space and either point- or region-based approach dictates the choice of relations and functions that we are presented with. Within the qualitative spatial reasoning paradigm, non-numerical predicates on regions are considered, most notably contact and connectedness. Traditionally spatial logics over languages containing relation and function symbols interpreted as relations and functions invariant under certain geometric transformations (Euclidean, topological, etc.) are called accordingly as e.g. Euclidean, topological (spatial) logic. We follow this convention here.[2] For example, consider an *affine* spatial logic constructed in the following manner. Start with a language with two primitive symbols $conv$ and $\leq$ let them denote the following predicates defined on regular open rational polygonal subsets of $\mathbb{R}^2$. The symbol $conv(a)$ is to be understood as "region a is convex" and the symbol $a \leq b$ as "region a is a subset of region b". It is an affine spatial logic, since convexity is an affine-invariant property. This spatial logic is in fact one that we are concerned the most with in this thesis.

The last choice made in constructing a spatial logic concerns the syntactical complexity of the language we want to use. It can be (most likely) first-order logic, propositional logic or a higher-order system. Also, languages of non-classical logics are sometimes considered.

Investigating spatial logics Having defined a spatial logic, we would like to explore its capabilities. We list some of the ways of doing so, summarised in the form of the following problems.

P1 How can we characterize the valid formulas of the spatial logic? That is, what is the theory of a given spatial logic?

P2 What is the expressive power of a spatial language? In particular, given a language, what other geometrical relations can we express in terms of primitive relations in that language?

[2] We note, however, a slight ambiguity here. First of all a relation/function invariant under one type of transformation can be nevertheless invariant under many others (e.g. trivially, any relation invariant under affine transformation is also invariant under Euclidean transformation). Secondly, a spatial logic can contain a combination of primitives invariant under different transformations.

P3 What is the computational complexity of a given spatial logic? Most first-order logics are, for obvious reasons, undecidable. However, restricting attention to certain fragments of those logics, might prove useful in terms of computational tractability.

All this gives rise to the interesting challenge of finding a spatial system balanced between expressive power and undecidability (here the prime example is Tarski's elementary geometry). Spatial logics can be interesting from a viewpoint of formal logic but there are also some more practical motivations for developing them. Most of the motivations come from computer science. The research in the field of *qualitative spatial reasoning*, developed within the area of Artificial Intelligence can serve as the first example. One approach uses a family of region connection calculi in their formalisation of spatial inference processes. Within qualitative paradigm no (numerical) information is required as to the distance between objects (thought of as regions, rather than points). Instead, their position is described by providing the qualitative information — for example which region is a part of which other and which regions are disjoint. We note in passing that Euclid himself does not make any use of numbers in the description of geometrical properties, thus his work might be interpreted as non-quantitative, yet point-based, in character. The theory of spatial databases provides the second more practical motivation for developing (region-based) spatial logic. In computer applications, spatial data is frequently stored in the form of polygons or polyhedra (that is, sets of points definable by Boolean combinations of linear inequalities). Development in this area of research gave rise to the concept of a *constraint database*.

Thesis structure The order of the presentation is as follows.

Chapter 2 contains the necessary mathematical background and notational conventions.

Chapter 3 presents a historical and philosophical background of logical investigations of affine geometry.

Chapter 4 is a presentation of more contemporary region-based spatial logics, both topological and affine.

Chapter 5 contains the main contribution of this thesis — an axiom system for spatial logics with convexity and inclusion predicates (thus dealing with problem **P1** regarding the investigations of the properties of a given spatial logic) together with some expressiveness results (**P2**).

Chapter 6 concludes the thesis and deals with some open problems (e.g. connected to **P3**).

We also include an index of chosen concepts and individuals mentioned in the thesis.

2

Mathematical Background

In this chapter we provide the reader with basic definitions and theorems used throughout the report. For more details consult [End01] [CK73] or [Mar01]. The basic notions used within set theory can be found in [Sup74]. When needed, specific references are provided in the respective sections.

We introduce the following notational conventions. We use

(i) boldface capital letters **A**, **B**, **C** etc. to denote points of a given mathematical space;

(ii) lowercase italicised letters (mostly) from a mid section of the alphabet l, m, n etc. to denote lines and half-planes of a given mathematical space;

(iii) lowercase italicised letters from the end of the alphabet x, y, z etc. to denote variables of a given language;

(iv) lowercase italicised letters from the beginning of the alphabet a, b, c etc. to denote elements of a given domain;

(v) lowercase Greek letters α, β, γ etc. to denote formulas from a given language;

(vi) uppercase Greek letters Σ, Γ, Ψ etc. to denote sets of formulas from a given language.

All the above can be combined with super- and sub- script notation. We reserve the right to change some of these conventions and to introduce new ones as we go.

2.1 Logic

Let $\mathcal{L}$ be a (first-order) language.

Definition 2.1.1 (Signature)**.** A *signature* Σ of $\mathcal{L}$ is a set of symbols given by specifying the following data:

(i) a set of m-placed function symbols $\mathcal{F}$ $(m \geq 1)$;

(ii) a set of n-placed relation symbols $\mathcal{R}$ $(n \geq 1)$;

(iii) a set of constant symbols $\mathcal{C}$.

Note that any or all of the sets $\mathcal{F}, \mathcal{R}, \mathcal{C}$ might be empty.

We often denote a (first-order) language with the signature Σ by $\mathcal{L}_\Sigma$. We sometimes refer to certain restrictions of a given first-order language $\mathcal{L}$: the quantifier-free fragment, the existential fragment (not containing the universal quantifier — also dubbed *constraint* language — denoted $\mathcal{L}^c_\Sigma$) and the universal fragment (not containing the existential quantifier).

Definition 2.1.2 ($\mathcal{L}$-term)**.** The set of $\mathcal{L}$*-terms* is the smallest set $Term$ such that:

(i) $c \in Term$ for each constant symbol $c \in \mathcal{C}$;

(ii) each variable symbol $v_i \in Term$, for $i = 1, 2, \ldots$;

(iii) if $t_1, \ldots, t_n \in Term$ and $f \in \mathcal{F}$, then $f(t_1, \ldots, t_n) \in Term$, where n is the arity of f.

Definition 2.1.3 ($\mathcal{L}$-formula)**.** We say that ϕ is an *atomic* $\mathcal{L}$-formula if ϕ is either

(i) $t_1 = t_2$, where t_1 and t_2 are terms, or

(ii) $R(t_1, \ldots, t_m)$, where $R \in \mathcal{R}$, and $t_1, \ldots, t_m$ are terms and m is the arity of R.

The set of $\mathcal{L}$*-formulas* is the smallest set $Formula$ containing the atomic formulas such that:

(i) if $\phi \in Formula$, then $\neg\phi$ is in $Formula$,

(ii) if $\phi, \psi \in Formula$, then $\phi \wedge \psi$ and $\phi \vee \psi \in Formula$, and

(iii) if $\phi \in Formula$, then $\exists_{v_i}\phi \in Formula$.

By the *scope* of the quantifier in a formula ϕ we mean the part of ϕ contained within a pair of brackets, leftmost of which is placed immediately after the quantifier. We drop the bracketing if it is clear from the context. We say that a variable v_i occurs *freely* in a formula ϕ, or that v_i is *free* in ϕ, if it is not within the scope of any quantifier $\exists v_i, \forall v_i$; otherwise we say that v_i is *bound*. We write $\phi(v_0, \ldots, v_n)$ (often abbreviated to $\phi(\overline{v})$ to denote a formula ϕ whose free variables form a subset of $\{v_0, \ldots, v_n\}$. We use the letters $x, y, z, \ldots$, possibly with the superscripts, to denote the variables.

We call an $\mathcal{L}$-formula an $\mathcal{L}$*-sentence* if it has no free variables.

Definition 2.1.4 ($\mathcal{L}$-Structure)**.** Let $\mathcal{L}$ be a language. An $\mathcal{L}$*-structure* $\mathcal{M}$ is given by the following data:

(i) a non-empty set Mcalled the *universe*, domain or underlying set of $\mathcal{M}$;

(ii) a function $f^{\mathcal{M}} : M^n \to M$ for each n-ary $f \in \mathcal{F}$;

(iii) a set $R^{\mathcal{M}} \subseteq M^m$ for each m-ary $R \in \mathcal{R}$;

(iv) an element $c^{\mathcal{M}} \in M$ for each $c \in C$.

We often write

$$\mathcal{M} = \langle M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C} \rangle.$$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the *interpretations* of the symbols f, R, c. By abusing the notation we will drop the superscript whenever it is clear from the context that we are talking about interpretations. We will use the notation A, B, M, N, etc. to refer to the underlying sets of the structures $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$, etc.

Definition 2.1.5 (Assignment)**.** Let $\langle x_0, x_1, \ldots \rangle$ be an infinite sequence of variables. An infinite sequence $\langle a_0, a_1, \ldots \rangle$ of elements of M is called an $\mathcal{M}$-*assignment*. Intuitively, we think of elements of an $\mathcal{M}$-assignment as assigning the value a_i to the free variable x_i. Given a term t and model $\mathcal{M}$, the *interpretation* of t in $\mathcal{M}$ under the assignment $\langle a_0, a_1, \ldots \rangle$ is defined in the obvious way.

Definition 2.1.6 (Truth in a model)**.** Let ϕ be a formula with free variables from $\overline{v} = \langle v_{i_1}, \dots v_{i_n} \rangle$, and let $\overline{a} = \langle a_{i_1}, \dots, a_{i_n} \rangle \in M^n$. We inductively define $\mathcal{M} \models \phi[\overline{a}]$ as follows.

(i) if ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi[\overline{a}]$ if $t_1^{\mathcal{M}}[\overline{a}] = t_2^{\mathcal{M}}[\overline{a}]$;

(ii) if ϕ is $R(t_1, \dots, t_m)$, then $\mathcal{M} \models \phi[\overline{a}]$ if $(t_1^{\mathcal{M}}[\overline{a}], \dots, t_m^{\mathcal{M}}[\overline{a}]) \in R^{\mathcal{M}}$;

(iii) if ϕ is $\neg\psi$, then $\mathcal{M} \models \phi[\overline{a}]$ if $\mathcal{M} \not\models \phi[\overline{a}]$;

(iv) if ϕ is $\psi \wedge \theta$, then $\mathcal{M} \models \phi[\overline{a}]$ if $\mathcal{M} \models \psi[\overline{a}]$ and $\mathcal{M} \models \theta[\overline{a}]$;

(v) if ϕ is $\psi \vee \theta$, then $\mathcal{M} \models \phi[\overline{a}]$ if $\mathcal{M} \models \psi[\overline{a}]$ or $\mathcal{M} \models \theta[\overline{a}]$;

(vi) if ϕ is $\exists v_j \phi(\overline{v}, v_j)$, then $\mathcal{M} \models \phi[\overline{a}]$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi[\overline{a}, b]$;

(vii) if ϕ is $\forall v_j \phi(\overline{v}, v_j)$, then $\mathcal{M} \models \phi[\overline{a}]$ if $\mathcal{M} \models \psi[\overline{a}, b]$ for all $b \in M$.

If $\mathcal{M} \models \phi[\overline{a}]$ we say that $\mathcal{M}$ *satisfies* $\phi[\overline{a}]$ or $\phi[\overline{a}]$ is *true* in $\mathcal{M}$.

Definition 2.1.7 ($\mathcal{L}$-Theory)**.** An $\mathcal{L}$-*theory* T is a set of $\mathcal{L}$-sentences. We say that a structure $\mathcal{M}$ is a *model* of T or that T has a model $\mathcal{M}$ and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. By the *theory* of $\mathcal{M}$ we mean the set $\{\phi \mid \mathcal{M} \models \phi\}$. We often write $Th(\mathcal{M})$ to denote the theory of $\mathcal{M}$.

Definition 2.1.8 ($\mathcal{L}$-Embedding)**.** Suppose that $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is an injective function $\eta : M \to N$ preserving the interpretation of all the symbols of $\mathcal{L}$. More precisely:

(i) $\eta(f^{\mathcal{M}}[a_1, \dots, a_m]) = f^{\mathcal{N}}(\eta(a_1), \dots, \eta(a_n))$ for all $f \in \mathcal{F}$ and $a_1, \dots, a_n \in \mathcal{M}$, where m is the arity of f;

(ii) $\langle a_1, \dots, a_n \rangle \in R^{\mathcal{M}}$ if and only if $\langle \eta(a_1), \dots, \eta(a_n) \rangle \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \dots, a_n \in M$ and n is the arity of R;

(iii) $\eta(c^{\mathcal{M}}) = c^{N}$ for $c \in \mathcal{C}$.

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-*isomorphism*. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or that $\mathcal{N}$ is an *extension* of $\mathcal{M}$.

Definition 2.1.9 (Elementary Equivalence)**.** We say that two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* and write $\mathcal{M} \equiv \mathcal{N}$ if $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$ for all $\mathcal{L}$-sentences ϕ.

We have the following result.

Theorem 2.1.1. *Suppose that $j : M \to N$ is an isomorphism. Then, $\mathcal{M} \equiv \mathcal{N}$.*

We define all set-theoretic notions in the usual way.

2.2 Boolean Algebra

Definition 2.2.1 (Boolean Algebra)**.** A *Boolean Algebra* is a structure

$$\mathcal{B} = \langle B, +, \cdot, -, 0, 1 \rangle$$

with two binary operations $+$ and $\cdot$, unary operation $-$ and two constants 0 and 1, such that for all $x, y, z \in B$:

$$x + -x = 1;$$

$$x \cdot -x = 0;$$

$$x + (y + z) = (x + y) + z;$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z;$$

$$x + y = y + x;$$

$$x \cdot y = y \cdot x;$$

$$x + (x \cdot y) = x;$$

$$x \cdot (x + y) = x;$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z);$$

$$x + (y \cdot z) = (x + y) \cdot (x + z).$$

A Boolean algebra $\mathcal{B}$ with B having only one element is called a *trivial* Boolean algebra or a *degenerate* Boolean algebra.

Definition 2.2.2 (Partially ordered set)**.** The structure $\mathcal{P} = \langle P, \leq \rangle$, such that P is a set and $\leq$ is a *partial order* is called a *partially ordered set* or a *poset*.

We can define a partial order $\leq$ on a Boolean algebra $\mathcal{B}$ by

$$x \leq y \text{ if and only if } x = x \cdot y \text{ if and only if } y = x + y.$$

An *atom* in a Boolean algebra is a nonzero element x such that there is no element y such that $0 < y < x$. A Boolean algebra is *atomic* if every nonzero element of the algebra is above an atom. We sometimes represent a Boolean algebra $\mathcal{B}$ as a partial order $\mathcal{B} = \langle B, \leq \rangle$ instead of the standard $\mathcal{B} = \langle B, +, \cdot, -, 0, 1 \rangle$.

Definition 2.2.3 (Complete Boolean Algebra)**.** A Boolean algebra $\mathcal{B} = \langle B, \leq \rangle$ is called *complete* if for every $A \subseteq B$, $\inf A$ and $\sup A$ exist.

Definition 2.2.4 (Subalgebra)**.** Let $\mathcal{B}$ be a Boolean algebra and $A \subseteq B$. We say that $\mathcal{A}$ is a Boolean *subalgebra* of $\mathcal{B}$ if $+, -, \cdot, 0, 1$ have the same meaning in $\mathcal{A}$ as they do in $\mathcal{B}$. We say that $\mathcal{A}$ is a *dense* subalgebra of $\mathcal{B}$ if, for every $x \in B$ with $0 < x$, there exists $y \in A$ such that $0 < y \leq x$.

2.3 Theory of Fields

Definition 2.3.1 (Monoid)**.** A *monoid* is a structure $\mathfrak{S} = \langle S, +, 0 \rangle$ with $+$ a binary operation and 0 a constant such that for all $x \in S$ the following holds.

$$x + (y + z) \equiv (x + y) + z.$$

$$x + 0 \equiv x, \; 0 + x \equiv x.$$

When $+$ is used to denote the binary relation in a monoid $\mathcal{S}$, it is customary to refer to $\mathcal{S}$ as an *additive* monoid. We note that sometimes $\cdot$ is used in stead of $+$. Then, the monoid is termed *multiplicative*. This convention carries over to groups.

Definition 2.3.2 (Group)**.** Let $\mathfrak{G} = \langle G, +, 0\rangle$ be a monoid. We call $\mathfrak{G}$ a *group* if for all $x \in G$ there exists $y \in G$ such that the following holds.

$$(x + y \equiv 0 \wedge y + x \equiv 0)$$

Definition 2.3.3 (Abelian Group)**.** Let $\mathfrak{G} = \langle G, +, 0\rangle$ be a group. We call $\mathfrak{G}$ an *Abelian* group if for all $x, y \in G$ the following holds.

$$x + y \equiv y + x$$

Definition 2.3.4 (Ring)**.** Let $\langle R, +, 0\rangle$ be an Abelian group and let $\langle R, \cdot, 1\rangle$ be a monoid. A structure $\mathfrak{R} = \langle R, +, \cdot, 0, 1\rangle$ is a *ring* if for all $x, y \in R$ the following holds.

$$x \cdot (y + z) \equiv (x \cdot y) + (x \cdot z)$$

Note that 1 and 0 need not be distinct. A ring is *commutative* if the multiplicative monoid is.

Definition 2.3.5 (Division Ring)**.** Let $\mathfrak{R} = \langle R, +, \cdot, 0, 1\rangle$ be a ring. We say that $\mathfrak{R}$ is a *division ring* if

$$0 \not\equiv 1;$$

and for all $x \in R$ there exists $y \in R$ such that

$$(x \cdot y \equiv 1 \wedge y \cdot x \equiv 1)$$

We denote division rings by $\mathbb{D}$.

Definition 2.3.6 (Field)**.** By a *field* $\mathfrak{F}$ we mean a commutative division ring. By an *ordered field* we mean a field $\mathfrak{F}$ together with a total order $\leq$ on F satisfying the following conditions (universal quantification is implied).

$$x \leq y \rightarrow x + z \leq y + z$$

$$0 \leq x \wedge 0 \leq y \rightarrow 0 \leq x \cdot y.$$

Definition 2.3.7 (Real Closed Field)**.** Let $\mathfrak{F}$ be an ordered field. $\mathfrak{F}$ is called *Euclidean* if every non-negative element in F is a square. An Euclidean field is called *real closed* if every polynomial of an odd degree with coefficients in F has a zero in F.

2.4 Topology

For more in-depth study of the field please consult [Kel75].

Definition 2.4.1 (Topological Space)**.** A *topological space* X is a set, with a specified family of subsets τ s.t.:

1. $\emptyset, X \in \tau$,
2. if $U_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} U_j \in \tau$,
3. $U \in \tau$ and $V \in \tau$, then $U \cap V \in \tau$.

The specified family of open sets τ is called the *topology* on X. A topological space is sometimes referred to as (X, τ), where X is a set and τ is a topology on X. If the topology is clear from the context we refer to the topological space (X, τ) as X.

Definition 2.4.2 (Continuous Function)**.** Let X, Y be topological spaces. A function $f : X \rightarrow Y$, where $X, Y \subseteq \mathbb{R}^n$, is continuous if and only if for each open subset V of Y, the subset $f^{-1}(V)$ is open in X.

Let us now introduce some basic concepts used within topology.

Let (X, τ) be a topological space and $V \subseteq X$. By the *complement* of V, written $\mathcal{C}(V)$, we mean the following.

$$\mathcal{C}(V) = \{x \mid x \in X, x \notin V\}.$$

Definition 2.4.3 (Closed Set)**.** Let X be a topological space. A set $A \subseteq$ X is *closed* if the complement of A is open.

The following is a very important notion from out point of view.

Definition 2.4.4 (Regular open/closed set)**.** Let X be a topological space. For every $u \subseteq X$, the set $r = (u)^{-^0}$ is called *regular open* and the set $r = (u)^{0^-}$ is called *regular closed*.

Definition 2.4.5 (Interior)**.** Let (X, τ) be a topological space and $V \subseteq X$. We define the *interior* of V, written V^0, as being the largest open subset of X, or member of τ, included within V.

Definition 2.4.6 (Closure)**.** Let (X, τ) be a topological space and $V \subseteq X$. We define the *closure* of V, written V^-, as the smallest closed subset of X, or member of set of complements of τ, which includes V.

Remark. The following equivalence holds: $V^- = -((-V)^0)$.

Definition 2.4.7 (Boundary)**.** Let (X, τ) be a topological space and $V \subseteq X$. The *boundary* of V, written $b(V)$ is defined as follows:

$$b(V) = V^- \cap -(V^0).$$

Definition 2.4.8. Let $\langle X, \tau \rangle$ be a topological space. A collection B of open subsets of X is said to be a *basis* for the topology τ if every open set is a union of members of B.

The *separation axioms* are additional conditions which may be required to a topological space in order to ensure that some particular types of sets can be separated by open sets, thus avoiding certain pathological cases. The following lists names and associated conditions for topological spaces, which are most important from our point of view.

Name	**Definition**
Semi-regular space	has a basis of regular open sets
Weakly regular space	semi-regular and for any non-empty open set u, there exists a non-empty open set v with $v^- \subseteq u$

2.5 Affine and Projective Geometry

The very general notion of a mathematical space is notoriously hard to capture precisely. Very vaguely, a space consists of a set and a construction imposing some structure on that set. We base our description of affine and projective spaces primarily on [Ben95]. For this section only we adopt the following notational conventions: $l(A, B)$ denotes the line through points A, B and $\|$ reads *is parallel to* (two lines are said to be parallel if their intersection is empty).

Definition 2.5.1 ([Ben95], p. 123)**.** An *affine space* $\mathfrak{A}$ is an ordered triple $\langle P, L, E\rangle$, where P is a nonempty set whose elements are called points and L and E are nonempty collections of subsets of P called lines and planes, respectively, satisfying the following:

1. If P and Q are distinct points, there is a unique line l such that $P \in l$ and $Q \in l$;
2. If P, Q and R are distinct noncollinear points, there is a unique plane containing them;
3. If P is a point not contained in a line l, there is a unique line m such that $P \in m$ and $m \cap l = \emptyset$;
4. If l, m and k are distinct lines with $l \cap m = \emptyset$ and $m \cap k = \emptyset$, then $l \cap k = \emptyset$.

Definition 2.5.2. An *affine plane* $\mathfrak{A}$ is an ordered pair $\langle P, L\rangle$, where P is a nonempty set whose elements are called points and L is a nonempty collection of subsets of P called lines, satisfying the following:

1. If P and Q are distinct points, there is a unique line l such that $P \in l$ and $Q \in l$;
2. If P is a point not contained in the line l, there is a unique line m such that $P \in m$ and $m \cap l = \emptyset$;
3. There are at least two points on each line;
4. There are at least two lines.

Familiar examples of affine planes include $\mathbb{R}^2$ and the rational coordinate plane $\mathbb{Q}^2$.

Definition 2.5.3 ([Ben95], p. 144)**.** A *projective space* $\mathfrak{P}$ is an ordered pair $\langle P, L\rangle$, where P is a nonempty set whose elements are called points and L is a nonempty collection of subsets of P called lines satisfying the following:

1. If P and Q are distinct points, there is a unique line l such that $P \in l$ and $Q \in l$;
2. If A, B, C and D are distinct points such that there is a point E in $l(A, B) \cap l(C, D)$, then there is a point F in $l(A, C) \cap l(B, D)$ [Pasch's Axiom];
3. There are at least three points on each line;
4. Not all points are collinear.

Definition 2.5.4. A *projective plane* $\mathfrak{P}$ is an ordered pair $\langle P, L\rangle$, where P is a nonempty set whose elements are called points and L is a nonempty collection of subsets of P called lines satisfying the following:

1. If P and Q are distinct points, there is a unique line l such that $P \in l$ and $Q \in l$;
2. If l and m are distinct lines in L, then $l \cap m \neq \emptyset$;
3. There are at least three points on each line;
4. There are at least two lines.

Two usual examples of projective planes are the *Euclidean hemisphere* and the so-called *real projective plane*, denoted $\mathbb{PR}^2$ (see [Ben95], p. 42). We now show how $\mathbb{PR}^2$ is defined.

Definition 2.5.5. A *projective point* is a line in $\mathbb{R}^3$ that passes through the origin of $\mathbb{R}^3$. The *real projective plane* $\mathbb{PR}^2$ is the set of all such points.

The expression $[a, b, c]$, in which the numbers a, b, c are not all zero, represents the point P in $\mathbb{PR}^2$ which consists of the unique line in $\mathbb{R}^3$ that passes through $(0, 0, 0)$ and (a, b, c). We refer to $[a, b, c]$ as *homogeneous coordinates* of P.

Affine and Projective Planes Given an affine plane $\langle P, L\rangle$ we extend it in the following way. For each pencil ϕ of parallel lines we define a symbol X_ϕ called a point at infinity. Let $P' = P \cup \{X_\phi \mid \phi \text{ is a pencil of parallel lines in } L\}$ and for each $l \in L$ such that $l \in \phi$, define $l' = l \cup \{X_\phi\}$. Let $l_\infty = \{X_\phi \mid \phi \text{ is a pencil of parallel lines in } L\}$. Define $L' = \{l' \mid l \in L\} \cup \{l_\infty\}$.

We have the following two results. [Ben95], p. 43-44.

Theorem 2.5.1. *$\langle P', L'\rangle$ is a projective plane whenever $\langle P, L\rangle$ is an affine plane.*

The proof is a straightforward check that all the axioms of projective plane hold in $\langle P', L'\rangle$. Now consider the following construction. Let $\langle P', L'\rangle$ be a projective plane. Choose $l \in L'$ and call it l_∞. Let $P = \{p \mid p \in P' \wedge p \notin l_\infty\}$. Now for each line l' different than l_∞ define $l = \{p \mid p \in l' \wedge p \notin l_\infty\}$. Define $L = \{l \mid l' \in L'\}$ (note that we have fixed the notation).

Theorem 2.5.2. *If $\langle P', L'\rangle$ is a projective plane, then relative to any line l_∞ in L', $\langle P, L\rangle$ is an affine plane.*

The proof is analogous. This time we are checking that all the axioms of the affine plane hold in $\langle P, L\rangle$.

Recall that an $n \times n$ matrix A is invertible if there exists a $n \times n$ matrix B with $AB = I$, where I is the identity matrix; A is orthogonal if $AA^T = I$, where A^T is the transpose of A (Cf. [Poo06]).

In the following definitions we confine our attention to the Euclidean plane. In describing transformations we follow [Tea94].

Definition 2.5.6. An *affine transformation* of $\mathbb{R}^2$ is a function $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\tau(x) = \mathbf{A}x + b,$$

where $\mathbf{A}$ is an invertible 2×2 matrix and $b \in \mathbb{R}^2$.

The following is standard.

Theorem 2.5.3. *An affine transformation maps straight lines to straight lines, preserves parallelism and ratios of lengths along parallel straight lines. The set of affine transformations forms a group under the operation of composition of functions.*

We also give an example of more familiar (perhaps) *Euclidean* transformation. By an isometry of $\mathbb{R}^2$ we mean a function from $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving distances. It is a standard result that there are four isometries: a translation, a rotation, a reflection, and a glide reflection.

Definition 2.5.7. A *Euclidean transformation* of $\mathbb{R}^2$ is a function $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\tau(x) = \mathbf{U}x + b,$$

where $\mathbf{U}$ is an orthogonal 2×2 matrix and $b \in \mathbb{R}^2$.

Theorem 2.5.4. *In $\mathbb{R}^2$, every isometry is a Euclidean transformation and every Euclidean transformation is an isometry. The set of all Euclidean transformations forms a group under the operation of composition of functions.*

Obviously, every Euclidean transformation of $\mathbb{R}^2$ is an affine transformation of $\mathbb{R}^2$ (every orthogonal matrix is invertible). Hence affine properties must be preserved under Euclidean transformations.

We now turn to the definition of a projective transformation. Let $[a, b, c] = \{\lambda(a, b, c) \mid \lambda \in \mathbb{R}\}$ be a point in $\mathbb{PR}^2$ and let $[A(a, b, c)] = \{\lambda(A(a, b, c)) \mid \lambda \in \mathbb{R}\}$ where A is an invertible 3×3 matrix.

Definition 2.5.8. A *projective transformation* of $\mathbb{PR}^2$ is a function $\tau : \mathbb{PR}^2 \to \mathbb{PR}^2$ of the form

$$\tau([a, b, c]) = [A(a, b, c)].$$

Analogously to the affine and Euclidean cases we have the following theorem.

Theorem 2.5.5. *Projective transformations preserve collinearity, coincidence and cross-ratio of points on a line. The set of all projective transformations forms a group under the operation of composition of functions.*

Affine Theorems in Euclidean plane This section lists a few important results in affine geometry (cf. [Ben95]). Most of these were originally formulated within the context of the Euclidean plane. In our presentation we assume that the underlying affine space is $\mathbb{R}^2$. Also, by $\overline{\mathbf{AB}}$ we mean the Euclidean distance between $\mathbf{A}$ and $\mathbf{B}$.

Theorem 2.5.6 (Pappus)**.** *If lines l and m meet at O, with P, Q, R in l and S, T, U in m, and if $l(P, T) \parallel l(Q, U)$ while $l(Q, S) \parallel l(R, T)$ then $l(P, S) \parallel l(R, U)$.*

Theorem 2.5.7 (Desargues I)**.** *Suppose that A, B, C are distinct noncollinear points with $l(A, A') \parallel l(B, B') \parallel l(C, C')$, $l(A, B) \parallel l(A', B')$ and $l(A, C) \parallel l(A', C')$. Then $l(B, C) \parallel l(B', C')$.*

Theorem 2.5.8 (Desargues II)**.** *Suppose that A, B, C are distinct noncollinear points with $l(A, A')$ $l(B, B')$ $l(C, C')$ meeting at a single point O, $l(A, B) \parallel l(A', B')$ and $l(A, C) \parallel l(A', C')$. Then $l(B, C) \parallel l(B', C')$.*

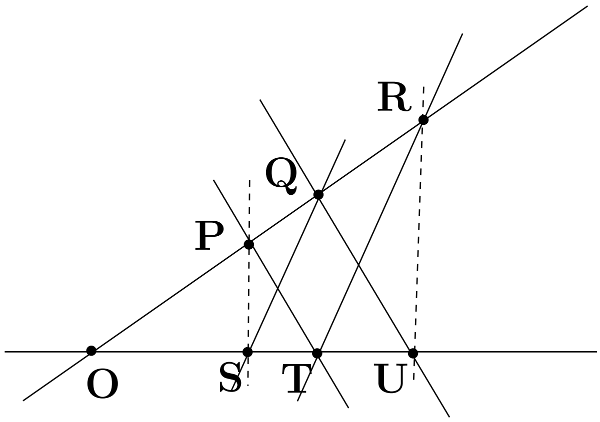


Figure 2.1: Pappus' Theorem in $\mathbb{R}^2$.

It is useful to view the Euclidean plane as the original example of an affine space. As the affine properties were abstracted away, it became obvious that certain desirable properties that held in the Euclidean plane might not hold for many other affine spaces. We introduce the following convention. If Pappus' Theorem holds in an affine space it is called *Pappian*. Similarly, if both Desargues' Theorems hold in an affine space it is called *Arguesian*. We note that Pappus theorem holds in any finite Arguesian plane. Also Pappus theorem implies Desargues' theorems ([Ben95], p. 66).

Definition 2.5.9. We say that $\overline{\mathbf{OC}}$ is the result of *addition* of $\overline{\mathbf{OA}}$ and $\overline{\mathbf{OB}}$ and write $\overline{\mathbf{OA}} + \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ if and only if the following lines can be found (see Fig. 2.3):

(a) l, l' meeting at a single point $\mathbf{O}$;

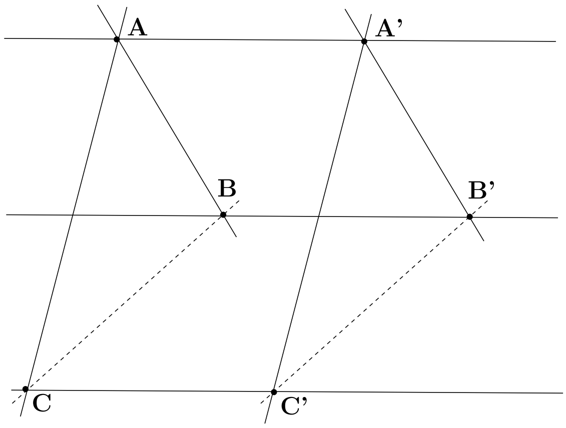
(b) m parallel to l';

(c) l_A such that $\ll l_A, l' \gg = \mathbf{A}$, parallel or coincident with l;

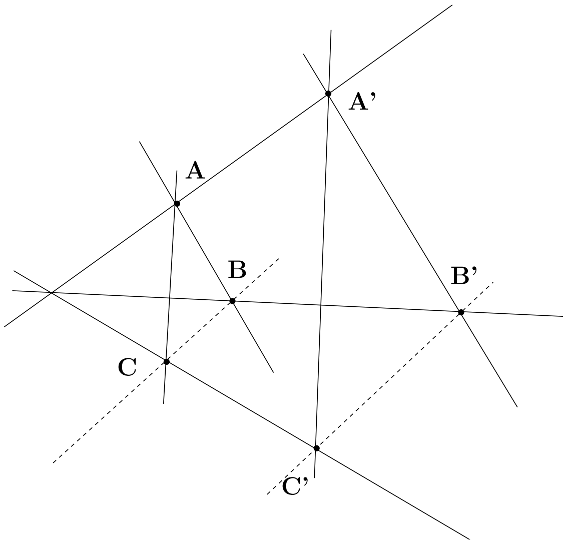
(d) l_B such that $\ll l_B, l' \gg = \mathbf{B}$, and such that l_B, l, m meet at a single point;

(e) l_C such that $\ll l_C, l' \gg = \mathbf{C}$, parallel or coincident with l_B.

