A MODEL-THEORETIC APPROACH TO MEREOTOPOLOGY

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Abstract

Classical approaches to a formal representation of space consider points to be primitive and regions to be sets of points. In contrast, mereotopology is an approach to the representation of space which considers regions to be primitive. In doing so, mereotopology becomes interesting to the Artificial Intelligence community for qualitative spatial representation and reasoning.

This thesis takes a model-theoretic approach to mereotopology. Various sets of regions are defined and investigated. The regular open semi-algebraic sets in the real plane are identified as a well-behaved set of regions. Several first-order languages are introduced whose variables range over regions. Predicate symbols are interpreted as mereological and topological relations such as 'region x is part of region y', 'regions x and y are in contact' or 'region x is connected'. The mereotopological languages are interpreted over various sets of regions, and thereby structures in the sense of model theory are introduced. The expressivity of the mereotopological languages and the properties of the model-theoretic structures are investigated. The theory of the mereotopological structure of regular open semi-algebraic regions is completely axiomatised in an extended first-order predicate calculus and is shown to be undecidable.

It is shown that the mereotopological structure of all semi-algebraic sets of the real plane can be interpreted in the mereotopological structure of the regular open semi-algebraic sets. Consequently, in spite of the absence of points in certain meretopological structures, one can refer to points in these structures.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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[1] http://www.lyx.org

I dedicate this thesis with love and gratitude to my wife Mareike Schoop.

Chapter 1

Introduction

Space is fundamental to the world we live in. Our ability to see, to hear, to touch and to move lets us perceive space. We distinguish a large variety of spatial concepts related to shape, location, size, distance etc. as shown by psychological experiments and the investigation of natural languages. Some spatial concepts are even used as metaphors together with non-spatial concepts, e.g. “deep thought” or “far-fetched argument”, which indicates that space plays an important role in our thinking. As a consequence, knowledge about space and how we relate to it is of considerable philosophical and practical interest.

1.1 What is mereotopology?

Topology is the mathematical discipline that investigates the properties of space which remain invariant under continuous change. Topology does not distinguish between a cup with one handle and a doughnut. This fact can be clarified by a plasticine model of a doughnut. The lump of plasticine can be continuously deformed, that is without tearing, perforating or joining ends, such that it takes the shape of a cup with handle. The lump of plasticine in the shape of a doughnut, however, cannot be deformed into the shape of a pretzel without the discontinuous deformation which is required to create additional “holes”. Classical topology, i.e. point-set-topology, takes the ontological view that points are the primary spatial entities and any other spatial entities are sets of points.

Like topology, *mereotopology* is a discipline that investigates the properties of space which remain invariant under continuous change. However, unlike classical topology, mereotopology takes the ontological view that the primary spatial entities are *regions* that have, in contrast to points, a spatial extension. It is helpful to think of regions as lumps of space that could be occupied by material objects. Properties of regions that are considered in mereotopology are, for example, 'region x is part of region y', 'region x is in contact with region y' or 'region x is connected'. The first of these properties is said to be *mereological,* while the latter two properties are said to be *topological.* Notwithstanding the distinction between mereological and topological properties, all properties considered in mereotopology are invariant under continuous change. However, it is mereology, i.e. the formal study of the relation between part and whole, together with topology that give mereotopology its name.

1.2 The motivation for mereotopology

There are three principal motivations for mereotopology:

1. Metaphysics and the foundations of geometry
2. Formal ontology
3. Qualitative spatial representation and reasoning (QSR)

Metaphysics is the philosophical study of being. Many philosophers have contributed a fair part of their work to the investigation of the nature of space: Aristotle, Newton, Kant, Brentano to name but a few. Ontological questions which have been asked are: Do points exist? If so, what is a point?

We do not perceive points. However, we perceive objects and therefore the "lumps of space", i.e. regions, they occupy. Is it possible to explain points in terms of regions? Is it possible to define classical geometry referring to regions only?

Formal ontology is the study of ontological principles that are independent of specific domains (Guarino, 1998). One of the aims of formal ontology is the development of common-sense ontologies of the everyday world of desks, mugs, holes, shadows etc. If points are assumed to be the primary spatial entities then desks, mugs, holes and shadows must be constructed from points as sets of points. Consequently, one has to deal with many of the problems inherent in set-theory. However, if objects of our everyday world are considered to be regions—although perhaps regions of various kinds—there is no necessity to construct entities as sets of primitive entities. Therefore, mereotopology is considered to provide the appropriate tools for formal ontology.

Finally, mereotopology is thought to be a promising approach to **qualitative spatial representation and reasoning** (QSR). To clarify the meaning of *qualitative* spatial representation and reasoning consider the following example. An Ordnance Survey map of London provides a more or less accurate graphical description of the absolute location of streets, buildings etc. with respect to a given coordinate system. Therefore, the map contains the quantitative information that, for example, the distance between Euston rail station and Heathrow airport is 5 miles. A London underground map, however, conveys no metric, but only topological, information. One can read from the map the qualitative information that Euston rail station is connected by tube to Leicester Square and that Leicester Square is connected by tube to Heathrow airport. Then one can infer that Euston rail station and Heathrow airport are connected by tube lines without using any quantitative information in the reasoning process. Many spatial reasoning problems require only qualitative spatial information. For example, assume a description of the relations between regions A, B, C and D is given by: A is inside B, B overlaps C, C touches D on the outside, D overlaps B, D is disjoint from A, and C overlaps A. Are there regions in the plane that satisfy this description? If so, we would like to see an example. If there are no such regions, we would like to have an explanation why there cannot be such regions.

Since mereotopology deals only with qualitative and more specifically topological spatial data, it is thought that mereotopology provides the means to efficient topological reasoning (Cohn et al., 1992). Furthermore, taking regions as primary spatial entities, mereotopology is considered to provide cognitively appropriate common-sense ontologies of everyday objects.

1.3 Issues in mereotopology

The prominent questions in mereotopology are:

1. Which regions have we got? That is, which regions do we take to exist?

2. Which mereological and topological properties do we take to be primitive?

These two questions are independent of the specific motivation for research in mereotopology. However, if one is motivated by the metaphysical questions then the next questions are:

I.1 How can we reconstruct points from regions?

I.2 How can we verify that reconstructed "points" are points?

If one is interested in formal ontology or topological representation and reasoning then the next questions are:

II.1 What can we express with the mereotopological properties we take to be primitive?

II.2 What is true of what we can say with respect to the regions we have?

II.3 Is there any algorithmic way to answer the previous question?

There are two main approaches to answering question 1. One is to take a set of entities from topology and let these entities represent regions. I will call such a set of entities a *spatial domain*. Thus, a spatial domain explicitly represents a set of regions. For example, the open sets of the topological space $\mathbb{R}^3$ can be taken to represent regions. An alternative approach is to represent regions implicitly by their mereological and topological properties. This latter approach is often realised by a set of axioms in a formal (first-order) language where variables range over regions. Therefore, for the latter approach, question 2 has to be answered first: What are the mereological and topological properties which we take to be primitive and which, therefore, correspond to predicate symbols in the formal language? For example, the binary mereological predicate symbol P and the topological predicate symbol C can be chosen to have the meaning 'region x is part of region y' and 'region x is in contact with region y' respectively. Such formal language will be called a *mereotopological language* and a set of sentences in a mereotopological language will be called a *mereotopological theory*.

Although the two approaches to the representation of regions are different, they complement each other. It is sensible to interpret a mereotopological theory over a spatial domain such that the variables range over the elements of the domain. In this case, the pair consisting of spatial domain and interpretation is said to be a *model* of the theory. Conversely, given an interpretation of a mereotopological language over a spatial domain, a mereotopological theory can be given to characterise some properties of the spatial domain.

Both approaches have drawbacks. A spatial domain has properties that are specific to the mathematical area the elements of the domain are taken from. Thus, the spatial domain has properties that are not necessarily properties of the regions the domain is intended to represent. A mereotopological theory, on the other hand, might be inconsistent and, therefore, does not represent regions at all, or the theory might be incomplete and, therefore, does not represent all properties of the set of regions.

However, both approaches taken together avoid these drawbacks. If we consider a variety of spatial domains that are taken to represent the same set of regions then it is possible to concentrate on the properties which are common to all of the spatial domains. The question is how we can identify spatial domains which represent the same set of regions. One solution is to consider the properties that can be expressed in the mereotopological language and which hold with respect to a given spatial domain. Then two spatial domains represent the same set of regions if their properties that can be expressed in the chosen mereotopological language are the same. Therefore, the questions for the expressivity of the mereotopological language (question II.1) and the properties which hold with respect to the spatial domain (question II.2) need to be answered. Thus, the answers to questions II.1 and II.2 lead to a unified approach to the representation of regions. Furthermore, the approach provides criteria for consistency and completeness of mereotopological theories. A mereotopological theory is consistent if it admits a model, i.e. a spatial domain the theory is interpreted over. A mereotopological theory is complete if all models of the theory satisfy the same sentences of the mereotopological language the theory is formulated in.

Mathematical model theory is the branch of mathematics with investigates (first-order) languages, theories in these languages and the models of theories. Therefore, mathematical model theory provides exactly the methods which are needed to answer the above questions in mereotopology.

1.4 Aim and objectives

The aim of the present work is to investigate the properties of regions represented by spatial domains and mereotopological theories, thereby giving answers to the above questions. Methods and results from topology, graph theory and model theory will be employed to achieve this aim.

The objectives are:

- to define a variety of spatial domains over familiar topological spaces,
- to choose interesting mereotopological languages which will be interpreted over the spatial domains, thereby defining models called *mereotopologies*,
- to investigate the expressivity of the mereotopological languages and the model theoretic properties of the mereotopologies,
- to determine the sentences in a mereotopological language which are true with respect to a selected mereotopology by a complete axiomatisation,
- to investigate whether the truth of sentences in a mereotopological language with respect to a selected mereotopology is decidable by an algorithm,
- to utilise the expressivity of a mereotopological language to "reconstruct" points and other spatial entities.

1.5 Thesis overview

The following chapter elaborates on the issues raised in the introduction and places mereotopology in its philosophical, mathematical and historical context.

Chapter 3 gives an answer to question 1, introducing a variety of spatial domains over the real plane and the real sphere. The properties of these domains are investigated.

In chapter 4, various mereotopological languages with predicate symbols expressing parthood, connection, contact and boundedness are introduced. The languages are interpreted over the spatial domains defined in chapter 3, and several model-theoretic structures called mereotopologies are introduced. Their properties, as well as the expressivity of the mereotopological languages, are investigated. Thus, chapter 4 answers questions 2 and II.1.

Chapter 5 provides a complete axiomatisation of the mereotopology of regular open semi-algebraic sets in the real plane. Thus, question II.2 is answered with respect to this specific mereotopology.

Chapter 6 is concerned with the feasibility of spatial reasoning in mereotopology. It is shown that reasoning about non-trivial mereotopologies is undecidable. Thus, this chapter settles question II.3.

In chapter 7, the intended ontological simplicity of mereotopology is questioned. It is argued that, under certain circumstances, a sparse spatial domain implicitly defines a richer spatial domain. It is shown that a mereotopology, although it is point-free, may represent points in a certain sense. The technique used to reconstruct points is a partial answer to question I.1.

The conclusion gives an interpretation of the results and indicates future work.

A glossary at the end of this thesis explains the frequently used concepts.

1.6 Notation and prerequisites

The reader is expected to have a basic knowledge of graph theory, topology, logic and model theory. For example, Armstrong's textbook (Armstrong, 1979) covers most of topology used in this thesis. The necessary background in logic and model theory is given, for example, in (Mendelson, 1997). Any concepts but the basic ones are introduced where necessary. These are also explained in a glossary at the end of the thesis. The following subsections present notations and basic concepts used in this thesis which are adjusted to the present purposes or are non-standard.

1.6.1 General

Let A and B be subsets of the set X. Then $||A||$ denotes the cardinality of the set A, $A \setminus B$ denotes the difference, and $\overline{A}$ denotes the set-theoretic complement with respect to X. The power-set of A is denoted by $\wp(A)$.

1.6.2 Topology

Let U be a subset of a topological space $X = (X, \tau)$. The closure of U is denoted by $[U]$ and the interior of U is denoted by U°. This somewhat unusual notation is chosen to reserve $\overline{U}$ for the set-theoretical complement of U and $\bar{r}$ for the tuple-notation in logic (see below).

In a metric space X with metric ρ, $B_{\epsilon}(p)$ denotes the open ball with radius ϵ around the point p, i.e. $B_{\epsilon}(p) = \{q \in X | \rho(p, q) < \epsilon\}$.

A continuous function γ from the unit interval with its usual topology into X is a *path*. The range $\gamma([0, 1])$ of the path γ is called the *locus* of γ and is denoted by $|\gamma|$. The *endpoints* of a path are the points $\gamma(0)$ and $\gamma(1)$. An *arc* (or *Jordan arc*) is an injective path. The *interior* of an arc γ is the set $|\gamma| \setminus \{\gamma(0), \gamma(1)\}$. A *simple closed curve* (or *Jordan curve*) is a continuous function from the unit-circle with its usual topology into a topological space X.

1.6.3 Graph theory

A *graph* Γ is a pair (V, E) where V, the set of *vertices*, is a finite set, and E, the set of *edges*, is a subset of $\{\{v_1, v_2\} | v_1, v_2 \in V, v_1 \neq v_2\}$. Thus, the graphs considered in this thesis do not have multiple edges or loops. Given a graph Γ,

$V(\Gamma)$ denotes the set of vertices and $E(\Gamma)$ denotes the set of edges. Two graphs of importance in this thesis are the graphs K_5 and $K_{3,3}$:

$$K_5 = (\{v_1, \ldots, v_5\}, \{\{v_i, v_j\} | 1 \leq i < j \leq 5\})$$

$$K_{3,3} = (\{v_1, \ldots, v_6\}, \{\{v_i, v_j\} | 1 \leq i \leq 3, 4 \leq j \leq 6\})$$

A subset V' of $V(\Gamma)$ induces a *subgraph* Γ' of Γ, written $\Gamma' \subseteq \Gamma$, with $\Gamma' = \{V', E(\Gamma) \cap \wp(V')\}$. Given a subset V' of $V(\Gamma)$, $\Gamma \setminus V'$ denotes the subgraph of Γ induced by $V(\Gamma) \setminus V'$. In this case, the vertices V' are said to be *deleted* from Γ. If $e = \{v_1, v_2\}$ is an edge of $\Gamma = (V, E)$ then Γ/e denotes the graph obtained from Γ by *contracting* the edge e to the vertex v_1. More precisely, Γ/e is the graph $(V \setminus \{v_2\}, (E \cup \{\{v_1, v\} | \{v_2, v\} \in E, v \neq v_1\}) \setminus \{\{v_2, v\} | v \in V\})$. Any graph Γ' obtained from a graph Γ by repeated deletion of vertices or contractions of edges is called a *minor* of Γ. Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $\sigma : V_1 \cup E_1 \rightarrow V_2 \cup E_2$ such that for all $v \in V_1$, $\sigma(v) \in V_2$ and for all $\{v_1, v_2\} \in E_1$, $\{\sigma(v_1), \sigma(v_2)\} \in E_2$. If σ is an isomorphism between Γ_1 and Γ_2 then I write $\sigma : \Gamma_1 \rightarrow \Gamma_2$.

A *path of length n joining vertices v_1 and v_2* is a graph with n vertices $v_1, \ldots, v_n$ and the edge set $\{\{v_i, v_{i+1}\} | 1 \leq i < n\}$. A *cycle* is a graph with vertices $v_1, \ldots, v_n$ and the edge set $\{\{v_i, v_{i+1}\} | 1 \leq i < n\} \cup \{\{v_1, v_n\}\}$. Two paths $\Gamma_1, \Gamma_2 \subseteq \Gamma$ joining vertices v_1 and v_2 are said to be *independent* (in Γ) if $V(\Gamma_1) \cap V(\Gamma_2) = \{v_1, v_2\}$. A graph Γ is *connected* if for any two distinct vertices $v_1, v_2 \in V(\Gamma)$ there exists a path $\Gamma' \subseteq \Gamma$ joining v_1 and v_2. A graph Γ is *n-connected* if any two distinct vertices are joined by at least n independent paths in Γ.