Definition 2.5.10. We say that $\overline{\mathbf{OC}}$ is the result of *multiplication* of $\overline{\mathbf{OA}}$ and $\overline{\mathbf{OB}}$ and write $\overline{\mathbf{OA}} \cdot \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ if and only if the following lines can be found (see Fig. 2.4):



(a) Desargues' Theorem I



(b) Desargues' Theorem II

Figure 2.2: Two Desargues' Theorems in $\mathbb{R}^2$. In a projective space these merge into one theorem.

(a) l_1, l_2, l_3 bounding a triangle;

(b) l_A such that $\ll l_A, l_3 \gg = \mathbf{A}$, parallel or coincident with l_2;

(c) l_B such that $\ll l_B, l_3 \gg = \mathbf{B}$, and such that l_B, l_1, l_2 meet at a single point;

(d) l_C such that $\ll l_C, l_3 \gg = \mathbf{C}$, parallel or coincident with l_B and such that l_C, l_A, l_1 meet at a single point.

The coordinatisation of affine planes forms an important part of the field. The following sequence of results describes relation between division rings,

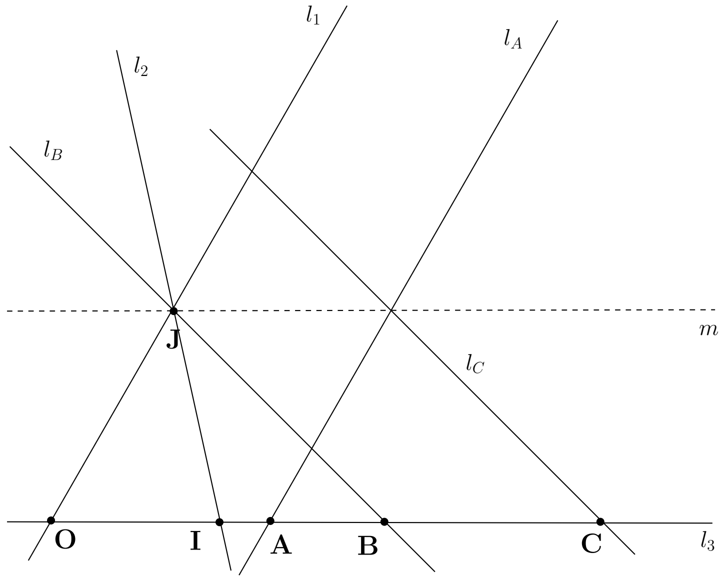


Figure 2.3: $\overline{\mathbf{OA}} + \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$.

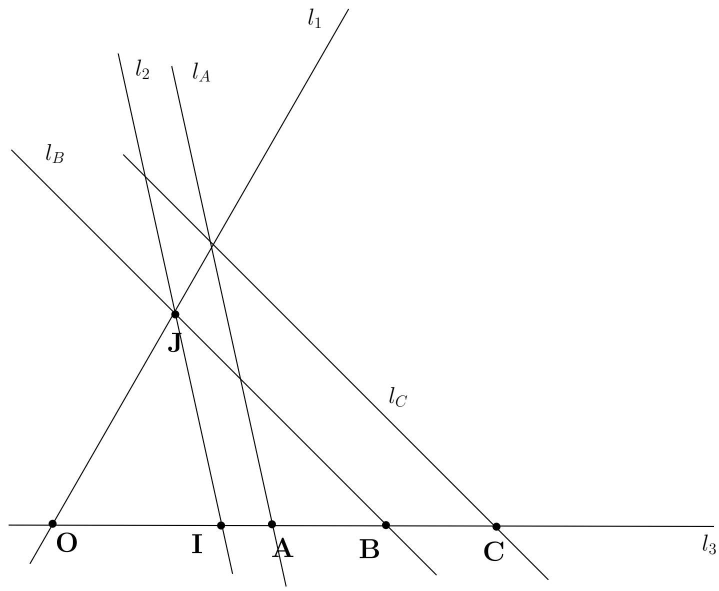


Figure 2.4: $\overline{\mathbf{OA}} \cdot \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$.

affine planes and coordinatisation. We start with the *fundamental theorem of affine geometry*, due to Hilbert.

Theorem 2.5.9 (Fundamental theorem)**.** *Relative to two fixed points O and I any line in an Arguesian affine plane is a division ring.*

Here, addition and multiplication are defined on the line as in Def. 2.5.9 and Def. 2.5.10 and O and I are the additive and multiplicative identities, respectively. The following is the strengthening of the fundamental theorem.

Theorem 2.5.10. *Relative to two fixed points O and I any line in a Pappian affine plane is a field.*

The next two theorems describe the specific conditions relating coordinatisation of affine planes.

Theorem 2.5.11. *For every division ring $\mathbb{D}$ a coordinate plane $\mathbb{D}^2$ is an Arguesian affine plane.*

Theorem 2.5.12. *Every Arguesian affine plane can be considered as a coordinate plane upon renaming its points as ordered pairs of (division) ring elements and associating a linear equation with each line.*

In this context we also note the following theorem by Wedderburn.

Theorem 2.5.13. *Every finite division ring is a field.*

We finish this section with an important construct in affine and projective geometry. We shall return to the following notions for example in section 3.2.2 while describing the historical development of affine and projective geometries in Chapter 3.

Definition 2.5.11. Given three collinear points A, B, C let L be a point not lying on the line through A, B, C. Let any line through C meet $l(L, A), l(L, B)$ at M, N respectively. If $l(A, N)$ and $l(B, M)$ meet at K and $l(L, K)$ meets $l(A, B)$ at D, then the above construction is called the *harmonic ratio* of A, B, C, D and D is called the *harmonic conjugate* of C with respect to A and B.

Definition 2.5.12. Let A, B, C, D be four collinear points. The *anharmonic ratio* $(A, B; C, D)$ is defined as follows $\frac{\overline{AC} \cdot \overline{BD}}{\overline{BC} \cdot \overline{AD}}$.

The importance of harmonic ratio lies in its non-metrical character. If projective geometry can be thought of as non-metrical in essence, the harmonic conjugate is a projective invariant that does not involve any numerical values in its definition. It can be stated in terms of anharmonic ratio (also called a cross-ratio) in the following way: $(A, B; C, D) = -1$.

2.6 Convexity

Definition 2.6.1. A non-empty set $S \subseteq \mathbb{R}^2$ is called *convex* if, for all (ζ_1, ζ_2), $(\zeta_1', \zeta_2') \in S$ and for all $\alpha \in [0, 1]$ we have:

$$(\alpha \cdot \zeta_1 + (1 - \alpha) \cdot \zeta_1', \alpha \cdot \zeta_2 + (1 - \alpha) \cdot \zeta_2') \in S.$$

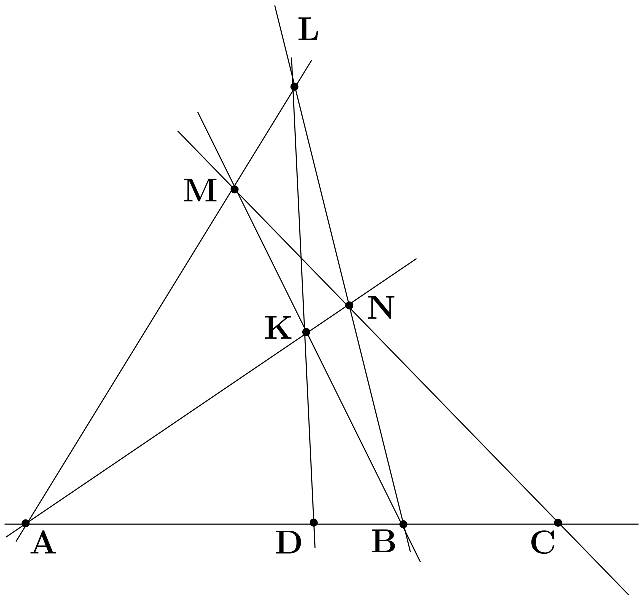
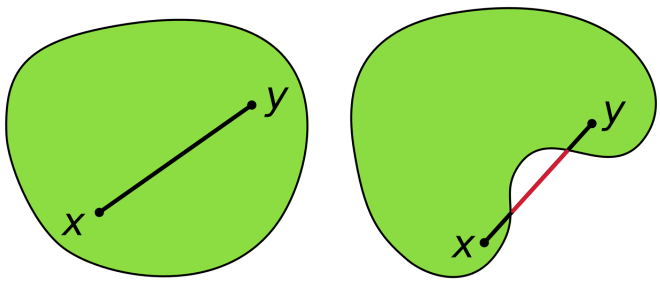


Figure 2.5: Point D is the harmonic conjugate of C with respect to A and B.

The empty set is taken to be non-convex.

In other words for any two points a, b in S the straight line segment between a and b is also in S. That is, for every $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$, we have that $\lambda a + \mu b \in S$.



(a) Convex set (b) Non-convex set

Figure 2.6: An example of convex and non-convex sets in $\mathbb{R}^2$. (Image courtesy of Wikipedia. Creative Commons License.)

We now list several important properties related to convexity (cf. [Web94], [Egg58]).

Lemma 2.6.1. *Let $A \subseteq \mathbb{R}^2$ be a convex set and let τ be an affine transformation. Then $\tau(A)$ is convex.*

Proof. Let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. If $x, y \in \tau(A)$ then $x = \tau(a), y = \tau(b)$ for some $a, b \in A$. Since A is convex, $\lambda a + \mu b \in A$. Since τ is an affine transformation (and hence linear) $\lambda x + \mu y = \lambda\tau(a) + \mu\tau(b) = \tau(\lambda a + \mu b)$. Thus $\lambda x + \mu y \in \tau(A)$. □

Definition 2.6.2. Let $A \subset \mathbb{R}^n$. By the *convex hull* of A, denoted $ch(A)$ we mean the intersection of all convex sets in $\mathbb{R}^n$ containing A.

Obviously for any set its convex hull is unique.

Theorem 2.6.1 (Helly)**.** *Let A be a finite class of N convex sets in $\mathbb{R}^n$ such that $N \geq n+1$ and each $n+1$-element subclass of A has a non-empty intersection. Then all N elements of A have a non-empty intersection.*

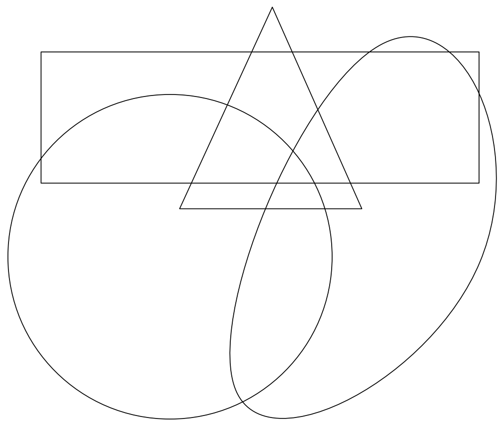


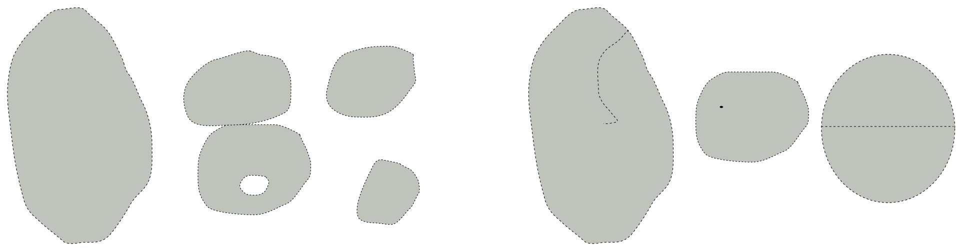
Figure 2.7: A very simple example of Helly's Theorem in $\mathbb{R}^2$.

2.7 Spatial Logic

This section contains definitions and basic results related to region-based spatial logics. Roughly speaking, given a language $\mathcal{L}$ an $\mathcal{L}-$model $\mathfrak{M}$interpreting the primitives of $\mathcal{L}$ geometrically counts as spatial logic. We choose to refer to elements of the domain of a region-based spatial logic as *regions*. We are, however, interested in ruling out as many *degenerate* sets (e.g. of fractal nature) as possible. There are two main reasons for doing so. Firstly, we want regions to model objects of every-day experience. Secondly, good region candidates should be characterised by some sort of regularity both in terms of shape and in terms of their properties. We hope that what follows makes these abstract criteria more concrete. We recall the definition of a *regular open* set.

Definition 2.7.1 (Regular open/closed set)**.** Let X be a topological space. For every $u \subseteq X$, the set $r = (u)^{-0}$ is called *regular open* and the set $r = (u)^{0-}$ is called *regular closed*.

So what do we mean by shape regularity? Let us consider the example of regular open subsets of $\mathbb{R}^2$. These, informally speaking, do not have "cracks" or "pin-holes" that can characterise an arbitrary subset of $\mathbb{R}^2$ (see 2.8).



(a) regular open sets (b) non-regular open sets

Figure 2.8: Examples of Regular and Non-regular Open Sets of $\mathbb{R}^2$ (after [PH07]).

The following theorem (see [Kop89]) exemplifies what we mean by regularity in terms of properties.

Theorem 2.7.1. *The set of regular open sets in X forms a Boolean algebra $RO(X)$ with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $x \cdot y = x \cap y$, $x + y = (x \cup y)^{-0}$ and $-x = (X \setminus x)^0$.*

That is, the behaviour of the members of $RO(X)$ conform to certain rules, the rules of the Boolean Algebra. Let us consider a specific example. Let $X = \mathbb{R}^2$, then the product of two regular open sets is their set-theoretic intersection, the sum of two regular opens is, roughly speaking, the union of the considered sets with internal boundaries removed. Note that we interpret partial order as set-theoretic relation $\subseteq$. (The example behaviour of sum in $RO(\mathbb{R}^2)$ is shown in Figure 2.9.) We might want to take into consideration certain types of regular open sets. In what follows we present three main candidates.

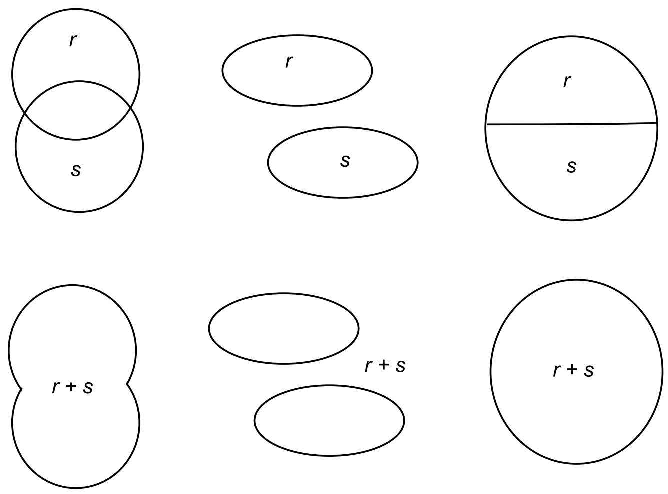


Figure 2.9: The behaviour of sum in $RO(\mathbb{R}^2)$, see [PH07].

Definition 2.7.2. Consider the language $\mathcal{L}_\Sigma$ with $\Sigma = \{<, +, \cdot, 0, 1\}$ with standard arithmetic interpretation. A set $u \subseteq \mathbb{R}^n$ is called *semi-algebraic* if there exists an $\mathcal{L}_\Sigma$-formula $\phi(\bar{x}, \bar{y})$ in $n + m$ variables $\bar{x}, \bar{y}$ and m-tuple of real numbers $\bar{b}$ such that

$$u = \{\bar{a} \in \mathbb{R}^n \mid \text{ the } (n+m)\text{−tuple satisfies the formula } \phi(\bar{x}, \bar{y})\}.$$

We are only interested in certain regular-open semi-algebraic subsets of $\mathbb{R}^n$ (the set of all semi-algebraic subsets of $\mathbb{R}^n$ is denoted by $ROS(\mathbb{R}^n)$). Observe that any $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ cuts it into two regular open half-spaces. Hence the following is well-defined.

Definition 2.7.3 (Polytope)**.** A *basic polytope* in $\mathbb{R}^n$ is the product, in $RO(\mathbb{R}^n)$, of finitely many open half-spaces. A *polytope* in $\mathbb{R}^n$ is the sum, in $RO(\mathbb{R}^n)$, of any finite set of basic polytopes.

We denote the set of polytopes in $\mathbb{R}^n$ by $ROP(\mathbb{R}^n)$. We refer to polytopes in $\mathbb{R}^2$ as *polygons* and polytopes in $\mathbb{R}^3$ as *polyhedra*. If the lines bounding these half-spaces have either rational or algebraic coefficients, we end up with two more region candidates: *rational* and *algebraic* polytopes. We denote the set of rational polytopes in $\mathbb{R}^n$ by $ROQ(\mathbb{R}^n)$ and the set of algebraic polytopes in $\mathbb{R}^n$ by $ROA(\mathbb{R}^n)$. Observe that $ROQ(\mathbb{R}^n) \subset ROP(\mathbb{R}^n)$, $ROA(\mathbb{R}^n) \subset ROP(\mathbb{R}^n)$ and $ROP(\mathbb{R}^n) \subset ROS(\mathbb{R}^n)$. It is an easy result that $ROS(\mathbb{R}^n)$ is a Boolean subalgebra of $RO(\mathbb{R}^n)$. Hence we have the following result.

Theorem 2.7.2. *$ROX(\mathbb{R}^n)$ is a Boolean subalgebra of $RO(\mathbb{R}^n)$, where $X \in \{P, Q, A, S\}$.*

We stress that if in the above case we take $X = \emptyset$ we obtain $RO(\mathbb{R}^2)$. In what follows we mainly focus on spatial logics with the domain $ROQ(\mathbb{R}^2)$ and $ROA(\mathbb{R}^2)$.

We introduce the following convention. By a topological spatial logic we mean a language that contains primitives interpreted as relations or functions invariant under topological transformations. Similarly, by an affine spatial logic we mean a language with primitives interpreted as relations or functions invariant under affine transformations, and so on. If it does not lead to confusion, we sometimes refer to a given spatial logic using one of its primitive relations or functions. One example is convexity spatial logic, so called, because it contains a predicate interpreted as convexity.

2.7.1 Convexity logic

This section introduces convexity spatial logics, the subject of investigation of this thesis. Let $\mathcal{L}_{conv,\leq}$ be the first-order language with two primitive predicates: binary $\leq$ and unary $conv$; and two constant symbols: 0 and 1. First consider the following structure $\mathfrak{M} = \langle RO(\mathbb{R}^2), \leq^{\mathfrak{M}}, conv^{\mathfrak{M}}, 0^{\mathfrak{M}}, 1^{\mathfrak{M}} \rangle$ where the primitives are given the following interpretation.

$$\leq^{\mathfrak{M}} = \{\langle a, b\rangle \in RO(\mathbb{R}^2) \times RO(\mathbb{R}^2) \mid a \subseteq b\};$$

$$conv^{\mathfrak{M}} = \{a \in RO(\mathbb{R}^2) \mid a \text{ is convex}\};$$

$$0^{\mathfrak{M}} = \emptyset;$$

$$1^{\mathfrak{M}} = \mathbb{R}^2.$$

We shall not work directly with $\mathfrak{M}$ but with certain substructures of $\mathfrak{M}$.

Definition 2.7.4. We define the model $\mathfrak{M}_X$ to be a substructure of $\mathfrak{M}$ with the domain $ROX(\mathbb{R}^2)$, where $X \in \{P, Q, A\}$. Note that in case when $X = \emptyset$ we obtain $\mathfrak{M}$.

We sometimes refer to $\mathfrak{M}_P, \mathfrak{M}_Q, \mathfrak{M}_A$ as the *polygonal*, *rational* and *algebraic* model, respectively.

3
History

3.1 Introduction

This chapter concerns the historical and philosophical background of logical investigations of affine geometry. For the sake of presentation we have divided our historical analysis into three periods. The *early* period, concerning mostly non-logical geometrical or philosophical ideas, covers roughly the time up until the first decade of the 20^{th}century. The *transitional* period, so-called because the methods of modern mathematical logic were still being developed then, spans from the end of the early period to the beginning of the second half of the 20^{th}century. The *modern* period covers the time from the 50s to the 90s of the past century. Obviously this division is only conventional and very imprecise: a lot of work done within the transitional period can be justly claimed to belong to the modern period in terms of approach.

The end of the 19^{th}century saw a rapid development of foundations of geometry. Out of the medley collectively refered to as *geometria situ* three separate geometrical theories began to emerge: topology and projective and affine geometry. The mathematical terminology relating to these theories was not fixed yet, so the same ideas came in many guises. This arises when one analyses the historical sources. For example, what Russell and Whitehead call *descriptive* geometry, is in fact referred to these days as *affine* geometry. The name *descriptive* seems to come from the very fact that within this approach the quantitative notions are replaced with qualitative ones. To add confusion, Russell refers to some projective properties by calling them descriptive many times, see e.g. [Rus97], p. 29. Also, Whitehead describes descriptive geometry as any geometry in which two straight lines do not necessarily intersect and hence he does not explicitly refer to the notion of descriptive space being qualitative in character. In the sequel, we shall give precedence to the term *affine*, as the term *descriptive* is nothing more than a historical curiosity. Obviously, whenever we do mention the term *descriptive* geometry" — for reasons of historical accuracy, say — it is to be read *affine* as these terms are treated coextensive. This also applies to any other term which is now known by a different name to the present day mathematical community.

We should note that since research on *topological* formalisms is the dominant theme in spatial logics and intertwines closely with research on affine spatial logics, we decided to give a brief overview of historical development of topological ideas. However, it is not the main theme of our investigations. To the best of our knowledge there is no such survey relating affine and projective ideas in the context of modern formal logic.

We wish to emphasise two main ideas coming out of our historical analysis. The first is the emergence of point-free, region-based approach to geometry, most notably in Whitehead's and Leśniewski's works. The second, perhaps more surprisingly, is the emergence of the qualitative approach to geometry. This we observed in the works of Bertrand Russell and, less explicitly, in Whitehead's proposition.

3.2 The Early Period

Euclid's *Elements* are usually thought to be the most influential book in the history of geometry. Euclid's work set the scene for the development of geometry for the next centuries to come. Geometry was presented there as based on a set of assumptions (postulates) and one primitive notion (that of a point). Arguably the most controversial of these postulates states that,

> given a line l and a point P not on that line there exists exactly one line m through P which is parallel to l.[1]

This postulate is variously called, the parallel postulate, Euclid's axiom or simply the fifth postulate. We use all of these in our description. This section presents the developments in geometry and philosophy of mathematics in the $18^{th}$and 19^{th}centuries. Since our interest lies in setting geometric ideas in the context of logic, we decided to follow Bertrand Russell's description of

[1]There are many equivalent formulations of this postulate. Euclid himself chose a different one, in terms of right angles.

this period.[2]

3.2.1 Idealism vs. Empiricism

Immanuel Kant's ideas on the philosophy of geometry had a tremendous impact on the whole field, so it is worth rehearsing them here. Philosophy from that period saw a shift from *ontological* to *epistemological* issues. Questions regarding human knowledge became more and more prominent. One of the most fundamental questions was the one asking about the origin of human knowledge. In general, one can recognise two approaches to this problem. One is that the nature of all human knowledge is empirical. This view, usually referred to as *empiricism* was predominant (especially prior to Kant) on the British Isles and advocated by philosophers such as David Hume or John Locke. The second possibility is to claim that at least certain knowledge of the real world is independent of human experience. This was the view held by Kant and it is usually branded as *idealism* or *realism*. Kant is famous for his somewhat idiosyncratic philosophical jargon. In his main philosophical works he deals with a combination of two categories of propositions. The first category concerns relation of a predicate and a subject of a proposition, This divides all propositions into *analytic* and synthetic. A proposition is analytic if its predicate is contained in its subject (*All bachelors are unmarried* is a well-known example); it is synthetic if it is not analytic (e.g. *All bachelors are unhappy*). The criterion for a second category concerns conditions of validity of a proposition. Here Kant divides all propositions into *a priori* (necessary), where the truth conditions do not depend on our experience, and *a posteriori*, where they do. This gives us a fourfold classification of prepositions: a priori analytic, a priori synthetic, a posteriori analytic and a posteriori synthetic. One of Kant's greatest contributions to the epistemology is his argument for the existence of synthetic a priori propositions. As an example he uses mathematics and specifically, geometry. Kant claims that the properties of the external world as we perceive it are not independent from us; in fact we perceive reality through the categories imposed by our intellect. Geometry for Kant is a science of space.[3] There is nothing contingent in this science.

[2]Russell's book *Foundations of Geometry* — based on his doctoral dissertation — concerns mainly the period in history of mathematics a few decades after Immanuel Kant, when the new idea of non-Euclidean geometry was born and developed.

[3]We note in passing that the influence of Kant on mathematics goes beyond the realm of geometry. Sir W.R. Hamilton, the discoverer of quaternions, thought of algebra, in Kantian

The statements of geometry, he says, are necessary (and hence a priori) and yet they extend our knowledge (and hence are synthetic). For him, space and time are forms of intellect and do not belong to the external world. This is the famous "Copernican revolution in philosophy". We should mention that in his text, Russell clearly distinguishes between logical (epistemological) and psychological components in Kant's analysis. Russell states that his interests lie in the logical part.

3.2.2 Three periods of geometry

Russell follows Klein[4] in dividing the history of geometry after Kant into three periods: synthetic, metrical and projective.

The synthetic period

For centuries mathematicians were trying to deduce the fifth postulate from the others. The early 19^{th}century saw a different approach to the problem. Instead of trying to prove the fifth postulate mathematicians, most notably Johann Bolyai and Lobatchevsky, negated the postulate and tried to deduce contradiction from the resulting system. This however proved impossible and the obtained systems were shown to be consistent (in a pre-logical sense). That is how the idea of non-Euclidean geometry obtained a solid mathematical footing. The period in which foundations of non-Euclidean geometry had been laid Russell calls *synthetic*. The name alludes to the fact that all the results were developed within the axiomatic (also called synthetic) paradigm.[5]

Gauss is often considered an originator of the modern idea of the non-Euclidean geometry. He did not publish any mathematical treatise on the topic; however his ideas influenced Wolfgang Bolyai, a Hungarian mathematician to work on these issues. It was Bolyai's son, Johann, who in an 1832 publication essentially brought into being the field of non-Euclidean geometry. Working in parallel, or in fact slightly ahead of Bolyai, was a Russian mathematician N. Lobatchevsky, who presented the following version of the negation of Euclid's postulate:

> With respect to a given straight line, all others in the same plane, may be divided into two classes, those which cut the given straight

terms, as a science of *time*.

[4][Rus97], Preface.

[5][Rus97], p. 13.

> line and those who do not cut it; a line which is the limit between the two classes is called parallel to the given straight line. It follows that, from any external point two parallels can be drawn, one in each direction.[6]

The constructions presented independently by Bolyai and Lobatchevsky underlie in fact just one type of the non-Euclidean geometries: hyperbolic. One of the well-known properties of hyperbolic geometries is that the sum of internal angles in a triangle is less than π.[7] According to Russell, the emergence of hyperbolic geometry in the synthetic period is in fact a mere a by-product of establishing the independence of the fifth postulate from the others.[8]

The metrical period

The main figures in the second period — Riemann and Helmholtz — were more influenced by Gauss and Herbart, a German philosopher, than by Bolyai or Lobatchevsky.[9] This period breaks with the synthetic paradigm. The motivation for geometers becomes even more philosophical and the main aim was to show the *empirical* nature of the accepted axioms.[10] The method was to define space in more general terms and abstracting from intuitions, develop and apply new mathematical tools to deal with these generalised notions. The most important innovations were a concept of a *manifold*, a *dimension* and a *curvature of a space*. Riemann

> regarded space as [...] a magnitude, or assemblage of magnitudes, in which the main problem consists in assigning quantities to the different elements or points, without regard to the *qualitative* nature of the qualities assigned.[11]

thus justifying the name *metrical* assigned to this period. In spite of the intentions of Riemann and Helmholtz, Russell tries to show that their method con-

[6][Rus97], p. 11.

[7]We note in passing that there are several models of hyperbolic geometry and the so-called inversive geometry serves as one of these. It is sometimes presented as a close relative of the projective geometry. It is interesting to observe that Russell makes no mention of the inversive geometry.We do not intend to introduce this notion here. For an introduction to inversive geometry see [Cox71].

[8][Rus97], p. 12.

[9][Rus97], p. 13.

[10][Rus97], p. 13.

[11]My italics. [Rus97], p. 15.

tains a non-empirical component. He points out that implicit in the system are three *a priori* axioms: the axiom of free mobility, the axiom stating that the number of dimensions is finite and is an integer and the axiom stating that two points are in a unique relation — distance.[12] Russell regards Helmholtz as more important philosophically than mathematically for the development of metrical period.[13] (Note that Helmholtz was a scientist working in many fields and his empirical approach to geometry stems from his physiological studies.) Helmholtz gives a list of axioms, empirical in nature, from which all the main results of Riemann follow. This is however harshly criticised by Russell. The last metrical mathematician mentioned by him is Beltrami. His work is more important mathematically (he deals with the notion of *negative* curvature of space) than philosophically.[14]

The projective period

The last period distinguished by Russell does away with the notion of distance altogether. It is dubbed *projective* and is by far the most important according to Russell.[15] He complains that this period did not have a philosophical exponent comparable to Riemann or Helmholtz. As we indicated, in his book Russell does not clearly distinguish between what we now call projective and affine geometries. Since at the time topology as a separate branch of mathematics had not yet been clearly defined, Russell makes no mention of it either (even though he is clearly conversant with homeomorphism-like transformation — see [Rus97], p.18). In one of his later texts he does distinguish between projective and affine geometries but notes that there is not much of a difference between these two.[16] It is not quite clear to which of the three mentioned branches of geometry his remarks could be applied more accurately.

Arthur Cayley is usually credited as the initiator of a modern approach to the projective geometry. A staunch supporter of Euclidean geometry, he saw his goal in establishing the notion of distance using purely *descriptive* terms.

[12][Rus97], p. 22.
[13][Rus97], p. 23.
[14][Rus97], p. 25-27.
[15][Rus97], Preface; also various comments throughout the text.
[16]Russell was a prolific author. He wrote several books touching on the foundations of mathematics, all of them with confusingly similar titles (perhaps that is where he also follows Kant's example). In this case we mean his *Principles of Mathematics*.

His importance lies in showing how ”metrical is only a part of projective”.[17] Russell devotes a considerable amount of time and energy to explaining how the quantitative notions used in projective geometry are merely notational conventions, and that there is no real meaningful connexion, under the threat of vicious circularity, between them and the notion of distance as known in ordinary metrical geometry. In his explanation Russell refers to the use of the notion of an anharmonic ratio as an invariant in projective geometry.[18] Problems of that nature pushed mathematicians to search for a non-metrical projective invariant. Finally, the notion of harmonic ratio was introduced (cf. our Definition 2.5.11).[19] In projective geometry one cannot distinguish between a collection of fewer than four points from any other on the same line.[20] We note that as a result, Russell considers the now standard construction relating the Euclidean space with projective geometry by means of the line at infinity as a purely technical result, with no philosophical significance.

We note that Russell argues against the first of Kantian distinctions of propositions: on analytic and synthetic. Kant developed his ideas in times when logic was synonymous with Aristotelian syllogistics, says Russell, and syllogistics had a big disadvantage when compared to modern logic: it concerned only the propositions of the form S x P, where S, P are the subject and the object and x represents the copula, possibly with the addition of negation. Since formal logic is able to represent more complicated sentence structure, the Kantian distinction cannot apply. As regards the distinction between a priori and a posteriori, according to Russell, the three implicit axioms mentioned in the section describing the metrical period form a basis of any geometry. Since projective geometry is logically prior to metrical, Russell rephrases these axioms in projective terms. He claims that these axioms are logically necessary and independent from any experience and hence fall into category of a priori propositions.[21]

[17] [Rus97], p. 29.
[18] [Rus97], p. 31-32.
[19] [Rus97], p. 35.
[20] [Rus97], p. 36.
[21] [Rus97], p. 52.

3.3 The Transitional Period

From our perspective it is important not only how the ideas underpinning affine geometry were developed. Since our research concerns region-based logics, it is interesting to observe how and when the region-based paradigm had begun to emerge. That is why we further separate the description of the transitional and the modern period into point-related and region-related.

3.3.1 Point-related research

Whitehead and the foundations of affine geometry

We emphasise that Whitehead's exposition is, for the most part, not stated in the formalised language of today's mathematics. Also, even though he is rightly thought of as one of the forefathers of formal logic, it should be remembered that his approach predates model theory. Whitehead mentions three axiom systems for affine geometry: Russell's, Peano's and Veblen's and describes the last two of these in some detail.

Peano's axioms The first system described by Whitehead is the one proposed by G. Peano and based on Pasch's considerations. It uses one primitive relation — betweenness. The axiom system comprises three parts containing axioms for the line, the plane and the three-dimensional space, respectively. In short, the axioms for the line state that: there is at least one point; for any two distinct points there exists a point between these; if a point lies between two other points A and B, it also lies between B and A, provided it is distinct from both A and B. There are also axioms securing "technical" properties of line segments.[22] Regarding the axioms for the plane, the first of these state that for a given line, there exists at least one point not on that line. The meaning of the other two axioms together is that in any triangle a line l containing one of the triangle vertices intersects a line containing another vertex if and only if l intersects the opposite side of the triangle. Finally, there are three axioms for the three-dimensional space. The first of these says that given a plane there always exists a point which external to it. The second, again, is

[22] By a "technical" property we mean (subjectively) a property that is not easily explained in simple terms and is most cases required in proving that some other, more natural property holds.

an axiom securing certain ”technical” property. Whitehead (p. 2–3) claims that the axioms

> [...] secure the ordinary properties of a straight line with respect to the order of parts on it, and also with respect to the division of a line into three parts by any two points on it and into two parts by any single point.

The second axiom system mentioned in the book is due to O. Veblen. Its primitive is a three-place relation denoted by Whitehead as *order*. For the description of this system consult [Whi07] or [Veb03]. According to Whitehead Veblen's axiom system is equivalent to the axiom system of Peano. Also (p. 9)

> both Peano and Veblen give an axiom securing the Dedekind's property. Also Veblen gives an axiom securing [...] Euclid['s axiom].

Here, the Dedekind's property is the following:

> If all points on a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.

We note that this property is not first-order expressible. Tarski and colleagues use a version of this property in their respective axiomatisations in a form of an axiom schema.[23]

Relation between affine and projective spaces Whitehead devotes a substantial part of his book to as he puts it ”enunciation of relations between [affine] and projective spaces”.[24]

Whitehead defines *projective space* as a non-empty set P, elements of which satisfy certain axioms. The first group of axioms he calls ”axioms of classification”. These involve points, lines and relations between points and lines. According to Whitehead all these hold both in descriptive and projective space.

[23]See for example [TG99].

[24]We note that Szczerba and Tarski's paper (cf. [ST79]) is influenced by Whitehead's constructions. See also section 3.4 for the description of [ST79].

The only axiom from this group distinctive of projective space is the one stating that if A and B are non-collinear points, and A' is a point on the line BC distinct from B and C, and B' is a point on the line AC, distinct from A and C, then the lines AA' and BB' have a point in common.

Next Whitehead gives an axioms he calls *Fano's axiom*, stating that if the point D is the harmonic conjugate[25] of the point C with respect to the points A and B, then C and D are distinct.[26] We note that Whitehead also gives "axioms of dimension" similar to those used by Tarski (cf. Section 3.4) to limit the dimension of a considered space. The last group of axioms are called by Whitehead the "axioms of order". Here, the harmonic conjugate is used to define ordering of points on a line.

Whitehead defines a convex subset of a projective space as follows. Let P be a projective space and let $S \subseteq P$. We say that S is *convex* if (i) S does not include any straight line; (ii) given any two points in S, S contains one of the two segments between them. Then he shows that points and lines belonging to a convex subset of a projective space can be shown to satisfy the axioms of affine space.[27] Whitehead also shows how to find a convex subset of a projective space, where the Euclidean axiom holds true. He then shows the independence of Dedekind's axiom by considering convex region whose elements are points with algebraic coordinates.[28]

Whitehead also describes a method for constructing a projective space, given an affine space. This has become a standard method and is described by us in Chapter 2 (also cf. [Ben95], p. 41–46). Historically, it was Klein, after von Staudt, who developed it in the 1870s. In this context, Whitehead also acknowledges work done by Pasch.[29] Whitehead shows that the axioms of projective geometry hold for *projective* points, lines and planes, as defined by him.[30] As a concluding remark Whitehead states:

> Thus all the axioms for projective geometry including [...] the Dedekind property are satisfied by the projective points and the projective lines. Furthermore the proper projective points evi-

[25] For the definition of harmonic conjugate see Definition 2.5.11
[26] [Whi13], p. 24.
[27] The construction is somehow involved, please consult [Whi07], p. 14.
[28] [Whi07], p. 14.
[29] [Whi07], p. 15.
[30] [Whi07], p. 28.

> dently form a convex region in the projective space formed by the projective points. Also the geometry of this convex region [...] corresponds step by step with the geometry of the original descriptive space. Thus the geometry of descriptive space can always be investigated by considering it as convex region in a projective space.[31]

Russell and the foundations of geometry

It is a widely held view that, after the emergence of non-Euclidean geometries Kant's ideas in regard to the philosophy of mathematics lost much of their validity. As we signalled earlier Russell's original goal was to critically analyse approaches to the philosophy of geometry after Kant, and propose his reinterpretation of that philosophy. He proposes a more fine-grained approach, where he modifies Kant's ideas and in his view saves some of Kant's original insights. In [Rus97] Russell claims that even if one cannot claim anymore that Euclidean geometry is a priori, there still is a part of geometry which should be considered to be so. Russell's idea is the following. He refers to the distinction (presented by us above) on synthetic, metrical and projective periods in the history of geometry and focuses on the metrical and projective approaches. He introduces two sets of statements, which he calls the axioms of projective and metrical (parts of) geometry respectively. The axioms of projective geometry according to Russell are as follows:

> I We can distinguish different parts of space, but all parts are qualitatively similar, and are distinguished only by the immediate fact that they lie outside one another.
>
> II Space is continuous and infinitely divisible; the result of infinite division, the zero of extension, is called a point.
>
> III Any two points determine a unique figure, called a straight line, any three in general determine a unique figure, the plane. Any four determine a corresponding figure of three dimensions, and for aught that appears to the contrary, the same may be true of any number of points. But this process comes to an end, sooner or later, with some number of points which determine the whole of space.[32]

[31] [Whi07], p. 32-33.
[32] [Rus97], p. 132.

Russell adds some vague remarks regarding the accuracy of these axioms:

> This statement of the axioms is not intended to have any exclusive precision [...]. [It] includes, if I am not mistaken, everything essential to projective geometry, and everything required to prove the principle of projective transformation.[33]

Russell considers the non-quantitative projective approach to be superior to the metrical one. Both projective and metrical axioms are, claims Russell, a priori. He also claims that the axioms of projective geometry are equivalent to the axioms of metrical geometry:

I′ the axiom of free mobility, replacing the axiom of homogeneity [I], in his own words: Spatial magnitudes can be moved from place to place without distortion.

II′ the axiom of dimensions: space must have a finite integral number of dimensions.[34]

III′ the axiom of distance, replacing axiom [III]: every point must have to every other point one, and only one, relation independent of the rest of space. This relation is the distance between the two points.

The main difference is that the latter are stated in a way that incorporates the notion of distance. However, there is a part of metrical geometry which is not a priori. We need axioms allowing us to distinguish between Euclidean and non-Euclidean geometry. And these axioms are of empirical nature.[35] Hence the above can be thought of as the a priori *metrical* part of Euclidean and non-Euclidean geometry alike.[36]

Later, Russell considerably changes his views on the foundations of geometry. For example, he does not defend Kant any more. Also, he does not

[33] [Rus97], p. 132.

[34] Perhaps a counterpart of the axiom [II]?

[35] [Rus97], p. 148. Cf. also (p. 6):

> [The axioms of] projective geometry [...] [are] completely a priori. In metrical geometry [...] the axioms will fall into two classes: (1) those common to Euclidean and non-Euclidean spaces. [...] [These are found to be] a priori. (2) Those axioms which distinguish Euclidean from non-Euclidean spaces. These will be regarded as wholly empirical.

[36] [Rus97], p. 149, 150-177.

divide the philosophy of geometry into non-Euclidean, metrical and projective periods anymore (nor does he make any mention of his earlier work on the topic!). He does now distinguish between projective and affine geometries (which he calls descriptive) but he also says that the differences are minuscule and inessential.[37] He introduces a new threefold categorisation of the development of geometry. First, he says, came the non-Euclideans, then the ideas developed by Cantor and Dedekind on the nature of continuity, and lastly the study of order is distinguished.[38]

Perhaps influenced by the distinctively logical approach of the Italian geometers like Pieri or Peano, Russell's treatment of geometry is now phrased more in the spirit of formal logic. Every geometry, he says, begins with the notion of point as undefinable. Projective geometry adds the notion of a straight line and a symmetrical relation between two points; in descriptive geometry this relation is asymmetrical (thus allowing ordering of the points on the line); metrical geometry adds a third relation between points - distance. This way of thinking about geometries can be found in later, much more technical, logical works of Tarski and colleagues,[39] where one develops geometrical theories with strong emphasis on the language in which to describe them. Russell goes on to define harmonic conjugate and thus introduce order in projective geometry (cf. paragraphs relating Whitehead's approach described in this chapter). He adds then Pieri's order axioms and his formulation of Dedekind's continuity property.[40] Finally Russell adds a dimension axiom.[41]

Descriptive geometry[42] begins with the notion of point and a relation of betweenness. A description of the semi-axiomatised properties of betweenness is given, after Peano and Vailati. Then the dimension and continuity axioms are added.[43] The treatment of descriptive geometry ends with the

[37] [Rus56], p.393-394.

[38] [Rus56], p. 381-383. It seems like these ideas were around at the time of [Rus97]. Is it possible that Russell, definitely a man of great sagacity, was simply ignorant of these?

[39] Curiously, Russell treats betweenness as a two-placed relation. For Tarski's works cf. [Tar56], [Tar59], [ST79] and many others.

[40] For Dedekind's property see the paragraph in this section on Whitehead and the foundations of affine geometry. For Pieri's work on the foundations of geometry cf. our remarks relating the axiomatisation of the geometry of solids in section 3.4.2.

[41] Cf. [Rus56], p. 386-388. *Nota bene*, formalised versions of these axioms are used later by Tarski e.g. in [Tar56]. Cf. also our earlier remarks on the Peano's axiom system.

[42] Also called by him *a geometry of position*, which echoes common origin of, nascent at the time of writing of [Rus56], topology and affine geometry.

[43] [Rus56], p. 394-399.

(standard) analysis of the relation of projective and descriptive geometries.

Russell begins his treatment of metrical geometry with a preliminary analysis of Euclid's work on geometry (which he calls *elementary* geometry). He then remarks

> [...] enough has been said to show that Euclid is not faultless, and that his explicit axioms are very insufficient.[44]

and moves on to describe his approach to metrical geometry. Russell lists *some* of the properties of the new primitive notion: distance.[45] Russell goes on to report on alternative approach by taking *motion* as an indefinable notion (as developed by Pieri). This section finishes with a description of methods of defining straight line, angle and order in metrical geometry; and the differences in two- and three-dimensional cases.[46]

In one sense at least, Russell's views did not change considerably. In [Rus56] as in [Rus97] he strongly emphasises the difference between qualitative and non-qualitative geometries, and sees the former as the true basis of any geometric enquiry.[47] As it was mentioned, he treats projective and descriptive geometries as essentially the same and non-qualitative in nature. Also, the introduction of the notion of distance in projective and descriptive geometries is, according to Russell, entirely superficial.[48]

Hilbert

Hilbert's *Foundations of Geometry* is perhaps the most important work on Euclidean geometry since the *Elements*. The book contains a number of axioms grouped into five categories ([Hil50], p. 2–16.). The first group, the axioms of connection, contains for example the following axiom.

> If two points **A** and **B** of a straight line a lie in a plane α, then every point of a lies in α.

The second group, the axioms of order, is later used by Tarski in his formalisation (see Section 3.4).[49] For example

[44] Perhaps alluding to the fact that for example Pasch's axiom is independent from Euclid's postulates, is used in the Elements but is missing from the list of axioms. Cf. [Rus56], p. 407.

[45] Cf. [Rus56], p. 407-408.

[46] [Rus56], p. 410-415, 416-417.

[47] Cf. [Rus56], p. 419-428.

[48] *Frivolous* is the word he uses. Cf. [Rus56], p. 425.

[49] Hilbert credits Pasch for the axioms in this group, [Hil50], p. 3.

> If **A**, **B**, **C** are points of a straight line and **B** lies between **A** and **C**, then **B** lies also between **C** and **A**.

Next is the axiom of parallels (Euclid's axiom). Then Hilbert distinguishes the axioms of congruence. These relate to axioms of the equidistance relation used by Tarski. For example

> If a segment **AB** is congruent to the segment **A'B'** and to the segment **A"B"** then the segment **A'B'** is also congruent to the segment **A"B"**.

Lastly Hilbert presents the axiom of continuity (also called Archimedean axiom[50]) and the axiom of completeness, which has a rather interesting form:

> To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible to extension, if we regard the five groups of axioms as valid.

It seems like what Hilbert tries to secure axiomatically here is precisely a subject of Tarski's intense research culminating in the representation theorem (see Section 3.4).

There is no doubt that Hilbert's book is a major stepping stone in the quest of logically strict analysis of the axiom system of Euclidean geometry. Hilbert makes an effort to formally analyse his axiom system in terms of its consistency and independence of the axioms ([Hil50], p. 17–23). Even though not fully formalised in the modern sense, the axioms and their consequences are presented with high degree of mathematical rigour. There are major caveats with that statement, however. For example the axiom of completeness, notwithstanding its special and very curious status, is formed with no regard to the underlying language of description, hence if we were to formalise this axiom system today, its formulation would surpass the limits of first-order language. Hence it seems justifiable to conclude that Hilbert's work was an important step towards, rather than an example of, what we would call today a logical analysis of an axiom system.

[50]We do not wish to state this axiom here. We note in passing that it is distinct from but related to the Archimedean property in the context of the theory of fields.

3.3.2 Region-related research

Whitehead, Regions and Topology

Whitehead is widely credited with the advancing of the region-based approach. The theory of what he calls a relation of extensive connection forms a part of his multifarious philosophical treatise *Process and Reality. An Essay in Cosmology*. As he himself admits, he has no intention of analysing the notion of extensive connection from a more formal point of view ([Whi29], p. 416.) Whitehead starts with a relation of connection, left undefined and then introduces a host of new relations.

Definition 3.3.1. Two regions A and B are said to be *medially connected* if there exists a region C such that both A and B are connected to C.

Definition 3.3.2 (de Laguna)**.** A region A is said to *include* a region B if every region connected with B is also connected with A.

Definition 3.3.3. A region A is said to *overlap* with a region B if there exists a region C such that both A and B include C.

He then lists a number of properties that these new relations should satisfy and after defining several other notions finally introduces the notion of external connection.

Definition 3.3.4. Two regions A and B are *externally connected* if A and B are connected and do not overlap.

By means of external connection he is now able to introduce two more relations: *tangential* and *non-tangential* inclusion.

Whitehead credits de Laguna with influence over his earlier ideas, which eventually led to the adoption of the notion of external connection as the central one ([Whi29], p. 420). He goes on to construct a theory of sets based on his definition of inclusion ([Whi29], p. 421–426.) We do not attempt to elaborate on the role of the theory of external connection in Whitehead's deeply nuanced philosophical views. We only note that in *Process and Reality* he deals with a broad spectrum of problems, including relation of an organism to an environment, the theory of feelings and the nature of God. As we pointed out so many times already, it is important to remember that the apparatus of topology and modern formal logic was not in place at the time of publication of his book, also these mathematical ideas are entangled in his intricate philosophical system and are not presented with the terseness normally encountered in mathematics. Perhaps that is why his work remained within the realm of philosophical investigations and it took another half a century to revive these ideas in a more formalised, strict setting. One clearly sees how Whitehead's *topological* ideas influenced modern spatial formalisms to come. As we will outline in the next section his intuitions were formalised and analysed by others in the after-war period. Whitehead's theory of sets is often compared to another alternative set theory developed by a reclusive Polish logician Stanisław Leśniewski. His proposition is dealt with in the following section.

Mereology

Leśniewski wrote little, and most of what we know about his work comes from his students. The theory of sets he was dealing with is dubbed *mereology* (from Greek: the theory of parts). Leśniewski presented his system in a mixture of idiomatic Polish and German and formalisms related to notational conventions of *Principia Mathematica*. His major publications come in the decade following the publication of *Process and Reality*, that is from the 1930s onwards. It was a time when formal logic was coming of age and the Polish logical school was at its best. All this and the fact that his results were analysed, extended and presented by a new generation of Polish logicians, made his work relatively better developed than that of Whitehead.

In essence, Leśniewski tried to formalize the notion of a *collective* set (as opposed to, considered as standard in modern set theory, *distributive* set). Let X be a collective set. We think of X as a certain whole, consisting of some parts, called the elements of X. For example, a book can be thought of as a collective set of pages, a library as a collective set of books etc.

Leśniewski used his system to analyze the parthood relation (in a collective sense). In his times the predominant way of formalising one's intuitions was by axiomatizing them. In a standard approach to mereology, the parthood relation ($\sqsubseteq$) is axiomatised as a *partial ordering*. We can introduce some additional mereological relations defined via the $\sqsubseteq$, describing situation where sets overlap, underlap, one is a proper part of another etc. Following [Var96] we define mereology as follows.

Definition 3.3.5 (Mereology)**.** Consider $\mathcal{L}_{\sqsubseteq}$. Let $O(x_1, x_2) = \exists z(z \sqsubseteq x_1 \wedge z \sqsubseteq x_2)$ ($O(x_1, x_2)$ reads x_1 overlaps x_2) and let $\phi(x)$ be a formula in which x occurs free. A theory of general mereology consists of the following axioms.

$$x \sqsubseteq x \qquad \text{(reflexivity)}$$

$$x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y \qquad \text{(antisymmetry)}$$

$$x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq x \qquad \text{(transitivity)}$$

$$x \not\sqsubseteq y \rightarrow \exists z(z \sqsubseteq x \wedge \neg O(z, y)) \qquad \text{(supplement axiom)}$$

$$\exists x \phi(x) \rightarrow \exists z \forall y(O(y, z) \leftrightarrow \phi(x) \wedge O(y, x)) \qquad \text{(fusion axiom)}$$

Tarski proved that every model of (atomic) general mereology is isomorphic to an atomic complete quasi-Boolean algebra (that is a Boolean algebra with the bottom element removed). He also proved that a model of atomless general mereology is given by a complete quasi-Boolean algebra on the set of regular open subsets of the Euclidean space. For more results on mereology please consult [Grz55], [Pie00] and [Gor03].

Summary

Clearly, one can divide the development of Russell's geometric ideas into two periods: the first, more philosophical, influenced by Kant, and the second, more logical in spirit, influenced by works of Cantor, Dedekind, Pieri or Peano. In the first period, axioms given by Russell are not really fit for use in axiomatisation of any geometrical space in question. Russell does not really look at the issue from a logical point of view. Obviously, the notion of interpretation had not been developed at that time, but Russell seems to be indifferent to the precise definition of the language in which to express the axiomatisation. It is not at all clear in what sense these axioms are sound and complete in respect to the considered spaces, especially in view of Russell's comments that the projective and metric axioms are essentially the same. The second period, on the contrary, is much more logical in spirit. He does care about the language in which to phrase the axioms, for instance. This is more of an achievement of the mathematicians that Russell is referring to in

[Rus56] but perhaps Russell should be credited at least with bringing these ideas together in an unified description of these developments. This in turn, served as a basis of more formal analysis in the following decades. We emphasise once more the superiority of non-quantitative part of geometry over the metrical part.

Whitehead and Russell's contribution to the formalisation of affine and projective spaces is that of compilers rather than authors. By providing a comprehensive overview of known results they made these accessible to the wider mathematical and philosophical audience. Thus, these publications paved the way of the research to come. In the modern period these ideas were developed within a framework of modern mathematical logic. Neither Russell nor Whitehead published more extensively on these topics later on. This might explain why their work, especially on the foundations of affine and projective geometry, is relatively less known. Whitehead is widely credited as being one of the first mathematicians to focus on region-based theories. It has to be said that his work on region-based theories is riddled with philosophy and lacks mathematical rigour.

Hilbert's work was a major influence on the next generation of mathematicians. This includes Tarski and his co-workers who developed tools of model theory and formal logic allowing proper analysis of a given axiom system. Hilbert's influence lies also in the method. As we pointed out he was one of the first mathematicians to analyse the axioms he proposed in what we would now call a logical manner. It should be stressed that Hilbert's *Foundations of Geometry* is far more rigorous mathematically than any other source we discussed in this section.

3.4 The Modern Period

In the description of the modern period we also found it important to bring out the emerging two approaches. Hence the modern period is also subdivided into point-related and region-related research.

3.4.1 Point-related research

Elementary geometry

In his classic paper [Tar59] Tarski considered system called by him *elementary geometry*. By this he understood (informally) a part of Euclidean geometry that can be formulated without using any set-theoretic notions, and hence devoid of variables of higher orders. As we pointed out, Hilbert's influence on Tarski's work is obvious here. For the most part it seems like Tarski's elementary geometry is the first-order part of formalisation proposed by Hilbert in his *Foundations*.[51] Consider the first-order language $\mathcal{L}_{\beta,\delta}$. Tarski proposed the following axiom system stated in the language $\mathcal{L}_{\beta,\delta}$ (universal quantifiers omitted).

$$(\beta(x,y,x) \to x = y) \qquad \text{identity for } \beta$$

$$(\beta(x,y,u) \wedge \beta(y,z,u) \to \beta(x,y,z)) \qquad \text{transitivity for } \beta$$

$$(\beta(x,y,z) \wedge \beta(x,y,w) \wedge x \neq y \to \beta(x,z,w) \vee \beta(x,w,z)) \qquad \text{connectivity for } \beta$$

$$\delta(x,y,y,x) \qquad \text{reflexivity for } \delta$$

$$(\delta(x,y,z,z) \to x = y) \qquad \text{identity for } \delta$$

$$(\delta(x,y,z,u) \wedge \delta(x,y,v,w) \to \delta(z,u,v,w)) \qquad \text{transitivity for } \delta$$

$$\exists v(\beta(x,t,u) \wedge \beta(y,u,z) \to \beta(x,v,y) \wedge \beta(z,t,v)) \qquad \text{Pasch's Axiom}$$

$$\exists v \exists w(\beta(x,u,t) \wedge \beta(y,u,z) \wedge x \neq y \to \beta(x,z,v) \wedge \beta(x,y,w) \wedge \beta(v,t,w)) \qquad \text{Euclid's Axiom}$$

[51] Hilbert's axiom system was, in effect, expressed in second-order language.

$$(\delta(x, y, x,' y') \wedge \delta(y, z, y,' z') \wedge \delta(x, u, x,' u') \wedge \delta(y, u, y', u') \wedge \beta(x, y, z) \wedge \beta(x', y', z') \wedge x \neq y \rightarrow$$
$$\delta(z, u, z', u'))$$ Five Segment Axiom

$$\exists z(\beta(x, y, z) \wedge \delta(y, z, u, w))$$ Axiom of Segment Construction

$$\exists xyz(\neg\beta(x, y, z) \wedge \neg\beta(y, z, x) \wedge \neg\beta(z, x, y))$$ Lower Dimension Axiom

$$(\delta(x, u, x, v) \wedge \delta(y, u, y, v) \wedge \delta(z, u, z, v) \wedge u \neq v) \rightarrow$$
$$(\beta(x, y, z) \vee \beta(y, z, x) \vee \beta(z, x, y))$$ Upper Dimension Axiom

All sentences of the form

$$\exists p \forall xy(\phi \wedge \psi \rightarrow \beta(p, x, y)) \rightarrow \exists p \forall xy(\phi \wedge \psi \rightarrow \beta(x, p, y)),$$

where p, y do not occur free in ϕ and p, x do not occur free in ψ (Continuity Schema).

Intuitive meanings of the above can be find in [TG99]. The continuity schema asserts that any two sets X and Y such that the elements of X precede the elements of Y with respect to some point a are separated by a point b, where X, Y are first-order definable.[52] We also note that the upper dimension axiom asserts that for any three points equidistant from each of two distinct points have to be collinear. This fails in dimensions greater than two (hence the name). Coupled with the lower dimension axiom this pinpoints the dimension. Up until this point Tarski's approach seems to be very similar to the one observed in Whitehead or Russell. The first difference is in precisely defining what *is* elementary geometry. Namely, it is defined as the smallest set of sentences (a theory) containing the above axioms and closed under the logical rules of inference.[53] A part of the problem with the previous attempts was that the *language* in which to state formalisms was not strictly defined.

[52]See [TG99].

[53]Hence, with that reading, elementary geometry and the theory of elementary geometry mean the same.

Tarski used well-defined first-order language with just two primitives. However, Tarski's greatest contribution is setting the problem within the context of model theory. This added precision to the question of "meaning" of the above axioms. With Whitehead and Russell the meaning was assumed to be known but never defined precisely. Rephrasing the problem by asking *what is a model for the theory of elementary geometry*, together with the full technical apparatus of model theory at our disposal, meant we could investigate the problem with far greater precision. We start with defining a candidate interpretation.

Definition 3.4.1. Let $\mathfrak{F} = \langle F, \cdot, +, \leq \rangle$ be a real closed field. Consider a structure

$$\mathfrak{C} = \langle C, \beta^{\mathfrak{C}}, \delta^{\mathfrak{C}} \rangle,$$

where $C = F \times F$ and

$$\beta^{\mathfrak{C}} = \{ \langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle, \langle z_1, z_2 \rangle \rangle \mid x_1, x_2, y_1, y_2, z_1, z_2 \in F$$
$$\text{and } (x_1 - y_1) \cdot (y_2 - z_2) = (x_2 - y_2) \cdot (y_1 - z_1)$$
$$\text{and } 0 \leq (x_1 - y_1) \cdot (y_1 - z_1)$$
$$\text{and } 0 \leq (x_2 - y_2) \cdot (y_2 - z_2) \}$$

$$\delta^{\mathfrak{C}} = \{ \langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle, \langle z_1, z_2, \rangle \langle u_1, u_2 \rangle \rangle \mid x_1, x_2, y_1, y_2, z_1, z_2, u_1, u_2 \in F$$
$$\text{and } (x_1 - y_1)^2 + (x_2 - y_2)^2 = (z_1 - u_1)^2 + (z_2 - u_2)^2 \}.$$

we call $\mathfrak{C}$ a *(two dimensional) Cartesian space* over $\mathfrak{F}$.

(In particular, if we set $\mathfrak{F}$ to have values from $\mathbb{R}$, we obtain the ordinary two dimensional Euclidean space.)

This allows us to attach precise meanings to the symbols of $\mathcal{L}_{\beta,\delta}$. Informally variables are set to range over points of the vector space over a field; $\beta(x, y, z)$ reads "y lies between x and z" ($z = x$ or $z = y$ not excluded) and $\delta(x, y, z, u)$

reads ″x is as distant from y as z is from u″. And only now can we say what the axioms can be thought to express. However, there is still no particular reason why we should pay more attention to the above interpretation, rather than to any other we choose to think of. That is why Tarski proves this, remarkable, theorem.

Theorem 3.4.1 (Representation Theorem)**.** *For $\mathfrak{M}$ to be a model of the theory of elementary geometry it is necessary and sufficient that $\mathfrak{M}$ be isomorphic with the Cartesian space over some real closed field $\mathfrak{F}$.*

Proof. See [Tar59]. □

Tarski calls it the *representation theorem* for the theory of elementary geometry as it answers the question of characterising *all* models of the theory. Observe that now not only is the problem of ″meaning″ settled in a formally satisfying manner. The representation theorem gives a precise answer to the question of the *standard* interpretation of elementary geometry. We can also ask other important questions regarding the theory of elementary geometry. One very interesting question relates the decidability of the satisfiability problem. And the answer is quite surprising.

Theorem 3.4.2. *The theory of elementary geometry is decidable but not finitely axiomatisable.*

Proof. See [Tar59]. □

The fact that elementary geometry is not finitely axiomatisable means any other axiom system proposed must contain either axiom schema(ta) or infinitary rule(s) of inference. In fact, later on Tarski's axiomatisation was refined by others; obviously none of these axiomatisations were finite (see for example [TG99]).

General Affine Geometry

Tarski in [Tar59] was able to formalise large part of Euclidean geometry using a language with only two primitives: betweenness (β) and equidistance (δ). This paper started off a bigger research program of formalising other parts of geometry. The paper [ST79] is a part of that research program. It is concerned with logical analysis of what is termed as *general affine geometry*.

Unfortunately the situation here is much more complex than in the case of elementary geometry. In particular the question of the representation theorem turns out to be much more elusive. The main result of [ST79] is a *near*-representation theorem for the general affine geometry of two dimensions. We also note that despite the similarities in method the results in this paper are obtained using far more sophisticated and difficult techniques than those used in [Tar59].

Definition 3.4.2. Consider $\mathfrak{L}_\beta$. By *general affine geometry* we mean a theory based on the following axioms (universal quantification omitted):

$$(\beta(x,y,x) \rightarrow x = y) \qquad \text{Identity}$$

$$((\beta(x,y,z) \wedge \beta(y,z,w) \wedge y \neq z) \rightarrow \beta(x,y,w)) \qquad \text{Transitivity}$$

$$(\beta(x,y,z) \wedge \beta(x,y,w) \wedge x \neq y \rightarrow \beta(x,z,w) \vee \beta(x,w,z)) \qquad \text{Connectivity}$$

$$\exists x(\beta(x,z,y) \wedge x \neq y) \qquad \text{Extension Axiom}$$

$$\exists v(\beta(x,t,u) \wedge \beta(y,u,z) \rightarrow \beta(x,v,y) \wedge \beta(z,t,v)) \qquad \text{Pasch's Axiom}$$

$$\begin{aligned}&(\beta(p,x,x') \wedge \beta(p,y,y') \wedge \beta(p,z,z') \wedge \beta(x,y,z'') \wedge \beta(x',y',z'') \wedge \beta(y,z,x'') \wedge \\ &\beta(y',z',x'') \wedge \beta(x,z,y'') \wedge \beta(x',z',y'') \wedge \neg\beta(p,x,y) \wedge \neg\beta(x,y,p) \wedge \neg\beta(y,p,a) \wedge \\ &\neg\beta(p,y,z) \wedge \neg\beta(y,z,p) \wedge \neg\beta(z,p,y) \wedge \neg\beta(p,z,x) \wedge \neg\beta(z,x,p) \wedge \neg\beta(x,p,z) \rightarrow \\ &\beta(x'',y'',z''))\end{aligned} \qquad \text{Desargues' Axiom}$$

$$\exists xyz(\neg\beta(x,y,z) \wedge \neg\beta(y,z,x) \wedge \neg\beta(z,x,y)) \qquad \text{Lower Dimension}$$

$$\exists q(\beta(y,q,z) \wedge (\beta(x,q,p) \vee \beta(q,p,x) \vee \beta(p,x,q)) \vee$$
$$(\beta(z,q,x) \wedge (\beta(y,q,p) \vee \beta(q,p,y) \vee \beta(p,y,q))$$
$$\vee(\beta(x,q,y) \wedge (\beta(z,q,p) \vee \beta(q,p,z) \vee \beta(p,z,q))$$

Upper Dimension

All sentences of the form

$$\exists p \forall xy(\phi \wedge \psi \to \beta(p,x,y)) \to \exists p \forall xy(\phi \wedge \psi \to \beta(x,p,y)),$$

where p, y do not occur free in ϕ and p, x do not occur free in ψ (Continuity Schema).

As acknowledged by the authors, this axiom system (as well as certain technical results within the paper) was influenced by Whitehead's considerations on affine geometry. We note also the similarities between this axiom system and the axiom system for elementary geometry ([Tar59]).

Definition 3.4.3. Let $\mathfrak{F} = \langle F, +, \cdot, \leq \rangle$ be an ordered field. Consider the two-dimensional linear space over $\mathfrak{F}$, that is the set $F \times F$ and two operations $\langle x, y \rangle \oplus \langle x', y' \rangle = \langle x + x', y + y' \rangle$ and $\langle x, y \rangle \otimes z = \langle x \cdot z, y \cdot z \rangle$.

By the *affine* plane over $\mathfrak{F}$ we mean the following structure

$$\mathfrak{A}(\mathfrak{F}) = \langle A_{\mathfrak{F}}, B_{\mathfrak{F}} \rangle,$$

where $A_{\mathfrak{F}} = F \times F$ and $B_{\mathfrak{F}}$ is a ternary relation on $A_{\mathfrak{F}}$ defined by the following stipulation: for any three points $a, b, c \in A_{\mathfrak{F}}$, $B_{\mathfrak{F}}(a, b, c)$ if, for some $x \in F$, $0 \leq x \leq 1$ and $b = [a \otimes (1 - x)] \oplus (c \otimes x)$.

We can clearly see the parallels between this and the original Tarski's paper now. Also, Whitehead's way of defining an affine space as a convex subset of a projective space is used in [ST79] within the model theoretic setting. A natural definition of convexity in terms of betweenness is given: a (non-linear) set S is convex if for all points $a, c \in S$ if a point b is such that $B_{\mathfrak{F}}(a, b, c)$, then $b \in S$. A set is weakly convex if for every four points $a, b, c, d \in S$, if there exists a point p such that $B_{\mathfrak{F}}(a, p, b)$ and $B_{\mathfrak{F}}(c, p, d)$ then $p \in S$.

Definition 3.4.4. Let S be any subset of $A_{\mathfrak{F}}$. The structure formed by the set S and the relation $B_{\mathfrak{F}}$ restricted to the points of S, denoted $\mathfrak{A}(\mathfrak{F}, S)$, is called the S-restricted affine plane over $\mathfrak{F}$.

Coordinatisation We pointed out in Section 2.5 that a coordinatisation of an affine space is an important result in affine geometry. It is not surprising then that the paper [ST79] uses coordinatisation of affine general geometry in their formalisation. It is done so by first of all showing how to define a real-closed field.

Let $\mathfrak{A} = \langle A, B \rangle$ be a model of general affine geometry. We define

$$\mathfrak{H}_{\mathfrak{A}} = \langle H = [e_0, e_\infty), 0 = e_0, 1 = e_1, +, \cdot, \leq \rangle,$$

where e_0, e_∞, e_1 are distinct elements of A satisfying certain conditions.[54] We further define

$$\mathfrak{G}_{\mathfrak{A}} = \langle H \times H, \langle 0, 0 \rangle, \langle 1, 1 \rangle, \oplus, \otimes, \leq \rangle,$$

where $\langle p, q \rangle \oplus \langle r, s \rangle = \langle p + r, q + s \rangle$; $\langle p, q \rangle \otimes \langle r, s \rangle = \langle p \cdot r + q \cdot s, p \cdot s + q \cdot r \rangle$; $\langle p, q \rangle \leq \langle r, s \rangle$ if $p + s \leq q + r \rangle$.