A *plane graph* Γ is a tuple (V, E) where V is a finite set of points in $\mathbb{R}^2$ or $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ and E is a finite set of arcs in $\mathbb{R}^2$ or S^2 respectively such that the endpoints of an arc in E are elements of V, no two arcs in E have the same endpoints, and the interiors of the arcs in E are disjoint. To avoid unnecessary notational overhead, a plane graph $\Gamma = (V, E)$ will be considered as the set $\bigcup V \cup \bigcup \{|\gamma| \, | \gamma \in E\}$ as well as the (abstract) graph $(V(\Gamma), \{\{\gamma(0), \gamma(1)\} | \gamma \in E\})$. The *faces* of a plane graph Γ are the components $\mathbb{R}^2 \setminus \Gamma$ or $S^2 \setminus \Gamma$ respectively. A graph Γ is *planar* if there exists a plane graph I which is isomorphic to Γ. In this case, the graph I is said to be a plane embedding of Γ. I will appeal to the following characterisation of planar graphs.

Theorem (Wagner 1937). *A graph* Γ *is planar if and only if neither* K_5 *nor* $K_{3,3}$ *is a minor of* Γ.

1.6.4 Logic and model theory

Given a signature Σ, the usual first-order language with signature Σ will be denoted by $\mathcal{L}(\Sigma)$. Gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}, \mathfrak{N}$ etc. denote $\mathcal{L}$-structures in the sense of model theory. Their domains are denoted by A, B, M, N etc. respectively. The domains which are of specific interest to mereotopology are denoted in bold type, e.g. **P** and **S**.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with domain A. A formula $\phi(x_1, \ldots, x_n)$ is said to *define* the set $\{(a_1, \ldots, a_n) \in A^n | \mathfrak{A} \models \phi[a_1, \ldots, a_n]\}$ in $\mathfrak{A}$. A subset B of A^n $(n \geq 1)$ is said to be $\mathcal{L}$*-definable in* $\mathfrak{A}$ if there exists a formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}$ such that $\phi(x_1, \ldots, x_n)$ defines B. The set B is $\mathcal{L}$*-definable with parameters in* $\mathfrak{A}$ if there exists a formula $\phi(x_1, \ldots, x_k)$ $(k \geq n)$ and a $(k-n)$-tuple $b_1, \ldots, b_{k-n}$ in A such that $B = \{(a_1, \ldots, a_n) | \mathfrak{A} \models \phi[a_1, \ldots, a_n, b_1, \ldots, b_{k-n}]\}$.

A set of sentences (in $\mathcal{L}$) is called a *theory* (in $\mathcal{L}$). The *theory of* $\mathfrak{A}$, denoted $Th(\mathfrak{A})$, is the set of all sentences in $\mathcal{L}$ which hold in $\mathfrak{A}$. A *type* $\chi(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ is a maximal consistent set of formulae in the variables $x_1, \ldots, x_n$. The *type of a tuple* $(a_1, \ldots, a_n) \in A^n$ is the type $\{\phi(x_1, \ldots, x_n)$ in $\mathcal{L} | \mathfrak{A} \models \phi[a_1, \ldots, a_n]\}$.

To simplify notation, a tuple of variables $x_1, \ldots, x_n$ as well as a tuple of elements $a_1, \ldots, a_n$ will be abbreviated to $\bar{x}$ and $\bar{a}$ respectively. The size of such abbreviated tuple either will be known from the context or will be of no importance.

Chapter 2

Mereotopology and related fields of study

The previous chapter introduced mereotopology as the region-based approach to topology. This chapter places mereotopology in its mathematical, philosophical and historical context. I will elaborate on the issues of mereotopology which have been raised in the introduction and present their treatment in the relevant literature. The focus will be on mereotopology motivated by the need for a qualitative representation of spatial knowledge in many application areas of Artificial Intelligence (AI). Since the reconstruction of points from regions is the origin of mereotopology, the reconstruction of points will be presented in a historical review.

2.1 The motivation for mereotopology

In section 1.2 of the introductory chapter, I explained that mereotopology is motivated on the one hand by the aim to reconstruct points from regions and on the other hand by the need for a qualitative representation of space. The former motivation will be considered in more detail in section 2.7. Here, I will elaborate on the latter.

Since space is a fundamental part of the world we live in, problem solving requires in many cases a qualitative representation and manipulation of spatial data. Qualitative spatial representation and reasoning (QSR), a well established subfield of AI, aims at solving problems which deal with qualitative spatial data (for an overview of QSR see (Hernández, 1994) and (Cohn, 1997)). Mereotopology as one specific approach to QSR aims at a cognitively realistic and useful qualitative representation of everyday objects in space. Therefore, mereotopology takes regions as primary entities and considers their mereological and topological properties. Points are not necessarily excluded as entities of mereotopology. However, in mereotopology, points must not be the primary entities from which other entities are constructed. The following themes which justify this view reoccur in the literature (Fleck, 1996; Galton, 1996; Gotts et al., 1996; Smith, 1998).

Firstly, there are considerations for the efficiency of spatial representation and reasoning. If points are the only primitive spatial entities then regions must be constructed from points. Thus, regions are higher-order entities and any formal system dealing with regions has to quantify over these higher-order entities. The necessity to deal with higher-order systems is believed to make efficient spatial reasoning impossible.

Secondly, if points are the only primary spatial entities then any set of points can be a "region". Consequently, there are "regions" we do not encounter in the everyday world: sets which are scattered in infinitely many "pieces", extremely convoluted lines, fractals etc. For example, a straight line can be continuously changed into a "line" that is so convoluted that it does not fit anymore in our intuitive classification of lines and areas. The development of such a "line" is depicted in figure 2.1. Each line segment between two of the indicated points of the generator is replaced by a smaller version of the generator itself to get the first step towards the final "line". Each line segment is again replaced by an even smaller version of the generator to get the second step. This procedure, infinitely repeated, defines a fractal whose Hausdorff dimension is 1.5 (Falconer, 1990). Thus, the set of points defined by the fractal neither belongs to the class of lines nor to the class of areas as one intuitively understands them.

Thirdly, taking regions as sets of points plays further havoc with common sense, as the following two puzzles show. Consider a ball of points, for instance the unit ball in the Euclidean space $\mathbb{R}^3$ with centre point $(0, 0, 0)$. It is common sense that a ball can be divided into two congruent halves. However, only one of the two halves can contain the centre point and, therefore, the halves cannot be congruent. For the second puzzle, consider a chess board with its black and white squares. Is the point at which two black squares and two white squares meet black or white? If the point is black then it belongs to the black squares and

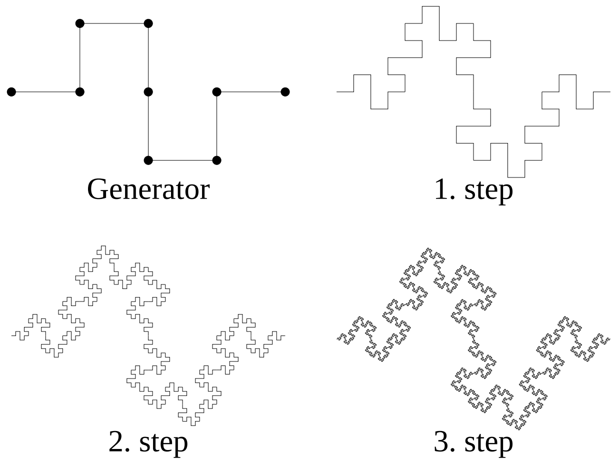


Figure 2.1: A set of points which is neither a line nor an area, but a fractal of Hausdorff dimension 1.5

the two white squares do not touch. If the point is white then it belongs to the white squares and the two black squares do not touch. However, it is common sense that there is no difference between the black and the white squares except for their colours. This second puzzle leads us to the problem of boundaries which is the fourth and last of the main reasons which exclude points as the primary entities in mereotopology.

Topology in its set-theoretical variation teaches us to classify sets of points into open sets, closed sets, boundaries etc. This classification forces fundamental questions on us which several philosophers have attempted to solve (cf. Varzi, 1997): Does a region include its boundary? Do regions share parts of their boundaries if they are in contact? What is the ontological difference between a region including its boundary and the region's interior?

According to Varzi (1997), Leonardo da Vinci assumes that two objects in contact have a common boundary which, however, is no material part of either of the objects in contact. Da Vinci argues that if a boundary participated in an object then it would be of substance and therefore separate objects. Brentano believes that there are two boundaries, one belonging to each of the entities in contact such that the boundaries share the same location. Bolzano states that contact is only possible between an object with a boundary and another without a boundary.

Any choice of these three possibilities seems to be more or less arbitrary. Some mereotopologists believe that boundaries—and, consequently, unpleasant

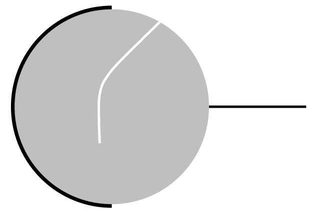


Figure 2.2: A region including a part of its boundary, a crack and a spike

ontological questions—can be avoided by adopting a mereotopological approach to the representation of space. Others believe that mereotopology is the very tool to deal with boundaries. The following section explains how this apparent contradiction comes about.

2.2 Boundaries and the distinction between open and closed regions

The AI community employs mereotopology for a qualitative and cognitively appropriate representation of the mereological and topological properties of objects in space. Experience tells us that any material object is three-dimensional throughout and has no "cracks" or "spikes". Therefore, a region such as the one depicted in figure 2.2 has no interpretation with respect to objects. Moreover, a representation of objects in space aimed at problem solving in AI most often does not require the notion of boundaries. Consider, for example, a robot which is programmed to find its way around a room without bumping into things. Certainly, the robot has to recognise obstacles but has no need to "know" about boundaries to achieve its task. Thus, there are applications of qualitative spatial representation and reasoning where boundaries unnecessarily complicate the situation.

Even mereotopological approaches which are not aimed at practical applications but simply at a representation of space can dispense with boundaries and the distinction between open and closed regions. For the sake of the argument, let a 'region' be considered as a chunk of space that is potentially occupied by one or several material objects. This definition presupposes space to be absolute. That is, space is assumed to exist independently of any other entity. The alternative would be to regard space as relative, i.e. exhibited only as a property of other entities and their relation to each other. Whether space is absolute or relative is the issue of an ongoing philosophical debate (see for example (Nerlich, 1994)) but does not concern the present work. It is simply convenient to consider space to be absolute for the development of spatial domains and mereotopological theories aimed at qualitative spatial representation and reasoning.

There are two assumptions we usually make about material objects which distinguish them from regions as potential receptacles of objects. Firstly, no two material objects can occupy the same region at the same time,[1] and secondly, not every sum of two material objects exists. For example, a desk together with a chair are nothing but the collection of the objects desk and chair. A bread-knife, however, is assembled from a handle and a blade. Knife, handle and blade are distinct material objects but the sum of handle and blade forms the object knife. See (Varzi, 1998) for the problems which such sums of objects pose to mereotopology. In contrast to objects, regions can overlap. Furthermore, regions can be summed together. The space occupied by a desk and a chair is just the region occupied by the desk summed together with the region occupied by the chair. Therefore, a mereotopological approach to the representation of objects has to have different properties than a mereotopological approach to the representation of regions as potential receptacles of objects. Moreover, I argue below that a mereotopological approach to the representation of regions as chunks of space has to dispense with boundaries (although nothing prevents us from talking about them, cf. chapter 7), while mereotopological approaches which include boundaries are usually aimed at the representation of objects in space.

Some formal ontologists employ mereotopology explicitly to represent the properties of objects. Formal ontology is the discipline that investigates the principles that underlie every ontology independently of the specific domain of application. Mereotopology, as the formal study which combines mereological and topological notions, is thought to be a valuable tool for formal ontology (Guarino, 1998). Moreover, mereotopology is seen by some as the true theory of part and whole. While mereology studies the relation between parts and the entities they are parts of, mereotopology provides a concept of a whole in the sense of 'consisting of one piece', i.e. being topologically connected, which is considered to be one criterion to identify wholeness (Varzi, 1997). So far objects which we encounter

[1] Observations like "a statue shares its localisation with the bronze it is made from" (see Casati and Varzi, 1996) do not interfere here, since an object 'statue', as considered here, has the properties of being a statue and being made from bronze. Neither statue nor bronze are objects in their own right and therefore do not share the same part of space.

in our everyday life have been in the centre of investigation in formal ontology, although abstract entities such as geographical entities have been considered as well (Casati et al., 1998). The consideration of material objects and geographical entities inevitably requires the consideration of spatial entities such as regions and boundaries.

Smith and Varzi (Smith, 1995, 1996, 1998; Smith and Varzi, 1997; Varzi, 1997) explicitly consider objects and develop a mereotopological theory which is informed by the the philosophical investigation of boundaries, continuity and contact by Brentano. They argue that the distinction between open and closed regions is consistent with common sense. Holes serve as examples. A hole has a parasitic existence, being dependent on the object it is a hole of. If a material object is accepted to include its boundary then a hole is defined by the boundary, but has no part in the boundary. Therefore, a hole does not include its boundary, while the host object does so. Hence the open-closed distinction makes sense (cf. Casati and Varzi, 1994). Moreover, Smith and Varzi argue that the introduction of one type of boundary is not sufficient for a mereotopological theory of objects. The distinction of two kinds of boundary makes it possible to distinguish two kinds of contact. Two material objects cannot be in contact, since both include their boundaries which they cannot share. Therefore, apparent contact of material objects, such as John and Mary holding hands, is a metrical notion of closeness but no genuine contact.

Smith and Varzi distinguish between *bona-fide* (or physical) and *fiat* (or human-demarcation-induced) boundaries, not only of objects but also of geographic entities (for the latter see also (Casati et al., 1998)). A spatial entity has a bona-fide boundary if its boundary is an interruption of the continuity of the entity against its surroundings. For example, a wine glass standing on a table is visibly separated from the table and the air that surrounds it. Therefore, the boundary of the glass is a bona-fide boundary. However, although we take the stem and cup of the glass to exist as separate entities, there is not necessarily a marked boundary between stem and cup. Nevertheless, the existence of a boundary separating stem and cup is taken for granted. Such boundary, therefore, is a fiat-boundary. One example for a geographic entity with bona-fide boundary is Great Britain, whose boundary is defined by the discontinuity between land and sea. However, the boundaries of post-code areas, or the boundary of the US state of Wyoming are fiat boundaries.

The introduction of fiat boundaries enables the contact between objects of the same kind. Two objects with fiat boundaries are in contact if they share a part of their fiat boundaries. Hence, two post-code areas can be in contact. An object with a bona-fide boundary, however, can only be in contact with another object without a boundary. For instance, a hole is in contact with its host.

Asher and Vieu (1995) also argue in favour of two different kinds of contact. They introduce the notions of 'weak contact' and 'strong contact' that are motivated by natural language usage. Asher and View argue that given a wine glass standing on a table, the wine glass and the table are in *weak contact* since the glass and the table come vanishingly close but do not share their boundary. Stem and cup of the wine glass, however, share a part of their boundary and therefore are said to be in *strong contact*.

However, I argue that although the distinction between types of boundaries and types of contact may be appropriate for objects, it goes amiss with regions considered as potential receptacles of objects. To oppose Smith and Varzi's argumentation with respect to regions I give the following example. Consider a mould that is a material object with a hole that is to be filled. Following Smith and Varzi's argumentation, the hole and the mould are in contact. Assume now that the hole is filled with bronze to form a new object. Since the new object is a material object, it is not in genuine contact with the mould. However, what is the difference between the region occupied by the hole and the region occupied by the newly formed object? Common sense says that the newly formed object fills exactly the space the hole filled before. Therefore, the region occupied by the hole and the region occupied by the newly formed object must be the same. The region occupied by the mould itself has not changed during the creation of the new object. Therefore, considering regions alone there cannot be different types of contact.

Casati and Varzi (1996, 1997) develop a mereotopological theory that caters for regions *and* objects. Their aim is to develop a theory of localisation. Therefore, their formal language contains a binary predicate symbol L representing localisation. The formula $\mathrm{L}(x, y)$ has the intended meaning '(entity) x is exactly located at (region) y'. The relation L is axiomatised as a mereological relation relating entities and regions. It turns out that a unary predicate defining the set of regions would serve equally well to define the theory of localisation.