We define the equivalence relation in the following way: $\langle p, q \rangle = \langle r, s \rangle$ if and only if $p+s = q+r$. Finally we construct the corresponding quotient structure:

$$\mathfrak{F}_{\mathfrak{A}} = \langle F_{\mathfrak{A}}, 0, 1, +, \cdot, \leq \rangle = \mathfrak{G}_{\mathfrak{A}} /_{=}.$$

We have the following theorem.

Theorem 3.4.3. *If $\mathfrak{A}$ is a model of general affine geometry then $\mathfrak{F}_{\mathfrak{A}}$ is a field; $\{x : 0 \leq x\}$ is a subuniverse of $\mathfrak{F}_{\mathfrak{A}}$ and $\mathfrak{H}_{\mathfrak{A}}$ is isomorphic to the subalgebra of $\mathfrak{F}_{\mathfrak{A}}$ induced by $\{x : 0 \leq x\}$.*

Now we want to show how facts about $\mathfrak{H}_{\mathfrak{A}}$ and $\mathfrak{F}_{\mathfrak{A}}$ can be expressed in the language of $\mathfrak{A}$. In order to do that we define a mapping

$$H : \mathfrak{H}_{\mathfrak{A}} \to \mathfrak{A}$$

such that

$$\mathfrak{H}_{\mathfrak{A}} \models \phi[v_0, \ldots, v_n] \text{ if and only if } \mathfrak{A} \models H(\phi)[e_0, e_1, e_\omega, e_\infty, v_0, \ldots, v_n].$$

[54]Since the construction is somewhat involved we do not wish to present the details here. For our purposes it suffices to say that a coordinate system is defined using e_0, e_∞, e_1 and few other points. Cf. [ST79].

We also define a mapping

$$HF : \mathfrak{F}_{\mathfrak{A}} \to \mathfrak{H}_{\mathfrak{A}}$$

such that

$$\mathfrak{F}_{\mathfrak{A}} \models \phi[\langle x_0, y_0\rangle, \ldots, \langle x_n, y_n\rangle] \text{ if and only if}$$
$$\mathfrak{A} \models HF(\phi)[e_0, e_1, e_\omega, e_\infty, x_0, y_0 \ldots, x_n, y_n].$$

We have the following theorem.

Theorem 3.4.4. *If $\mathfrak{A}$ is a model of general affine geometry, then $\mathfrak{F}_{\mathfrak{A}}$ is a real-closed field.*

Proof. The proof uses the above mappings and a fact that $\mathfrak{F}_{\mathfrak{A}}$ is a model of all formulas of the form of axiom 9 and then it is shown that multiplication is commutative. □

Szczerba and Tarski aim to solve the representation problem for general affine geometry as in the case of elementary geometry. The first theorem shows that every model of affine geometry is isomorphic to some S-restricted affine plane induced by that model.

Theorem 3.4.5. *If $\mathfrak{A}$ is a model of general affine geometry then there is a set S, non-linear and convex in $\mathfrak{A}(\mathfrak{F}_{\mathfrak{A}})$ and such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}(\mathfrak{F}_{\mathfrak{A}}, S)$.*

The next result shows that the converse of the above theorem is only true in certain cases and fails in general.

Theorem 3.4.6. *Every S-restricted affine space over $\mathbb{R}$ is a model of general affine geometry, provided that S is convex in $\mathfrak{A}(\mathbb{R})$. If a real-closed ordered field $\mathfrak{F}$ is not isomorphic to $\mathbb{R}$, then there is a convex set S in $\mathfrak{A}(\mathfrak{F})$ such that $\mathfrak{A}(\mathfrak{F}, S)$ is not a model of general affine geometry.*

So the situation is much more complex than in the case of elementary geometry. We recall that elementary geometry was showed to be decidable. Here, again the situation is much worse.

Theorem 3.4.7. *The theory of general affine geometry is undecidable and not finitely axiomatisable.*

An extended analysis of some problems raised in [ST79] were subsequently considered in [PS79]. Observe that in terms of the Russellian distinctions presented in Section 3.3.1 Tarski and Szczerba are looking at the non-metrical (for Russell *a priori*) part of geometry. In our research, we are interested in essentially the same part of geometry but viewed from a region-based point of view.[55]

3.4.2 Region-related research

In this section we describe early region-based spatial logics. We focus on two important examples. The first is Tarski's geometry of solids and the second is Clarke's calculus of individuals. We note that there is a marked difference between these two in terms of approach. Tarski develops ideas of his supervisor Leśniewski. However, the geometry of solids is presented in a manner similar to elementary geometry or general affine geometry, described in the previous section. Clarke's develops Whitehead's ideas. It has to be pointed out that his system is also much in the spirit of Whitehead's pre-model theoretic considerations.

The Geometry of solids

This is one of the very first examples of region-based theory from the literature. The language of the geometry of solids is many-sorted. There are two types of variables over which quantification is allowed. First-order variables denoted $x_1, \ldots, x_n$, and second-order variables, denoted $X_1, \ldots, X_n$. There are two primitive relation symbols – binary $\sqsubseteq$ and unary γ. Informally, the variables are set to range over subsets of $\mathbb{R}^3$ and sets of subsets of $\mathbb{R}^3$, respectively. Primitives are to be read in the following way. The relation symbol $\sqsubseteq$ is interpreted as the set theoretic inclusion and the predicate symbol γ as the property of being a sphere.

The following are the auxiliary relations defined in terms of $\sqsubseteq$. Since we have not introduced the semantics yet, from a formal point of view these should be viewed as nothing more than formulas for which we chose to assign convenient name tags to guide our intuition.

1. We say that x is *disconnected* from y and write $D(x, y)$ if and only if $\neg\exists z(z \sqsubseteq x \land z \sqsubseteq y)$,
2. We say that x is a *proper part* of y and write $PP(x, y)$ if and only if $x \sqsubseteq y \land x \neq y$,

[55]See Chapter 5. Of course, we deal with more specific example of affine space, the real plane.

3. We say that x is a *sum* of all elements of X and write $S(x, X)$ if and only if $\forall y(y \in X \rightarrow y \sqsubseteq x) \wedge \neg\exists z \forall y(z \sqsubseteq x \wedge y \in X \wedge D(z, y))$

At this point further notational conventions are introduced using $\sqsubseteq, D, PP$ and S.

1. We say that x is *internally tangent* to y and write $IT(x, y)$ if and only if $\forall x_1 \forall x_2(\gamma(x) \wedge \gamma(y) \wedge PP(x, y) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge x \sqsubseteq x_1 \wedge x \sqsubseteq x_2 \wedge x_1 \sqsubseteq y \wedge x_2 \sqsubseteq y \rightarrow x_1 \sqsubseteq x_2 \vee x_2 \sqsubseteq x_1)$,
2. We say that x is *externally tangent* to y and write $ET(x, y)$ if and only if $\forall x_1 \forall x_2(\gamma(x) \wedge \gamma(y) \wedge D(x, y) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge x \sqsubseteq x_1 \wedge x \sqsubseteq x_2 \wedge D(x_1, y) \wedge D(x_2, y) \rightarrow x_1 \sqsubseteq x_2 \vee x_2 \sqsubseteq x_1)$,
3. We say that x, y are *internally diametrical* to z and write $ID(x, y, z)$ if and only if $\forall x_1 \forall x_2(\gamma(x) \wedge \gamma(y) \wedge \gamma(z) \wedge IT(x, z) \wedge IT(y, z) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge D(x_1, z) \wedge D(x_2, z) \wedge ET(x, x_1) \wedge ET(y, x_2) \rightarrow D(x_1, x_2))$,
4. We say that x, y are *externally diametrical* to z and write $ED(x, y, z)$ if and only if $\forall x_1 \forall x_2(\gamma(x) \wedge \gamma(y) \wedge \gamma(z) \wedge ET(x, z) \wedge ET(y, z) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge D(x_1, z) \wedge D(x_2, z) \wedge x \sqsubseteq x_1 \wedge y \sqsubseteq x_2 \rightarrow D(x_1, x_2))$.

Definition 3.4.5. We say that two spheres x, y are *concentric* and write $con(x, y)$ if and only if one of the following conditions holds:

(i) $x = y$,

(ii) $\forall x_1 \forall x_2(PP(x, y) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge ED(x_1, x_2, x) \wedge IT(x_1, x_2, y)$
$\rightarrow ID(x_1, x_2, y))$,

(iii) $\forall x_1 \forall x_2(PP(y, x) \wedge \gamma(x_1) \wedge \gamma(x_2) \wedge ED(x_1, x_2, y) \wedge IT(x_1, x_2, x)$
$\rightarrow ID(x_1, x_2, x))$.

We are getting closer to the axiom system proposed by Tarski for the geometry of solids. The following definitions are used in the axiomatisation of the the geometry of solids. Firstly, let us note that a *point* is defined as a set of balls concentric with a given ball.

Definition 3.4.6. We say that points X, Y are *equidistant from* point Z and write $equid(X, Y, Z)$ if and only if

$X = Y = Z \vee \exists z(z \in Z) \wedge \exists z \neg\exists x(z \in Z \wedge ((x \in X \vee x \in Y) \wedge x \sqsubseteq z \wedge D(z, x)))$.

We note that it is possible now to include the postulates of elementary the geometry of three dimensions, by applying results from [Tar59].

Definition 3.4.7. We say that x is a *solid* and write $sol(x)$ if and only if

$$\exists X \forall y (y \in X \wedge S(x, X)).$$

Definition 3.4.8. We say that a point X is *interior* to a solid x and write $int(X, x)$ if and only if

$$\exists y (y \in X \wedge y \sqsubseteq x)$$

.

Axiomatisation The theory of the geometry of solids has the following axiomatisation.

1. $\sqsubseteq$ is transitive;
2. for every non-empty X there exists exactly one x which is a sum of all elements of X;
3. Pieri's axioms of the Euclidean Geometry of $\mathbb{R}^3$;[56]
4. if x is a solid, the class of all interior points of x is a non-empty regular open set;
5. if the class of points is a non-empty regular open set, there exists a solid x such that X is the class of all its interior points;
6. if x and y are solids, and all the interior points of x are at the same time interior to x, then x is a part of y.

Semantics Tarski presented the following results concerning the interpretations of the theory of the geometry of solids. Consider a structure $\mathfrak{M}_T$ where the first-order variables range over regular open subsets of $\mathbb{R}^3$ (regions) and second-order variables range over sets of regions and where primitives are interpreted as a binary inclusion relation ($\sqsubseteq$) and the unary property of *being a sphere* (γ).

[56] We were not able to locate an English (or Italian for that matter) version of Pieri's works containing this mentioned axiomatisation. For an informal description see [Smi10], p. 479–483. We note that Pieri's axiom system involved just the equidistance relation. The paper [Smi10] claims (p. 483) that Tarski preferred Pieri's system to the one of Hilbert.

Theorem 3.4.8. *The structure $\mathfrak{M}_T$ is a model of the theory of the geometry of solids.*

So only now do the name tags attached previously to formulas have become meaningful. Next, Tarski was able to prove a very strong representation theorem that allows us to regard $\mathfrak{M}_T$ the standard model for the theory of the geometry of solids.

Theorem 3.4.9. *The theory of the geometry of solids is categorical, that is it has only one model up to isomorphism.*

We note the difference between the geometry of solids and elementary geometry. This representation theorem is much stronger. It says that every model of the geometry of solids is isomorphic to $\mathfrak{M}_T$. Observe that the theory of elementary geometry is not categorical. Tarski also showed that even though the geometry of solids has a region-based interpretation, it is possible to *construct* points from regions and *simulate* statements about points using statements about regions. Tarski's paper is somehow sketchy, in particular many proofs are merely outlined. The interested reader is referred to the article [PG] containing an exhaustive analysis of Tarski's geometry of solids.

Calculus of Individuals

According to [Cla81] the drawback of Whitehead's approach is not only that it is far from being strictly formalized: it is also contradictory. One of the axioms of the original system states that no region is connected to itself, whereas the opposite can be inferred from the other axioms. Clarke's goals were to present a strict formalization of Whitehead's ideas and to avoid the contradiction.

Even though *calculus of individuals* is presented with no intended interpretation, it is hinted that, following Whitehead, variables can be thought of as ranging over regions and the only primitive relation $C(x, y)$ can be interpreted as x shares a common point with y. Auxiliary relations $\leq$ and O interpreted as inclusion and *overlaps* relation (two regions overlap if their *interiors* share a common point), respectively. Axiomatisation of this first-order theory is presented in [Cla81] and analyzed in [BG91]. In Clarke's *calculus* the *weak* contact relation is axiomatised as follows.

Definition 3.4.9 (Weak Contact Relation)**.** We say that a binary relation C is a *weak contact* relation if it satisfies the following.

1. $\forall x\, C(x, x)$,
2. $\forall x \forall y\, C(x, y) \rightarrow C(y, x)$,
3. $\forall x \forall y \forall z (C(z, x) \leftrightarrow C(z, y)) \rightarrow x = y$.

Note in particular the first axiom which stands in disagreement with Whitehead's original assumption. Using C numerous other relations can be "defined" (we emphasise that definitions are only possible with the notion of interpretation clearly defined first).

1. $DC(x, y)$ if and only if $\neg C(x, y)$ (disconnected);
2. $x \leq y$ if and only if $\forall z(C(z, x) \rightarrow C(z, y))$ (part);
3. $x < y$ if and only if $x \leq y \wedge \neg y \leq x$ (proper part);
4. $O(x, y)$ if and only if $\exists z(z \leq x \wedge z \leq y)$ (overlaps);
5. $DR(x, y)$ if and only if $\neg O(x, y)$ (discrete);
6. $EC(x, y)$ if and only if $C(x, y) \wedge \neg O(x, y)$ (externally connected);
7. $TP(x, y)$ if and only if $x \leq y \wedge \exists z(EC(z, x) \wedge EC(z, y))$ (tangential part);
8. $NTP(x, y)$ if and only if $x \leq y \wedge \neg \exists z(EC(z, x) \wedge EC(z, y))$ (nontangential part).

Semantics As one of advantages of his system Clarke states the possibility of distinguishing between *contact* and *overlap* relations. The paper [BG91] is devoted to the quest for a model of Clarke's axiom system. In this paper the pair $\langle R, C \rangle$, where R is a nonempty set and C is a contact relation axiomatised as above is called a *connection structure*. It is shown that a connection structure enhanced with the rest of Clarke's axioms is isomorphic to the atomless Boolean algebra. As a result the following is proved.

Theorem 3.4.10. *The class of regular open subsets of the Euclidean space is a model of Clarke's system.*

Proof. See [BG91]. □

It is also shown that *contact* and *overlap* relations in fact coincide, making the calculus of individuals less useful – for example EC relation defined by

Clarke is not satisfied by any pair of regions and NTP and TP relations are both equivalent to $\leq$ relation.

3.5 Summary

The modern period was characterised by the rise of new paradigm in formal logic. The greatest achievement of Tarski and others is the novelty of the *method* they used rather than anything else. The axioms in [Tar59] and [ST79] come for the most part either from Hilbert or Whitehead. On the other hand, Clarke's investigations belong more to the previous period. True, he has improved on Whitehead vague remarks, it was however another generation of researches that took the notion of interpretation of the calculus of individuals more seriously.

4

Contemporary Spatial Formalisms

4.1 Introduction

This chapter focuses on recent developments in region-based spatial logics. As mentioned in the previous chapter, Tarski's original contribution was to initiate a research program on logical analysis of geometry within the model-theoretic approach. That allowed mathematically precise investigation of a number of spatial formalisms. There are two important observations we wish to make here. Firstly, the model-theoretic approach was mostly applied to point-based spatial logics, like elementary geometry. There has not been much interest in region-based spatial logics until relatively recently, when the so-called *qualitative spatial reasoning* research program began emerging within the artificial intelligence community. It was developed initially without much emphasis on the notion of interpretation — these logics were developed axiomatically. The turn of the century saw a gradual change from axiomatic to model-theoretic approach in the context of region-based spatial logics. The second point we wish to make is that there is a slight but important difference between Tarski's and the modern approach to defining logics. Both elementary geometry and general affine geometry were constructed starting from a set of axioms to which the (mathematically precise) notion of interpretation was applied. These days the order is reversed - one starts with a model, giving us the interpretation of primitives of a considered language and proceeds to investigate the properties of this model and its theory. Now, if we so wish, we might try to axiomatise the theory in an attempt to fathom its properties. And there is no requirement for these axioms to be "natural" or intuitive in any way.[1] What we are looking for is a sound and complete (if not entirely

[1] More on that topic in the context of spatial logics can be found in [PH01].

elegant) axiom system. Hence, the axiomatic method becomes subservient to the model-theoretic one.

We divided this chapter into two parts. The first part concerns the initial stage of research, which we chose to call the axiomatic approach, whereas the second part concerns the results within the model-theoretic approach. Within both parts we differentiate between affine and topological spatial logics and present their development in parallel.

4.2 The Axiomatic approach

4.2.1 Topological Spatial Logics

In [Var96] Region Connection Calculus (RCC) is described as an extension of a sub-theory of general mereology by the axioms of the contact relation C. The axioms proposed were a modified version of those investigated by Clarke (cf. [CBGG97]). Historically, RCC was intended as a correction of Clarke's calculus of individuals. The language of RCC comprises two binary relation symbols C and $\leq$. The paper [CBGG97] states explicitly that RCC should be interpreted topologically. However, the question of interpretation of RCC was answered in a satisfying way in [DW03].

Definition 4.2.1 (Boolean Contact Algebra)**.** Let BA be the theory of Boolean Algebras and let $\mathcal{L}$ be an extension of the language of Boolean algebras by the binary relation symbol C, then the theory extending BA with the following axioms is called the theory of Boolean Contact Algebras.

1. $C(x, y) \rightarrow x \neq 0 \wedge y \neq 0$,
2. $x \neq 0 \rightarrow C(x, x)$,
3. $C(x, y) \rightarrow C(y, x)$,
4. $C(x, y) \wedge y \leq z \rightarrow C(x, z)$,
5. $C(x, y + z) \rightarrow C(x, y) \vee C(x, z)$,

We are also interested in the following properties.

(Ext) $(C(x, z) \rightarrow C(y, z)) \rightarrow x \leq y$,

(Con) $x \neq 0 \wedge y \neq 0 \wedge x + y = 1 \rightarrow C(x, y)$.

We call C a *contact relation*. The following theorem achieves a similar goal to Tarski's representation theorem in the case of Elementary Geometry.

Theorem 4.2.1 (Representation Theorem)**.** *Each BCA B is isomorphic to a structure $\langle B, C\rangle$ where B is a dense substructure of regular closed algebra $RC(X)$ over some weakly-regular τ_1-space X and C is the contact relation on $RC(X)$.*

Proof. See [DW03]. □

The following two theorems are, in effect, completeness and soundness theorems for RCC.

Theorem 4.2.2. *If X is a weakly regular τ_1-space and $\langle B, C\rangle$ is a dense substructure of the regular closed algebra $RC(X)$ and C is the contact relation on $RC(X)$, then $\langle B, C\rangle$ is a Boolean Contact Algebra.*

As we already noted the axioms of RCC are the modified version of those presented by Clarke (cf. [CBGG97]). These are, essentially, the axioms of BCA presented above. This brings us to the following result.

Theorem 4.2.3. *Each RCC-model is isomorphic to a substructure of the regular closed sets of a connected weakly regular τ_1-space.*

We now turn our attention to a particular example of an RCC-related formalism, winning much of the attention of the researchers, called $RCC8$. It was originally presented as defined within RCC but has been since separated and is nowadays presented in a way in which the contact relation is not explicitly used (see e.g. [RCC92]).

RCC8 $RCC8$ is defined as the constraint language over the signature comprising the following binary relation symbols: DC (disconnected), EC (externally connected), PO (partially overlaps), EQ (equals), TPP (tangential proper part), TPP^{-1} (reverse tangential proper part), $NTPP$ (nontangential proper part), $NTPP^{-1}$ (reverse nontangential proper part). As in the case of RCC the question of formally defining the interpretation had been left open for a long time. The authors of [RCC92] write:

> The [...] primitives [of $RCC8$] include physical objects, regions and other sets of entities [p. 3]

and later add that "a spatial interpretation is assumed in a pictorial model" similar to our Figure 4.1 which shows a graphical interpretation of the basic $RCC8$-relations if we take variables to range over regular closed subsets of $\mathbb{R}^2$.

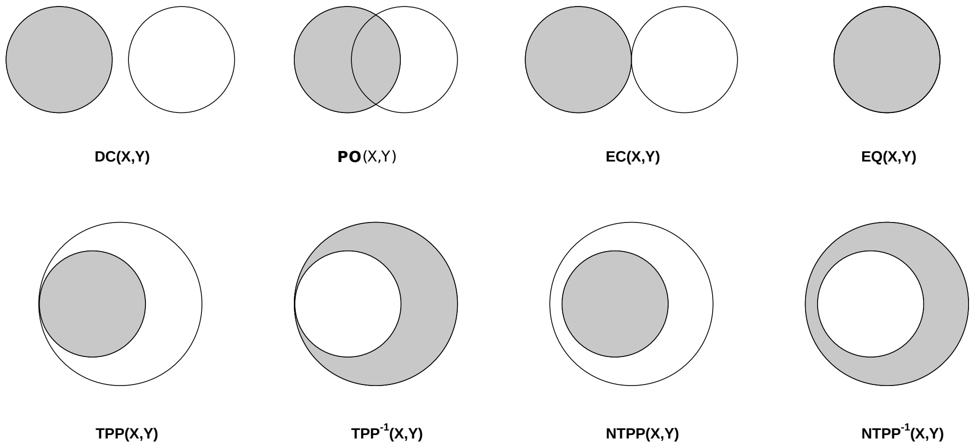


Figure 4.1: Basic $RCC8$ relations for variables ranging over regular closed subsets of $\mathbb{R}^2$.

4.2.2 Affine Spatial Logics

Convexity has played an important role in the development of qualitative spatial reasoning from the very beginning. It seems to be the case however that the research on convexity formalisms was treated as secondary to the research on the topological qualitative spatial reasoning. In [RCC92], which deals with RCC-type formalisms, a section is devoted to an extension of these formalisms with a function convexhull(x) having the obvious interpretation. "An axiomatisation" of this new function is proposed ($DR[a, b]$ reads: a is discrete from b).[2]

1. $\forall x(P(x, \text{convexhull}(x)))$
2. $\forall x(P(\text{convexhull}(\text{convexhull}(x)), \text{convexhull}(x)))$
3. $\forall xyz((P(x, \text{convexhull}(y)) \wedge (P(y, \text{convexhull}(z)))) \rightarrow (P(x, \text{convexhull}(z))))$

[2] That is a and b can only share boundary points. See [RCC92], p. 3–4.

4. $\forall xy((P(x, \text{convexhull}(y)) \wedge (P(y, \text{convexhull}(x)))) \rightarrow O(x, y))$

5. $\forall xy((DR(x, \text{convexhull}(y)) \wedge (DR(y, \text{convexhull}(x)))) \rightarrow DR(\text{convexhull}(x), \text{convexhull}(y)))$

The introduction of this new primitive is motivated by inexpressiveness of topological spatial logics in terms of describing certain *everyday* geometric relations (see [RC92]). As an example consider a relation of being inside a region, without being part of it (e.g. water in a bottle). This can be formalised as follows. A region r is said to be inside region s if r is a part of a convex hull of s but not part of s itself. The convex hull function is used to define this and other related properties.

$$\text{INSIDE}(x, y) := \text{DR}(x, y) \wedge \text{P}(x, \text{convexhull}(y))$$

$$\text{P-INSIDE}(x, y) := \text{DR}(x, y) \wedge \text{PO}(x, \text{convexhull}(y))$$

$$\text{OUTSIDE}(x, y) := \text{DR}(x, \text{convexhull}(y))$$

with obvious interpretations (P-INSIDE stands for partially inside). Inverse relations are defined accordingly.

The paper [Coh95] contains another attempt at capturing the properties of the convex hull function axiomatically.

1. $\forall x(\text{convexhull}(\text{convexhull}(x))) = \text{convexhull}(x))$

2. $\forall x(x \neq \text{convexhull(x)} \rightarrow TPP(x, \text{convexhull}(x)))$

3. $\forall xy(P(x, y) \rightarrow P(\text{convexhull}(x), \text{convexhull}(y)))$

4. $\forall xy P(\text{convexhull}(x) + \text{convexhull}(y), \text{convexhull}(x + y))$

5. $\forall xy(\text{convexhull}(x) = \text{convexhull}(y) \rightarrow C(x, y))$

6. $\forall xy(\text{convexhull}(x) \cdot \text{convexhull}(y) = \text{convexhull}(\text{convexhull}(x) \cdot \text{convexhull}(y)))$

7. $\forall xy(\text{DC}(x, y) \rightarrow \neg conv(x + y))$

8. $\forall xy(\text{NTPP}(x, y) \rightarrow \neg conv(y + (-x)))$

9. $\forall xy(conv(x) \wedge conv(y) \rightarrow conv(x \cdot y))$

10. $\forall xyz(\text{EC}(x, y) \wedge conv(x+y) \wedge \text{EC}(y, z) \wedge conv(y+z) \wedge \text{DC}(x, z) \rightarrow conv(y))$

where $conv(x) :=$ convexhull$(x) = x$.
This is then used to investigate further the expressive power of the thus extended RCC8.

Yet another attempt at axiomatising the convex hull function are presented in [Ben94].

1. $\forall x(\text{TP}(x, \text{convexhull}(x)))$;
2. $\forall x(\text{convexhull}(\text{convexhull}(x)) = \text{convexhull}(x))$;
3. $\forall xy(P(x, y) \rightarrow P(\text{convexhull}(x), \text{convexhull}(y)))$;
4. $\forall xy(\text{convexhull}(x) = \text{convexhull}(y) \rightarrow C(x, y)$.

Also, [Ben96] tries to give a modal interpretation of the convex hull function in the spirit of Tarski and McKinsey's work on relation between the $S4$ modal operator and the topological interior operator (see [TM44]). Bennett proposes the following translation of the above axioms (here $\circ$ is the convexity operator; $\square$ is the interior operator and $\blacksquare$ is S5 modal operator).

1. $\blacksquare(X \rightarrow \circ X) \wedge \neg\blacksquare(X \rightarrow \square\blacksquare X)$
2. $\circ \circ X \leftrightarrow \circ X$
3. $\blacksquare(\circ X \leftrightarrow \circ Y) \rightarrow \neg\blacksquare\neg(X \wedge Y)$
4. $\blacksquare(X \rightarrow Y) \rightarrow (\circ X \rightarrow \circ Y)$

Additionally, a new axiom is also considered

5. $\forall xy(\text{convexhull}(x) \cdot \text{convexhull}(y) =$
   $\text{convexhull}(\text{convexhull}(x) \cdot \text{convexhull}(y)))$

and translated in the following way

5 $\circ(\circ X \wedge \circ Y) \leftrightarrow (\circ X \wedge \circ Y)$

4.3 The Model-theoretic approach

Recall the early topological spatial logics described in the previous section: RCC and $RCC8$. We mentioned that $RCC8$ was first defined within RCC. With the topological interpretation in place one can now ask the question: what does it *really* mean that the primitives of $RCC8$ are definable in RCC? It was soon realised that the answer to this question depends heavily on the underlying topological space. It turns out that $RCC8$ primitives are indeed definable in the language of RCC over many of the topological spaces of interest. If that space is weakly regular τ_1 and connected we take the contact relation to be interpreted as follows (s and t are regular closed sets): $C[s,t]$ if and only if $s \cap t \neq \emptyset$ and the $RCC8$ primitives are given the following interpretation.

$DC[s,t]$ if and only if $s \cap t = \emptyset$;

$EC[s,t]$ if and only if $s \cap t \neq \emptyset$ and $(s)^0 \cap (t)^0 = \emptyset$;

$PO[s,t]$ if and only if $t^0 \cap s^0 \neq \emptyset$ and $t \cap -s \neq \emptyset$ and $-t \cap s \neq \emptyset$;

$EQ[s,t]$ if and only if $s = t$;

$TPP[s,t]$ if and only if $s \subseteq t$ and $b(s) \cap b(t) \neq \emptyset$;

$TPP^{-1}[s,t]$ if and only if $TPP(t,s)$;

$NTPP[s,t]$ if and only if $s \subseteq t$ and $b(s) \cap b(t) = \emptyset$;

$NTPP^{-1}[s,t]$ if and only if $NTPP(t,s)$.

From a purely formal point of view then, there are good reasons to focus on the more expressive language of RCC. (Recall that this is the first-order language over the signature $\{+, \cdot, -, 0, 1, C\}$.) With the intended interpretation in mind we view it as the Boolean Algebra together with the (topological) contact relation. Hence the language can be divided into two parts: algebraic and topological. At this point we might want to consider other languages where the topological part is being replaced or appended with other topological relations or functions. We adopt this line of thinking here. The following sections describe a selection of languages designed according to that pattern. We divided our description into two parts. The first concerns first-order languages and the second part deals with constraint languages.

Recall that $\mathfrak{M}_P$ denotes the polygonal model and $\mathfrak{M}_Q$ the rational model.

4.3.1 Topological Spatial Logics

First-order languages We note that, given the intended interpretation, we can replace $+, \cdot, -$ with the order relation $\leq$ (cf. Section 2.7). We adopt this convention here. Consider the following signatures:

$$\Sigma_1 = \{\leq, C\},$$
$$\Sigma_2 = \{\leq, c\},$$
$$\Sigma_3 = \{\leq, C, c\}.$$

We consider the first-order languages over these signatures, denoted in the usual way. These languages are interpreted over a specific topological space, the real plane. We now have to choose a plausible region-candidate for the variables to range over. For the reasons outlined above we wish to consider subalgebras of the set of regular open/closed subsets of $\mathbb{R}^2$.[3] We limit ourselves to the the following list of region-candidates (using the notation introduced in Section 2.7): $RO(\mathbb{R}^2)$, $ROP(\mathbb{R}^2)$ and $ROQ(\mathbb{R}^2)$. Finally we give the following interpretation to the primitives: let s, t be regions

$C[s, t]$ if and only if $(s)^0 \cap (t)^0 \neq \emptyset$;

$c[s]$ iff s is a connected set;

$s \leq t$ iff $s \subseteq t$.

We first turn to the task of showing how sensitive different languages are to changes in the domain. What we refer to as topological logics were historically often called *mereotopologies* and were viewed as a combination of mere-

[3]The choice between regular open or closed subsets is arbitrary.

ological and topological ideas. There is, however, a more technical definition of mereotopology. For example, Pratt-Hartmann (e.g. [PH07]) defines it as certain type of Boolean Algebra. We introduce this technical notion here as it is used in the results we wish to present.

Definition 4.3.1 (Mereotopology)**.** Let X be a topological space. A *mereotopology* over X is a Boolean sub-algebra M of $RO(X)$ such that, if o is an open subset of X and $p \in o$, there exists $r \in M$ such that $p \in r \subseteq o$.

Definition 4.3.2. A mereotopology M is *finitely decomposable* if every region in M is the sum of finitely many connected regions in M.

All the regular open algebras of interest are in fact mereotopologies (consult [PH07] for proofs), hence we often refer to a topological logic as mereotopology and by specifying its domain (e.g. $RO(\mathbb{R}^2)$, $ROQ(\mathbb{R}^2)$).
Consider $\mathcal{L}_{\leq,C,c}$. The following example outlines the difference between the theory of the mereotopology over $RO(\mathbb{R}^2)$ and the theories of the mereotopologies over $ROP(\mathbb{R}^2)$ and $ROQ(\mathbb{R}^2)$. Consider the following $\mathcal{L}_{\leq,C,c}$-sentence.

$$\forall x \forall y (C(x,y) \rightarrow \exists z (c(z) \wedge z \leq y \wedge C(x,z))).$$

It "says" that, if a region contacts another region, then it contacts some connected part of it. This sentence is satisfied by any finitely decomposable mereotopology (such as $ROP(\mathbb{R}^2)$); it is not however satisfiable over $RO(\mathbb{R}^2)$ (for details see [PH07]). Let us consider yet another example — $\mathcal{L}_{\leq,c}$. It turns out that also here different domains determine different theories. The following shows that the mereotopologies $RO(\mathbb{R}^2)$ and and $ROP(\mathbb{R}^2)$, $ROQ(\mathbb{R}^2)$ have different $\mathcal{L}_{\leq,c}$-theories. Consider the following $\mathcal{L}_{\leq,c}$-sentence.

$$\forall x_1 \forall x_2 \forall x_3 (c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1 + x_2 + x_3) \rightarrow (c(x_1 + x_2) \vee c(x_1 + x_3))).$$

It "says" that if three connected regions have a connected sum, then the first must form a connected sum with one of the other two. [PH07] shows that this sentence is true not only in $ROP(\mathbb{R}^2)$ but also in $ROQ(\mathbb{R}^2)$, whereas it is not true in $RO(\mathbb{R}^2)$. Figure 4.2 shows a $RO(\mathbb{R}^2)$-construction consisting of regions r, s, t defined as follows.

$$r = \{(x,y) \mid -1 < x < 0;\ -1 - x < y < y < 1 + x\};$$
$$s = \{(x,y) \mid 0 < x < 1;\ -1 - x < y < sin(1/x)\};$$
$$t = \{(x,y) \mid 0 < x < 1;\ sin(1/x) < y < 1 + x\}.$$

It is clear that the sum of r, s, t is the large triangle (as depicted in the Figure 4.2) and so the sum of r, s, t is connected but neither $r+s$ nor $r+t$ is connected.

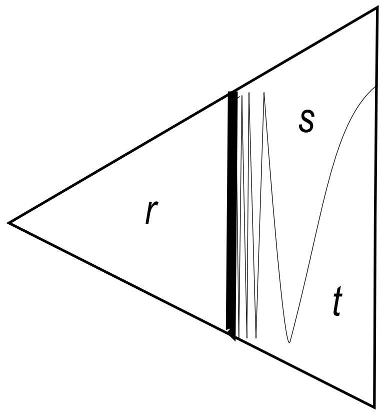


Figure 4.2: Three connected regions in $\mathbb{R}^2$, see [PH07]

One can also try and compare the expressive power of different topological languages. What we essentially do is to fix the interpretation and, given two languages, see if one language's primitives are expressible in the other. For example say we wish to compare languages $\mathcal{L}_{\leq,C}$ and $\mathcal{L}_{\leq,c}$ interpreted geometrically. The first result shows that in fact instead of considering $\mathcal{L}_{\leq,C}$ we could confine our attention to $\mathcal{L}_C$.

Theorem 4.3.1 ([PH07]). *Let M be a mereotopology over a weakly regular space X, and let $r_1, r_2 \in M$. Then $r_1 \leq r_2$ if and only if $M \models \phi_{\leq}[r_1, r_2]$, where $\phi_{\leq}(x_1, x_2)$ is the $\mathcal{L}_C$-formula*

$$\forall z(C(x_1, z) \to C(x_2, z)).$$

Proof. See [PH07]. □

A mereotopology M is said to *respect components* if every component of every region in M is also in M. Most of the mereotopologies discussed in this section respect components.

Theorem 4.3.2 ([PH07]). *Let M be a mereotopology over a regular topological space X such that M respects components, and let $r \in M$. Then r is connected if and only if $M \models \phi_c[r]$, where $\phi_c(x)$ is the $\mathcal{L}_C$-formula*

$$\forall x_1 \forall x_2 (x_1 > 0 \wedge x_2 > 0 \wedge x_1 \cdot x_2 = 0 \wedge x_1 + x_2 = x \rightarrow \exists x_1' \exists x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge C(x_1', x_2') \wedge \neg C(x_1' + x_2', -x))).$$

Proof. see [PH07]. □

The two presented results ensure that for mereotopologies over regular topological spaces which respect components $\mathcal{L}_C$ is at least as expressive as $\mathcal{L}_{\leq,c}$. The converse is true for some *well-behaved* mereotopologies (we refer the reader to [PH07]).

We note that one can also investigate expressiveness results of a more absolute character. Given a topological logic and homeomorphic tuples of regions, we might ask, do these tuples satisfy the same formulas? And conversely, can we find a formula satisfiable only by homeomorphic regions? We do not investigate this problem here (cf. [PH07]). However we do provide similar sort of results for convexity logic in Section 5.3.

One might wonder how the axiomatisation attempts described above fit into model-theoretic approach to spatial logics. As we mentioned earlier, there is a place for axiomatisation here. Constructing an axiom system is one of the ways of exploring the properties of a given theory. Take $\mathcal{L}_{\leq,c}$, where variables range over regular open polygonal subsets of $\mathbb{R}^2$. The paper [PHS98] axiomatises the $\mathcal{L}_{\leq,c}$-theory of $ROP(\mathbb{R}^2)$, which was later refined in [PH07]. It includes the axioms of the non-trivial Boolean Algebra together with axioms expressing certain topological properties (not chosen for their intuitiveness). Among others these axioms include: an axiom ensuring that two connected regions with a non-empty intersection have a connected sum; an axiom schema stating that if connected regions form a connected sum, then at least one of these regions is such that its removal from the sum leaves out another connected sum. There are also axioms securing certain "technical" properties, aiding the construction of the completeness proof. Importantly, an *infinite* inference rule is used.

$$\frac{\{\forall x(\psi_n(x) \rightarrow \delta(x)) | n \geq 1\}}{\forall x \delta(x)},$$

where for each $n \geq 1$, $\psi_n(x)$ stands for a formula

$$\exists z_1 \ldots \exists z_n (\bigwedge_{1 \leq i \leq n} c(z_i) \wedge (x = \sum_{1 \leq \leq n} z_i)).$$

Intuitively, this rule states that every region is a sum of connected regions. Please consult [PH07] for a detailed description of the system and soundness and completeness results. We note that we have taken up a similar approach to axiomatising convexity logic, presented in the next chapter.

The constraint topological languages Since most of the first-order logics are undecidable,[4] it is reasonable to try and restrict the languages in hope to restore decidability. In what follows we describe several constraint languages with emphasis on complexity results. Recall the signatures $\Sigma_2 = \{\leq, c\}$ and $\Sigma_3 = \{\leq, C, c\}$ from the previous section. We introduce two more, closely related, signatures $\Sigma_2' = \{\leq, c^-\}$ and $\Sigma_3' = \{\leq, C, c^-\}$. This section concerns the complexity results relating the constraint languages over these signatures with topological interpretation. The paper [KPHZ10] deals with much greater selection of interpretations; we however focus on the situation where these constraint languages are interpreted over the real plane. The only new symbol $c^-[r]$ is interpreted as: r has a connected closure.

Theorem 4.3.3. *The satisfiability problem for* $\mathcal{L}^c_{\leq,c^-}$, $\mathcal{L}^c_{\leq,C,c^-}$ *and* $\mathcal{L}^c_{\leq,C,c}$ *over* $RO(\mathbb{R}^2)$ *and* $ROP(\mathbb{R}^2)$ *is* $\mathcal{EXPTIME}$*-hard.*

Proof. See [KPHWZ10] □

Thus we are left with just one language $\mathcal{L}^c_{\leq,c}$, dealt with in the following theorem.

Theorem 4.3.4. *The satisfiability problem for* $\mathcal{L}^c_{\leq,c}$ *over* $ROP(\mathbb{R}^2)$ *is* $\mathcal{EXPTIME}$*-hard.*

By a *graph model* we mean a pair $\mathfrak{G} = \langle G, \sigma\rangle$, where $G = \langle V, E\rangle$ is a graph and σ is a function mapping $\mathcal{L}^c_{\leq,c}$-variables to a subset of V. The relation symbol $\leq$ is interpreted as the inclusion relation on the power set of V and c is interpreted as (graph-theoretic) connectedness. The proof proceeds as follows. First it is shown that an $\mathcal{L}^c_{\leq,c}$ -formula is satisfiable over $ROP(\mathbb{R}^2)$ if and only if it is true in a connected *planar* graph model. Then, using the techniques developed in [KPHWZ10] it is shown that it is $\mathcal{EXPTIME}$-hard to decide whether an $\mathcal{L}^c_{\leq,c}$ -formula is true in a planar graph model.

We wish to mention complexity results for two other constraint languages.[5] The satisfiability of $RCC8$ over regular closed subsets of any topological

[4] For the discussion of the first-order undecidability of RCC and BCA cf. [Gri08], p. 47–52, and [CBGG97].

[5] For other complexity results please consult [KPHZ10], [KPHWZ10] and [PH02].

space X is shown to be $\mathcal{NP}$-complete. The complexity does not change if we take $X = \mathbb{R}^2$ (see [KPHZ10]). Note that $RCC8$ was modelled after Allen's interval algebra (see [All83] for the survey) and it was hoped that similar techniques used in tackling problems from Allen's algebra would prove useful in case of $RCC8$. In particular, [RN97] presents an algorithm for solving instances of $RCC8$ satisfiability problem — it is translated as a constraint satisfaction problem. A composition function, mapping $RCC8$-relations to $RCC8$-terms, is introduced (defined as a composition table look-up). For the details of the construction the reader is referred to [RN97]. This approach did not prove to be as fruitful as in the case of Allen's algebra, since $RCC8$-terms enhanced with the composition function do not form a relation algebra.[6] A particular modification of $RCC8$, called $BRCC8$, is of special interest here. In [WZ00] the $RCC8$-relation symbols were enhanced with *Boolean region terms*: binary $+, \cdot$ and unary $-$. The resulting system—$BRCC8$—has the following interpretation. Variables range over regular *closed* sets: $RC(X)$; $RCC8$-relation symbols are interpreted as above ; $s + t$ interpreted $s \cup t$; $s \cdot t$ interpreted $(s \cap t)^0$ and $-t$ if and only if $(X/t)^0$. Formulas are defined analogously to the $RCC8$ case, the only difference is that now the basic relations can hold between sums, products and complements of regions. If we take the topological space $X = \mathbb{R}^n$, for any $n \geq 1$ the satisfiability problem turns out to be $\mathcal{PSPACE}$-complete.

4.3.2 Affine Spatial Logics

This section deals primarily with affine *convexity* spatial logics. We introduce four signatures containing $conv$ and some choice of "topological" symbols.

$$\Sigma_1 = \{conv\} \cup \{R \mid R \text{ is an RCC8 relation symbol}\},$$
$$\Sigma_2 = \{conv, EC, PP\},$$
$$\Sigma_3 = \{conv, C\},$$
$$\Sigma_4 = \{conv, \leq\}.$$

We denote the constraint languages over the signatures Σ_1 and Σ_2 by $\mathcal{L}^c_{conv,RCC8}$ and $\mathcal{L}^c_{conv,EC,PP}$ and the first-order languages over Σ_3 and Σ_4 by

[6] For a more detailed treatment of $RCC8$ and related systems reader is referred to [RN07], [Ren02] and [Gri08].

$\mathcal{L}_{conv,C}$ and $\mathcal{L}_{conv,\leq}$. These languages are interpreted over the real plane with variables ranging over the elements of $RO(\mathbb{R}^2)$ (in some, indicated, cases we restrict our attention to $ROP(\mathbb{R}^2)$ and $ROQ(\mathbb{R}^2)$). All RCC8 relation symbols are interpreted in the standard way; $EC[a,b]$ iff a and b are externally connected; $PP[a,b]$ iff a is a proper part of b; C is the contact relation; $conv$ is the convexity property. We recall that $\mathcal{L}_{conv,\leq}$ is the language we introduced in Section 2.7.

The paper [DGC99] is one of the first within the QSR paradigm to deal primarily with convexity logic. There, the languages $\mathcal{L}^c_{conv,RCC8}$ and $\mathcal{L}^c_{conv,EC,PP}$ are investigated. Two main results are presented there. The first is a complexity result for $\mathcal{L}^c_{conv,RCC8}$. A procedure is described such that given a set of constraints S in $\mathcal{L}^c_{conv,RCC8}$ (assuming the above interpretation), this procedure generates a set of constraints S' such that S is satisfiable by a tuple of regular open subsets of $\mathbb{R}^2$ if S' is satisfiable by a tuple of regular open *polygons* of $\mathbb{R}^2$ with a boundable number of vertices. Hence the satisfiability problem for $\mathcal{L}^c_{conv,RCC8}$ interpreted over $RO(\mathbb{R}^2)$ is reduced to the satisfiability problem for $\mathcal{L}^c_{conv,RCC8}$ interpreted over $RCP(\mathbb{R}^2)$. S' in turn can be reduced in a similar fashion to the set S'' of formulas in the language of fields reconstructed in $\mathcal{L}^c_{conv,RCC8}$ (using the fact that one can "talk" about coordinatisation in $\mathcal{L}^c_{conv,RCC8}$ — we discuss coordinatisation when describing $\mathcal{L}^c_{conv,EC,PP}$). Decidability – as well as the complexity bounds – then follows from Tarski's well known work ([Tar48]).[7]

We now give a brief overview of the procedure. Let r, s be regions related by one of the $\mathcal{L}^c_{conv,RCC8}$ primitive relations.[8] In essence, the procedure consists of selecting witness triangular subregions $r' \subset r$ and $s' \subset s$ in a way specific to the primitive relation involved. Consider a constraint expressing a fact that r is related to other regions by n primitive relations. The procedure will then select n witness regions for r. We now map r to a new region r' which is the convex hull of the sum of all n witness regions involved. Repeating the procedure for every region in the considered constraint, we end

[7]And the satisfiablity problem for the theory of fields is known to be at least doubly exponential. See [DH88].

[8]We note that the paper [BC99] proposes a model building algorithm for determining consistency of RCC8-relations and shows how to extend this procedure to incorporate convexity. This procedure is related to that of [DGC99] but it is claimed to be less computationally heavy (p. 3). In the case of convexity [BC99] uses the property that for a convex region r and a point p not in that region, p lies outside any triangular region whose vertices are contained in r (by convexity this triangular region is also contained in r). For the details of the construction please consult [BC99].

up with a new set of polygonal regions. We refer the reader to the full explanation of the procedure to the original paper.

The second result of [DGC99] relates to the language $\mathcal{L}^c_{conv,EC,PP}$. A crucial step, as in the case of $\mathcal{L}^c_{conv,RCC8}$ is to show that $\mathcal{L}^c_{conv,EC,PP}$ is expressive enough to allow talking about coordinatisation. This is indeed the case: it is shown that statements about points and lines can be simulated with statements about regions. This includes the properties of collinearity and non-collinearity and the betweenness relation (by the formula containing three n-sequences of variables bet$(\bar{x}, \bar{y}, \bar{z})$, [DGC99], p. 251–254).We have the following theorem.

Theorem 4.3.5. *If a region s is an affine transformation of region r, then for any $\mathcal{L}^c_{conv,EC,PP}$-formula ϕ we have that $RC(\mathbb{R}^2) \models \phi[r]$ if and only if $RC(\mathbb{R}^2) \models \phi[s]$.*

An analogous theorem, relating the language $\mathcal{L}_{conv,\leq}$ is proved in [Pra99]. The main results in this paper concern three interpretations, where variables are set to range over $RO(\mathbb{R}^2)$, $ROP(\mathbb{R}^2)$ or $ROQ(\mathbb{R}^2)$.
The paper [Pra99] extends the results from [DGC99] by investigating if the converse of Theorem 4.3.5 holds. In the case of $\mathfrak{M}_Q$ the converse theorem is shown to be true.

Theorem 4.3.6. *Every n-tuple in $ROQ(\mathbb{R}^2)$ satisfies an $\mathcal{L}_{conv,\leq}$-formula φ with the following property: any two n-tuples satisfying φ are affine-equivalent.*

The proof relies on constructing certain formulas that allow us to "talk" about rational polygons and fixing of their bounding lines in certain manner. We use these, appropriately called, *fixing formulas* in our own construction in Chapter 5. However, the converse of Theorem 4.3.5 is false in the case of $\mathfrak{M}_P$.

Recall that a model $\mathfrak{A}$ is *prime* if for any model $\mathfrak{B}$ such that $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}$, the model $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$. It is shown that the model $\mathfrak{M}_Q$ is prime and that the models $\mathfrak{M}, \mathfrak{M}_Q, \mathfrak{M}_P$ all have different theories.
The paper [Dav06] deals with the language $\mathcal{L}_{conv,C}$. It is shown that a number of topological relations are $\mathcal{L}_{conv,C}$-definable: P (part of) , PP (proper part of), c (connectedness), O (overlap), EC (externally connect) with obvious interpretations. As an example, consider the formula $\phi(x)$ defined to be $\forall y \forall z \forall w((C(w,x) \leftrightarrow C(w,y) \vee C(w,z)) \rightarrow C(y,z))$. The formula ϕ says that if a region is a sum of two others, then these two have to contact each other.

Hence, $\phi[r]$ is satisfied if and only if r is connected. Consider the following lemma.

Lemma 4.3.1. *If a and b are convex solids such that a is in contact with b, then a and b meet in a single point if and only if the following holds: for all regions $c, d \in U$, if $c \subset a$ and $d \subset b$, b is in contact with c and a is in contact with d, then c is in contact with d.*

This lemma is used to show that there exists an $\mathcal{L}_{conv,C}$-formula ψ such that ψ is satisfiable by two regions if and only if these regions meet at a single point. Using this fact and some related definitions it is shown that an affine coordinate system is also $\mathcal{L}_{conv,C}$-definable. Then using standard techniques (not dissimilar to those utilised by [ST79]) real addition and multiplication are defined relative to a given affine coordinate system. All this is needed to prove the following theorems.

Theorem 4.3.7. *Let U be a class of closed regions in the plane that includes all simple polygons. Let $\phi(x_1, \ldots, x_n)$ be an analytical and affine-invariant relation over U. Then ϕ is first-order definable in the structure $\langle U, C, \mathit{Convex}\rangle$.*

Theorem 4.3.8. *Let ϕ be as in Theorem 4.3.7. Then ϕ is first-order definable in $\langle RC(\mathbb{R}^2), C, \mathit{Convex}\rangle$ if and only if ϕ an analytical and affine-invariant relation over $RC(\mathbb{R}^2)$.*

(Analogous results hold when U is the set of all rational regular closed polygons, cf. [Dav06]).

An analytical relation is one that can be defined in the structure $\mathfrak{S} = \langle \mathbb{N} \cup \mathbb{N}^\omega, +, \cdot, []\rangle$, where $\mathbb{N}^\omega$ is the set of infinite sequences of natural numbers and given a sequence $\bar{s}$ and a number p; $\bar{s}[p]$ means "p^{th}element of $\bar{s}$". In order to prove the results, it is shown that infinite sequences of points, the indexing function on such sequences and the closure function (mapping an infinite sequence of points to a region) can all be defined in $\mathcal{L}_{conv,C}$.

On a final note we wish to observe that the language $\mathcal{L}_{conv,\leq}$ turns out to be the most expressive — to a certain extent — of all those considered in this section.

Theorem 4.3.9. *The relations $C(x, y)$ and $EC(x, y)$ are $\mathcal{L}_{conv,\leq}$-definable in $\mathfrak{M}_Q$ and $\mathfrak{M}_P$.*

It follows that all the $RCC8$ relations are $\mathcal{L}_{conv,\leq}$-definable over these domains. It follows that the expressiveness results from [DGC99] and [Dav06] carry over to $\mathcal{L}_{conv,\leq}$. This theorem is also used to prove the undecidability of $\mathfrak{M}_P$ and $\mathfrak{M}_Q$. From this undecidability of $\mathfrak{M}$ follows easily.

4.4 The Modal approach

In this section we gather interesting results on modal affine and projective spatial logics. We note that the results presented here are only tangentially related to our investigations.

Venema The paper [Ven99] deals with *projective* spatial logic. In this paper a modal language is introduced for talking about projective planes. Let $\Sigma = \{\Diamond_P, \Diamond_L\}$. Take VAR_P and VAR_L be two countably infinite disjoint sets elements of which are called *point* and *line* variables respectively. Next we define FOR_P and FOR_L. The set FOR_P is the smallest set satisfying the following conditions: (i) every $p \in VAR_P$ is in FOR_P; (ii) if $p \in FOR_P$, then $\neg p \in FOR_P$; (iii) if $p_1, p_2 \in FOR_P$ then $p_1 \wedge p_2 \in FOR_P$; (iv) if $l \in FOR_L$ then $\Diamond_P l \in FOR_P$. Analogously, the set FOR_L is the smallest set satisfying the following conditions: (i) every $l \in VAR_L$ is in FOR_L; (ii) if $l \in FOR_L$, then $\neg l \in FOR_L$; (iii) if $l_1, l_2 \in FOR_L$ then $l_1 \wedge l_2 \in FOR_L$; (iv) if $p \in FOR_P$ then $\Diamond_L p \in FOR_L$.

Note that to avoid notational clutter we do not sort the logical connectives.

Hence L is a two-sorted modal language, where we distinguish between point and line formulas. In order to define semantics for this language we need to introduce a few more notions.

Definition 4.4.1. A *two-sorted frame* is a two-sorted structure $\mathcal{F} = \langle P, L, I\rangle$ such that $P \cap L = \emptyset$ and $I \subseteq P \times L$. Elements of P and L are called points and lines respectively; I is called the *incidence relation*.

A projective plane is thought of as a two-sorted frame $\mathcal{F} = \langle P, L, I\rangle$ satisfying the following properties.

1. Each pair of distinct points is connected by exactly one line;
2. each pair of distinct lines intersects in exactly one point;

3. there are at least four points such that no three of them are incident with one and the same line.

We let PP denote the class of projective planes. According to [Ven99] we obtain an equivalent definition of a projective plane if we replace (3) with

$(3')$ there are at least four lines such that no three of them are incident with one and the same point.

Finally, we can define semantics for L.

Definition 4.4.2. Let $\mathcal{F} = \langle P, L, I \rangle$ be some frame. A *valuation* on $\mathcal{F}$ is a map assigning subsets of P to point variables and subsets of L to line variables. A *(two-sorted) model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ such that $\mathcal{F}$ is a frame and V is a valuation. Given a model $\mathcal{M}$ we define the notion of truth as follows:

$\mathcal{M}, s \models p$ if $s \in V(p)$;

$\mathcal{M}, k \models l$ if $k \in V(l)$;

$\mathcal{M}, x \models \neg\phi$ if not $\mathcal{M}, x \models \phi$,

$\mathcal{M}, x \models \phi \wedge \psi$ if $\mathcal{M}, x \models \phi$ and $\mathcal{M}, x \models \psi$;

$\mathcal{M}, s \models \Diamond_P L$ if there is some k with $s\ I\ k$ and $\mathcal{M}, k \models L$;

$\mathcal{M}, k \models \Diamond_L P$ if there is some s with $s\ I\ k$ and $\mathcal{M}, s \models P$,

where $s \in P$, $k \in L$, $p \in VAR_P$, $l \in VAR_L$, x is either in P or L.

The expressiveness of this language is explored to some degree. For example, it is shown that the theorem of Pappus is expressible (Theorem 2.5.6 in Chapter 2).[9] More precisely:

Lemma 4.4.1. *Let $\mathcal{F}$ be a projective plane. Then there exists a formula ψ such that the Pappus theorem holds in $\mathcal{F}$ if and only if $\mathcal{F} \models \psi$.*

Next, Venema proposes the projective calculus AXP consisting the following axioms and rules of inference.

1. all classical tautologies;

[9]Cf. [Ven99], p. 6–8

2. $\Box_P(l_1 \rightarrow l_2) \rightarrow (\Box_P l_1 \rightarrow \Box_P l_2)$;

3. $\Box_L(p_1 \rightarrow p_2) \rightarrow (\Box_L p_1 \rightarrow \Box_L p_2)$;

4. $p \rightarrow \Box_P \Diamond_L p$;

5. $l \rightarrow \Box_L \Diamond_P l$;

6. $\Diamond_P \top$;

7. $\Diamond_L \top$;

8. $\Diamond_a \Diamond_a p \rightarrow \Diamond p$;

9. $\Diamond_b \Diamond_b l \rightarrow \Diamond l$.

Here, $\Box_i \phi = \neg \Diamond_i \neg \phi$ and $\Diamond_a \phi = \Diamond_P \Diamond_L \phi$, $\Diamond_b \phi = \Diamond_L \Diamond_P \phi$.

(MP) Modus Ponens: $\phi \rightarrow \psi$, ϕ / ψ.
(N) Necessitation for $\Box_P$ and $\Box_L$: ϕ / $\Box_P \phi$;ψ / $\Box_L \psi$, where ϕ is a point formula and ψ is a line formula.
(SUB) Substitution uniformly replacing in any formula ϕ some propositional variable by some formula of the same sort: ϕ / $s(\phi)$.

The meaning of axioms 1–3 is obvious. Informally, axioms 4–5 form the modal way of stating that the accessibility relations connected to the diamonds $\Diamond_P$ and $\Diamond_L$ are each other's converse. The axiom 6 states that each point is incident with at least one line. The axiom 8 is the transitivity axiom for $\Diamond$. The axioms 5 and 7 are the obvious duals of 6 and 8 (cf. [Ven99], p. 13). This projective calculus determines the consequence relation $\vdash_{AXP}$.

Theorem 4.4.1. *The calculus AXP is strongly sound and complete with respect to the class PP of projective planes. That is, for any set Φ of formulas and any formula ϕ (of the same sort) we have*

$$\Phi \vdash_{AXP} \phi \text{ if and only if } \Phi \models_{PP} \phi.$$

It turns out the PP-satisfiability problem (PP-SAT) for $\mathcal{L}_\Sigma$, that is the problem of determining for a given $\mathcal{L}_\Sigma$-formula ϕ if ϕ is satisfiable in some projective plane is decidable. The paper [Ven99] also explores the computational complexity of PP-SAT and obtains the following result.

Theorem 4.4.2. *PP-SAT is $\mathcal{NEXPTIME}$-complete.*

Proof. See [Ven99]. □

Hodkinson One reason why [Ven99] is important for our purposes is that Hodkinson et al model their ideas on this paper and use similar framework to talk about *affine* modal logic.

Definition 4.4.3. Let $\Sigma = \{\Box_P, \Box_L, \Box_{PL}\}$ and let syntax be defined as in [Ven99] with the obvious modifications and with the following addition. We extend the definition of FOR_L with the following condition: (v) if $p \in FOR_L$ then $\Box_{PL} p \in FOR_L$.

Definition 4.4.4. A two-sorted *affine frame* is a two-sorted structure $F = \langle P, L, \epsilon, \| \rangle$ such that $P \cap L = \emptyset$, $\epsilon \subseteq P \times L$, and $\| \subseteq L \times L$. Elements of P and L are called *points* and *lines*, respectively; ϵ is called the *incidence* relation and $\|$ is called the *parallel* relation.

A *model* and *valuation* are defined as in [Ven99]. The definition of truth in a model differs in the case of modalities in the following way.

Definition 4.4.5. Let $s \in P$ and $l \in L$

$\mathcal{M}, s \models \Box_P L$ if $\mathcal{M}, m \models L$ for every $m \in L$ with $s \epsilon m$;

$\mathcal{M}, l \models \Box_L P$ if $\mathcal{M}, t \models P$ for every $t \in P$ with $t \epsilon l$;

$\mathcal{M}, l \models \Box_{PL} L$ if $\mathcal{M}, m \models L$ for every $m \in L$ with $l \parallel m$.

The definitions of validity and satisfiability are standard.

Next [HH08] defines an affine plane.

Definition 4.4.6. A (two-sorted) affine frame A = $\langle P, L, \epsilon, \| \rangle$ is said to be an *affine plane* if:

1 For any two distinct points $s, t \in P$, there is exactly one line $l \in L$ such that $s \epsilon l$ and $t \epsilon l$.

2 For all $l, m \in L$, we have $l \parallel m$ iff $l = m$ or there is no $s \in P$ with $s \epsilon l$ and $s \epsilon m$.

3 For any $l \in L$ and $s \in P$, there is exactly one line $m \in L$ such that $s \epsilon m$ and $m \parallel l$.

4 There are distinct $s, t, u \in P$ such that for no $l \in L$ do we have $s\epsilon l$, $t\epsilon l$, and $u\epsilon l$.

The logic of affine planes is defined to be the set of all point and line formulas that are valid in every affine plane. We note that Hodkinson et al. also gives a translation of the modal formulas into first-order logic (see [HH08], p. 949). Next [HH08] defines a formula ψ such that $\neg\psi$ is valid in every affine plane. Certain properties of this formula are used in the proof of the following, main theorem of the paper (cf. [HH08], p. 950–951).

Theorem 4.4.3. *The modal logic of affine planes is not finitely axiomatisable.*

The chapter [BGKV07] is a very good source on standard results in affine logics. It refers, *inter alia*, to the results presented in [ST79], [Tar59] and more indirectly to [Whi07] and [Rus97], [Rus56]. After rehearsing basic concepts in affine and projective geometries, including coordinatisation of affine spaces, the first-order theories are considered where [ST79] is reviewed. From our point of view, the important contribution of [BGKV07] is the description of two-sorted modal logic for plane affine geometry, but mainly because we are also interested in – a specific example of – plane affine geometry (cf. [HH08]).

Definition 4.4.7. A two sorted frame $\mathcal{F} = \langle P, L, I\rangle$ is *affine* if the following are satisfied.

1. $\forall X \forall y \exists z(XIz \wedge y \parallel z)$;
2. $\forall x \forall y \forall z(x \parallel y \wedge y \parallel z \rightarrow x \parallel z)$,

where $X \in P$, $x, y, z \in L$ and $x \parallel y$ iff for all $Z \in P$ if ZIx and ZIy, then $x = y$. We call $\parallel$ a *parallelism* relation. We also define a *strong* parallelism relation, denoted $\parallel_s$: $x \parallel_s y$ iff for all $Z \in P$ not ZIx or not ZIy. We append the definitions in [HH08] in the following way. Let $\Sigma = \{\Box_P, \Box_L, [\parallel_s]\}$ and let syntax be defined as in [Ven99] with the following addition. We extend the definition of FOR_L with the following condition: (v) if $p \in FOR_L$ then $\Diamond_{PL} l \in FOR_L$. The semantics is analogous to the one in [HH08]. The main difference is the interpretation of $[\parallel_s]$: $M, x \models [\parallel_s]\alpha$ iff for all $y \in L$ such that $x \parallel_s y$ $M, y \models \alpha$.

The main result is the following theorem.

Theorem 4.4.4. *The satisfiability problem for M is $\mathcal{NEXPTIME}$-hard.*

Proof. See [BG02]. □

4.5 Summary

The main theme of this chapter was the model-theoretic approach to spatial formalisms. Historically it was the research on the qualitative spatial reasoning that revived the interest in the topological work of Whitehead and Clarke. Initially, the research was carried out with the notion of interpretation being only vaguely defined. It was soon realized, that in order to meaningfully investigate region-based spatial logics the tools developed in model theory are required. We mentioned that within the model-theoretic approach axiomatisation is an important way of fathoming the properties of a given theory. However, since the axiom system is no longer a starting point, it does not really matter if it contains any "intuitively" valid assertions. We gave an example of such an axiom system in the case of topological spatial logic. The next chapter presents, among others, the main contribution of this thesis: an axiom system for an affine spatial logic.

5 Axiomatisation

5.1 Introduction

Recall that $\mathfrak{M}_X$ denotes the structure $\langle ROX(\mathbb{R}^2), conv, \leq\rangle$, with $X \in \{P, Q, A\}$. If $X = \emptyset$ we obtain $\mathfrak{M} = \langle RO(\mathbb{R}^2), conv, \leq\rangle$. This chapter contains the main contribution of this thesis — the axiom system for the $\mathcal{L}_{conv,\leq}$-theory of $\mathfrak{M}_Q$. It is divided into two parts. The first part contains expressiveness results relating to the rational model (and often generalising to other domains). The second part describes the proposed axiom system and provides proofs of the soundness and completeness theorems.

5.2 Expressiveness

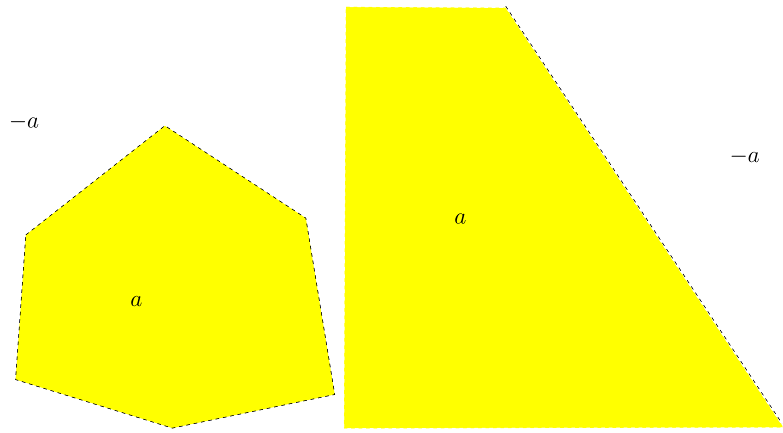
5.2.1 Basic Expressiveness

Most of the results from this section are either directly taken from, or are built on, the results presented in [Pra99]. We present an account of this work in Chapter 4.

Lemma 5.2.1. *Let $l \in ROX(\mathbb{R}^2)$. Then $\mathfrak{M}_X \models hp[l]$ if and only if l is a half-plane, where $hp(x)$ is the formula:*

$$conv(x) \wedge conv(-x).$$

Proof. It is enough to observe that for any convex region l its complement $-l$ is also convex if and only if l is a half-plane. See Figure 5.2.1 □



(a) $conv(a)$ but $\neg conv(-a)$ (b) $conv(a)$ and $conv(-a)$

Figure 5.1: Using convexity to define a half-plane in $\mathbb{R}^2$.

We use letters l, m, n etc. (possibly with subscripts) to denote half-planes but sometimes we use the same symbols to denote the lines bounding these half-planes. We hope no confusion arises.[1]

Lemma 5.2.2. *Let $l_1, l_2 \in ROX(\mathbb{R}^2)$. Then l_1 and l_2 are half-planes with lines bounding them being coincident if and only if*

$$\mathfrak{M}_X \models \alpha[l_1, l_2],$$

where $\alpha(x_1, x_2)$ is the formula:

$$hp(x_1) \wedge hp(x_2) \wedge (x_1 = x_2 \vee x_1 = -x_2).$$

Proof. Clearly, two lines are coincident just in case they bound the same half-planes. □

It is easy to see (cf. Figure 5.2) that for any two half-planes l_1, l_2, their bounding lines are parallel or coextensive if and only if at least one of $l_1 \cdot l_2 = \emptyset$, $l_1 \cdot -l_2 = \emptyset$, $-l_1 \cdot l_2 = \emptyset$, $-l_1 \cdot -l_2 = \emptyset$ holds. This observation allows us to talk about parallel lines in $\mathfrak{M}_X$.