Whether one intends to represents objects in space or space by itself, the principal question in any approach to mereotopology is: Which regions exist? In mereotopology, the set of regions is either explicitly represented by a *spatial domain* or it is represented by a *mereotopological theory* as explained in the following two sections.

2.3 Spatial domains

A set of elements which are taken to represent regions will be called a *spatial domain*. Most spatial domains are constructed from a tried and tested representation of space. Such representations are the two- and three-dimensional Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. The primary entities of Euclidean spaces, however, are points. Therefore, any representation of a region over a Euclidean space is necessarily a set of points. This fact seems to contradict the very idea of mereotopology. However, a set of points simply *represents* a region. Thus, not points but sets of points are the primary entities in a spatial domain. A commonly used repository of sets which are taken to represent regions are the *regular open* or *regular closed* sets over the Euclidean spaces. A regular open set is identical to the interior of its closure, and a regular closed set is identical to the closure of its interior. Therefore, regular sets do not have “spikes” or “cracks”. Moreover, the regular open sets and the regular closed sets of a topological space form a Boolean algebra with respect to the inclusion relation. The regular open sets of Euclidean spaces will be considered in detail in chapter 3.

Few mereotopologists introduce spatial domains. Dornheim (1998) introduces a spatial domain constructed over the real plane $\mathbb{R}^2$. He defines simple regions as the bounded (regular) closed sets in $\mathbb{R}^2$ whose boundary is a simple polygon. His spatial domain consists of all finite unions of simple regions, the complements of these unions in the Boolean algebra of regular closed sets, the empty set and $\mathbb{R}^2$. Thus, the spatial domain forms a Boolean subalgebra of the Boolean algebra of regular closed sets in $\mathbb{R}^2$, and at most one component of a region is unbounded.

Pratt and Lemon (1997) and Pratt and Schoop (1998, 1999) consider the spatial domains of regular open semi-linear and semi-algebraic sets over the real open and closed planes. These spatial domains will be presented in chapter 3.

Gotts (1994, 1996a) suggests closed and bounded manifolds with well-behaved boundaries as well as regular closed sets of a regular connected space as representations of regions.

Asher and Vieu (1995) do not define a spatial domain explicitly, but they do define a class of spatial domains by properties expressed in the mathematical meta-language of topology. The spatial domains are essentially the semi-regular sets of a topological space; i.e. for every region x, $[x] = [x^\circ]$ or $x^\circ = [x]^\circ$. Thus, the regions do not have "spikes" or "cracks", but may contain or exclude parts of their boundary. An additional constraint on Asher and Vieu's spatial domains is that the inclusion relation on regions must not be dense.

All these spatial domains have in common that, on the one hand, they are restricted to relatively well-behaved sets and, on the other hand, they attempt to provide sufficiently many regions for the representation of objects in space.

2.4 Mereotopological theories

A *mereotopological language* is a first-order language with a number of predicate symbols with a mereological or topological interpretation. A *mereotopological theory* is a set of sentences in a mereotopological language. Several predicate symbols with mereological or topological interpretation have been presented in the literature (see table 2.1). Predicate symbols with a geometric interpretation have been suggested as well (e.g. to express convexity (Cohn, 1995) or congruence (Borgo et al., 1996a)). However, such symbols will not be considered in this thesis.

In the following, I will present some of the mereotopological theories that have been presented in the literature. Since most mereotopological theories are formulated in the language with predicate symbols P and C, and since these theories share a number of basic axioms, I will present a representative mereotopological theory and point out the differences to specific other mereotopological theories.

The simplest mereotopological theory, called Ground Mereotopology (MT),[2] consists of the following three axioms:

C1 $\forall x(\mathrm{C}(x, x))$ C-reflexivity

C2 $\forall x \forall y(\mathrm{C}(x, y) \rightarrow \mathrm{C}(y, x))$ C-symmetry

C3 $\forall x \forall y(\mathrm{P}(x, y) \rightarrow \forall z(\mathrm{C}(z, x) \rightarrow \mathrm{C}(z, y)))$ P implies C

[2]The names of the mereotopological theories in this section are those used in Varzi (1996).

Symbol	Reading
$\mathrm{P}(x, y)$	region x is part of region y (Whitehead, 1929; Clarke, 1981, and many more)
$\mathrm{C}(x, y)$	regions x and y are in contact (Whitehead, 1929; Clarke, 1981, and many more)
$\mathrm{c}(x)$	region x is connected (Pratt and Lemon, 1997; Pratt and Schoop, 1998)
$x \leq y$	region x is part of region y (Pratt and Lemon, 1997; Pratt and Schoop, 1998)
$\mathrm{SR}(x)$	region x is a simple region, i.e. x is connected (Borgo et al., 1996a)
$l(x, y)$	regions x and y are separated (Grzegorczyk, 1960; Eschenbach, 1994)
$\mathrm{R}(x)$	x is a region (Eschenbach and Heydrich, 1995)

Table 2.1: Mereotopological predicate symbols

Thus, the symbol C represents a reflexive and symmetric binary relation, and if region x is part of region y then regions x and y are in contact. So far, there is nothing spatial about this theory. However, the symbols can clearly be given a spatial interpretation. Consider the open discs in $\mathbb{R}^2$ as regions and assume that a region r is part of a region s if and only if $r \subseteq s$, and regions r and s are in contact if and only if the closures of r and s have a point in common. Then it is easy to see that the following predicates defined via P and C also have spatial interpretations (cf. figure 2.3):

$\mathrm{PP}(x, y) =_{df} \mathrm{P}(x, y) \wedge \neg\, \mathrm{P}(y, x)$	x is a proper part of y
$\mathrm{O}(x, y) =_{df} \exists z(\mathrm{P}(z, x) \wedge \mathrm{P}(z, y))$	x and y overlap
$\mathrm{PO}(x, y) =_{df} \mathrm{O}(x, y) \wedge \neg\, \mathrm{P}(x, y) \wedge \neg\, \mathrm{P}(y, x)$	x and y overlap partially
$\mathrm{EC}(x, y) =_{df} \mathrm{C}(x, y) \wedge \neg\, \mathrm{O}(x, y)$	x and y are externally connected
$\mathrm{DC}(x, y) =_{df} \neg\, \mathrm{C}(x, y)$	x and y are disconnected
$\mathrm{TPP}(x, y) =_{df} \mathrm{PP}(x, y) \wedge \exists z(\mathrm{EC}(z, x) \wedge \mathrm{EC}(z, y))$	x is a tangential proper part of y
$\mathrm{NTPP}(x, y) =_{df} \mathrm{PP}(x, y) \wedge \neg\exists z(\mathrm{EC}(z, x) \wedge \mathrm{EC}(z, y))$	x is a non-tangential proper part of y

Region x is a proper part of region y, in symbols $\mathrm{PP}(x, y)$, if x is a part of

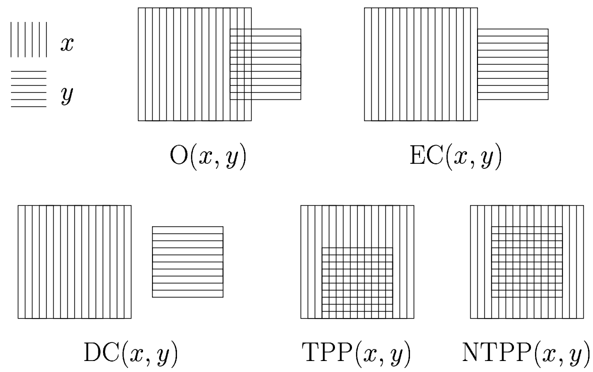


Figure 2.3: Visualisation of various binary relations definable by the predicates $P(x, y)$ and $C(x, y)$

y but y is not a part of x. Two regions overlap, in symbols $O(x, y)$, if there is a third region which is part of both. The overlap-relation is refined by the relation of partial overlap ($PO(x, y)$): regions x and y overlap but neither is x a part of y nor is y a part of x. These three relations are defined via the mereological relation P. The other three relations rely on the topological relation C as well. Two regions are in external contact, in symbols $EC(x, y)$, if they are in contact but do not overlap. External contact facilitates a refinement of 'proper part' into 'tangential proper part' and 'non-tangential proper part'. If region x is a proper part of region y and there is a region z which is in external contact with x and y, then x is a tangential proper part of y. If there is no such z then x is a non-tangential proper part of y. Thus, the two predicates P and C can distinguish a variety of spatial relations that can be seen as the analogues to some of Allen's primitive relations between temporal intervals (Allen, 1981).

Note that interpreting $C(x, y)$ as '(the closures of) regions x and y share a point' excludes by axiom C1 the empty set from the spatial domain. In many mereotopological approaches an empty region or null-element is excluded from the spatial domain on ontological grounds.

There are two different ways to extend the mereotopological theory MT. Either the mereological predicate P is defined in terms of the topological predicate C, or both predicates P and C are primitives. The next two sections present these two different extensions of MT.

2.4.1 Mereology subsumed by topology

The extension of the mereotopological theory MT by the following axiom treats the mereological predicate $\mathrm{P}(x,y)$ as defined by $\forall z(\mathrm{C}(z,x) \rightarrow \mathrm{C}(z,y))$.

C4 $\forall x \forall y(\forall z(\mathrm{C}(z,x) \rightarrow \mathrm{C}(z,y)) \rightarrow \mathrm{P}(x,y))$ C defines P

The theory C1-C4 has been named Strong Mereotopology (SMT). If the following axiom is added to SMT we get Strong Extensional Mereotopology (SEMT).

C5 $\forall x \forall y(\forall z(\mathrm{C}(z,x) \leftrightarrow \mathrm{C}(z,y)) \rightarrow x = y)$ C-extensionality

Thus, regions are extensional with respect to the contact predicate: if two regions x and y are in contact with the same regions then x and y must be identical.

Given a spatial domain M defined over a topological space, the predicate C has usually one of the following three interpretations:

1. $\{(r,s) \in M^2 | [r] \cap [s] \neq \emptyset\}$
2. $\{(r,s) \in M^2 | [r] \cap s \neq \emptyset \text{ or } r \cap [s] \neq \emptyset\}$
3. $\{(r,s) \in M^2 | r \cap s \neq \emptyset\}$

Given interpretation 1 or 2 in the presence of axiom C5, the closures of any two regions must be distinct. Therefore, a model of SEMT with interpretation 1 or 2 does not admit boundaries or the distinction between open and closed regions. For this reason, SEMT is popular with mereotopologists in AI. Given interpretation 3 and a model of SEMT where the spatial domain consists only of open sets, mereotopology is possibly reduced to mereology, since in this case the contact relation coincides with the overlap relation. However, the relations do not necessarily coincide. It is certainly possible that two open sets intersect but do not have a common part in a thinly populated spatial domain. For a deeper investigation of these three possible interpretations of C see (Cohn and Varzi, 1998).

It is easy to see that the following sentences are theorems of SEMT.

P1 $\forall x(\mathrm{P}(x,x))$ P-reflexivity

P2 $\forall x \forall y(\mathrm{P}(x,y) \wedge \mathrm{P}(y,x) \rightarrow x = y)$ P-antisymmetry

P3 $\forall x \forall y \forall z(\mathrm{P}(x,y) \wedge \mathrm{P}(y,z) \rightarrow \mathrm{P}(x,z))$ P-transitivity

P4 $\forall x \forall y(\forall z(\mathrm{P}(z,x) \leftrightarrow \mathrm{P}(z,y)) \rightarrow x = y)$ P-extensionality

Sentences P1-P4 form essentially the theory of Extensional Mereology as developed by Leśniewski (1929) (see also (Simons, 1987)). Sentences P1-P3 form the main axioms of every mereological theory. Therefore, in SEMT, topology can be seen to subsume mereology.

Strong (extensional) mereotopology can be extended by the following existential axioms. Generally, it is assumed that a maximal region, called *universe,* exists. The existence of such region is expressed by axiom P5:

P5 $\exists x \forall y(\mathrm{P}(y,x))$ existence of universe

It follows from axiom C4 that a universe is unique. It is easy to see that in SEMT axiom P5 is equivalent to the sentence $\exists x \forall y(\mathrm{C}(y,x))$ which in some mereotopological theories replaces P5.

Intuitively, if regions x and y exist then there also exists a region which is the "union" or "sum" of regions x and y. The space occupied by a desk together with the space occupied by a chair is simply the space occupied by the desk and the chair. Likewise, if two shadows overlap, they have a complete shadow in common. Thus, an "intersection" or "product" of regions and, in the presence of a universe, a "complement" sensibly exist. In many mereotopological theories, sum, product and complement of regions are introduced as definite descriptions:

D1 $x + y =_{df} \imath z \forall w(\mathrm{C}(w,z) \leftrightarrow (\mathrm{C}(w,x) \vee \mathrm{C}(w,y))$ sum

D2 $x \times y =_{df} \imath z \forall w(\mathrm{C}(w,z) \leftrightarrow (\mathrm{C}(w,x) \wedge \mathrm{C}(w,y))$ product

D3 $-x =_{df} \imath z \forall w(\mathrm{C}(w,z) \leftrightarrow \neg \mathrm{P}(w,x))$ complement

where a formula $\psi[\imath x \phi(x)]$ is substituted by $\exists y(\forall x(\phi(x) \leftrightarrow x = y) \wedge \psi[y])$. The existence of the sum and product of two regions is asserted by axioms such as P6, P7 and P8:

P6 $\forall x \forall y \exists z(z = x + y)$ existence of sum

P7 $\forall x \forall y(\mathrm{O}(x,y) \rightarrow \exists z(z = x \times y))$ existence of product

P8 $\forall x \exists y(\exists z(\neg \mathrm{P}(z,x)) \rightarrow y = -x)$ existence of complement

Note that axioms P7 and P8 are conditional, since an empty region does not exist in models of SEMT. The product of two regions exists only if they overlap, and the complement exists only for regions which are not the universe. A model satisfying axioms C1-C5 and P5-P8 is closed under finite sums, products and complements. Such a model is a Boolean algebra with its bottom-element removed (Biacino and Gerla, 1991). While most extensions of strong mereotopology in the literature agree on axioms C1-C5, P5 and P6, the definitions of product and complement of regions differ.

The most prominent extensions of strong mereotopology are those by Clarke (1981), Randell et al. (1992b) and Asher and Vieu (1995). The latter two approaches are inspired by Clarke's work. Clarke extends the 'calculus of individuals' (Leonard and Goodman, 1940) to include a notion of contact. His mereotopological theory consists essentially of SEMT, axiom P6 and the following two axioms:

P7$_{Clarke}$ $\forall x \forall y \forall z \Big(((\mathrm{C}(z,x) \rightarrow \mathrm{O}(z,x)) \wedge (\mathrm{C}(z,y) \rightarrow \mathrm{O}(z,y)) \rightarrow (\mathrm{C}(z, x \times y) \rightarrow (\mathrm{O}(z, x \times y)) \Big)$ existence of product

P9$_{Clarke}$ $\forall x \exists y (\mathrm{NTP}(y,x))$ no atoms
where $\mathrm{NTP}(x,y)$ stands for $\mathrm{P}(x,y) \wedge \neg \exists z (\mathrm{EC}(z,x) \wedge \mathrm{EC}(z,y))$

Clarke considers the distinction between $\mathrm{C}(x,y)$ and $\mathrm{O}(x,y)$ to be a virtue of his theory. Unfortunately, the axioms do not force this distinction. In addition, since by theorem P1, x is part of x, Clarke's definition of complement (D3 above) implies that no region is in contact with its complement.

The Region Connection Calculus (RCC) (Randell et al., 1992a,b; Bennett, 1995, 1996a; Gotts et al., 1996; Cohn et al., 1997) overcomes these idiosyncrasies. The mereotopological theory RCC consists of SEMT together with axioms P6, P7 and P8. However, in RCC the product and complement of regions are defined differently compared to D2 and D3:

D2$_{RCC}$ $x \times y =_{df} \imath z \forall w (\mathrm{C}(w,z) \leftrightarrow \exists v (\mathrm{P}(v,x) \wedge \mathrm{P}(v,y) \wedge \mathrm{C}(v,w)))$ product

D3$_{RCC}$ $-x =_{df} \imath z \forall w ((\mathrm{C}(w,z) \leftrightarrow \neg\, \mathrm{NTPP}(w,x)) \wedge (\mathrm{O}(w,z) \leftrightarrow \neg\, \mathrm{P}(z,x)))$ complement

It follows from axiom P8 and theorem P1 of the RCC-theory that a region x is in contact with $-x$ but does not overlap $-x$. Hence, C and O are distinct predicates in RCC.