Lemma 5.2.3. *Let $l_1, l_2 \in ROX(\mathbb{R}^2)$. Then there exists a formula $par(x_1, x_2)$ such that $\mathfrak{M}_X \models par[l_1, l_2]$ if and only if l_1 and l_2 are half-planes and lines bounding them are parallel.*

[1]Obviously, with each line there are two half-planes associated.

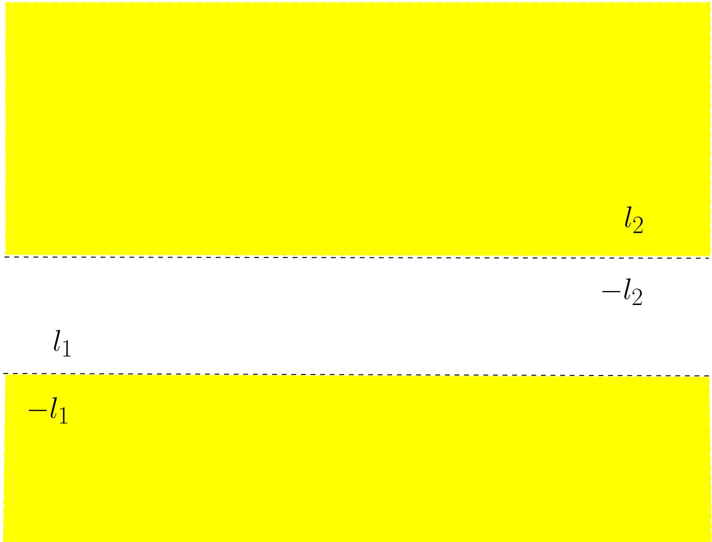


Figure 5.2: Lines bounding l_1 and l_2 are parallel: half-planes $-l_1$ and l_2 have an empty intersection.

Proof. Consider the formula: $par(x_1, x_2) :=$

$$hp(x_1) \wedge hp(x_2) \wedge x_1 \neq \pm x_2 \wedge (x_1 \cdot x_2 = 0 \vee x_1 \cdot -x_2 = 0 \vee -x_1 \cdot x_2 = 0 \vee -x_1 \cdot -x_2 = 0).$$

□

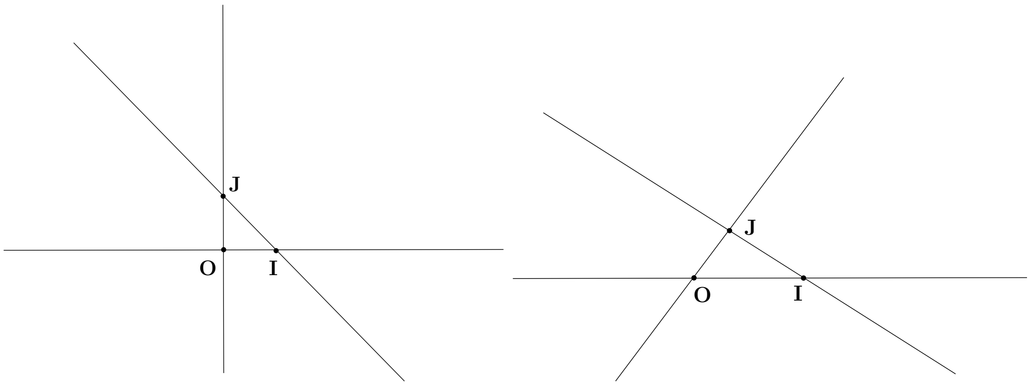
Definition 5.2.1. Let l, m, n be any non-parallel, non-coincident lines with $l \cap m = \mathbf{O}$, $l \cap n = \mathbf{I}$ and $m \cap n = \mathbf{J}$. We say that l, m, n form a *coordinate system* or a *coordinate frame* and call l the *abscissa*, m the *ordinata* and refer to point $\mathbf{O}$ as the *origin* and to segments $\overline{\mathbf{OI}}$ and $\overline{\mathbf{OJ}}$ as the *units of measurement* on the lines they belong to.

Given n variables $x_1, \ldots, x_n$ and their complements there are 2^n possible n-element products of $\pm x_i$ for $1 \leq i \leq n$. We denote them by:

$$\prod_{1 \leq i \leq n}^{(1)} \pm x_i, \ldots, \prod_{1 \leq i \leq n}^{(2^n)} \pm x_i.$$

The index set of the set of all possible n-element products of $\pm x_i$ is denoted by $\mathbb{P}_n$. Note that in this case phrases like complement and product are used informally to denote respective function symbols in the language.

Lemma 5.2.4. *There exist formulas* $coor(x_1, x_2, x_3)$ *and* $\Gamma(x_1, x_2, x_3)$ *such that*



(a) (b)

Figure 5.3: Example coordinate systems.

$\mathcal{M}_X \models \Gamma[l_1, l_2, l_3]$ if and only if l_1, l_2, l_3 are half-planes such that lines bounding them meet at a single point; and $\mathcal{M}_X \models coor[l_1, l_2, l_3]$ if and only if l_1, l_2, l_3 are half-planes such that lines bounding them form a coordinate system.

Proof. Consider the formula

$$\eta(x_1, x_2, x_3) := \bigwedge_{1\le i\le 3} hp(x_i) \wedge \bigwedge_{1\le i<j\le 3} \neg par(x_i, x_j).$$

Now put

$$coord(x_1, x_2, x_3) := \eta(x_1, x_2, x_2) \wedge \neg \bigvee_{1\le i<j\le 8} (\prod_{1\le k\le 3}^{(i)} \pm x_k = 0 \wedge \prod_{1\le k\le 3}^{(j)} \pm x_k = 0)$$

and

$$\Gamma(x_1, x_2, x_3) := \eta(x_1, x_2, x_2) \wedge \bigvee_{1\le i<j\le 8} (\prod_{1\le k\le 3}^{(i)} \pm x_k = 0 \wedge \prod_{1\le k\le 3}^{(j)} \pm x_k = 0).$$

Observe that l_1, l_2, l_3 satisfy *coord* just in case no two products $\prod_{1\le k\le 3}^{(i)} \pm l_k$, $\prod_{1\le k\le 3}^{(j)} \pm l_k$ are empty. Similarily, l_1, l_2, l_3 satisfy Γ just in case there are (at least) two empty products $\prod_{1\le k\le 3}^{(i)} \pm l_k$, $\prod_{1\le k\le 3}^{(j)} \pm l_k$. This, together with the fact that three non-parallel lines can only divide the plane into 6 or 7 regions depending on whether these lines meet pairwise at three distinct points or meet at a single point yields the result (see also Fig. 5.4). □

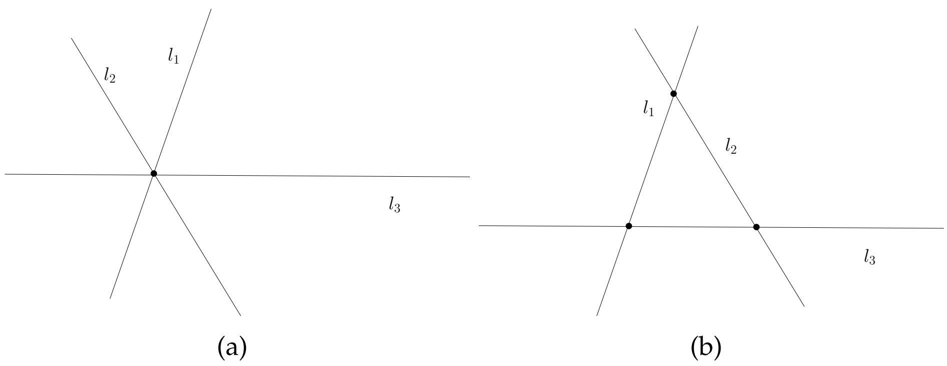


Figure 5.4: Examples of $\Gamma[l_1, l_2, l_3]$ and $\Delta[l_1, l_2, l_3]$ respectively.

Given $\mathfrak{M}_X \models coord[l, m, n]$ we adopt the convention that the line bounding the first half-plane is the ordinata and the line bounding the third half-plane is the abscissa. In general this will be represented by the numbering of the elements, like so: $\mathfrak{M}_X \models coord[l_1, l_2, l_3]$ but occasionally we purposefully disrupt the numbering and write, for example, $\mathfrak{M}_X \models coord[l_1, l_3, l_2]$.

We now present a number of expressiveness results extending [Pra99].

Definition 5.2.2. Let $l, m \in ROX(\mathbb{R}^2)$ be half-planes. We say that the lines bounding them form a *general line pair* if the lines intersect at the unique point **P**.

Lemma 5.2.5. *Let $l_1, l_2 \in ROX(\mathbb{R}^2)$. There exists a formula $\ll x_1, x_2 \gg$ such that $\mathfrak{M}_X \models \ll l_1, l_2 \gg$ if and only if l_1, l_2 are half-planes and lines bounding them form a general line pair.*

Proof. Consider the formula (see Figure 5.5): $\ll x_1, x_2 \gg := hp(x_1) \wedge hp(x_2) \wedge \pm x_1 . \pm x_2 \neq 0$. □

Lemma 5.2.6. *Let $l_1, l_2, l_3, l_4 \in ROX(\mathbb{R}^2)$. There exists a formula $\ll x_1, x_2 \gg \doteq \ll x_3, x_4 \gg$ such that $\mathfrak{M}_X \models \ll l_1, l_2 \gg \doteq \ll l_3, l_4 \gg$ if and only if l_1, l_2 and l_3, l_4 are half planes such that lines bounding them form general line pairs which determine the same point.*

Proof. Consider the following formula:

$$\ll x_1, x_2 \gg \doteq \ll x_3, x_4 \gg := \Gamma(x_1, x_2, x_3) \wedge \Gamma(x_1, x_2, x_4).$$

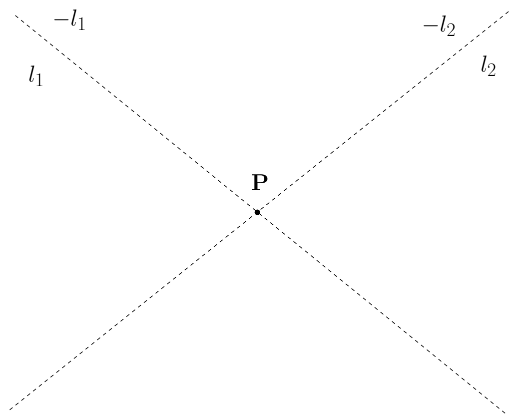


Figure 5.5: The line bounding l_1 intersects the line bounding l_2 at a point $\mathbf{P}$. Observe that all half-planes bounded by these lines have non-empty intersections.

□

Recall the definitions of addition (Def. 2.5.9) and multiplication (Def. 2.5.10 in an affine plane, presented in Section 2.5).

Definition 5.2.3 ([Ben95])**.** We say that $\overline{\mathbf{OC}}$ is the result of *addition* of $\overline{\mathbf{OA}}$ and $\overline{\mathbf{OB}}$ and write $\overline{\mathbf{OA}} + \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ if and only if the following lines can be found (see Fig. 2.3):

(a) l, l' meeting at a single point $\mathbf{O}$;

(b) m parallel to l';

(c) l_A such that $\ll l_A, l' \gg = \mathbf{A}$, parallel or coincident with l;

(d) l_B such that $\ll l_B, l' \gg = \mathbf{B}$, and such that l_B, l, m meet at a single point;

(e) l_C such that $\ll l_C, l' \gg = \mathbf{C}$, parallel or coincident with l_B.

Definition 5.2.4 ([Ben95])**.** We say that $\overline{\mathbf{OC}}$ is the result of *multiplication* of $\overline{\mathbf{OA}}$ and $\overline{\mathbf{OB}}$ and write $\overline{\mathbf{OA}} \cdot \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$ if and only if the following lines can be found (see Fig. 2.4):

(a) l_1, l_2, l_3 bounding a triangle;

(b) l_A such that $\ll l_A, l_3 \gg = \mathbf{A}$, parallel or coincident with l_2;

(c) l_B such that $\ll l_B, l_3 \gg = \mathbf{B}$, and such that l_B, l_1, l_2 meet at a single point;

(d) l_C such that $\ll l_C, l_3 \gg = \mathbf{C}$, parallel or coincident with l_B and such that l_C, l_A, l_1 meet at a single point.

We now show that addition and multiplication are definable in relation to a given coordinate frame.

Lemma 5.2.7. *There exists a formula*

$$add(x_1, x_2, x_3, x_a, x_b, x_c)$$

such that

$$\mathcal{M}_X \models add[l_1, l_2, l_3, a, b, c]$$

if and only if the following conditions all hold:

(i) l_1, l_2, l_3 *are half-planes such that lines bounding them form a coordinate system with* l_3 *the abscissa and* l_1 *the ordinata;*

(ii) a *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{A}$*;*

(iii) b *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{B}$*;*

(iv) c *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{C}$*;*

(v) $\overline{\mathbf{OA}} + \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$.

Proof. The following shows that the construction from Definition 2.5.9 is expressible in $\mathcal{L}_{conv,\leq}$. Consider the following formula

$$\exists y \; (coord(x_1, x_2, x_3) \wedge (par(x_a, x_1) \vee \alpha(x_a, x_1)) \wedge \Gamma(x_b, x_1, y) \wedge \ll x_3, x_b \gg \\ \wedge (par(x_b, x_c) \vee \alpha(x_b, x_c)) \wedge \Gamma(y, x_a, x_c)).$$

□

Lemma 5.2.8. *There exists a formula*

$$mult(x_1, x_2, x_3, x_a, x_b, x_c)$$

such that

$$\mathcal{M}_X \models mult[l_1, l_2, l_3, a, b, c]$$

if and only if the following conditions all hold:

(i) l_1, l_2, l_3 *are half-planes such that lines bounding them form a coordinate system, with* l_3 *the abscissa and* l_1 *the ordinata;*

(ii) a *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{A}$*;*

(iii) b *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{B}$*;*

(iv) c *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{C}$*;*

(v) $\overline{\mathbf{OA}} \cdot \overline{\mathbf{OB}} = \overline{\mathbf{OC}}$.

Proof. The following shows that the construction from Definition 2.5.10 is expressible in $\mathcal{L}_{conv,\leq}$. Consider the following formula:

$$coord(x_1, x_2, x_3) \wedge (par(x_a, x_2) \vee \alpha(x_a, x_2)) \wedge \Gamma(x_1, x_2, x_b) \wedge \ll x_3, x_b \gg$$
$$\wedge \Gamma(x_c, x_a, x_1) \wedge (par(x_c, x_b) \vee \alpha(x_c, x_b)).$$

□

We obtain an easy consequence of Lemma 5.2.8.

Lemma 5.2.9. *For each natural number* n *there exists a formula*

$$power_n(x_1, x_2, x_3, x, z_n)$$

such that

$$\mathcal{M}_X \models power_n[l_1, l_2, l_3, a, b_n]$$

if and only if the following conditions hold:

(i) l_1, l_2, l_3 *are half-planes such that lines bounding them form a coordinate system, with all the intersection points defined as above;*

(ii) a *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{A}$*;*

(iii) b *is a half-plane such that the line bounding it intersects the line bounding* l_3 *at some point* $\mathbf{B}$*;*

(iv) $\overline{\mathbf{OA}}^n = \overline{\mathbf{OB}}$.

Proof. An easy application of the multiplication formula:

$$\exists z_1 \ldots \exists z_{n-1}(mult(x_1, x_2, x_3, x, x, z_1) \wedge \bigwedge_{2 \leq i \leq n} mult(x_1, x_2, x_3, x, z_{i-1}, z_i))$$

□

Rational Fixing Formulas We now describe a series of results, most of them directly used in the axiomatisation. The most important of these results is the existence of formulas allowing us to *fix* every rational half-plane. Our main contribution here is that we spell out many of the proofs in [Pra99].

Lemma 5.2.10. *Let l_1, l_2, l_3 be rational lines forming a coordinate system with points* $\mathbf{O}, \mathbf{I}, \mathbf{J}$ *as in Definition 5.2.1. Let m_1, m_2, m_3 be rational lines such that the following conditions all hold (see Figure 5.6 for an example):*

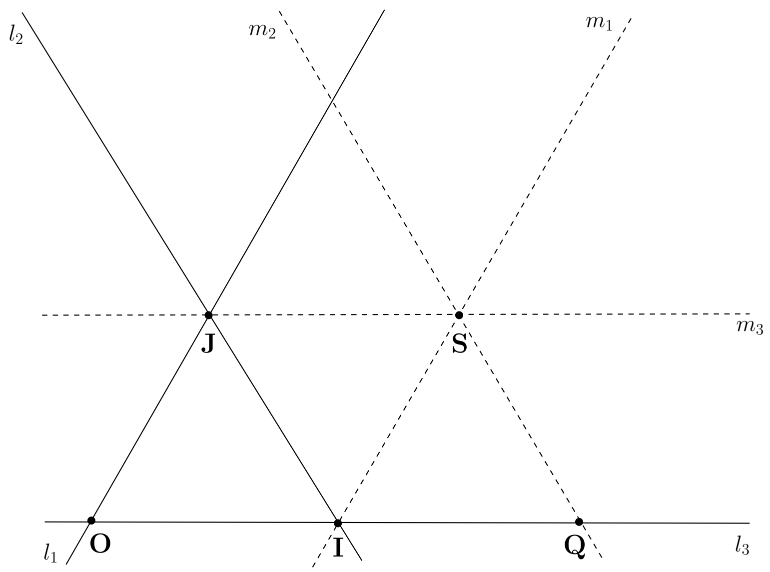


Figure 5.6: Coordinate frame with $\overline{\mathbf{OI}} = \overline{\mathbf{IQ}}$.

- *for each l_i and m_i: $l_i \parallel m_i$,*
- $m_1 \cap m_2 \cap m_3 = \mathbf{S}$*,*
- $l_1 \cap l_2 \cap m_3 = \mathbf{J}$*,*
- $l_2 \cap l_3 \cap m_1 = \mathbf{I}$*,*

- $l_3 \cap m_2 = \mathbf{Q}$,

then $\overline{\mathbf{OI}} = \overline{\mathbf{IQ}}$.

Proof. See [Pra99].

Lemma 5.2.11. *Assume* $l_1, l_2, l_3, m \in ROQ(\mathbb{R}^2)$ *are half-planes and let* l_1, l_2, l_3 *form a coordinate frame. There exists a formula* $\phi_n(x_1, x_2, x_3, y)$ *such that for any rational line* m *intersecting the line bounding* l_3 *at a point* $\mathbf{K}$, $\mathcal{M}_Q \models \phi_n[l_1, l_2, l_3, m]$ *if and only if* $\overline{\mathbf{OK}} = n\overline{\mathbf{OI}}$, *where* n *is an integer.*

Proof. Construction from Lemma 5.2.10 is expressible in $\mathcal{L}_{\leq,conv}$ (see 5.6 and [Pra99]). We obtain the desired result by repeating this construction several times (see Figure 3 for an example). Let $\psi_n(u_1, \ldots, u_n, k, m)$ be the following formula:

$$\bigwedge_{\substack{1\leq j\leq n-2,\\ j \text{ odd}}} par(u_j, u_{j+2}) \wedge \bigwedge_{\substack{1\leq j\leq n-2,\\ j \text{ even}}} par(u_j, u_{j+2}) \wedge$$

$$\bigwedge_{\substack{1\leq j\leq n-1,\\ j \text{ odd}}} \ll u_j, m \gg \doteq \ll u_{j+1}, m \gg \wedge \bigwedge_{\substack{1\leq j\leq n-1,\\ j \text{ even}}} \ll u_j, k \gg \doteq \ll u_{j+1}, k \gg,$$

now put $\phi_n(x_1, x_2, x_3, y) := \exists u_1 \ldots \exists u_n \exists m$

$$(coor(x_1, x_2, x_3) \wedge par(m, x_3) \wedge u_1 = x_1 \wedge u_2 = x_2 \wedge$$

$$\psi_n(u_1, \ldots, u_n, x_3, m) \wedge \ll u_n, x_3 \gg \doteq \ll y, x_3 \gg \wedge \ll m, x_1 \gg \doteq \ll m, x_2 \gg).$$

□

Lemma 5.2.12. *Lemma Assume* $l_1, l_2, l_3, m \in ROQ(\mathbb{R}^2)$ *are half-planes and let* l_1, l_2, l_3 *form a coordinate frame. There exists a formula* $\phi_q(x_1, x_2, x_3, y)$ *such that for any rational line* m *intersecting the line bounding* l_3 *at a point* $\mathbf{K}$, $\mathcal{M}_Q \models \phi_q[l_1, l_2, l_3, m]$ *if and only if* $\overline{\mathbf{OK}} = q\overline{\mathbf{OI}}$, *where* q *is a rational number.*

Proof. Assuming the terminology from the previous lemma, we are required to show that there exists a formula satisfied in the rational model if and only

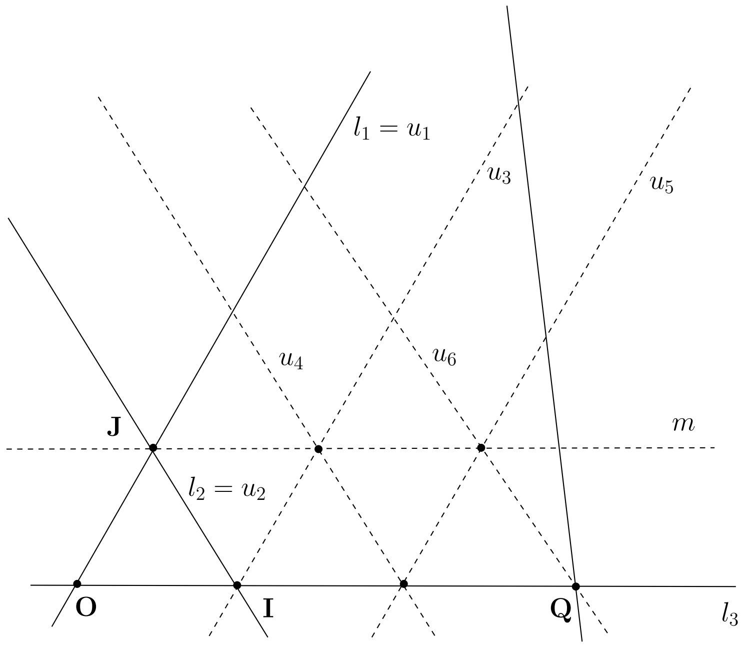


Figure 5.7: An example configuration: $\overline{\mathbf{OQ}} = 3\overline{\mathbf{OI}}$

if $\overline{\mathbf{OK}} = \dfrac{n}{m}\overline{\mathbf{OI}}$, hence

$$m\overline{\mathbf{OK}} = n\overline{\mathbf{OI}}.$$

The last equation holds if and only if we can find some $\mathbf{I}'$ such that

$$\overline{\mathbf{OK}} = n\overline{\mathbf{OI'}}$$

and

$$\overline{\mathbf{OI}} = m\overline{\mathbf{OI'}}$$

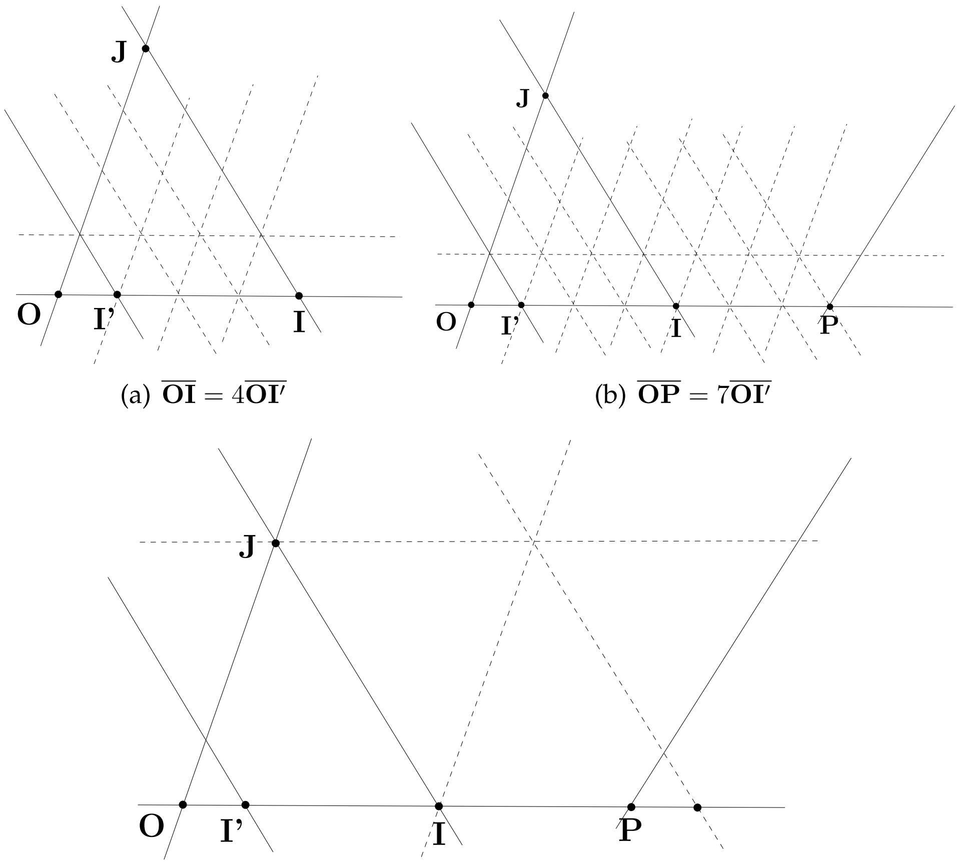
as then we obtain

$$m\overline{\mathbf{OK}} = mn\overline{\mathbf{OI'}} = nm\overline{\mathbf{OI'}} = n\overline{\mathbf{OI}}.$$

The following shows that there exists a formula expressing the above condition (see also Figure 5.8). Let $\phi_{\frac{n}{m}}(x_1, x_2, x_3, v) :=$

$$\exists z(\phi_m(x_1, z, x_3, x_2) \wedge \phi_n(x_1, z, x_3, v)).$$

□



(a) $\overline{\mathbf{OI}} = 4\overline{\mathbf{OI'}}$ (b) $\overline{\mathbf{OP}} = 7\overline{\mathbf{OI'}}$

(c) $\overline{\mathbf{OP}} = \frac{7}{4}\overline{\mathbf{OI}}$

Figure 5.8: A rational case.

Lemma 5.2.13. *Let l_1, l_2, l_3, m be half-planes in $ROQ(\mathbb{R}^2)$, such that*

$$\mathcal{M}_Q \models coord[l_1, l_2, l_3].$$

Then there exists a formula satisfied in the rational model by the tuple l_1, l_2, l_3, m such that for any rational half-plane m' the tuple l_1, l_2, l_3, m' satisfies this formula if and only if lines bounding m and m' are coincident.

Proof. Let m be any half plane. Observe that for the line bounding m one of these cases holds (see Figure 5.9):

1. the line bounding m is parallel or coincident with the line bounding l_3;
2. the line bounding m is parallel or coincident with the line bounding l_1;
3. the line bounding m intersects l_1 and l_3 at the same point **O** and is parallel to the line bounding l_2;

4. the line bounding m intersects l_1 and l_3 at the same point $\mathbf{O}$ and intersects the line bounding l_2 at some point $\mathbf{P}$;

5. the line bounding m intersects the lines bounding l_3 and l_1 at some rational points $\mathbf{P}$ and $\mathbf{Q}$ respectively;

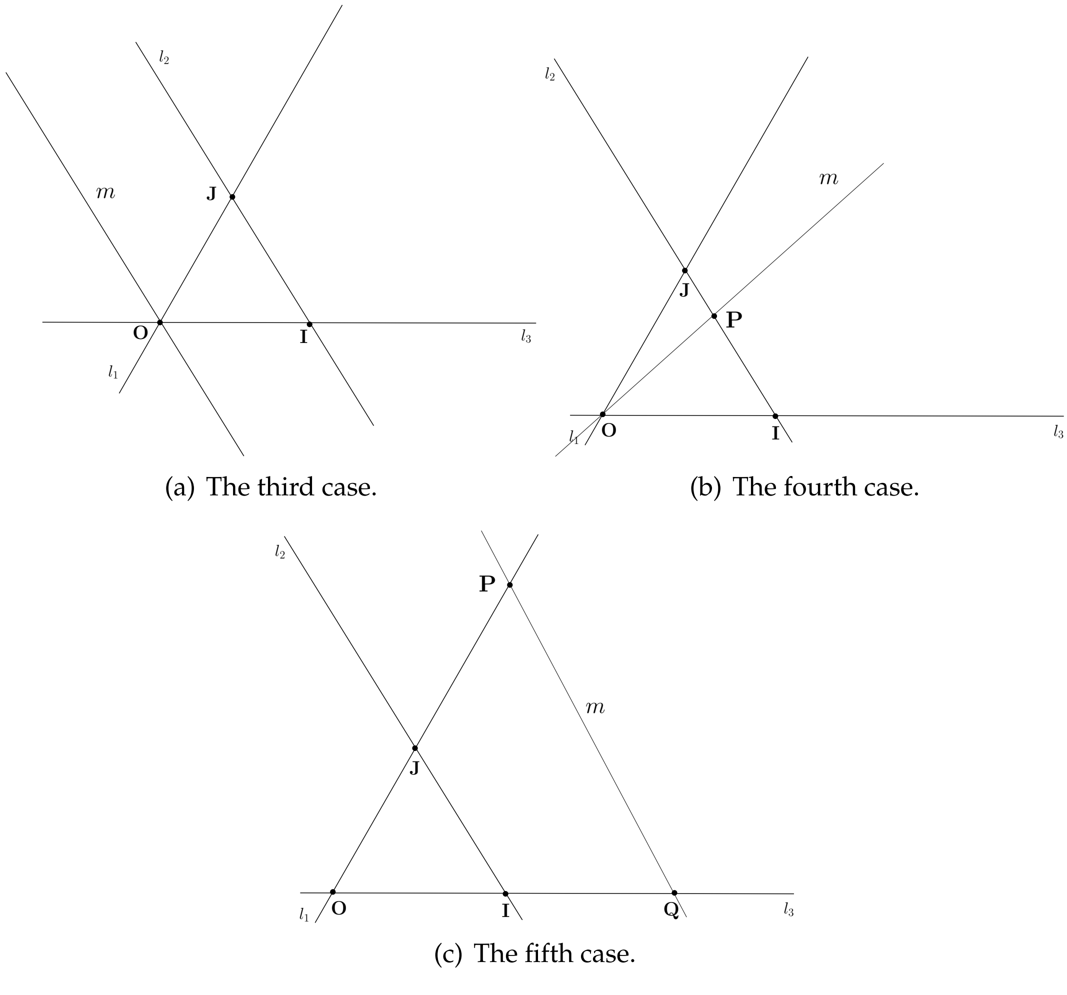


Figure 5.9: Building rational fixing formulas. The last three cases.

What is left, is to show that these cases are expressible in our language.

Consider the following formulas.

1′. $\exists z(\phi_q(x_3, z, x_1, x_2) \wedge \phi_r(x_3, z, x_1, w)) \wedge (par(w, x_3) \vee \alpha(w, x_3))$;

2′. $\exists z(\phi_q(x_1, z, x_3, x_2) \wedge \phi_r(x_1, z, x_3, w)) \wedge (par(w, x_1) \vee \alpha(w, x_1))$;

3′. $\phi_0(x_1, x_2, x_3, w) \wedge par(w, x_2)$;

4′. $\exists z(\phi_q(x_3, z, x_2, x_1) \wedge \phi_r(x_3, z, x_2, w)) \wedge \ll w, x_1 \gg = \ll w, x_3 \gg$;

5′. $\exists z(\phi_q(x_1, z, x_3, x_2) \wedge \phi_r(x_1, z, x_3, w)) \wedge \exists z(\phi_s(x_3, z, x_1, x_2) \wedge \phi_t(x_3, z, x_1, w))$.

It is easy to see that l_1, l_2, l_3, m satisfy one of the formulas (1′-5′) if and only if the corresponding case (1-5) holds. In each of these cases the line bounding m is uniquely determined in a sense that there is no other line bounding a half-plane m' and satisfying exactly the conditions specified for line bounding m, which is not coincident with m. □

We see that we can fix the line bounding a half-plane with reference to a given coordinate frame. Note however that for each line there exist two half-planes (each being the complement of the other). In order to pin-point one of the two half-planes we introduce the following notational shorthand. Let $\prod^{+}(x_1, x_2, x_3, x) := x_1.x_2.x_3.x \neq 0$ and $\prod^{-}(x_1, x_2, x_3, x) := x_1.x_2.x_3.x = 0$ and let $j \in \{+, -\}$. Now, the idea is to extend the formulas (1′–5′) with formulas of the form $\prod^{+}$ or $\prod^{-}$ indicating one of the sides of the considered line. For example, let l_1, l_2, l_3 be rational half-planes forming a coordinate frame and let m be a half-plane bounded by the line coincident with the line bounding l_1. Hence we have that $\mathfrak{M}_Q \models \alpha[l_1, m]$ (recall that α "says" that l_1 and m are coincident). This allows us to fix the line bounding m, but m itself can lie on either side of that line. In order to tell m and its complement apart we add an extra information with the help of $\prod^{j}$ $j \in \{+, -\}$. We specify a region contained in m but not contained in its complement. Most cases it will be the region $l_1 \cdot l_2 \cdot l_3$.Note however that in cases when m intersects $l_1 \cdot l_2 \cdot l_3$ we specify $l_1 \cdot l_2 \cdot n$ in $\prod^{j}$ instead, where n is a half-plane bounded by line that together with l_1, l_3 serves as an "auxilliary" coordinate frame (as in Lemma 5.2.12, see also Fig. 5.10). Hence assuming that m does contain $l_1 \cdot l_2 \cdot l_3$ we end up with $\alpha(x_1, x) \wedge \prod^{+}(x_1, x_2, x_3, x) \neq 0$. The full list of the amended formulas is as follows (assuming variable names are not changed).