Following the intuition that space can be divided into smaller and smaller regions, the following axiom of the atomless RCC-theory ensures that there are no atoms with respect to the mereological predicate P.

P9 $\forall x \exists y(\mathrm{NTPP}(y, x))$ no atoms

In RCC the intended meaning of the symbol C is 'the *closures* of regions x and y have a point in common'. Therefore, by the extensionality axiom C5, a spatial domain constructed over $\mathbb{R}^2$ or $\mathbb{R}^3$ that satisfies the axioms of RCC consists either of regular open or of regular closed sets. Hence, with this specific interpretation of C the interpretation of P coincides with set inclusion.

Düntsch et al. (1998b) and Stell and Worboys (1997) show that any model of RCC is an atomless Boolean algebra with the bottom-element removed. Gotts (1996a) shows that the regular closed sets of a connected space obeying the T_3-separation axiom provide a model for RCC.

The RCC-theory as defined so far is obviously not complete, and thus does not characterise the intended meaning of the topological predicate C. For example, the theory does not pin down the dimension of the universe, i.e. the space itself, though Gotts (1994, 1996c) makes suggestions as to how specific dimensions can be incorporated in RCC. Furthermore, RCC does not determine how regions are distributed in space. It is easy to see that a model of RCC is given by the set of regular open sets in the real plane $\mathbb{R}^2$ which have finitely many components, each of which either is an open disc having the centre point $(0, 0)$ and a radius smaller than 1, or is an annulus with the centre point $(0, 0)$ and an outer radius smaller than 1. Bennett (1995, 1996a) aims to remedy the incompleteness of RCC and extends it by further existential axioms. However, the new axioms are complex and do not provide a complete theory.

The mereotopological theory of Asher and Vieu (1995) includes all axioms of RCC except for axiom P9. However, complementation is defined differently:

$\mathrm{D3}_{AV}$ $-x =_{df} \iota z \forall w(\mathrm{C}(w, z) \leftrightarrow \exists v(\neg \mathrm{C}(v, x) \wedge \mathrm{C}(v, w)))$ complement

Hence, as in Clarke's theory a region is not in contact with its complement. However, the axiom $\exists x \exists y(\mathrm{EC}(x, y))$ of Asher and Vieu's theory forces the contact-relation and the overlap-relation to be distinct. In addition to the contact predicate, Asher and Vieu define a predicate for *weak contact*:

D4 $\mathrm{WC}(x, y) =_{df} \neg \mathrm{C}(x, y) \wedge \forall z((\mathrm{P}(x, z) \wedge z = i(z)) \rightarrow \mathrm{C}(y, c(z)))$

where $i(x)$ and $c(x)$ are defined thus

D5 $i(x) =_{df} \imath z \forall w(\mathrm{C}(w,z) \leftrightarrow \exists v(\mathrm{NTP}(v,x)))$ interior

D6 $c(x) =_{df} \imath z((\forall y(\mathrm{C}(x,y)) \rightarrow z = x) \vee z = -i(-x))$ closure

Note that the closure of the universe is the universe itself, since the complement of the interior is not defined. A simple model of Asher and Vieu's mereotopological theory is given by the spatial domain $M = \{(0,2),(0,1),(0,1],(1,2),[1,2)\} \subseteq \wp(\mathbb{R})$ where C is interpreted as $\{(x,y) \in M^2 | x \cap y \neq \emptyset\}$. Thus, $(0,1]$ and $[1,2)$ are the only regions in contact which do not overlap. Interior and closure as defined in D5 and D6 are given by the topological interior and closure with respect to the subspace $(0,2)$. Hence, $(0,1]$ and $(1,2)$, and $(0,1)$ and $[1,2)$ are in weak contact.

The mereotopological theories which will be considered in this thesis are extensions of SEMT, although not all theories will employ the very expressive topological notion of contact.

2.4.2 Mereology and topology as partners

The characteristic of strong mereotopology which distinguishes it from other mereotopological theories is that the mereological relation is defined by the topological relation. However, in some mereotopological approaches the mereotopological relation and the topological relation are treated as separate though interlinked relations. Thus, in such approaches, mereology and topology can be seen as partners. A mereotopological theory following such approach includes the axioms of ground mereotopology, MT, but excludes the axiom C4 which defines the mereological relation P via the relation C. Since the sentences P1-P3 are not theorems of MT, in this approach they have to be included as axioms in the theory. Sum, product and complement of regions are defined in terms of the mereological predicate:

D1′ $x \oplus y =_{df} \imath z \forall w(\mathrm{O}(w,z) \leftrightarrow (\mathrm{O}(w,x) \vee \mathrm{O}(w,y))$ sum

D2′ $x \otimes y =_{df} \imath z \forall w(\mathrm{P}(w,z) \leftrightarrow (\mathrm{P}(w,x) \wedge \mathrm{P}(w,y))$ product

D3′ $\ominus x =_{df} \imath z \forall w(\mathrm{P}(w,z) \leftrightarrow \neg\, \mathrm{O}(w,x))$ complement

The axioms stating the existence of sums, products and complements have to be adapted accordingly:

P6′ $\forall x \forall y(\exists z(z = x \oplus y))$ existence of sum

P7′ $\forall x \forall y(\mathrm{O}(x, y) \rightarrow \exists z(z = x \otimes y))$ existence of product

P8′ $\forall x \forall y(\exists z(\neg \mathrm{P}(z, x)) \rightarrow \exists z(z = \ominus x))$ existence of complement

There are now two possibilities to introduce extensionality. Extensionality can be based purely on the predicate C using axiom C4 or it can be based on the predicate P using axiom P4. It is easy to see that given axiom C3, P4 implies C4.

The theory consisting of MT, P1-P5, P6′, P7′ and P8′ is called Closed Extensional Mereotopology (CEMT). General Extensional Mereotopology (GEMT) consists of the axioms C1-C3, P1-P5 and the following axiom schema:

P9 For each $n \geq 1$, each formula $\phi(x_1, \ldots, x_n)$ the axiom

$$\forall x_2 \ldots \forall x_n \Big(\exists x_1(\phi(x_1, \ldots, x_n)) \rightarrow \exists z \forall y \Big(\mathrm{O}(y, z) \leftrightarrow \exists x_1(\phi(x_1, \ldots, x_n) \wedge \mathrm{O}(y, x))\Big)\Big)$$

fusion

This axiom schema ensures that the sum of every infinite set of regions which can be defined with parameters exists. Note that there are more subsets of an infinite spatial domain than there are sets definable with parameters, since the mereotopological language is a finitary first-order language. Lacking expressivity results for mereotopological languages, it is not clear which regions the fusion axiom schema forces to exist. They might exhibit pathological behaviour and therefore might not be appropriate for a representation of everyday objects.

The fusion axiom schema can be used to introduce topological concepts such as interior, closure, boundary etc. to a mereotopological theory. The quasi-topological operators for interior, exterior, closure and boundary of regions are defined thus:

D5 $i(x) =_{df} \iota z \forall y(\mathrm{O}(y, z) \leftrightarrow \exists w(\mathrm{NTPP}(w, x) \wedge \mathrm{O}(y, w))$ interior

D6 $e(x) =_{df} i(\ominus x)$ exterior

D7 $c(x) =_{df} \ominus e(x)$ closure

D8 $b(x) =_{df} \ominus(i(x) \oplus e(x))$ boundary

Note, however, that although by the fusion axiom schema of GEMT the interior and closure of a region exist, they need not be distinct. Consider, for example, the spatial domain of regular open sets of a Euclidean space where the contact relation is interpreted as "the closures of the two regions share a point" and the mereological relation is interpreted as set-theoretical inclusion. Certainly, the interior of a region r is the region itself. The complement of r is the largest regular open set disjoint from r. Therefore, the closure of r is r itself. Further axioms are necessary if interior and closure of regions are intended to be distinct.

The mereotopological theory of Borgo et al. (1996a,b, 1997) extends CEMT. Their mereotopological language employs the mereological predicate symbol P, a unary predicate symbol SR with the reading 'x is a simple region' meaning 'region x is connected' and a binary predicate symbol CG with the reading 'x and y are congruent'. Given the geometrical interpretation of the latter predicate symbol, the mereotopological language is certainly very expressive. The authors give an axiom system which aims to give a "good" characterisation of the intended models which contain regular three-dimensional regions. Unfortunately, these intended models are not formally defined. The mereological part of the axiom system is based on closed extensional mereology (axioms P1-P5, P6′, P7′, P8′). The topological part states the existence of connected regions and their interior parts. The contact predicate C is defined in terms of P and SR. On the morphological level, Hilbert's work on the axiomatisation of congruence and the correspondence between points and spheres is exploited. Spheres are defined following Tarski's definitions (Tarski, 1956). The congruence of spheres allows Borgo et al. to define the congruence and convexity of arbitrary regions.

The approach to mereotopology which will be presented in the following claims that topology subsumes mereology in certain circumstances. However, it will be argued here that also this approach can be considered as taking mereology and topology as equivalent partners.

Eschenbach and Heydrich (1995) introduce a mereotopological language where the binary predicate symbol C is replaced by a unary predicate symbol R with the intended meaning 'x is a region'. Thus, Eschenbach and Heydrich assume implicitly that the domain, over which the variables of the mereotopological language are interpreted, contains a whole variety of entities, some of which are regions. The presented mereotopological theory consists of axioms P1-P5 and P9 together with the definitions of $\mathrm{C}(x, y)$ and $\mathrm{EC}(x, y)$ given by $\mathrm{O}(x, y) \wedge \mathrm{R}(x) \wedge \mathrm{R}(y)$ and

$\mathrm{C}(x,y) \wedge \forall z(\mathrm{P}(z,x) \wedge \mathrm{P}(z,y) \rightarrow \neg\,\mathrm{R}(z))$ respectively. Thus, the formulae C1-C3 are provable from axioms P1-P3. Therefore, so it is claimed, topology is nothing but a subtheory of mereology where quantification is restricted to "regions", i.e. to the elements satisfying the predicate $\mathrm{R}(x)$. However, the classification of entities by the predicate $\mathrm{R}(x)$ is not a mereological notion. Here, "regions" are implicitly intended to be extended lumps of space. A point or a boundary, therefore, is not a "region". This very separation of extended lumps of space to other spatial entities gives the predicate $\mathrm{R}(x)$ clearly a *topological* meaning. If all entities satisfy $\mathrm{R}(x)$ then contact coincides with overlap and we are left with a purely mereological theory.

2.4.3 Mereotopological theories taking dimensions or boundaries into account

Few mereotopological approaches take regions of various dimensions into account.

Gotts (1996b) presents a mereotopological theory based on the single mereotopological predicate $INCH(x,y)$ with the meaning 'x INcludes a CHunk of y'. Intuitively, two regions r and s stand in the $INCH$-relation if the dimension of $r \cap s$ is not smaller than the dimension of s. A spatial domain of well-behaved closed manifolds is intended to provide a model of the theory. However, no formal results of the paper are based on this interpretation. It follows from the axioms of Gotts' mereotopological theory that regions can be classified into classes of equi-dimensional regions and that these classes form a linear order with a minimal element. A part-whole predicate P and a contact predicate C are definable. Thus, given an appropriate interpretation, the $INCH$-predicate is at least as expressive as the usual mereotopological predicates.

Galton (1996) presents a modification of generalised extensional mereotopology to axiomatise spatial domains of regions of various dimensions. Galton employs a binary predicate symbol B, where $\mathrm{B}(x,y)$ is read as 'region x is part of the boundary of region y', and the symbol P with its usual mereological interpretation. However, two regions stand in the mereological relation only if they are of the same dimension. Galton argues that a point might be incident in a line but does not contribute the line's extent and therefore is not a part of the line. Given this assumption, the set of points can be defined on the grounds of the mereological notion: $PT(x) \equiv \forall y(\mathrm{P}(y,x) \rightarrow \mathrm{P}(x,y))$. A relation defining pairs of regions of the same dimension can be defined using P. Furthermore, since a part of the boundary of a region x is of lower dimension than x, Galton is able to define a binary relation defining the pairs (x, y) where x is of lower dimension than y. The axioms of the mereotopological theory ensure that the dimensions form a strict linear order. However, the mereotopological theory does not force the existence of any element of any dimension. If a spatial domain is chosen to include regions of various dimensions, for example the closed one- or two-dimensional sets in the plane, then the whole plane is a region and the sum of any definable set of region exists by the fusion axiom of the theory. Elements of mixed dimensionality, however, do not have to exist, since a line lies in the plane but is not part of the plane.

The mereotopological theories of Smith and Varzi which are explicitly tailored to boundaries were already discussed in section 2.2.

2.5 The expressivity of mereotopological languages

There are at least two kinds of expressivity results. Firstly, one can ask which sets of regions can be defined in a given mereotopological language (absolute expressivity). Secondly, one can ask whether some mereotopological language can differentiate regions that another language cannot tell apart (relative expressivity). For example, consider the spatial domain $RO(\mathbb{R}^2)$ of regular open sets of the real plane and the mereotopological language with C as its only mereotopological predicate symbol. It is easy to show that the formula $\forall z(\mathrm{C}(x, z) \rightarrow \mathrm{C}(y, z))$ defines the inclusion relation if the relation C is interpreted by the set $\{(r, s) \in RO(\mathbb{R}^2) | [r] \cap [s] \neq \emptyset\}$ (see lemma 4.1.2 below). However, if the set of all subsets of $\mathbb{R}^2$ is taken as a spatial domain, then the inclusion relation is not definable by the predicate symbol C with the given interpretation (see lemma 7.0.1 below). Consequently, the mereotopological language with predicates C and P, where C is interpreted as before and P is interpreted as the inclusion relation, is more expressive than the mereotopological language with C as its only mereotopological predicate symbol with respect to $\wp(\mathbb{R}^2)$. However, both languages have the same expressivity with respect to the spatial domain $RO(\mathbb{R}^2)$.

Since only a small number of authors have considered formally defined spatial domains, formal expressivity results are limited. Pratt et al. (Pratt and Lemon, 1997; Pratt and Schoop, 1998) introduce several spatial domains and thus are able to prove relative and absolute expressivity results for two basic mereotopological languages (Pratt and Schoop, 1999). This work is included and extended in sections 4.1 and 4.4 of this thesis. Other authors, who only give an informal interpretation of their mereotopological languages, have presented very complex formulae which are intended to define relevant mereotopological notions. Gotts et al. (Gotts, 1994; Gotts et al., 1996) construct formulae that are intended to distinguish between a ball and a torus. The mereotopological language of Borgo et al. (1996a,b, 1997) is especially expressive since not only predicate symbols with mereotopological but also with geometrical interpretation (congruence) are employed. Borgo et al. provide formulae defining spheres and convex regions. Most often, these "experimental" expressivity results are plausible. Nevertheless, they need to be approached with caution since no spatial domain is formally defined.

Düntsch et al. (1998a) investigate the expressivity of the eight basic binary relations =, DC, EC, PO, TPP, NTPP, TPP^{-1} and NTPP^{-1} (cf. page 31) in the three-variable fragment of first-order logic. Düntsch et al. show that the relations DC and PO can be refined into five relations.

Formal languages and their expressive power have been investigated in the more general setting of topology and geometry. Several modal languages have been employed to capture spatial notions (e.g. Rescher and Garson, 1968; Asher and Vieu, 1995; Balbiani et al., 1996; Dabrowski et al., 1996) with varying degrees of success (see Lemon and Pratt, 1997). Henson et al. (1977) consider the lattice of closed sets of a topological space as spatial domain. They declare a topological property to be "first-order definable" if it is definable in the first-order language of lattices interpreted over the closed sets. Under these assumptions, a surprising number of topological properties are first-order definable. For example, some of the separation axioms as well as the connectedness of a topological space are first-order definable. Bankston (1984, 1990) develops the notion of a first-order representation which maps topological spaces to $\mathcal{L}$-structures such that homeomorphic spaces get mapped to isomorphic structures. This notion allows him to compare the expressive power of first-order languages with respect to classes of topological spaces. For example, Bankston shows that all compact 2-dimensional simplicial complexes are definable in the language of lattices where the lattices are given by the closed sets of metrisable topological spaces.

First-order languages specifically aimed at the application in spatial databases have been investigated (Kuijpers et al., 1999; Kuijpers and den Bussche, 1999). This work is closely related to research in o-minimal structures (see section 2.8.1 below).

In order to reference all entities of a topological space, i.e. points *and* sets, a (monadic) second-order language $\mathcal{L}_t$ was studied in (Flum and Ziegler, 1980; Ziegler, 1985). Flum and Ziegler show that some of the separation axioms of topology are $\mathcal{L}_t$-definable (cf. section 2.8.3).

2.6 The complexity and decidability of mereotopological theories

The undecidability of many mereotopological theories was essentially shown by Grzegorczyk (1951). In particular, the theory of his "algebra of bodies" is relevant to mereotopology (cf. chapter 6), where the algebra of bodies is essentially the set of regular open sets of a Euclidean space of dimension two or higher. Dornheim (1998) shows the undecidability of a mereotopological theory that admits a model which is ontologically much sparser than the "algebra of bodies".

Reasoning in theories in the three-variable fragment of first-order logic has been shown to be decidable and even tractable. In the centre of the complexity analysis of restricted mereotopological languages stands the Region Connection Calculus with its eight pairwise disjoint and mutually exhaustive basic relations =, DC, EC, PO, TPP, NTPP, TPP^{-1} and NTPP^{-1} that are defined via the symbol C (Cohn et al., 1995) (cf. section 2.4, page 31). The set of these eight relations is known as RCC-8. The reduced set of binary relations =,DC $\cup$ EC, PO, TPP $\cup$ NTPP, TPP^{-1} and NTPP^{-1} is known as RCC-5. The formulae of RCC-5 and RCC-8 respectively are first order formulae which use the relation symbols and have at most three variables, one of which must be bound. Bennett (1994) shows that every theory in RCC-8 can be transformed in an equi-satisfiable theory of propositional intuitionistic logic with respect to a given standard interpretation. Renz and Nebel (Nebel, 1995; Renz, 1998; Renz and Nebel, 1997, 1998a,b) show the satisfiability problem in any RCC-8 theory where no formula contains an existential quantification to be solvable in polynomial time. The satisfiability of any unrestricted theory of RCC-5 or RCC-8, however, is NP-hard. Furthermore, Renz and Nebel identify a maximal tractable subclass of RCC-8 containing 148 binary relations including the eight basic ones. (A subclass C of RCC-8 is given by a subset of the powerset of RCC-8 where each element $R \in C$ defines the relation $\bigcup R$.)

Jonsson and Drakengren (1997) classify all subclasses of RCC-5 with respect to the complexity of their satisfiability problem. They identify the four maximal tractable subclasses of RCC-5.

2.7 The origin of mereotopology

Since Euclid, 'point' has been treated as a *primitive* concept of geometry. Is this a necessity? Are there ways to construct the familiar geometry without referring to the concept 'point'? What do we mean by 'point' anyway? These questions led philosophers and mathematicians in the beginning of the 20th century to look for an alternative formalisation of geometry. The first step in this direction was to identify alternative primitives. We do not perceive points in space. However, we perceive physical objects. Having in mind this cognitive difference, a number of new primitives were suggested: 'solid bodies' (Huntington, 1913), 'regions' Whitehead (1920), 'solids' (de Laguna, 1922), 'volumes' (Nicod, 1930), 'lumps' (Menger, 1940), 'bodies' (Grzegorczyk, 1960) etc. I will use the term 'region' to refer to any of these.

An alternative formalisation of geometry based on the primitive 'region' has to enable a reconstruction of points. Otherwise, the formalisation cannot be an alternative formalisation of the familiar geometry. Therefore, the reconstruction of points is a necessary part in an alternative approach to geometry.

Whitehead (1919, 1920) develops a 'theory of extensive abstraction' to reconstruct points from regions. He uses the binary relation 'region x extends over region y', the converse of the relation 'region y is part of region x', to define sets of sets of regions which are intended to serve as a substitute for points. Hence, Whitehead introduces the concept of an *abstractive set*: A set R of regions is an abstractive set[3] if

(i) of any two members of R, one contains the other as part

(ii) there is no region which is a common part of every member of R.

Then an abstractive set "converges to the ideal of all nature with no ... extension" (Whitehead, 1920, p. 61). However, abstractive sets cannot be taken

[3] Actually, Whitehead introduces abstractive sets over durations and events and only later constructs abstractive sets of regions (Whitehead, 1920, pp. 60,79).

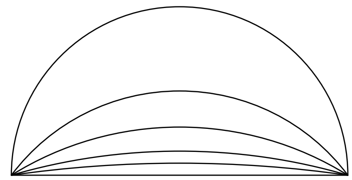


Figure 2.4: An abstractive set

as a substitute for points, since there exist several abstractive sets converging to the same point. Thus, there would be too many points. Therefore, Whitehead introduces further concepts: An abstractive set P is said to *cover* an abstractive set Q if every member of P contains as part some member of Q. Then two abstractive sets have the same *abstractive force*, if each covers the other. This relation is an equivalence relation and its equivalence classes are the *abstractive elements.* However, there are still more abstractive elements than points, since an abstractive set does not necessarily converge to a point, but may also converge to a line (cf. figure 2.4). Whitehead realizes that his primitive relation of 'extending over' cannot remedy this problem.

It remains for de Laguna (1922) and Nicod (1930) to show independently a way to the solution. Both introduce a notion of contact between regions. Moreover, Nicod is the first to interpret regions as regular closed sets of a topological space.

De Laguna proposes an improvement of Whitehead's idea by considering the binary relation 'region x can connect regions y and z' which he uses to define the relation 'region x is part of region y'. De Laguna gives the relation 'region x can connect regions y and z' the following meaning: after some translation and rotation of region x, x has at least one point in common with y and at least one point in common with z. Obviously, this primitive relation conveys more than purely topological information; if x can connect regions y and z then the maximal "diameter" of x is smaller or equal to the minimal distance of y and z. Nevertheless, de Laguna's paper can be seen as the birth of mereotopology, since for the first time mereological and topological notions are employed together.

Following Whitehead, de Laguna defines points, lines and surfaces as abstractive elements. De Laguna can ensure by employing the can-connect-relation that abstractive sets converge to a point. It is worth pointing out that lines and surfaces in de Laguna's construction are not defined as sets of points but that points, lines and surfaces belong to the same class of entities. Since de Laguna's primitive relation carries topological as well as geometrical information, he is able to define geometrical concepts such as collinearity. However, he does not provide a complete formalisation of geometry.

Nicod takes an approach which is very similar to de Laguna's but employs a purely topological relation to define points.

Whitehead (1929) reviews his earlier work and employs a single primitive binary relation with the intended meaning 'region x is externally in contact with region y'. However, he does not strive for a complete formalisation of geometry in terms of regions and gives a long list of assumptions which are to capture his intuition about regions. This list of assumptions is inconsistent as Clarke (1981) points out.

Grzegorczyk (1960) reconstructs points from regions using a single primitive relation 'x and y are separated'. Biacino and Gerla (1996) show that Grzegorczyk's and Whitehead's definitions of point are equivalent. Clarke (1985) extends his earlier work (Clarke, 1981, cf. section 2.4.1) to reconstruct points. However, his definition of point causes the topological relation of contact and the mereological relation of overlap to coincide, thereby reducing his system to mereology (Biacino and Gerla, 1991). Probably the most famous reconstruction of points is that by Tarski (1956), who defines points as sets of concentric spheres. Huntington (1913) also employs spheres as his primitive entities. His points, however, are spheres which do not contain any other spheres. Therefore, points are primitive entities in his system and do not have to be reconstructed. Eschenbach (1994) bases her definition of point on the primitive mereological predicate 'x and y are discrete' and a primitive topological predicate 'x is a region'. Again points are not constructed but are primitive entities which do not have proper parts.

The most recent work on the reconstruction of points is done by Roeper (1997). He uses two primitive relations: the mereotopological primitive relation 'region x and region y are in contact' and a topological primitive relation 'region x is limited'. Roeper identifies points with sets of ultrafilters in the Boolean algebra of regular closed sets in a locally compact Hausdorff space.

Although not directly related to mereotopology, it is worth pointing out that the reconstruction of Euclidean geometry based on regions has been developed further in physics. Schmidt (1979) employs the primitives 'region', 'inclusion' and 'transport' to construct a structure isomorphic to Euclidean three-dimensional space. Regions are taken to be rigid bodies, inclusion is the mereological notion between regions, and 'transport' describes the mappings formed by translations and rotations. For a summary of (Schmidt, 1979) see (Gerla, 1995).

2.8 Fields related to mereotopology

2.8.1 Tame topology and o-minimal structures

In 1984, Grothendieck wrote in §5 of his "Esquisse d'un programme" (Schneps and Lochak, 1997, p. 29 of the French original):

> ... "general topology" was developed ... by analysts and in order to meet the needs of analysis, not for topology per se, i.e. the study of the topological properties of the geometrical shapes. ... Even now, just as in the heroic times when one anxiously witnessed for the first time curves cheerfully filling squares and cubes, when one tries to do topological geometry in the technical context of topological spaces, one is confronted at each step with spurious difficulties related to wild phenomena.

Grothendieck outlined how a field of *tame topology* might be developed which is free from wild phenomena. He recognised the semi-algebraic, algebraic and sub-analytic sets (see e.g. Bochnak et al., 1998; Shiota, 1997) as examples of tame topological structures. These structures are tame in the sense that any semi-algebraic (algebraic, sub-analytic) set is the finite union of 'simpler' sets. Model-theorists have recently become interested in tame topology (van den Dries, 1996, 1998), since some model-theoretic structures have been identified whose definable sets form tame topological structures. For example, the real semi-algebraic sets are exactly the sets which are definable with parameters over $\mathbb{R}$ in the language of the real closed field. Similar results have been obtained for certain model-theoretic structures which have been called o-minimal (Knight et al., 1986; Pillay and Steinhorn, 1986; Pillay, 1987). A model-theoretic structure defined over a (usually dense) linear order R is called *o-minimal* (= order-minimal) if every subset of R definable with parameters is the finite union of open intervals and points of R.

This thesis is not concerned with tame topology or o-minimal structures as such. However, there is a definite link between mereotopology in the context of AI and tame topology in that mereotopologists in AI explicitly consider spatial structures that do not exhibit wild phenomena. Consequently, I will make heavy use of results in tame topology and, more specifically, semi-algebraic geometry.

2.8.2 Graph theory

There are a number of correspondences between graphs and arrangements of regions in space. Consequently, some satisfiability problems in mereotopology are equivalent to satisfiability problems in graph theory. As an example, consider the following satisfiability problem. Assume that $\mathrm{O}(x, y)$ stands for 'region x overlaps region y'. The question is whether there exist open-disc homeomorphs $r_1, \ldots, r_4, s_1, s_2, s_3$ in the real plane such that $\mathrm{O}(r_i, r_{i+1})$ $(1 \leq i \leq 3)$, $\mathrm{O}(r_4, r_1)$, not $\mathrm{O}(r_1, r_3)$, not $\mathrm{O}(r_2, r_4)$, $\mathrm{O}(r_i, s_j)$ $(1 \leq i \leq 4, 1 \leq j \leq 2)$, not $\mathrm{O}(s_i, s_j)$ $(1 \leq i < j \leq 3)$, $\mathrm{O}(s_3, r_1)$, $\mathrm{O}(s_3, r_3)$, not $\mathrm{O}(s_3, r_2)$ and not $\mathrm{O}(s_3, r_4)$. This sort of problem is known in graph theory as *string graph problem* since the description can be represented by a planar embedding of a graph where the vertices represent the regions and an edge joins two vertices if and only if the corresponding regions overlap. Recognising string graphs has been shown to be NP-hard (Kratochvíl, 1991).

Chen et al. (Grigni et al., 1995; Papadimitriou, 1997; Chen et al., 1998a,b) consider a problem that is slightly different from the string graph problem. They introduce the concept of a planar map graph. A planar map graph represents the overlap relation of *closed*-disc homeomorphs with disjoint interior in the plane. Chen et al. show that a graph is a planar map graph if and only if it is the half-square of a planar bipartite graph where a square of a planar graph Γ is a graph Γ^2 that has the same set of vertices as Γ and two vertices are linked by an edge in Γ^2 if there is a path of length two between the two vertices in Γ. A half square of a planar bipartite graph is simply the square of the graph restricted to one of the two partitions of vertices. It follows from standard results that the recognition problem for planar map graphs is in NP.

In this thesis, the complexity of satisfiability problems plays a minor role. However, correspondences between graphs and arrangements of regions in the real plane will be exploited in several ways.

2.8.3 Spatial reasoning in modal and higher order logics

Most mereotopological approaches are formulated in first-order logic. Unfortunately, a large number of first-order mereotopological theories have been shown to be undecidable (see section 2.6). Modal logics offer a chance to obtain decidable or even tractable mereotopological or spatial theories. Modal logics which attempt to capture spatial notions have been investigated by Lemon and Pratt (1997, 1999). They show that the modal logics of Rescher and Garson (1968), von Wright (1979), Jeansoulin and Mathieu (1995) and Bennett (1996b) do admit models which cannot be called spatial. Thus, in a sense these logics are not "spatially complete".

An interesting spatial modal logic is presented by Dabrowski et al. (1996). They extend the work of Moss and Parikh (1992) and introduce a bi-modal language for spatial representation and reasoning. Formal models are constructed from subset frames which are pairs $\mathcal{X} = \langle X, \mathcal{O} \rangle$ where X is a set of points and $\mathcal{O}$ is a set of non-empty subsets of X. The modal logic *topologic* is introduced. It is shown that *topologic* has the finite model property and therefore is decidable. *Topologic* is expected to be strong enough to support elementary topological reasoning.

Flum and Ziegler (Flum and Ziegler, 1980; Ziegler, 1985) interpret a monadic second-order language $\mathcal{L}_2$ over topological structures that are tuples $(\mathfrak{A}, \sigma)$ where $\mathfrak{A}$ is an $\mathcal{L}$-structure with domain A and $\sigma \subseteq \wp(A)$ is a topology on A. Although central model-theoretic theorems such as the compactness theorem, the completeness theorem and the Löwenheim-Skølem theorem hold for monadic second-order logic (interpreted as a two-sorted logic), they fail for $\mathcal{L}_2$ interpreted only over topological structures. Although there is no $\mathcal{L}_2$-formula ϕ such that $(\mathfrak{A}, \sigma) \models \phi$ if and only if σ is a topology on A, there exists a formula ϕ_{bas} such that $(\mathfrak{A}, \sigma) \models \phi_{bas}$ if and only if σ is the basis of a topology on A. Flum and Ziegler restrict $\mathcal{L}_2$ to a monadic second-order language $\mathcal{L}_t$ such that every sentence of $\mathcal{L}_t$ holds in a topological structure if and only if it holds with respect to some basis of the topology of the structure. Then compactness, completeness and Löwenheim-Skølem theorem hold for $\mathcal{L}_t$ interpreted only over topological structures.

2.9 Conclusion

In this chapter, I have elaborated on the issues in mereotopology and presented their treatment in the literature. It turns out that a rigorous definition and investigation of well-behaved collections of regions, i.e. spatial domains, has been neglected. The majority of publications in mereotopology are concerned with the invention of axioms to capture some properties of regions. However, only a minority of mereotopological theories are shown to be consistent and complete.

Chapter 3

Planar spatial domains and their properties

The goal of this chapter is to identify a spatial domain that on the one hand is well-behaved and on the other hand provides regions that are sufficient for a common-sense representation of objects in (two-dimensional) space. First, I will introduce a spatial domain that is less well-behaved. Then, I will restrict the domain further and further to achieve the goal.

3.1 The spatial domain of regular open sets

In the previous chapter, I discussed the ontological advantages of boundary-free spatial domains. Open sets can have "cracks" and closed sets can have "spikes" as depicted in figure 3.1(a) and (b). The set of regular open sets provides a representation of regions which do not differ with respect to their boundaries. A *regular open set* of a topological space is identical to the interior of its closure. A *regular closed set* is identical to the closure of its interior. Therefore, regular open and regular closed sets have neither "cracks" nor "spikes". Moreover, no two regular open (closed) sets differ only with respect to their boundaries. Thus, the regular open as well as the regular closed sets provide boundary-free spatial domains. In this section, I will investigate the spatial domain of regular open sets.