(i) if ϕ is of the form (1′-3′) then $\phi' := \phi \wedge \prod^{j}(x_1, x_2, x_3, w)$;

(ii) if ϕ is of the form (4′) then $\phi' := \phi \wedge \prod^{j}(x_3, z, x_2, w)$;

(iii) if ϕ is of the form (5′) then $\phi' := \phi \wedge \prod^{j}(x_1, z, x_3, w)$.

We call formulas (i)-(iii) the *rational fixing formulas*. All the above considerations allow us to prove the following result easily.

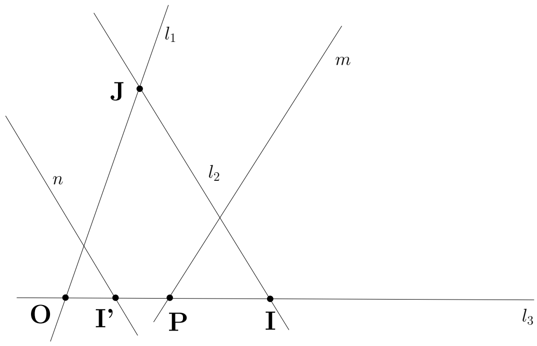


Figure 5.10: Lines l_1, l_3 and n form an auxilliary coordinate frame.

Lemma 5.2.14. *Let l_1, l_2, l_3, m be half-planes in $ROQ(\mathbb{R}^2)$, such that*

$$\mathcal{M}_Q \models coord[l_1, l_2, l_3].$$

Then there exists a formula satisfied in the rational model by the tuple l_1, l_2, l_3, m such that for any half-plane m' the tuple l_1, l_2, l_3, m' satisfies this formula if and only if $m = m'$.

Note that Theorem 4.3.6, presented in Chapter 4 is an immediate consequence of the above.

As we are going to make extensive use of the rational fixing formulas, we introduce a notational convention for them. Let $\mathfrak{M}_Q \models \tau^j_{(P,Q)}[l_1, l_2, l_3, m]$ be a formula of the form (i)–(iii) expressing the following: m is fixed with respect to the coordinate frame formed by l_1, l_2, l_3. The parameter j ($j \in \{+, -\}$) is a flag to say which side of the line bounding m we are interested in (expressed by a subformula of the form $\prod^j$). The parameters P, Q are members of $\mathbb{Q} \cup \{\infty\}$. If $P, Q \in \mathbb{Q}$ and neither P nor Q equals 0, we think of them as the rational points at which m intersects l_3 and l_1 respectively. This property is expressed by a subformula of the form (5′). If one of this parameters equals zero, then the line bounding m passes through the origin of the coordinate frame and, depending on the value of the other parameter, the line bounding m either intersects l_2 (if this other parameter is a rational number) or is parallel to l_2 (if this parameter equals ∞). This property is expressed by a subformula of the type (3′) or (4′). If $P = \infty$ then m is parallel to or coincides with l_3; and if $Q = \infty$, then m is parallel to or coincides with l_1. This property corresponds to a subformula of the type (1′) or (2′). Note that we can neither have both P and Q equal 0, nor both $P, Q = \infty$.

Let $\tau_1, \tau_2, \ldots$ be enumeration of the rational fixing formulas.

5.2.2 Other Expressiveness results

This section presents our original contribution in terms of expressivness of $\mathcal{L}_{conv,\leq}$

Betweenness We repeatedly discussed the notion of betweenness in this thesis. Most importantly betweenness is featured in papers by Tarski and colleagues ([Tar56],[ST79] see also Section 3.4).

There are many related axiomatic approaches to the topic (standard text-books include [BS55] and [Szm81]).

Definition 5.2.5. Let G be a non-empty set and let $B \subset G \times G \times G$. Relation B is a *betweenness relation* if for all $a, b, c \in G$:

(A) $B(a, b, c) \vee B(b, c, a) \vee B(c, a, b)$;

(B) $B(a, b, a) \rightarrow a = b$;

(C) $B(a, b, c) \rightarrow B(c, b, a)$;

(D) $B(a, b, c) \wedge B(a, c, d) \rightarrow B(b, c, d)$;

(E) $B(a, b, c) \wedge B(b, c, d) \wedge b \neq c \rightarrow B(a, c, d)$.

In this section we show that betweenness — in the above sense — can be defined in our language, without relativising to any coordinate frame.

Lemma 5.2.15. *Let **A**, **B** and **C** be any points in $\mathbb{R}^2$ and let m, l, l_1, l_2, l_3 be any line such that m parallel to l and let **bet**(**A**,**B**,**C**) be defined in a following manner (see Figure 5.11):*

(i) ***A*** *=* $\ll l_1, l \gg$*,* ***B*** *=* $\ll l_2, l \gg$ *and* ***C*** *=* $\ll l_3, l \gg$*;*

(ii) l_1, l_2, l_3 *meet* m *at a single point;*

(iii) either (a) lines l_1, l_2, l_3 *are pairwise coincident or (b)* l_1, l_2, l_3 *bound a triangle* T*,* l, l_1, l_3 *bound a triangle* $T_1 \subseteq T$ *and* l, l_2, l_3 *bound a triangle* $T_2 \subseteq T$*.*

*Then **bet**(**A**,**B**,**C**) is a betweenness relation.*

Proof. To see that the condition (A) is satisfied, consider a point **D** not on the line containing **A**, **B** and **C**. Clearly, the lines connecting **D** to **A**, **B** and **C** form a configuration satisfying (i)-(iii). The condition (B) is satisfied by (ii). Conditions (C), (D) and (E) follow easily. □

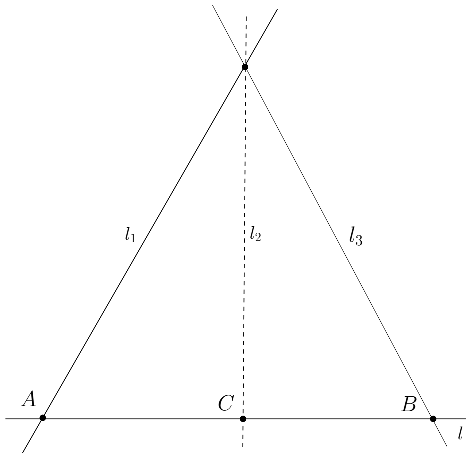


Figure 5.11: Point **C** lies between points **A** and **B**.

We obtain the following easily.

Theorem 5.2.1. *Let l, l_1, l_2, l_3 be half-planes in $ROX(\mathbb{R}^2)$. There exists a formula $\beta(x, x_1, x_2, x_3)$ such that $\mathfrak{M}_X \models \beta[l, l_1, l_2, l_3]$ just in case the point determined by $\ll l, l_2 \gg$ lies between the points determined by $\ll l, l_1 \gg$ and $\ll l, l_3 \gg$.*

Proof. Clearly, all the conditions mentioned in Lemma 5.2.15 are expressible in the language $\mathcal{L}_{conv,\leq}$. □

We note that it seems possible to show that the conditions (i)-(iii) capture the betweenness relation in the sense of [Tar56], [ST79] or [BS55].

As a consequence of the above lemma we obtain a series of results regarding $\mathfrak{M}_A$.

Algebraic fixing formulas We now focus on results that follow from the expressiveness results presented above. Observe that what we have established allows us to talk about polynomials of any degree with rational coefficients, relative to a coordinate system. We do so by combining formulas add, $mult$, $power_n$ and rational fixing formulas.

For example consider the formula $\phi(x) := \exists y \exists z \exists v$

$(power_2(x_1, x_2, x_3, x, y) \wedge \tau^{+}_{(4,\infty)}(x_1, x_2, x_3, z) \wedge mult(x_1, x_2, x_3, z, y, v) \wedge \tau^{+}_{(2,\infty)}(x_1, x_2, x_3, v))$.

This formula is satisfied by a if and only if, intuitively speaking, a is a half-plane bounded by a line crossing the abscissa of the coordinate system determined by l_1, l_2, l_3 at some point x that is a solution to the equation $4x^2 = 2$.

Lemma 5.2.16. *Assume $l_1, l_2, l_3, m \in ROA(\mathbb{R}^2)$ be half-planes and let l_1, l_2, l_3 form a coordinate frame. There exists a formula*

$$\varphi_P(x_1, x_2, x_3, y),$$

where P is a polynomial with rational coefficients $a_n, a_{n-1}, \ldots, a_0$, such that for any algebraic line m intersecting the line bounding l_3 at a point $\mathbf{K}$,

$$\mathcal{M}_Q \models \varphi_P[l_1, l_2, l_3, m]$$

if and only if

$$\mathbf{O} = a_n\overline{\mathbf{OK}}^n + a_{n-1}\overline{\mathbf{OK}}^{n-1} + \cdots + a_0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are rational numbers.

Proof. As indicated above, the condition $\mathbf{O} = a_n\overline{\mathbf{OK}}^n + a_{n-1}\overline{\mathbf{OK}}^{n-1} + \cdots + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are rational, is expressible in $\mathcal{L}_{conv,\leq}$ using a combination of the rational fixing formulas and formulas $mult$, add and $power_n$. □

Lemma 5.2.17. *Let l_1, l_2, l_3, m be half-planes in $ROA(\mathbb{R}^2)$, such that*

$$\mathcal{M}_A \models coord[l_1, l_2, l_3].$$

Then there exists a formula satisfied in the algebraic model by the tuple l_1, l_2, l_3, m such that for any half-plane m' the tuple l_1, l_2, l_3, m' satisfies this formula if and only if lines bounding m and m' are coincident.

Proof. This is very similar to Lemma 5.2.13. Observe that Lemma 5.2.16 gives us way to fix an algebraic line if its intersection with the abscissa is a solution to a given polynomial with rational coefficients. In general such a polynomial has more than one solution, hence we are not able to pinpoint a single algebraic line. However, since for any pair of algebraic numbers there exists a rational number separating them, given a list of algebraic numbers we can separate them all with rational numbers and refer to any algebraic number from the list uniquely by determining *between which rational numbers it lies*. The main difference between this lemma and Lemma 5.2.13 is that we need to show that this is expressible (observe that if $\mathcal{M}_A \models \beta[l_3, a, b, c]$ and $\mathcal{M}_A \models \tau_{<P,\infty>}[l_1, l_2, l_3, a]$ and $\mathcal{M}_A \models \tau_{<Q,\infty>}[l_1, l_2, l_3, c]$, then $P \leq Q$). But this is simple. By Lemma 5.2.16 m satisfies some $\varphi_P(x_1, x_2, x_3, y)$. To obtain the desired result, we need to extend this formula with

$$\tau_{(P,\infty)}(x_1, x_2, x_3, y_1) \ \wedge \ \tau_{(Q,\infty)}(x_1, x_2, x_3, y_2) \wedge$$

$$\beta(x_3, y_1, y, y_2) \ \wedge (\varphi_P(x_1, x_2, x_3, y') \rightarrow \neg\beta(x_3, y_1, y', y_2)).$$

□

In a manner similar to Lemma 5.2.14 we obtain the following result.

Lemma 5.2.18. *Let l_1, l_2, l_3, m be half-planes in $ROA(\mathbb{R}^2)$, such that*

$$\mathcal{M}_A \models coord[l_1, l_2, l_3].$$

Then there exists a formula satisfied in the algebraic model by the tuple l_1, l_2, l_3, m such that for any half-plane m' the tuple l_1, l_2, l_3, m' satisfies this formula if and only if $m = m'$.

Trivially, since $\mathbb{Q} \subset \mathbb{A}$, we have that $Th(ROQ(\mathbb{R}^2)) \neq Th(ROA(\mathbb{R}^2))$. This is shown by constructing a sentence true in $ROA(\mathbb{R}^2)$ but false in $ROQ(\mathbb{R}^2)$, which is an easy exercise (see also [Pra99]).

Another consequence of the above is that the primitive relations of elementary geometry: equidistance and betweenness are expressible in the rational model *relative* to a given coordinate system. In order to achieve that we need to coordinatise $\mathfrak{M}_Q$. In the process, we extend the results of Lemma 5.2.12 to any line parallel to the abscissa (see Figure 5.2.2 for an example). Let l_1, l_2, l_3 be rational lines forming a coordinate frame as in Definition 5.2.1. Let s be a line parallel to l_3 and let m be a rational line intersecting s at a point $\mathbf{S}$.

(i) take the line parallel to l_1 and containing $\mathbf{I}$ and mark its intersection with s as $\mathbf{I_S}$;

(ii) take the line parallel to l_2 and containing $\mathbf{I_S}$ and mark its intersection with l_1 as $\mathbf{J_S}$;

(iii) perform the operations described in Lemma 5.2.12 treating s as the abscissa and $\mathbf{I_S}$ and $\mathbf{J_S}$ as the respective units of measurement, obtaining a formula ϕ_Q, for some rational number Q.

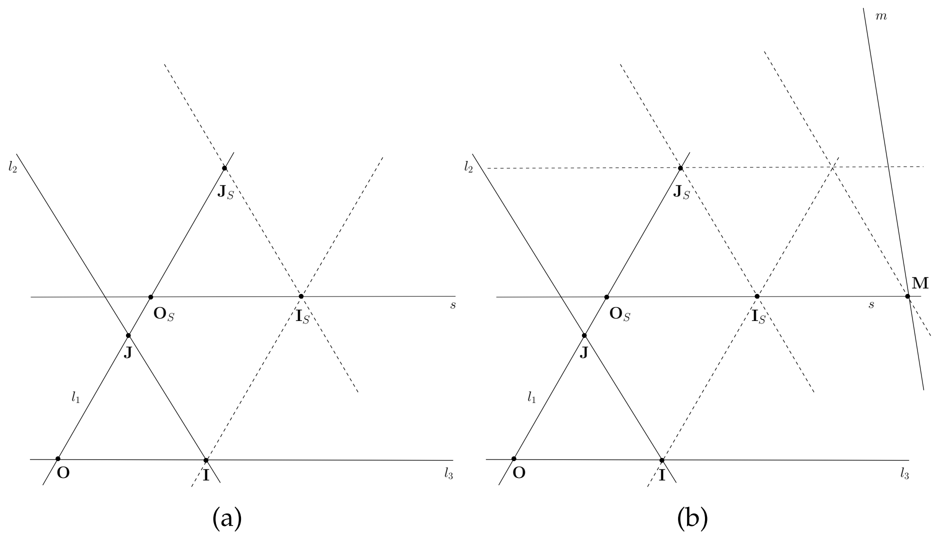


Figure 5.12: Extending Lemma 5.2.14.

Obviously, for any given rational point we can find a rational line containing it, which is not parallel to the abscissa in a given coordinate frame. This means that the following is a well defined notion.

Definition 5.2.6. Let l_1, l_2, l_3 be rational lines forming a coordinate frame. Let m be any rational line. We define a *coordinate function* ρ assigning pairs of rational numbers (P, Q) to any point of intersection $\mathbf{M}$ of m with a given rational line as follows.

(1) if $\mathbf{M} \in l_3$, then $\rho(M) = (P, 0)$, where $\phi_P[l_1, l_2, l_3, m]$ is the corresponding formula from Lemma 5.2.14;

(2) if $\mathbf{M} \in l_1$, then $\rho(M) = (0, P)$, where $\phi_P[l_3, l_2, l_1, m]$ is the corresponding formula from Lemma 5.2.14;

(3) if $M \in m_Q$, where m_Q is the line parallel to l_3 and intersecting l_1 at the point Q, then $\rho(M) = (P, Q)$, where $\phi_P[l_1, l_2, l_3, m]$ is the correspond-

ing formula from Lemma 5.2.14 and $\phi_Q[l_1, l_2', l_3', m]$ is the formula obtained by extending the results of Lemma 5.2.14 in a manner described in points (i)-(iii) above.

By using the fixing formulas involved in the definition of a coordinate function we can explicitly use definitions of betweenness and equidistance in a sense of Tarski (see Section 3.4) *with a reference to a given coordinate frame*, as all the above is expressible in $\mathfrak{M}_Q$. The details are routine.

We observe that with some modifications it should be possible to define a coordinate function for $\mathfrak{M}_A$.

Models Finally we show that convexity spatial logics of Euclidean spaces of different dimensionality have different theories. To this end we use the Helly's theorem presented in Section 2.6. Consider the following formula $\phi(x_1, \ldots, x_N) :=$

$$\bigwedge_{I \subseteq 2^S} \prod_{i \in I} x_i \neq 0 \wedge \bigwedge_{1 \leq j \leq N} conv(x_j) \rightarrow \prod_{1 \leq j \leq N} x_j \neq 0,$$

where $S = \{1, \ldots, N\}$.

This formula says in $ROX(\mathbb{R}^2)$ that regions $r_1, \ldots, r_N$ have non-empty intersection if each r_j is convex and for every subset of $\{r_1, \ldots, r_N\}$, its elements have a non-empty intersection.

Theorem 5.2.2. *For a given n there exists a set of formulas Φ_n expressing the Helly's theorem in $ROX(\mathbb{R}^n)$.*

Proof. Consider $\phi_N := \forall x_1 \ldots \forall x_N \phi(x_1, \ldots, x_N)$, where $\phi(x_1, \ldots, x_N)$is defined as above. We define $\Phi_n = \{\phi_N \mid N \geq n+1 \text{ and } |I| = n+1\}$. Note that the constraint $|I| = n+1$ refers to the condition in Helly's theorem that each $n+1$-element collection of sets indexed by S has a non-empty intersection, where n denotes the dimensionality. □

Let $\mathcal{M}_X^n$ denote $\langle ROX(\mathbb{R}^n), conv, \leq \rangle$.

Theorem 5.2.3. $Th(\mathcal{M}_X^n) \neq Th(\mathcal{M}_X^{n+1})$ *for all n.*

Proof. Observe that, by the virtue of Helly's theorem, for some $\phi_N \in \Phi_n$ we have $\mathcal{M}^n_X) \models \phi_N$ but $\mathcal{M}^{n+1}_X \not\models \phi_N$. $\square$

Let us consider a specific example of the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$. Take the real plane first. By Helly's theorem, for every collection of at least three convex sets such that each three of those have a non-empty intersection, we know that all members of the considered collection have a non-empty intersection. However, in $\mathbb{R}^3$ Helly's theorem holds for a collection of at least four convex sets such that each four of those have a non-empty intersection. Hence, it is contingent that, say, some five element collection of convex sets each three of which have a non-empty intersection, will turn out to have the property posited by Helly's theorem. It is necessarily true in $\mathbb{R}^2$. The above example is expressed in both cases by the same $\mathcal{L}_{conv,\leq}$-formula, which is satisfied in two-dimensional and not in three-dimensional model.

5.3 Axioms

In this section we propose an axiom system for the theory of $\mathfrak{M}_Q$, denoted $Th(\mathfrak{M}_Q)$.[2] Recall that $\tau_1, \tau_2, \ldots$ is an enumeration of the fixing formulas.

Let $S = \{1, \ldots, n\}$, fix $P \subseteq 2^S$ such that for every $i \in S$ there exists $I \in P$ such that $i \in I$. Let $y = bc(x_1, \ldots, x_n)$ be any formula of the form $y = \sum_{I \in P} \prod_{i \in I} x_i$. We call $y = bc(x_1, \ldots, x_n)$ a *Boolean combination* formula and bc a *Boolean term*.

Let $K \neq \emptyset$, $j \in \{+, -\}$ and let $P, Q \in \mathbb{Q} \cup \{\infty\}$. We propose the following axiomatisation of $Th(\mathfrak{M}_Q)$:

1. axioms of non-trivial Boolean Algebra;

2. $\exists x_1 \exists x_2 \exists x_3 coord(x_1, x_2, x_3)$;

3. $\neg conv(0)$;

[2]A version of this axiom system has been reported in [Try10].

4. $\forall x_1 \ldots \forall x_n \forall y((\bigwedge_{i \in S} hp(x_i) \wedge y = bc(x_1, \ldots, x_n)) \rightarrow (conv(y) \leftrightarrow \bigvee_{K \subseteq S} \prod_{k \in K} x_k = y))$, where $y = bc(x_1, \ldots, x_n)$ is a Boolean combination formula;

5. $\forall x_1 \forall x_2 \forall x_3 \forall y_1 \ldots \forall y_m(\bigwedge_{1 \leq i \leq m} \tau_i(x_1, x_2, x_3, y_i) \rightarrow \prod_{1 \leq i \leq m} y_i = 0)$, where any element of $ROQ(\mathbb{R}^2)$ bounded by the half-planes fixed by τ_i in reference to any coordinate system is empty;

6. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau^j_{(P,Q)}(x_1, x_2, x_3, y) \rightarrow \neg \tau^{j'}_{(P',Q')}(x_1, x_2, x_3, y))$, where $P \neq P'$ or $Q \neq Q'$ or $j \neq j'$;

7. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau^j_{(P,Q)}(x_1, x_2, x_3, y) \wedge \tau^{j'}_{(P',Q')}(x_1, x_2, x_3, y') \rightarrow y = y')$, where $P = P'$ and $Q = Q'$ and $j = j'$;

8. $\forall x_1 \forall x_2 \forall x_3(coord(x_1, x_2, x_3) \rightarrow \exists y(\tau_i(x_1, x_2, x_3, y)))$;

(R1):

$$\frac{\{\forall y \forall x_1 \forall x_2 \forall x_3(coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \rightarrow \psi(x_1, x_2, x_3, y))) \mid \tau \text{ a fixing formula}\}}{\forall y \forall x_1 \forall x_2 \forall x_3(coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \psi(x_1, x_2, x_3, y)).}$$

(R2):

$$\frac{\{\forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1 \leq i \leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n)) \rightarrow \psi(y)) \mid n \in \mathbb{N}, y = bc(x_1, \ldots, x_n) \text{ a Boolean combination formula}\}}{\forall y(\psi(y)).}$$

Our axiom system comprises two parts:

1. axioms and rules of inference of classical predicate calculus;
2. non-logical axioms **(1-8)** and rules of inference **(R1)** and **(R2)** above.

On an intuitive level, assuming our standard interpretation, the meaning of the above axioms is as follows. Axioms **1** make sure that the structure is a Boolean Algebra. Axiom **2** asserts that there are at least three regions such that lines bounding them form a coordinate frame. Axiom **3** states that 0 is not convex. Axiom **4** states that if a convex region is a Boolean combination of half-planes, then it has to be a product of some of these half-planes. Axioms **5** ensures that if τ_i fix half-planes (in $ROQ(\mathbb{R}^2)$) with an empty product, then elements fixed by τ_i interpreted in *any* model of the proposed axiom system, are forced to have a product equal to 0. Axioms **6** say that no half-plane can be fixed by two formulas τ and τ' differing on any of their parameters. Axioms **7** say that if two half-planes a and a' say, have fixing formulas with the same parameters, then $a = a'$. Axiom schema **8** ensures that, given a coordinate system and a fixing formula, there is a half-plane fixed by this formula in reference to this coordinate system. Infinitary rule **R1** states that every half-plane can be fixed in reference to a given coordinate system. Finally **R2** states that every region is a Boolean combination of some half-planes.

Let Φ be a set of $\mathcal{L}_{conv,\leq}$-sentences. A proof in the above axiom system is a sequence of $\mathcal{L}_{conv,\leq}$-formulas $\{\phi_\alpha\}_{\alpha<\beta}$ for some (not necessarily finite) ordinal β such that every ϕ_α is either an element of Φ; an axiom; or the result of applying a rule of inference to some formulas ϕ_γ with $\gamma < \alpha$. If ψ is the last line of such proof we write $\Phi \vdash \psi$. If $\Phi = \{\phi\}$ we write $\phi \vdash \psi$ and if $\Phi = \emptyset$ we write $\vdash \psi$ and call ψ a theorem. Denote the set of theorems by $T(\mathbf{Ax})$.

Theorem 5.3.1 (Deduction Theorem)**.** *Let ϕ be an $\mathcal{L}_{conv,\leq}$-sentence and ψ an $\mathcal{L}_{conv,\leq}$-formula such that $\phi \vdash \psi$. Then $\vdash \phi \to \psi$.*

Proof. By assumption, there is a proof with premises $\{\phi\}$ and the last line ψ and each formula β is either an axiom or a result of applying a rule of inference to earlier lines in the proof. There are two interesting cases:

(i) $\beta = \forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \to \psi(x_1, x_2, x_3))$ is derived from formulas

$$\forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \to \psi(x_1, x_2, x_3, y))$$

for all fixing formulas τ by **R1**;

(ii) $\beta = \forall y(\psi(y))$ is derived from formulas

$$\forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1 \leq i \leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n) \rightarrow \psi(y))$$

for all $n \in \mathbb{N}$ and all bc-formulas by **R2**. The first case: by the inductive hypothesis we have

$$\vdash \phi \rightarrow \forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y))$$

$$\rightarrow \psi(x_1, x_2, x_3, y))$$

for each τ and so

$$\vdash \forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y))$$

$$\rightarrow (\phi \rightarrow \psi(x_1, x_2, x_3, y))).$$

By **R1**

$$\vdash \forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow (\phi \rightarrow \psi(x_1, x_2, x_3)))$$

and so

$$\vdash \phi \rightarrow \forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \psi(x_1, x_2, x_3)).$$

The second case: by the inductive hypothesis we have

$$\vdash \phi \rightarrow \forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1 \leq i \leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n) \rightarrow \psi(y)))$$

for each $n \in \mathbb{N}$ and each bc-formula. Whence

$$\vdash \forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1 \leq i \leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n) \rightarrow \phi \rightarrow \psi(y))$$

and so by **R2** $\vdash \forall y(\phi \rightarrow \psi(y))$. Hence $\vdash \phi \rightarrow \forall y \psi(y))$. □

5.3.1 Soundness

In this section we prove the soundness theorem for our axiom system.

Definition 5.3.1. Let L be a (non-empty) set of half-planes and let A be any region. We say that A is a *product from L* if and only if $A = \prod_{1 \leq i \leq m} l_i$, where for all l_i either $l_i \in L$ or $-l_i \in L$. If in addition for all $l \in L$, $l = l_i$ or $l = -l_i$ for some i, then A is a *total* product from L.

We take all products to be non-empty and distinct.

Definition 5.3.2. Let A be a convex set. A line l is *internal* to A if and only if A has a non-empty intersection with both half-planes bounded by l.

Definition 5.3.3. Let A be a convex set. A line l is *external* to A if l is not a bounding line of A and A is a subset of one and only one of the half-planes bounded by l.

Lemma 5.3.1. *Let A be a convex sum of products from a set L of half-planes. Let l be a half-plane bounded by a line internal to A. Then for every P_i, a product from A having l as a bounding half-plane, there exists exactly one P_j from A such that $P_j \leq -l$ and P_j has the same segment of the line bounding l as its bounding line segment. We say that P_i and P_j form a* matching pair *(in A)* from l.

Proof. First of all observe that P_j has to belong to A. Since the line bounding $-l$ is internal to A we know that there exists $P_k \leq -l$ such that P_k is a product from A. Assume $P_k \neq P_j$ (othewise trivial). By convexity, all points lying on a straight line segment between P_k and P_i. have to belong to A.
Since A is a sum of products from L and all the non-empty products from L partition the plane, P_j is the only product from L containing this line segment and P_j has to belong to A. □

Lemma 5.3.2. *Let A be a convex sum of products from a set L of half-planes. If $P_i = l_1 \cdot \ldots \cdot l_{n(i)} \cdot l$ and $P_j = l'_1 \cdot \ldots \cdot l'_{m(j)} \cdot -l$ are products from A such that P_i, P_j form a matching pair, then $P_i = l_1 \cdot \ldots \cdot l_{n(i)} \cdot l'_1 \cdot \ldots \cdot l'_{m(j)} \cdot l$ and $P_j = l_1 \cdot \ldots \cdot l_{n(i)} \cdot l'_1 \cdot \ldots \cdot l'_{m(j)} \cdot -l$.*

Proof. For every line m external to P_i (P_j), either (a) m is a bounding line of P_j (P_i) or (b) m is external to P_j (P_i). Assume (b) and that P_i and P_j do not belong to the same half-plane bounded by m. Since P_i and P_j share a bounding line segment s, and since two product can only share one bounding line segment,

m separates P_i and P_j and P_i and P_j can only meet at one point at m. This is a contradiction. By a similar reckoning we obtain the conclusion assuming (a). □

Lemma 5.3.3. *Let A be a convex sum of products from a set L of half-planes. Let P_i and P_j form a matching pair in A. Then $P_i + P_j$ is a product from $L' = L \setminus \{l\}$, where l is a line internal to A containing the bounding line segment common to P_i and P_j.*

Proof. By Lemma 5.3.2 $P_i = l_1 \cdot \ldots \cdot l_{n(i)} \cdot l'_1 \cdot \ldots \cdot l'_{m(j)} \cdot l$ and $P_j = l_1 \cdot \ldots \cdot l_{n(i)} \cdot l'_1 \cdot \ldots \cdot l'_{m(j)} \cdot -l$.

Let us denote $\prod\limits_{1 \leq k \leq n(i)} l_k \cdot \prod\limits_{1 \leq k \leq m(j)} l'_k$ by C. We then obtain: $P_i = l \cdot C$ and $P_j = -l \cdot C$. Now, consider

$$P_i + P_j = l \cdot C + -l \cdot C$$

by the distributive law for Boolean algebras we obtain

$$l \cdot C + -l \cdot C = C \cdot (l + -l)$$

and finally

$$C \cdot (l + -l) = C \cdot 1 = C.$$

Observe that, since we eliminated the line bounding l and $-l$, C is a product of half-planes from some $L' = L \setminus \{l\}$. □

Lemma 5.3.4. *Let A be a convex sum of products from a set L of half-planes. Consider a line l internal to A. Let P be a product from A such that no half-plane bounded by l is a bounding half-plane of P. If Q is a product from L such that $Q \leq \lambda$ and $P = Q \cdot \lambda$, where λ is either of the half-planes bounded by l, then $P = Q$.*

Proof. Trivially, $Q = Q \cdot \lambda$. □

Theorem 5.3.2. *Let A be a convex k-element $(k > 1)$ sum of total products from a set L of half-planes with $|L| = n$ $(n > 0)$. Then A is a k'-element $(k' < k)$ sum of products from a set $L' \subset L$ with $|L'| = n - 1$.*

Proof. Since $k > 1$ and all the total products are distinct, there exists at least one line l internal to A. If $n = 1$, then $A = l + -l = 1$. By definition $1 = \prod \emptyset = A$. If $n > 1$, fix l_i from a set of $l_1, \ldots, l_s$ internal lines of A. By

Lemma 5.3.3, for all matching pairs P, Q from l_i there exists $P' = P + Q$ from $L' \subset L \setminus \{l\}$. By Lemma 5.3.4, for every $S = m_1 \cdot \ldots \cdot m_r \cdot \lambda$ such that $\lambda = l_i$ or $\lambda = -l_i$ and S does not belong to any matching pair from l_i we have that $S = m_1 \cdot \ldots \cdot m_r$. That is S equals some S' from $L' \subset L \setminus \{l\}$. □

Essentially Theorem 5.3.2 allows us to eliminate an internal line (see Fig. 5.13). Observe that given a region A which is a convex k-element sum ($k > 1$) of total products from some set L, we can apply Theorem 5.3.2 until A is a 1-element sum of total products from some $L' \subset L$. That is, A is a *product* from L: we can assume that all the remaining half-planes in L are the bounding half-planes for A. (If l is not such a half-plane then $l \geq A$ and hence A can be thought of as a product of some half-planes not including l.)

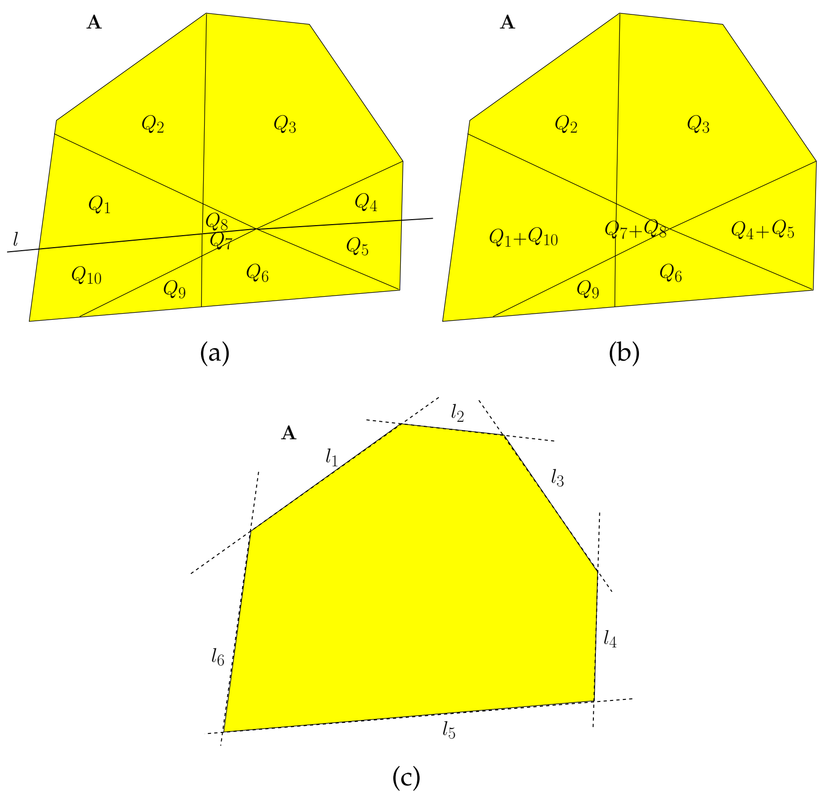


Figure 5.13: (a)-(b): Eliminating an internal line l from A, (c): A after all internal lines have been eliminated.