Remember that the closure of a set u is denoted by $[u]$, its interior by u°, its boundary by $\partial(u)$ and its set-theoretic complement by $\overline{u}$.

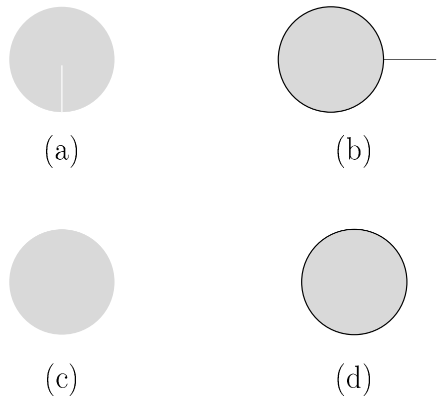


Figure 3.1: Open sets (left) and closed sets (right) in $\mathbb{R}^2$

Theorem 3.1.1 (Koppelberg (1989), Theorem 1.37). *Let X be a topological space and $RO(X)$ be the regular open sets of X. Then $RO(X)$ is a complete Boolean algebra under set-theoretical inclusion. The distinguished bottom-element 0 and top-element 1 and the operations $+, \cdot$ and $-$ for join, meet and complement respectively are given by*

$$0 = \emptyset, \qquad 1 = X,$$
$$u + v = [u \cup v]^\circ, \quad u \cdot v = u \cap v, \quad -u = (X \setminus u)^\circ$$

where $u, v \in RO(X)$. Furthermore, for $M \subseteq RO(X)$ the least upper bound, $\sum M$, is given by $[\bigcup M]^\circ$ and the greatest lower bound, $\prod M$, is given by $[\bigcap M]^\circ$.

The join will here be called *sum* and the meet will be called *product*. While the product of two regular open sets is simply their intersection, the sum of two regular open sets is, very roughly, the union of the sets where cracks are filled up. Figure 3.2 depicts three examples of sums.

The remainder of this section is concerned with the properties of regular open sets of a topological space X, and more specifically of $\mathbb{R}^2$. The following lemma will be used without further mention.

Lemma 3.1.2. *Let X and Y be topological spaces and $h : X \to Y$ be a homeomorphism. Let $u, v \in RO(X)$. Then $h(u) \in RO(Y)$, $h(-u) = -h(u)$, $h(u+v) = h(u) + h(v)$ and $h(u \cdot v) = h(u) \cdot h(v)$.*

Proof. Since for any set $u \subseteq X$, $h([u]) = [h(u)]$, $h(u^\circ) = h(u)^\circ$ and $h(\overline{u}) = \overline{h(u)}$, the lemma is straightforward. □

The following definition introduces a central notion in topology: connectedness.

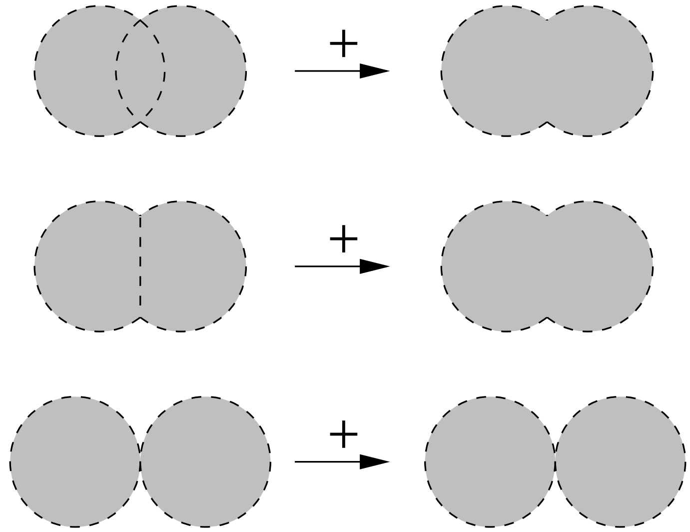


Figure 3.2: Three pairs of regular open sets and their sums

Definition 3.1.3. Let X be a topological space and $u \subseteq X$. Two non-empty disjoint sets $v_1, v_2 \subseteq X$ form a *separation* of u if $u = v_1 \cup v_2$. A subset u of X is *connected* if for all separations $v_1, v_2 \subseteq X$ of u either $[v_1] \cap v_2 \neq \emptyset$ or $v_1 \cap [v_2] \neq \emptyset$. A maximal connected subset of $u \subseteq X$ is said to be a *component* of u. The topological space X is *locally connected* if for every point $p \in X$ and every open neighbourhood u of p, there exists a connected open neighbourhood of p lying in u.

Lemma 3.1.4. *A component of a regular open set in a locally connected topological space X is regular open.*

Proof. Let v be a regular open set in X and let u be a component of v. It is a standard result that u is open (Armstrong, 1979, Chapter 3, Exercise 26). Then $u \subseteq [u]^\circ \subseteq [u]$. Since u is connected, $[u]^\circ$ is connected (see e.g. Newman, 1964, Chapter IV, Theorem 1.2). Since $u \subseteq [u]^\circ \subseteq [v]^\circ = v$ and u is a maximally connected subset of v, $u = [u]^\circ$ as required. □

Since the attention is restricted here to regular open sets, it is a helpful result that connectedness of a regular open set can be defined referring to regular open sets only.

Lemma 3.1.5. *Let X be a locally connected topological space. A non-empty regular open set $u \subseteq X$ is connected if and only if for every non-empty regular open sets $v_1, v_2 \subseteq X$ with $u = v_1 + v_2$, $u \cap [v_1] \cap [v_2] \neq \emptyset$.*

Proof. Assume u is disconnected. Let v_1 be a component of u. By lemma 3.1.4, v_1 is regular open. Let $v_2 = r \cdot (-v_1)$. Then (v_1, v_2) is a separation of u and $v_1 \cap [v_2] = \emptyset$. Since v_1 is a maximal connected subset of u, $u \cap [v_1] = v_1$. Hence, $u \cap [v_1] \cap [v_2] = \emptyset$.

Conversely, assume u is connected. Let $v_1, v_2 \subseteq X$ be regular open sets such that $u = v_1 + v_2$. Assume $v_1 \cap v_2 = \emptyset$, otherwise trivially $u \cap [v_1] \cap [v_2] \neq \emptyset$. Certainly, $u = v_1 \cup v_2 \cup (\partial(v_1) \cap u)$, whence the pair $(u \cap [v_1], v_2)$ forms a separation of u. Since u is connected and $[u \cap [v_1]] \cap v_2 \subseteq [u] \cap [v_1] \cap v_2 = [v_1] \cap v_2 = \emptyset$, $(u \cap [v_1]) \cap [v_2] \neq \emptyset$. □

Lemma 3.1.6. *Let X be a topological space and $u, v_1, v_2 \subseteq X$ be regular open sets such that u is non-empty and $u + v_1$ and $u + v_2$ are connected. Then $u + v_1 + v_2$ is connected.*

Proof. By (Newman, 1964, Chapter IV, Theorem 1.5), $(u + v_1) \cup (u + v_2)$ is connected. Since $(u + v_1) \cup (u + v_2) \subseteq [(u + v_1) \cup (u + v_2)]^\circ = u + v_1 + v_2 \subseteq [(u + v_1) \cup (u + v_2)]$, by (Newman, 1964, Chapter IV, Theorem 1.2), $u + v_1 + v_2$ is connected. □

Now I will concentrate on the Euclidean space $\mathbb{R}^2$. Let the set of regular open sets in $\mathbb{R}^2$ be denoted by $\mathbf{F}$ ($\mathbf{F}$ standing for *full* domain). The set $\mathbf{F}$ will be called the *spatial domain of regular open sets.* The spatial domain $\mathbf{F}$ provides regions of simple shape such as rectangles and discs. Moreover, $\mathbf{F}$ contains regions whose shape is extravagant and which, therefore, have properties which common sense would not expect from real-world objects.

Consider a regular open set r in the real plane whose boundary consists in part of a variation of the topologist's sine curve. An example is given in figure 3.3 where the oscillating curve is defined by $f(x) := \sin(10\pi/x)/(1+x)$ for $0 < x \leq \pi$. Consider the rectangular regular open set $u = (-\frac{1}{4}\pi, \pi) \times (-1, 1)$. The curve $f((0, \pi])$ splits the regular open set u into the regular open sets r, s and t as shown. The set $r + s + t = [r \cup s \cup t]^\circ = u$ is connected. However, neither $r + t$ nor $s + t$ is connected! Clearly, this somewhat astonishing result depends on the shape of the boundaries. One property which is closely related to the shape of a boundary is accessibility:

Definition 3.1.7. A boundary point p of a set $u \subseteq \mathbb{R}^n$ is *accessible (from u)* if there is a path $\gamma : [0, 1] \to u \cup \{p\}$ such that $\gamma([0, 1)) \subseteq u$ and $\gamma(1) = p$. The

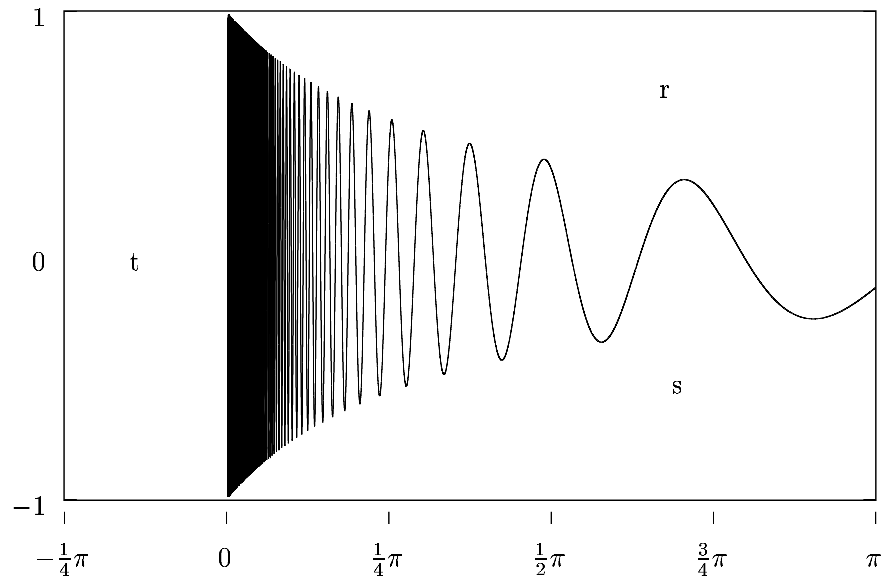


Figure 3.3: Three regions bounded by the topologist's sine curve

boundary of u is accessible (from u) if every boundary point of u is accessible (from u).

For instance, the points $\{0\} \times [-1, 1]$ on the y-axis in figure 3.3 belong to the boundary of r but they are not accessible from r. A large number of results of this thesis will depend on the accessibility of boundaries.

The region depicted in figure 3.4 shows that there are connected regular open sets which have boundary points that are neither accessible from the region itself nor from its complement. In this case, the region engulfs the non-accessible part of its boundary except for one point. Although the region is regular open there are boundary parts that cannot come into contact with any ball of arbitrary size which is disjoint from the region. In this respect, the region has similarities to open regions with "cracks". Thus, the region is not suitable for a representation of everyday objects.

It was shown in lemma 3.1.5 that the sum of two disjoint connected regular open sets r and s is connected if and only if r and s share a boundary point p which is an interior point of $r + s$. The question is whether at least one such point p must be accessible from r as well as s. The following example shows that this is not the case.

As in a construction of the middle third Cantor set on $[0, 1]$, let $E_0 = (\frac{1}{3}, \frac{2}{3})$, $E_1 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, $E_2 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27})$ etc. Since $\mathbf{F}$ is

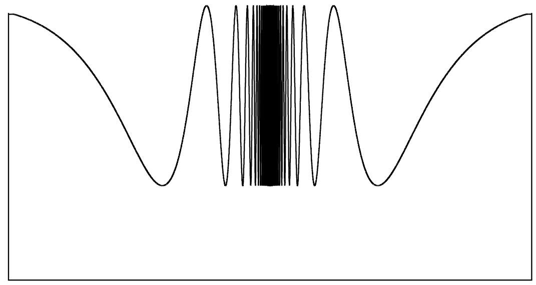


Figure 3.4: A connected regular open set sharing non-accessible boundary points with its complement

a complete Boolean algebra, $r = ((0,1) \times (2,2.5)) + \sum_{n\in\mathbb{N}}(E_{2n} \times (-1,2))$ and $s = ((0,1) \times (-2.5,-2)) + \sum_{n\in\mathbb{N}}(E_{2n+1} \times (-2,1))$ are regular open sets. The three initial steps of their development are shown in figure 3.5. The regions r and s will be called the *interlinked Cantor combs.* Certainly, r and s are connected. Furthermore, $r + s$ is connected, although r and s do not share any accessible boundary point.

3.1.1 Regular open sets with accessible boundaries

The next lemmas show that connected regular open sets in $\mathbb{R}^2$ with accessible boundaries have especially pleasing properties. Two new notions are required. Given a connected open set u and two boundary points p and q of u, an arc γ in $u \cup \{p\}$ with $\gamma(1) = p$ is called an *end-cut in u.* An arc γ in $u \cup \{p,q\}$ with $\gamma(0) = p$ and $\gamma(1) = q$ is called a *cross-cut in u.*

Lemma 3.1.8. *Let $r_1, r_2, r_3 \subseteq \mathbb{R}^2$ be mutually disjoint connected regular open sets with accessible boundaries. Then $[r_1] \cap [r_2] \cap [r_3]$ contains at most two points.*

Proof. Assume p_1, p_2 and p_3 are distinct points of $[r_1] \cap [r_2] \cap [r_3]$. Let q_j be a point in r_j $(1 \leq j \leq 3)$. Since the boundaries of r_1, r_2 and r_3 are accessible and the regions are connected, there exist end-cuts $\gamma_{i,j}$ $(1 \leq i \leq j \leq 3)$ from q_j to p_i in r_j. Since r_1, r_2 and r_3 are mutually disjoint, the arcs $\gamma_{i,j}$ form a plane embedding of the non-planar graph $K_{3,3}$ which is impossible. Therefore, $[r_1] \cap [r_2] \cap [r_3]$ contains at most two points. □

Note that this lemma fails for regular open sets in general as the example in figure 3.3 shows.

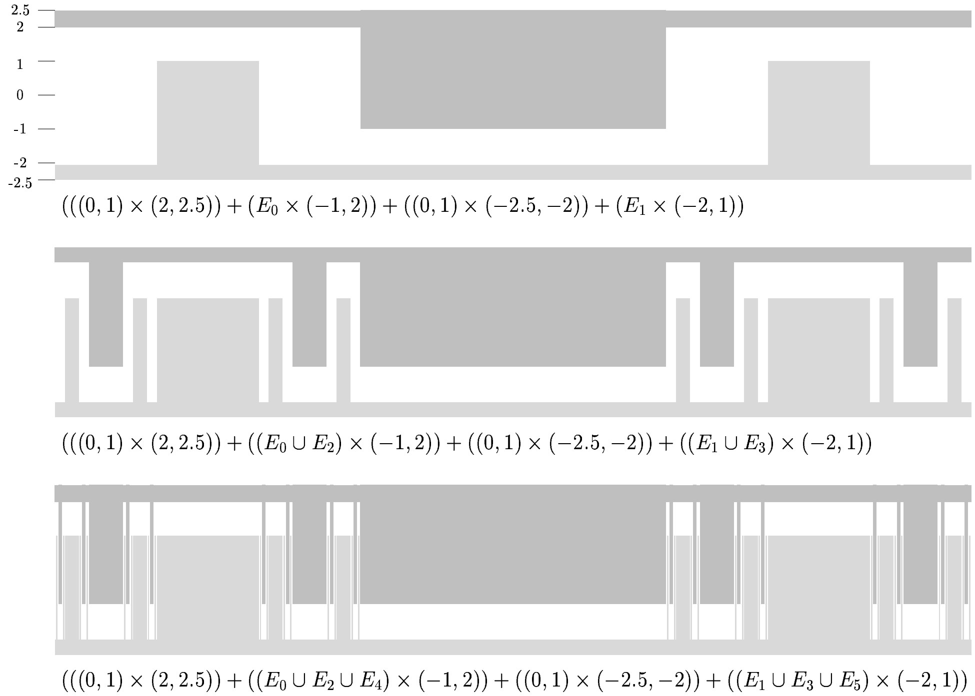


Figure 3.5: Three stages in the development of interlinked Cantor combs

Lemma 3.1.9. *Let $r, s_1, s_2 \subseteq \mathbb{R}^2$ be mutually disjoint regular open sets with accessible boundaries such that r has n components ($n \geq 1$), and s_1 and s_2 are connected. Then $||\partial(s_1) \cap \partial(s_2) \cap (r + s_1 + s_2)|| \leq n$.*

Proof. Suppose p and q are distinct points of $\partial(s_1) \cap \partial(s_2) \cap (r + s_1 + s_2)$. Since s_1 and s_2 are connected and have accessible boundaries, there are cross-cuts γ_1 and γ_2 from p to q in s_1 and s_2 respectively. Thus, $\gamma_1 \cup \gamma_2$ is a Jordan curve defining the open halves d_1 and d_2 of the plane. Since $s_1 + s_2$ is disconnected, it follows from lemma 3.1.5 that $p, q \in \partial(s_1 + s_2) \cap \partial(r)$.

If $n = 1$ then r lies in d_1 or d_2. Hence, p and q are not interior points of $r + s_1 + s_2$ contradiction the assumption. Hence, $||\partial(s_1) \cap \partial(s_2) \cap (r + s_1 + s_2)|| \leq 1$.

If $n > 1$ then note that by lemma 3.1.8 for each component t of r, $||\partial(t) \cap \partial(s_1) \cap \partial(s_2)|| \leq 2$. Moreover, if $p \in \partial(s_1) \cap \partial(s_2) \cap \partial(t)$ then by the above argument t lies in d_1, say. Since the boundary of r is accessible, some other component t' of r lies in d_2 such that $p \in \partial(t')$. Hence, $||\partial(s_1) \cap \partial(s_2) \cap (r + s_1 + s_2)|| \leq n$. □

Lemma 3.1.10. *Let $r_1, \ldots, r_n \subseteq \mathbb{R}^2$ ($n \geq 2$) be connected regular open sets with accessible boundaries such that $r_1 + \ldots + r_n$ is connected. Then for some i with $2 \leq i \leq n$, $r_1 + r_i$ is connected.*

Proof. If $r_1 \cdot r_i \neq \emptyset$ for some $i \in \{2, \ldots, n\}$ then by lemma 3.1.6, $r_1 + r_i$ is connected. Suppose $r_1 \cdot r_i \neq \emptyset$ and $r_1 + r_i$ is disconnected for $i \in \{2, \ldots, n\}$. By lemma 3.1.9, $||\partial(r_1) \cap \partial(r_i) \cap (r_1 + \ldots + r_n)|| \leq n - 2$ for each $i \in \{2, \ldots, n\}$. Hence, $||\partial(r_1) \cap (r_1 + \ldots + r_n)|| \leq (n-2)(n-1)$ contradicting the density of (accessible) boundary points (see e.g. Newman, 1964, p. 162). □

A correspondence between finite collections of connected regular open sets with accessible boundaries and finite graphs will be established in the next lemmas.

Definition 3.1.11. Given non-empty connected regions $r_1, \ldots, r_n \in \mathbf{S}$, the *binary connection graph* $\Gamma = (V, E)$ *on* $r_1, \ldots, r_n$ is given by the vertex set $V = \{r_1, \ldots, r_n\}$ and the edge set $E = \{\{r_i, r_j\} | 1 \leq i < j \leq n, r_i + r_j \text{ is connected}\}$.

The following graph theoretical result will be used in the sequel.

Proposition 3.1.12 (Diestel (1997), Proposition 1.4.4). *Given a connected graph* $\Gamma = (V, E)$*, there is a vertex* $v \in V$ *such that* $\Gamma \setminus \{v\}$ *is connected.*

Lemma 3.1.13. *Let* $r_1, \ldots, r_n \subseteq \mathbb{R}^2$ $(n \geq 1)$ *be non-empty connected regular open sets with accessible boundaries. Then* $r_1 + \ldots + r_n$ *is connected if and only if the binary connection graph on* $r_1, \ldots, r_n$ *is connected.*

Proof. Assume that the binary connection graph Γ on $r_1, \ldots, r_n$ is connected. I show by induction over n that $r_1 + \ldots + r_n$ is connected. For $n = 1$ the lemma holds trivially. Let $n > 1$. By proposition 3.1.12, for some i with $1 \leq i \leq n$, $\Gamma \setminus \{r_i\}$ is connected. Then by induction hypothesis $\sum_{1 \leq j \leq n, j \neq i} r_j$ is connected. Since Γ is connected, for some $k \neq i$, $\{r_i, r_k\} \subseteq E$ and thus $r_i + r_k$ is connected. Hence, by lemma 3.1.6, $r_1 + \ldots + r_n$ is connected.

Conversely, assume $r_1 + \ldots + r_n$ is connected. I show that, after renumbering $r_1, \ldots, r_n$ if necessary, the binary connection graph on $r_1, \ldots, r_i$ is connected for each $i \in \{1, \ldots, n\}$. I proceed by induction over i.

If $i = 1$ then then the hypothesis is trivially true. Let $i > 1$. By induction hypothesis the binary connection graph on $r_1, \ldots, r_{i-1}$ is connected. By the first part of this proof, $r_1 + \ldots + r_{i-1}$ is connected. Since $(r_1 + \ldots + r_{i-1}) + r_i + \ldots + r_n$ is connected, it follows from lemma 3.1.10 that for some r_j $(i \leq j \leq n)$, r_i say, $(r_1 + \ldots + r_{i-1}) + r_i$ is connected. Again by lemma 3.1.10, $r_i + r_j$ is connected for some r_j $(1 \leq j \leq i-1)$, r_1 say. Hence, $\{r_1, r_i\}$ is an edge in the binary connection graph on $r_1, \ldots, r_i$ which therefore is connected. □

Lemma 3.1.14. *Let $r_1, \ldots, r_n \subseteq \mathbb{R}^2$ be connected regular open sets with accessible boundaries such that $r_1 + \ldots + r_n$ is connected. Then, after renumbering if necessary, $r_2 + \ldots + r_n$ is connected.*

Proof. Let Γ be the binary connection graph on $r_1, \ldots, r_n$. By proposition 3.1.12, there exists i, $1 \leq i \leq n$, such that $\Gamma \setminus \{r_i\}$ is connected. Then by lemma 3.1.13 $\sum_{j=1, j\neq i}^{n} r_j$ is connected. □

Lemma 3.1.15. *Let $r_0, r_1, \ldots, r_n \subseteq \mathbb{R}^2$ be mutually disjoint connected non-empty regular open sets with accessible boundaries. Moreover, assume that the complements of $r_0, \ldots, r_n$ are non-empty and connected, and that $r_0 + \ldots + r_n = \mathbb{R}^2$. Then, after renumbering if necessary, $r_2 + \ldots + r_n$ is connected and has a connected complement.*

Proof. By lemma 3.1.10, there is r_i, r_1 say, such that $r_0 + r_1$ is connected. If $-(r_0 + r_i)$ is connected, the lemma holds. If $-(r_0 + r_i)$ is disconnected, let s_1 be a component of $-(r_0 + r_i)$. Then $r_0 + r_j$ is connected for some $r_j \subseteq s_1$. If $-(r_0 + r_j)$ is connected, we are done. Otherwise let s_2 be a component of $-(r_0 + r_j)$ such that $r_i \not\subseteq s_2$. Hence, $s_2 \subset s_1$. Proceeding in this way, the finiteness of $r_0, r_1, \ldots, r_n$ guarantees that one can find some r_k such that $r_0 + r_k$ and $-(r_0 + r_k)$ is connected. Then, after renumbering if necessary, $-(r_0 + r_k) = r_2 + \ldots + r_n$. □

The above lemmas show that regular open sets with accessible boundaries are well-behaved. However, the components of the intersection of two connected regular open sets with accessible boundaries do not necessarily have an accessible boundary (see figure 3.6). The next section makes an attempt to identify a well-behaved Boolean subalgebra of regular open sets where connected regions have accessible boundaries.

3.2 A spatial domain constructed from Jordan regions

For the introduction of the spatial domain of this section and for some proofs later on it will be convenient to consider spatial domains over compact topological spaces. Therefore, the *one-point compactification* (X^*, τ^*) of a topological space (X, τ) is introduced as follows. Let p_∞ be a point not in X. The point p_∞ is called *the point at infinity.* Let $X^* = X \cup \{p_\infty\}$ and declare $u \subseteq X^*$ as open,

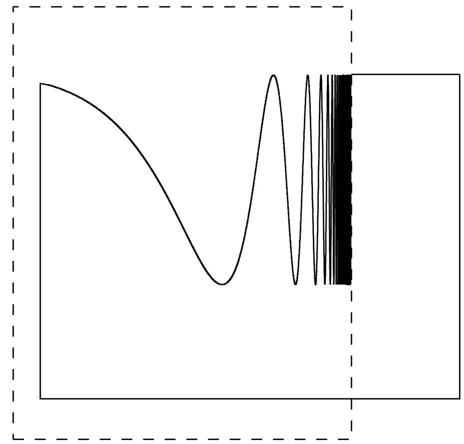


Figure 3.6: The intersection of two regular open sets with accessible boundaries

i.e. $u \in \tau^*$, if $u \in \tau$ or u is the complement in X^* of a set which is closed and compact in X. Then it is easy to see that for a locally compact space (X, τ), i.e. every point of X has a compact neighbourhood, (X^*, τ^*) is a compact space.

The real plane $\mathbb{R}^2$ is also called the *open plane* in contrast to its one-point compactification $(\mathbb{R}^2)^*$ which is known as the *closed plane*. Occasionally, I will make use of the standard result that $(\mathbb{R}^2)^*$ is homeomorphic to the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ where the homeomorphism is induced by stereographic projection (see e.g. Armstrong, 1979, Section 3.3, Problem 18). In the sequel, I write $\widetilde{\mathbb{R}^2}$ to refer indifferently to $\mathbb{R}^2$ or $(\mathbb{R}^2)^*$. The following proposition justifies the extension of the *-notation to any Boolean subalgebra of $RO(\mathbb{R}^2)$.

Proposition 3.2.1. *The function* $^*\colon RO(\mathbb{R}^2) \to RO((\mathbb{R}^2)^*)$ *with*

$$u^* = \begin{cases} u \cup \{p_\infty\} & \text{if } \mathbb{R}^2 \setminus u \text{ is compact in } \mathbb{R}^2 \\ u & \text{otherwise} \end{cases}$$

defines a Boolean algebra isomorphism such that for $u, v \in RO(\mathbb{R}^2)$, $u \subseteq v$ *if and only if* $u^* \subseteq v^*$, *and* u *is connected if and only if* u^* *is connected. Moreover,* u *is bounded if and only if* $p_\infty \notin [u^*]$.

Proof. Let $u \in RO(\mathbb{R}^2)$. Let w be the closure of u in $\mathbb{R}^2$ and let w_* be the closure of u in $(\mathbb{R}^2)^*$. Since every open neighbourhood of a point $p \in \mathbb{R}^2$ in $(\mathbb{R}^2)^*$ contains an open neighbourhood of p in $\mathbb{R}^2$, $w \subseteq w_*$.

Assume u is bounded. Then w is compact in $\mathbb{R}^2$. Hence, $(\mathbb{R}^2)^* \setminus w$ is open in $(\mathbb{R}^2)^*$. Thus, w is closed in $(\mathbb{R}^2)^*$, whence $w_* = w$. Since $\mathbb{R}^2 \setminus u$ is not compact, $u^* = u$. Hence $[u^*]^\circ = w^\circ = u = u^*$.

Assume u is unbounded and $\mathbb{R}^2 \setminus u$ is compact. Then $u^* = u \cup \{p_\infty\}$ is open in $(\mathbb{R}^2)^*$. Since p_∞ is an interior point of $u \cup p_\infty$, p_∞ is a boundary point of u in $(\mathbb{R}^2)^*$. Hence, $w_* = w \cup \{p_\infty\}$, whence $[u^*]^\circ = (w \cup \{p_\infty\})^\circ = u \cup \{p_\infty\} = u^*$.

Assume u is unbounded and $\mathbb{R}^2 \setminus u$ is not compact in $\mathbb{R}^2$. Then $-u$ is unbounded and $u \cup \{p_\infty\}$ is not open in $(\mathbb{R}^2)^*$. Since $[u] = \mathbb{R}^2 \setminus -u$ is not compact, $-u \cup \{p_\infty\}$ is not open in $(\mathbb{R}^2)^*$. However, u and $-u$ are open in $(\mathbb{R}^2)^*$, whence p_∞ is a boundary point of u and $-u$. Hence, $w_* = w \cup \{p_\infty\}$. Since $u^* = u$, p_∞ is an exterior point of u^*, whence $[u^*]^\circ = (w \cup \{p_\infty\})^\circ = u = u^*$.

Thus, the function * is well-defined. Note that the topology $\mathbb{R}^2$ is a subspace topology of $(\mathbb{R}^2)^*$ induced by $\mathbb{R}^2$. Hence, if u is regular open in $(\mathbb{R}^2)^*$, $u \setminus \{p_\infty\}$ is regular open in $\mathbb{R}^2$. Since in this case, $(u \setminus p_\infty)^* = u$, the function * is surjective.

Let $u, v \in RO(\mathbb{R}^2)$. It has to be shown that $(-u)^* = -u^*$, $(u \cdot v)^* = u^* \cdot v^*$ and $(u+v)^* = u^* + v^*$. Again let w be the closure of u in $\mathbb{R}^2$.

If u is bounded then $[u^*] = w$. Hence, $-u^* = (\mathbb{R}^2)^* \setminus [u^*] = (\mathbb{R}^2)^* \setminus w = (\mathbb{R}^2 \setminus w) \cup \{p_\infty\} = -u \cup \{p_\infty\} = (-u)^*$. If u is unbounded then $[u^*] = w \cup \{p_\infty\}$. Hence, $-u^* = (\mathbb{R}^2)^* \setminus [u^*] = (\mathbb{R}^2)^* \setminus (w \cup \{p_\infty\}) = \mathbb{R}^2 \setminus w = -u = (-u)^*$.

If $\mathbb{R}^2 \setminus u$ and $\mathbb{R}^2 \setminus v$ both are compact then $\mathbb{R}^2 \setminus u \cup \mathbb{R}^2 \setminus v = \mathbb{R}^2 \setminus (u \cap v)$ is compact. Then $u^* \cdot v^* = (u \cup \{p_\infty\}) \cap (v \cup \{p_\infty\}) = (u \cap v) \cup \{p_\infty\} = (u \cdot v)^*$. If $\mathbb{R}^2 \setminus u$ is not compact or $\mathbb{R}^2 \setminus v$ is not compact then $\mathbb{R}^2 \setminus u \cup \mathbb{R}^2 \setminus v = \mathbb{R}^2 \setminus (u \cap v)$ is not compact. Hence, $u^* \cdot v^* = (u \cdot v)^*$.

Since $u + v = -(-u \cdot -v)$, $(u+v)^* = u^* + v^*$.

It remains to show that $u \subseteq v$ iff $u^* \subseteq v^*$ and u is connected iff u^* is connected.

Assume $u \subseteq v$. If $\mathbb{R}^2 \setminus u$ is compact then $\mathbb{R}^2 \setminus v$ is compact. Hence, $u^* = u \cup \{p_\infty\} \subseteq v \cup \{p_\infty\} = v^*$. If $\mathbb{R}^2 \setminus u$ is not compact, then $u^* = u \subseteq v \subseteq v^*$. Assume $u^* \subseteq v^*$. Then $u^* \setminus \{p_\infty\} = u \subseteq v^* \setminus \{p_\infty\} = v$.