We have the following theorem.

Theorem 5.3.3. *Let $A \in ROX(\mathbb{R}^2)$ be any convex set and let $h_1, h_2, \ldots, h_n$ be half-planes in $ROX(\mathbb{R}^2)$. A is expressible as a sum of products of $h_1, h_2, \ldots, h_n$ if and* *only if* $A = \prod_{i \in K} h_i$ *for some* $K \subseteq \{1, \ldots, n\}$.

Proof. We just need to show that A can be thought of as a sum of total products from $H = \{h_1, h_2, \ldots, h_n\}$ but this is easy (see Fig. 5.14 for an example). If A is a one-element sum, the result is instantaneous. Otherwise the result follows by Theorem 5.3.2. The only if direction is trivial. □

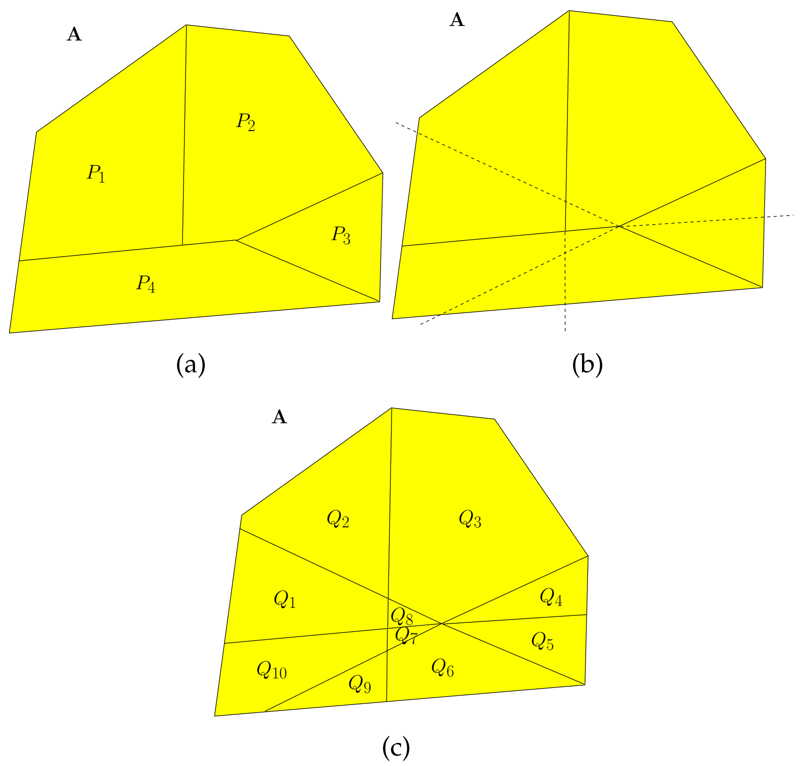


Figure 5.14: Making $A = P_1 + P_2 + P_3 + P_4$ into the sum of total products Q_1–Q_{10}.

Theorem 5.3.4. *The inference rules are truth-preserving.*

Proof. **R1**: Observe that given any coordinate system and a half-plane h, the position of h in reference to this coordinate system falls into five categories mentioned in the outline of the proof of Lemma 5.2.14. Since the intersection point of two non-parallel rational lines is a point with rational coordinates, clearly any such an arrangement is expressible by some fixing formula τ. The result then follows.

R2: The result is obvious, as every $r \in ROQ(\mathbb{R}^2)$ is a rational polygon and so it is a Boolean combination of some rational half-planes. □

We are ready to state the main result of this section.

Theorem 5.3.5 (Soundness Theorem). *Let ψ be an $\mathcal{L}_{conv,\leq}$-sentence. If $\psi \in T(\mathbf{Ax})$ then $\mathfrak{M}_Q \models \psi$.*

Proof. We are required to show that all the axioms are true in $\mathfrak{M}_Q$ and that the inference rules are truth preserving. It should be clear why Axioms **2**, **5-8** are true in $\mathfrak{M}_Q$. Since $ROQ(\mathbb{R}^2)$ is a Boolean Algebra, Axioms **1** hold. Since by definition 0 is non-convex, axiom **3** holds. Axioms **4** is true by virtue of Theorem 5.3.3 and rules **R1** and **R2** by Theorem 5.3.4. □

5.3.2 Completeness

In this section we prove the completeness theorem for our axiom system. We make extensive use of the following, classical results. Let $\Sigma(\bar{x})$ be a set of formulas in a language $\mathcal{L}$ with free variables in $\bar{x}$. An $\mathcal{L}$-structure $\mathfrak{A}$ is said to *realise* $\Sigma(\bar{x})$ if there exists a tuple $\bar{a}$ from A satisfying every $\sigma(\bar{x}) \in \Sigma(\bar{x})$. We say that $\mathfrak{A}$ *omits* $\Sigma(\bar{x})$ if $\mathfrak{A}$ does not realise $\Sigma(\bar{x})$. An $\mathcal{L}$-theory T is said to *locally realise* $\Sigma(\bar{x})$ if there is a formula $\phi(\bar{x})$ such that $\phi(\bar{x})$ is consistent with T and for all $\sigma(\bar{x}) \in \Sigma(\bar{x})$, $T \models \forall\bar{x}\,(\phi(\bar{x}) \rightarrow \sigma(\bar{x}))$. We say that T *locally omits* $\Sigma(\bar{x})$ if T does not locally realise $\Sigma(\bar{x})$. In other words, T locally omits $\Sigma(\bar{x})$ if for every formula $\phi(\bar{x})$ consistent with T there exists $\sigma(\bar{x}) \in \Sigma(\bar{x})$ such that $T \not\models \forall\bar{x}\,(\phi(\bar{x}) \rightarrow \sigma(\bar{x}))$.

We modify these standard notions as follows.

Definition 5.3.4. A theory T is said to *locally realise* $\Sigma(\bar{x})$ *given a formula* $\alpha(\bar{x})$ if there exists $\phi(\bar{x})$ such that $\phi(\bar{x}) \wedge \alpha(\bar{x})$ is consistent with T and for all $\sigma(\bar{x}) \in \Sigma(\bar{x})$,

$$T \models \forall\bar{x}\,(\phi(\bar{x}) \wedge \alpha(\bar{x}) \rightarrow \sigma(\bar{x}))\,.$$

Otherwise $\phi(\bar{x})$ locally omits $\Sigma(\bar{x})$ given $\alpha(\bar{x})$ in T.

We have the following theorem.

Theorem 5.3.6 (Conditional Omitting Types Theorem). *Let $\mathcal{L}$ be a countable language and let T be a consistent L-theory and $\alpha(\bar{x})$ an $\mathcal{L}$-formula. Let $\Sigma_1(\bar{x})$ be a set of $\mathcal{L}$-formulas in the same variables as $\alpha(\bar{x})$ and $\Sigma_2(x)$ be a set of $\mathcal{L}$-formulas in a single variable x.*

If T

(i) *locally omits* $\Sigma_1(\bar{x})$ *given* $\alpha(\bar{x})$*, and*

(ii) *locally omits* $\Sigma_2(y)$*,*

then T *has a countable model omitting* $\{\alpha(\bar{x}) \wedge \sigma(\bar{x}) : \sigma(\bar{x}) \in \Sigma_1(\bar{x})\}$ *and* $\Sigma_2(x)$*.*

Proof. Let $\mathcal{C} = c_0, c_1, \ldots,$ be a countable set of individual constants, and let $\phi_0, \phi_1, \ldots$ be the enumeration of all the sentences of the language formed by adding the constants $\mathcal{C}$ to $\mathcal{L}$. We define a sequence of consistent theories $T_0 \subseteq T_1 \subseteq \ldots$ with $T = T_0$. Suppose $T_m = T \cup \{\Psi_0, \ldots, \Psi_R\}$ has been defined, we show how to define T_{m+1}. Start by setting $T_{m+1} = T_m$. Let $\Psi = \Psi_0 \wedge \ldots \wedge \Psi_R$ and let $\bar{c} = c_0, \ldots, c_p$ be the constants appearing in Ψ, $\bar{z} = z_0, \ldots, z_p$ be fresh variables and $\Psi(\bar{z})$ the result of replacing $\bar{z}$ for $\bar{c}$ in Ψ. Note that $\Psi(\bar{z})$ is consistent with T.

(1) Let $\bar{w}$ be any tuple of the same arity as $\bar{x}$, formed from variables $\bar{z}, z_m$ (repetitions allowed). Note that z_m may or may not be one of the variables in $\bar{z}$. There are finitely many such tuples, say $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_s$. Let $\bar{d}_1, \ldots, \bar{d}_s$ be the corresponding tuples chosen from the constants $\bar{c}, c_m$. We construct a sequence of formulas $\pi_1, \ldots, \pi_s$ as follows. Consider the formula $\Psi(\bar{z}) \wedge \alpha(\bar{w}_1)$ and prefix it with quantifiers of the form $\exists z_i$, where z_i is a variable occurring in $\bar{z}$ but not in $\bar{w}_1$, denoted $\exists^{w_1}(\Psi(\bar{z}) \wedge \alpha(\bar{w}_1))$. If this is consistent with T, choose $\sigma_1(\bar{x}) \in \Sigma_1(\bar{x})$ (using the fact that T locally omits $\Sigma_1(\bar{x})$ given $\alpha(\bar{x})$) such that $\exists^{w_1}(\Psi(\bar{z}) \wedge \alpha(\bar{w}_1)) \wedge \neg\sigma_1(\bar{w}_1)$ is consistent with T. Otherwise set $\sigma_1 := \perp$. Then $\pi_1 := \exists^{w_1}(\Psi(\bar{z})) \wedge \neg\sigma_1(\bar{w}_1)$ is consistent with T. Suppose π_{i-1} has been defined, we show how to define π_i. First, remove any existential quantification from π_{i-1} and replace it by $\exists z_i$, where z_i is a variable occurring in $\bar{z}$ but not occurring in $\bar{w}_i$ (if it exists) to obtain a formula of the form $\exists^{w_i}(\Psi(\bar{z}) \wedge \neg\sigma_1(\bar{w}_1) \wedge \ldots \wedge \neg\sigma_{i-1}(\bar{w}_{i-1}))$. Consider the formula $\exists^{w_i}(\Psi(\bar{z}) \wedge \neg\sigma_1(\bar{w}_1) \wedge \ldots \wedge \neg\sigma_{i-1}(\bar{w}_{i-1}) \wedge \alpha(\bar{w}_i))$. If this is consistent with T, by the fact that T locally omits $\Sigma_1(\bar{x})$ given $\alpha(\bar{x})$, choose $\sigma_i(\bar{x}) \in \Sigma_1(\bar{x})$ such that $\exists^{w_i}(\Psi(\bar{z}) \wedge \neg\sigma_1(\bar{w}_1) \wedge \ldots \wedge \neg\sigma_{i-1}(\bar{w}_{i-1}) \wedge \alpha(\bar{w}_i)) \wedge \neg\sigma_i(w_i)$. Otherwise set $\sigma_i := \perp$. Then $\pi_i := \exists^{w_i}(\Psi(\bar{z}) \wedge \neg\sigma_1(\bar{w}_1) \wedge \ldots \wedge \neg\sigma_{i-1}(\bar{w}_{i-1})) \wedge \neg\sigma_i(\bar{w}_i)$ is consistent with T.

(2) Finally we arrive at the formula π_s. Remove any existential quantification from π_s and prefix the resulting formula with $\exists z_k, k \neq m$. Note that this formula, denoted $\pi_s(z_m)$ is also consistent with T. By the fact that T locally omits Σ_2, choose $\tau(x) \in \Sigma_2(x)$ such that $\pi_s(z_m) \wedge \neg\tau(z_m)$ is consistent with T.

Add $\neg\sigma_1(\bar{d}_1), \ldots, \neg\sigma_s(\bar{d}_s), \neg\tau(c_m)$ to T_{m+1}.

If ϕ_m is consistent with T_{m+1} add ϕ_m to T_{m+1}, add $\neg\phi_m$ otherwise. If $\phi_m := \exists n(\Psi(n))$ pick the first unused constant c_Q of $\mathcal{C}$ and add $\Psi(c_Q)$ to T_{m+1}. Thus T_{m+1} is a consistent theory.

Let $T^\star = \bigcup_{m \geq 0} T_m$. Then $T^\star$ is a complete theory consistent with $\alpha(\bar{x})$ which defines a model $\mathfrak{N} \models T^\star$ in the obvious way (see [CK73] p. 81).

Note that the construction from section **(1)** gives us that for each $\varphi(\bar{x})$, if $\varphi(\bar{x}) \wedge \alpha(\bar{x})$ is consistent with T we obtain $\varphi(\bar{x}) \wedge \alpha(\bar{x}) \wedge \neg\sigma(\bar{x})$, for some $\sigma \ \in \Sigma_1$. And we set $\sigma := \perp$ if $\varphi(\bar{x}) \wedge \alpha(\bar{x})$ is not consistent with T. Hence we have

$$T \models \exists \bar{x}(\varphi(\bar{x} \wedge (\neg\alpha(\bar{x}) \vee \neg\sigma(\bar{x})))),$$

for some $\sigma \ \in \Sigma_1$. We obtain:

$$T \not\models \forall \bar{x}(\varphi(\bar{x}) \rightarrow (\alpha(\bar{x}) \wedge \sigma(\bar{x}))),$$

for some $\sigma \ \in \Sigma_1$.

Hence $\mathfrak{N}$ omits the type $\{\alpha(\bar{x}) \wedge \sigma(\bar{x}) \mid \sigma(\bar{x}) \in \Sigma_1\}$. That is, $\mathfrak{N}$ omits Σ_1 given α.

It follows that $\mathfrak{N}$ omits Σ_2 by a similar but more standard argument (see [CK73], p. 95–97). □

We are now ready to state the main theorem of this section:

Theorem 5.3.7 (Completeness Theorem). *Let ψ be an $\mathcal{L}_{\leq,conv}$-sentence. If $\mathfrak{M}_Q \models \psi$ then $\psi \ \in T(\mathbf{Ax})$.*

Proof. Let ψ be an $\mathcal{L}_{\leq,conv}$-sentence such that $\psi \notin T(\mathbf{Ax})$. We are required to show that $\mathfrak{M}_Q \not\models \psi$. Now let

$$T = \{\phi : \neg\psi \vdash \phi\}.$$

By the deduction theorem T is consistent. Consider the following sets of formulas:

1. $\Sigma_1(x_1, x_2, x_3, y) = \{coord(x_1, x_2, x_3) \wedge hp(y) \wedge \neg\tau(x_1, x_2, x_3, y) : \tau$ a fixing formula$\}$,

2. $\Sigma_2(x) = \{\neg\exists y_1 \ldots \neg\exists y_n(\bigwedge_{1\leq i\leq n} hp(y_i) \wedge x = bc(y_1, \ldots, y_n)) : n \in \mathbb{N},\ x = bc(y_1, \ldots, y_n)$ a Boolean combination formula$\}$.

Suppose $\Theta(x_1, x_2, x_3, y)$ is a formula such that

$$\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y)$$

is consistent with T. We then have

$$T \not\models \forall y \forall x_1 \forall x_2 \forall x_3 \neg(\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y))$$

and

$$T \not\models \forall y \forall x_1 \forall x_2 \forall x_3\ (coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \neg\Theta(x_1, x_2, x_3, y)),$$

so by **R1**:

$$T \not\models \forall y \forall x_1 \forall x_2 \forall x_3((coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \\ \rightarrow \neg\Theta(x_1, x_2, x_3, y)),$$

for some τ.[3]

Hence $\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y)$ consistent with T implies

$$T \not\models \forall y \forall x_1 \forall x_2 \forall x_3(\Theta(x_1, x_2, x_3, y) \wedge (coord(x_1, x_2, x_3) \wedge hp(y) \\ \rightarrow \neg\tau(x_1, x_2, x_3, y)),$$

for some τ.

[3]Since if $T \models \forall y \forall x_1 \forall x_2 \forall x_3((coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \rightarrow \neg\Theta(x_1, x_2, x_3, y))$ for all τ, then by **R1** $T \models \forall y \forall x_1 \forall x_2 \forall x_3\ (coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \neg\Theta(x_1, x_2, x_3, y))$.

In other words, T locally omits $\Sigma(x_1, x_2, x_3, y) = \{\neg\tau(x_1, x_2, x_3, y) : \tau$ a fixing formula$\}$ given $coord(x_1, x_2, x_3) \wedge hp(y)$.

Now suppose $\Theta(x)$ is any formula consistent with T. We then have

$$T \not\models \forall x \neg\Theta(x)$$

and by **R2**: $T \not\models \forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1\leq i\leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n)) \rightarrow \neg\Theta(y))$ for some $n \in \mathbb{N}$ and some bc, so $\Theta(x)$ consistent with T implies

$$T \not\models \forall y(\Theta(y) \rightarrow \neg(\exists x_1 \ldots \exists x_n(\bigwedge_{1\leq i\leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n))$$

for some $n \in \mathbb{N}$ and some bc. In other words, T locally omits $\Sigma_2(y)$.

By the conditional omitting types theorem there exists a countable model $\mathfrak{A}$ of T omitting $\Sigma_1(x_1, x_2, x_3, y)$ and $\Sigma_2(y)$.

A more intuitive way of saying that $\mathfrak{A}$ omits Σ_1 and Σ_2 is that for every element a of A and any $l_1, l_2, l_3 \in A$ forming a coordinate frame, a can be expressed as a Boolean combination of some $b_1, \ldots, b_k \in A$ such that $\mathfrak{A} \models \bigwedge_{1\leq i\leq k} hp[b_i]$ and $\mathfrak{A} \models \bigwedge_{1\leq i\leq k} \tau_i[l_1, l_2, l_3, b_i]$, where τ_i is a fixing formula for b_i.

Since the carrier set of $\mathfrak{A}$ is countable we can enumerate its elements :

$$A = \{a_1, a_2, a_3, \ldots\}.$$

We fix this notation for the remainder of this section. Assume WLOG that $|A| > 2$ and that, by Axiom 2, a_1, a_2, a_3 are such that

$$\mathfrak{A} \models coord[a_1, a_2, a_3].$$

By the fact that $\mathfrak{A}$ omits Σ_2, for each $a_i \in A$ we have that

$$\mathfrak{A} \models a_i = bc(b^{(i)}_{N(1)}, \ldots, b^{(i)}_{N(i)}))$$

for some $b^{(i)}_{N(1)}, \ldots, b_{N(i)} \in A$ such that

$$\mathfrak{A} \models \bigwedge_{1 \leq j \leq N(i)} hp[b_j].$$

Since $\mathfrak{A}$ omits Σ_1, for each $j \in \{1, \ldots, N(i)\}$ there exists a fixing formula τ such that $\mathfrak{A} \models \tau[a_1, a_2, a_3, b_j]$.

We now proceed to define a mapping $e : A \to ROQ(\mathbb{R}^2)$.

Fix $h_1, h_2, h_3 \in ROQ(\mathbb{R}^2)$ such that

$$\mathfrak{M}_Q \models coord[h_1, h_2, h_3].$$

We start by defining a mapping $e^{(k)}$ for each initial segment $a_1, \ldots, a_k$ of elements of A. By the above considerations, let $a_i = bc(b^{(i)}_1, \ldots, b^{(i)}_{N(i)})$ for each $i \in \{1, \ldots, k\}$, we define $e^{(k)}(a_i) = h_i$, $i \in \{1, 2, 3\}$.

Note that by Lemma 5.2.14 there exists a unique half-plane $h^{(i)}_j \in ROQ(\mathbb{R}^2)$ such that

$$\mathfrak{M}_Q \models \tau^{(i)}_j[h_1, h_2, h_3, h^{(i)}_j].$$

We define $e^{(k)}(b^{(i)}_j) = h^{(i)}_j$ and extend $e^{(k)}$ homomorphically to all elements of the subalgebra generated by $a_1, \ldots, a_k$.

By saying that (a half-plane) b is involved in a construction of (a region) a we mean that $\mathfrak{A} \models a = bc[b_1, \ldots, b_{k-1}, b, b_{k+1}, \ldots, b_n]$ for some $k, n \in \mathbb{N}$, $k \leq n$.[4]

Lemma 5.3.5. *Let $a_1, \ldots, a_k$ be some initial segment of A. Then the mapping $e^{(k)}$ is well defined.*

Proof. Firstly note that for each half-plane b_i involved in a construction of any of $a_1, \ldots, a_k$ it follows from Axiom **6** that if $\mathfrak{A} \models \tau_{(P,Q)}[a_1, a_2, a_3, b_i]$ and $\mathfrak{A} \models \tau'_{(P',Q')}[a_1, a_2, a_3, b_i]$, then $P = P'$ and $Q = Q'$.

Now let b and b' be two half-planes involved in a construction of some a and a' respectively, such that $\mathfrak{A} \models \tau[a_1, a_2, a_3, b]$, $\mathfrak{A} \models \tau'[a_1, a_2, a_3, b']$ and $b = b'$.

[4]Obviously we use terms like "a half-plane" or "a region" in relation to $\mathfrak{A}$ just to guide intuition.

By the definition of $e^{(k)}$, b and b' are mapped to some h and h', respectively. We are required to show that $h = h'$. But this follows from the fact that $\mathfrak{A} \models \tau'[a_1, a_2, a_3, b]$ (and so the respective conditions, as described above, are satisfied) and Lemma 5.2.14. □

Lemma 5.3.6. *Let $a_1, \ldots, a_k$ be some initial segment of A. Then the mapping $e^{(k)}$ is injective.*

Proof. Let b and b' be two half-planes involved in a construction of any of $a_1, \ldots, a_k$. We need to show that if $e^{(k)}(b) = h$ and $e^{(k)}(b') = h'$ are such that $h = h'$ then $b = b'$. Let $\mathfrak{M}_Q \models \tau_{(P,Q)}[h_1, h_2, h_3, h]$ and $\mathfrak{M}_Q \models \tau'_{(P',Q')}[h_1, h_2, h_3, h']$. By Lemma 5.2.14 it follows that for these fixing formulas $P = P'$ and $Q = Q'$. The result then follows from Axiom **7**. □

Since we fixed $h_1, h_2, h_3 \in ROQ(\mathbb{R}^2)$, for any b, b' such that $e^{(k)}(b) = h$ and $e^{(k')}(b') = h'$ with $b = b'$ we have that $h = h'$. This follows by reckoning analogous to the one presented in the proof of Lemma 5.3.5.

Lemma 5.3.7. *For any initial segment $a_1, \ldots, a_k \in A$ the mapping $e^{(k)}$ is Boolean algebra isomorphism.*

Proof. It follows from axiom 5 that $e^{(k)}$ is a monomorphism. It is onto by definition. □

Lemma 5.3.8. *For any initial segment $a_1, \ldots, a_k \in A$ the mapping $e^{(k)}$ is an embedding.*

Proof. By Lemma 5.3.7 $e^{(k)}$ is a Boolean Algebra isomorphism. We are required to show that the following holds:

- $\mathfrak{A} \models conv[a_i]$ if and only if $\mathfrak{M}_Q \models conv[e^{(k)}(a_i)]$ where $1 \leq i \leq k$;

  Since each a_i is a Boolean combination of some $b_1, \ldots, b_{N(i)}$ we need to show that

  $$\mathfrak{A} \models conv[bc(b_1, \ldots, b_{N(i)})]$$

  if and only if

  $$\mathfrak{M}_Q \models conv[bc(e^{(k)}(b_1), \ldots, e^{(k)}(b_{N(i)}))].$$

  Suppose $\mathfrak{A} \models conv[a_i]$, by Axiom **3** $a_i \neq 0$ and by Axiom **4** we have $\mathfrak{A} \models a_i = \prod_{j \in I} b_j$, for some $I \subseteq \{1, \ldots, N(i)\}$. Therefore $e^{(k)}(a_i) \neq 0$

(Lemma 5.3.7) and by definition

$$e^{(k)}(a_i) = \prod_{i\ \in I} e^{(k)}(b_i).$$

Since $e^{(k)}(b_j)$ are half-planes (and as such convex)

$$\mathfrak{M}_Q \models conv[e^{(k)}(a_i)].$$

Conversely, suppose $\mathfrak{M}_Q \models conv[e^{(k)}(a_i)]$, then (by theorem 15)

$$e^{(k)}(a_i) = \prod_{1 \leq j \leq N(i)} e^{(k)}(b_j)$$

for some selection of half-planes $e(b_1), \ldots, e^{(k)}(b_{N(i)})$. Therefore, since $e^{(k)}$ is a Boolean algebra homomorphism

$$e^{(k)}(\prod_{1 \leq j \leq N(i)} b_j) = \prod_{1 \leq j \leq N(i)} e^{(k)}(b_j)$$

and since it is injective $a_i = \prod_{1 \leq i \leq k} b_i$, $a_i \neq 0$ with

$$\mathfrak{A} \models \bigwedge_{1 \leq j \leq N(i)} hp(b_j)$$

Hence, by Axioms **3** and **4**, we have $\mathfrak{A} \models conv[a_i]$. □

Lemma 5.3.9. *Let $e : A \to ROQ(\mathbb{R}^2)$ be defined as $e = \bigcup_{i=1}^{\infty} e^{(i)}$. Then e is an embedding.*

Proof. We need to show that e is injective. But this is obvious in view of Lemma 5.3.8. □

Lemma 5.3.10. *The mapping $e : A \to ROQ(\mathbb{R}^2)$ is an isomorphism.*

Proof. By Axiom **8** for any $P, Q \in \mathbb{Q}$ we can find an element $a \in A$ such that $\mathfrak{A} \models \tau_{(P,Q)}[a_1, a_2, a_3, a]$ and so e is onto. □

It follows that $\mathfrak{A}$ is isomorphic to $\mathfrak{M}_Q$ and so $\mathfrak{A} \models \phi$ if and only if $\mathfrak{M}_Q \models \phi$ for all $\mathcal{L}_{conv,\leq}$-sentences ϕ.

Recall the way $\mathfrak{A}$ is constructed. It follows that $\mathfrak{A} \models \neg\psi$ but then also $\mathfrak{M}_Q \models \neg\psi$ and so $\mathfrak{M}_Q \not\models \psi$, which concludes the completeness proof. □

5.4 Summary

The language $\mathcal{L}_{conv,\leq}$ over the considered spatial domains is very expressive. One can simulate statements about points and straight lines with statements about regions. This in turn allows the introduction of a coordinate frame and eventually, the rational fixing formulas. Taking this idea a bit further we saw that the algebraic fixing formulas can be defined in a similar manner. This allows one to distinguish between the rational and algebraic models. The fact that the Helly's theorem is "expressible" in $\mathfrak{M}_X$ allows one to show that $\mathcal{L}_{conv,\leq}$ is sensitive enough to detect changes in dimensionality. The fixing formulas are also crucial in axiomatising the theory of the rational model. We proposed an axiom system making a heavy use of the rational fixing formulas and containing two infinitary rules of inference. We also conjectured that with the use of similar techniques applied to the algebraic fixing formulas, it would be possible to axiomatise the algebraic model.

6
Conclusions and Further Work

We presented an axiom system for an affine spatial logic with convexity predicate and variables ranging over polygonal subsets of the real plane. Chapter 3 gave the historical and philosophical background of logical investigations of affine geometry. We described early attempts at the axiomatic characterisation of affine geometry by Whitehead and Russell. Russell's work is more philosophical in spirit. He starts with a Kantian view of mathematics and sees affine geometry as the a priori part of geometry. His views do evolve; what does not change is his conviction of the importance of non-qualitative notions in geometry. Whitehead's contribution is more technical, yet, as we pointed out, it comes prior to major developments in model theory. Whitehead is widely credited as the proponent (together with Leśniewski) of the region-based approach to geometry. Even though David Hilbert's contribution is much more in spirit of modern formal logic — he proposed an axiom system for Euclidean geometry — even his work cannot fully qualify as a spatial logic. From our perspective it is important that this early work on the foundations of geometry served as a major influence on the next generation of logicians. Alfred Tarski took Hilbert's ideas on axiomatisation of Euclidean geometry, extended them and presented them in a framework of modern logic. Tarski's major contribution is the change of emphasis from the geometry itself to the language that describes it. This allowed posing precise mathematical questions and obtaining precise answers. His work on Euclidean geometry was extended in his joint work with Lesław Szczerba on affine geometry. This work in turn is heavily influenced by Whitehead's considerations. Adopting the early Russellian approach, it can be said that Szczerba and Tarski axiomatised the a priori part of geometry, while Tarski's solo work concerned the a posteriori part. Regarding the region-based approach, we have Tarski's work (again!) on the one hand and Clarke's investigations on the other. The former was building on Leśniewski's ideas and proposed one of the first region-based spatial logics, the geometry of solids. The latter was extending Whitehead's topological ideas and constructed the calculus of individuals. Chapter 4 describes how the ideas of spatial logic were developed more recently. One can distinguish two approaches here. The first, we chose to call axiomatic, stems from the work of Whitehead and Clarke; the second, we called model-theoretic, owes more to Tarski. Both approaches are developed within, or in close proximity to, the qualitative spatial reasoning paradigm. The idea, most notably put forward by Randell Cui and Cohn in their seminal paper, is that non-quantitative analysis might be less error prone and more computationally robust than the standard numerical approach. Also, it is claimed that qualitative region-based reasoning is, in some sense, more fundamental (an example of human cognitive procedures is usually given). We should like to stress that despite the similarity, Russell's ideas do not seem to be a conscious influence here. The dominance of topological spatial logics in both axiomatic and model-theoretic approaches is evident. In terms of affine spatial logics we should mention Bennett and Cohn in the axiomatic approach and Davis and Pratt-Hartmann in the model-theoretic one. Chapter 5 describes our own contribution. We place ourselves within the model-theoretic approach and build on work by Davis and Pratt-Hartmann. In Russellian terms, we work — like Szczerba and Tarski — on the a priori fragment of geometry, but from a region-based perspective. In that sense we reach back to Whitehead's ideas and trough Clarke's work place ourselves within the qualitative spatial reasoning paradigm. Adopting the model-theoretic approach, one starts not with the syntactic notion of an axiom system, but rather with the semantic notion of an interpretation. Axiomatisation is treated secondary as a means of finding out more about the investigated logic. Make no mistake, this does not mean axiomatisation is unimportant; it merely indicates that it is viewed as one of many ways to explore a given logic and its theory.

Thesis contribution We started the thesis with a list of three groups of problems:

P1 How can we characterize the valid formulas of the spatial logic? That is, what is a theory of a given spatial logic? In many cases, it also makes sense to ask which sentences of the considered language are true in all structures of that class.

P2 What is the expressive power of a spatial language? In particular, given a language, what other geometrical relations can we express in terms of primitive relations in that language?

P3 What is the computational complexity of a given spatial logic? Most first order logics are, for obvious reasons, undecidable. However, restricting attention to certain fragments of those logics, might prove useful in terms of computational tractability.

In terms of these we have dealt with some success with the problems **P1** and **P2**. Our main contribution is as follows. We axiomatised the theory of $\langle ROQ(\mathbb{R}^2), conv, \leq\rangle$, where $ROQ(\mathbb{R}^2)$ is the set of regular open rational polygons of the real plane; $conv$ is the convexity property and $\leq$ is the inclusion relation. We proved soundness and completeness for our axiom system (**P1**). We have also proved several expressiveness results (**P2**). We showed that the betweenness and equidistance relations are definable in our logic in reference to a given coordinate frame. We also showed that betweenness can be defined without any reference to a coordinate frame. We showed that, roughly speaking, the property of being a root of a polynomial with rational coefficients can be defined in reference to a given coordinate frame. We also showed that Helly's theorem can be "expressed" in our logic. As a consequence we showed that models of different dimensions have different theories. We also presented explicit (not difficult) proofs of several expressiveness results from [Pra99] (Section 5.2).

Future Work Several natural extensions of our work suggest themselves at this point. We would like to investigate the possibility of axiomatising other affine spatial logics over the real plane. The most promising case is the one where $ROQ(\mathbb{R}^2)$ is replaced with $ROA(\mathbb{R}^2)$. We are strongly convinced that, given our expressiveness results, this axiomatisation is a straightforward application of the techniques used in axiomatising $ROQ(\mathbb{R}^2)$ (especially those utilised in the proof of the completeness theorem). The cases of other domains of interest remain somehow elusive, with $RO(\mathbb{R}^2)$ being perhaps the most difficult. Another way of extending our results would be to consider dimensions other than 2. Again, the cases of $ROQ(\mathbb{R}^n)$ and $ROA(\mathbb{R}^n)$ for $n > 2$

seem very promising. In terms of expressiveness we would like to investigate the possibility of defining other topological relations in our logic or in the indicated alternatives. Most notably we think that it should be possible to define the contact relation in the case of $RO(\mathbb{R}^2)$ (recall that [Pra99] defined the contact relation for the domains $ROP(\mathbb{R}^2)$ and $ROQ(\mathbb{R}^2)$). In terms of the above list of problems, **P3** proved to be the most elusive. The first-order theory of our convexity logic is undecidable and the existential fragment's complexity reduces to the result which is very hard to improve. Exploring the computational complexity of convexity spatial logics of different dimensions is one option here. Also, we might want to restrict ourselves even further and consider only special types of formulas (like Horn clauses) to see if there is any improvement in terms of computational complexity. Last but not least, it might prove useful to try and axiomatise these mentioned fragments of our convexity logic. This might be very helpful in implementing it in some programming language in a form of spatial reasoner, even despite its computational intractability.

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