Assume u is connected. Let $v_1, v_2 \in RO(\mathbb{R}^2)$ such that $u = v_1 + v_2$. Then by lemma 3.1.5 $[v_1] \cap [v_2] \cap u \neq \emptyset$. Since $v_1 \subseteq v_1^*$, $v_2 \subseteq v_2^*$ and $u \subseteq u^*$, $[v_1^*] \cap [v_2^*] \cap u^* \neq \emptyset$. Hence, u^* is connected. Let u^* be connected and let $v_1^*, v_2^* \in RO((\mathbb{R}^2)^*)$ such that $v_1^* + v_2^* = u^*$. Since $[v_1^*] \cap [v_2^*] \cap u^*$ is an infinite set, $[v_1] \cap [v_2] \cap u = ([v_1^*] \cap [v_2^*] \cap u^*) \setminus \{p_\infty\} \neq \emptyset$. Hence, by lemma 3.1.5, u is connected. □

Given any Boolean subalgebra $(M, +, \cdot, -, \emptyset, \mathbb{R}^2)$ of $RO(\mathbb{R}^2)$, I write M^* to refer to the set $\{u^* | u \in M\}$ and I write $\widetilde{M}$ to refer indifferently to M or M^*. The following lemma will be used without further mention.

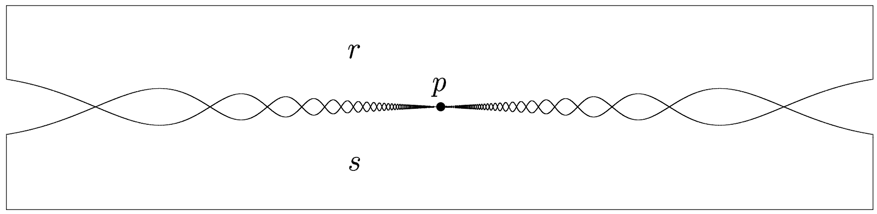


Figure 3.7: A region of the spatial domain $\widetilde{\mathbf{J}}$ with infinitely many components

Lemma 3.2.2. *Let X and Y be topological spaces and let $h\colon X \to Y$ be a homeomorphism. Then $h^*\colon X^* \to Y^*$ defined by*

$$h^*(x) = \begin{cases} p_\infty & \textit{if } x = p_\infty \\ h(x) & \textit{otherwise} \end{cases}$$

is a homeomorphism.

Proof. Certainly, h^* is bijective. Let U be an open set in X^*. For the continuity of $(h^*)^{-1}$, it has to be shown that $h^*(U)$ is open in Y^*. Either (i) U is open in X or (ii) $X \setminus U$ is closed and compact in X.

(i) If U is open in X then, since h^{-1} is continuous, $h(U)$ is open in Y. Hence, $h(U) = h^*(U)$ is open in Y^*.

(ii) If $X \setminus U$ is closed and compact in X then, since h is a homeomorphism, $h(X \setminus U)$ is closed and compact in Y. WLOG assume $p_\infty \in U$, otherwise (i) applies. Hence, $Y^* \setminus h(X \setminus U) = Y \setminus h(X \setminus U) \cup \{p_\infty\} = h(U \setminus \{p_\infty\}) \cup \{p_\infty\} = h^*(U)$ is open in Y^*. The continuity of h^* is shown by exchanging h and h^{-1}. □

Now I turn to the definition of a Boolean subalgebra of $RO(\mathbb{R}^2)$ which omits at least some pathological regular open sets. A set $u \subseteq \widetilde{\mathbb{R}^2}$ is said to be a *Jordan region* if it is bounded and its boundary is a simple closed curve. Let the spatial domain $\mathbf{J}^*$ be the set of finite sums of finite products of Jordan regions in $\mathbf{F}^*$. Then $\mathbf{J}^*$ is a Boolean subalgebra of $\mathbf{F}^*$. It follows from proposition 3.2.1 that the spatial domain $\mathbf{J} = \{u \setminus \{p_\infty\} | u \in \mathbf{J}^*\}$ is a Boolean subalgebra of $\mathbf{F}$.

Are the regions of $\widetilde{\mathbf{J}}$ well-behaved? It is easy to see that the domain $\widetilde{\mathbf{J}}$ contains regions with infinitely many components. Figure 3.7 shows two Jordan regions r and s whose boundaries are in part defined by the image of $\sin(\frac{1}{x})x$ and $-\sin(\frac{1}{x})x$ respectively. Then $r \cdot s$ has infinitely many components. Furthermore, the boundary point p of $r \cdot s$ is not accessible from $r \cdot s$. The example of the interlinked Cantor combs on page 57 showed that there are connected regions of $\widetilde{\mathbf{F}}$ whose sum

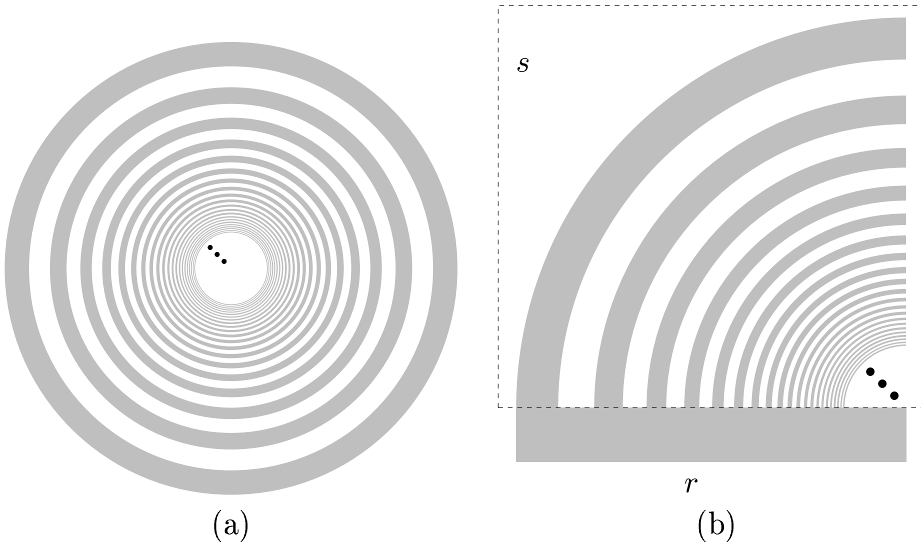


Figure 3.8: (a) A region of $\widetilde{\mathbf{J}}$ which shares one non-accessible boundary point with its complement; (b) the construction of the region in (a)

is connected but which nevertheless do not share any accessible boundary points. I conjecture that there are no such regions in the spatial domain $\widetilde{\mathbf{J}}$. However, there are regions in the spatial domain $\widetilde{\mathbf{J}}$ that have one non-accessible boundary point with their complement in common. An example of such region is shown in figure 3.8(a). The components of this region are an infinite number of concentric annuli with smaller and smaller radii. The region is constructed as the sum of four products of Jordan regions that are similar to regions r and s depicted in figure 3.8(b). It is not obvious that the region r depicted in figure 3.8(b) really is a Jordan region. The following notions and theorems help to prove that r is a Jordan region.

Definition 3.2.3. Let X be a metric space with metric ρ. A set $U \subseteq X$ is *locally connected* at the point p, if, given a positive ϵ, there exists a positive δ such that any two points of $U \cap B_\delta(p)$ lie in a connected set lying in $U \cap B_\epsilon(p)$ (cf. definition 3.1.3). The set U is *uniformly locally connected* if given a positive ϵ, there exists a positive δ such that all pairs of points p and q, that satisfy $\rho(p,q) < \delta$, are joined by a connected subset of the space of diameter less than ϵ, where the *diameter* of a set is the least upper bound of the distances between all pairs of points of the set.

Theorem 3.2.4 (Newman (1964), Chapter VI, Thrm 14.1). *All Jordan regions are uniformly locally connected.*

Theorem 3.2.5 (Newman (1964), Chap. VI, Thrm 13.1). *If the subset U of a metric space X is uniformly locally connected then U is locally connected at all points of $[U]$.*

Theorem 3.2.6 (Newman (1964), Chapter VI, Thrm 14.4). *If $U \subseteq X$ is locally connected at a point p of $\partial(U)$, then p is accessible from U.*

Corollary 3.2.7. *Jordan regions have accessible boundaries.*

Useful are also the following two converses of Jordan's curve theorem.

Theorem 3.2.8 (Newman (1964), Chapter VI, Thrm 16.1). *If a closed set in $(\mathbb{R}^2)^*$ has two connected complements from each of which it is accessible at every point, it is a simple closed curve.*

Theorem 3.2.9 (Newman (1964), Chapter VI, Theorem 16.2). *If a connected open set in $(\mathbb{R}^2)^*$ is uniformly locally connected, has a connected complement and a connected boundary, then its boundary is a simple closed curve, a point or the empty set.*

If is easy to see that region r in figure 3.8(b) is uniformly locally connected, has a connected complement and a connected boundary. By theorem 3.2.9, the boundary of region r is a simple closed curve. Hence r is a Jordan region and an element of $\mathbf{J}$.

The construction of regions with non-accessible boundary points can be taken further. Certainly there exists an element of $\mathbf{J}$ that has infinitely many non-accessible boundary points that have themselves a non-accessible boundary point as limit point. Therefore, the set of non-accessible boundary point is not necessarily discrete. However, I conjecture that connected elements of $\widetilde{\mathbf{J}}$ are well-behaved in the following sense.

Conjecture 3.2.10. *The boundary points of a connected region $r \in \widetilde{\mathbf{J}}$ are accessible from r.*

Since the boundary of a Jordan region is accessible by corollary 3.2.7, the above conjecture is true if the following conjecture can be proven.

Conjecture 3.2.11. *Every connected region of $\mathbf{J}^*$ is the sum of finitely many Jordan regions.*

The verification of the above conjecture would provide an alternative characterisation of Jordan regions:

Conjecture 3.2.12. *A non-empty bounded connected region of* $\widetilde{\mathbf{J}}$ *with non-empty connected complement is a Jordan region.*

Proof. Let $r \in \widetilde{\mathbf{J}}$ be a non-empty bounded connected set with non-empty connected complement. By conjecture 3.2.10, the closed set $\partial(r)$ is accessible from r and $-r$. Hence by theorem 3.2.8, $\partial(r)$ is a simple closed curve. Since r is bounded, r is a Jordan region. □

A further conjecture that will be of importance later on is:

Conjecture 3.2.13. *Let* $r \in \widetilde{\mathbf{J}}$ *and let* R *be a subset of the components of* r*. Then* $\sum R \in \widetilde{\mathbf{J}}$*.*

Although it is uncertain whether a component of a region in $\widetilde{\mathbf{J}}$ is an element of $\widetilde{\mathbf{J}}$, the following result shows that connectedness for regions of $\widetilde{\mathbf{J}}$ can be defined referring to regions in $\widetilde{\mathbf{J}}$ only.

Lemma 3.2.14. *A region* $r \in \widetilde{\mathbf{J}}$ *is connected if and only if for all non-empty regions* $s_1, s_2 \in \widetilde{\mathbf{J}}$ *with* $r = s_1 + s_2$, $r \cap [s_1] \cap [s_2] \neq \emptyset$*.*

Proof. The proof of the only-if-direction proceeds as for lemma 3.1.5. For the if-direction assume that r is disconnected. Certainly there exists a region $s \in \widetilde{\mathbf{J}}$ such that some but not all components of r lie in s. Then the pair $(r \cdot s, r \cdot -s)$ forms a separation of r and $(r \cdot s) \cap [r \cdot -s] = \emptyset$. Since $r \cdot s$ is the sum of components of r, $r \cap [r \cdot s] = r \cdot s$. Hence, $[r \cdot s] \cap [r \cdot -s] \cap r = \emptyset$. □

This section discussed the Jordan domain $\widetilde{\mathbf{J}}$ as a possibly well-behaved spatial domain. However, regions in $\widetilde{\mathbf{J}}$ may have infinitely many components and therefore non-accessible boundary points. In the next section, I will introduce better behaved spatial domains. The main characteristic of these spatial domains will be that every region has only finitely many components and that every boundary point of a region is accessible from the region.

$\{(x,y) \in \mathbb{R}^2 | x^2/25 + y^2/16 < 1$ and $x^2 + 4x + y^2 - 2y > -4$ and $x^2 - 4x + y^2 - 2y > -4$ and $(x^2 + y^2 - 2y \neq 8$ or $y > -1)\}$

Figure 3.9: A semi-algebraic set in the plane (after Bochnak et al., 1998)

3.3 The spatial domain of regular open semi-algebraic sets

There are Boolean subalgebras of $\widetilde{\mathbf{J}}$ which are especially well-behaved. Two such subalgebras, the regular open semi-algebraic sets and the regular open semi-linear sets in the real plane, will be introduced in the sequel. I start off with the definition of semi-algebraic sets. Semi-linear sets will be introduces in section 3.4.

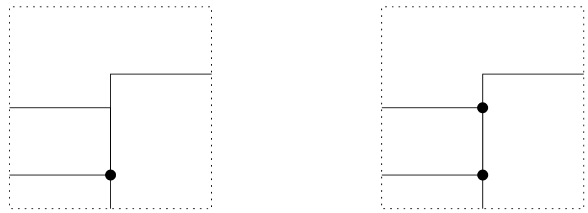
3.3.1 Semi-algebraic sets and their properties

The following definitions, propositions and theorems are adapted from (van den Dries, 1998) and (Bochnak et al., 1998) and will be employed in later chapters. The relevant proofs can be found in the references just mentioned.

Definition 3.3.1. The *semi-algebraic sets* in $\mathbb{R}^n$ form the smallest set $C \subseteq \wp(\mathbb{R}^n)$ such that C contains all sets $\{\bar{x} \in \mathbb{R}^n | f(\bar{x}) > 0\}$ where f is a polynomial with n parameters from $\mathbb{R}$ and and C is closed under finite union, finite intersection and complementation.

Incidentally, the semi-algebraic sets are exactly the subsets of $\mathbb{R}^n$ $(n \geq 1)$ that are definable with parameters over $\mathbb{R}$ in the language of the ordered real closed field with constants (van den Dries, 1998, Chapter 2, Corollary 2.11). Figure 3.9 shows an example of a semi-algebraic set.

Let $\mathbf{S}$ denote the set of all regular open semi-algebraic sets in $\mathbb{R}^2$. According to the convention which was introduced above, $\mathbf{S}^*$ denotes the set of regular open semi-algebraic sets in $(\mathbb{R}^2)^*$ where $(\mathbb{R}^2)^*$ is now considered as the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$ which itself is a semi-algebraic set in



(a) A decompostion but not a stratification

(b) A stratification

Figure 3.10: Two decompositions of a square

$\mathbb{R}^3$. The set of all semi-algebraic sets in $\mathbb{R}^2$ will play a major role in chapter 7. I continue with important results about semi-algebraic sets.

Proposition 3.3.2 (Bochnak et al. (1998), Proposition 2.2.2).
The closure, interior and boundary of a semi-algebraic set in $\mathbb{R}^n$ are semi-algebraic.

A semi-algebraic set $A \subseteq \mathbb{R}^n$ will be called a *k-cell* ($0 \leq k \leq n$) if it is homeomorphic to $(0,1)^k$ where $(0,1)^0$ stands for a point. Clearly, every k-cell is a semi-algebraic set. A finite set $\mathcal{C}$ of mutually disjoint 0-, 1-, ... and n-cells in $\mathbb{R}^n$ is a *decomposition* of a set $A \subseteq \mathbb{R}^n$ if $A = \bigcup \mathcal{C}$. A decomposition $\mathcal{C}$ of A is a *stratification* of A if for each $C \in \mathcal{C}$, $A \cap ([C] \setminus C)$ is the union of elements of $\mathcal{C}$. Figure 3.10 shows two decompositions, one of which is a stratification. The following cell stratification theorem, which is a consequence of (van den Dries, 1998, Chapter 4, Proposition 1.13), indicates the especially well-behaved nature of semi-algebraic sets.

Cell stratification theorem. *Every semi-algebraic set in $\mathbb{R}^n$ has a stratification.*

Thus, every semi-algebraic set is the disjoint union of finitely many cells.

Definition 3.3.3 (Bochnak et al. (1998), Def. 2.4.2). A semi-algebraic set $A \subseteq \mathbb{R}^n$ is said to be *semi-algebraically connected* if for all semi-algebraic disjoint closed sets B_1 and B_2 in $\mathbb{R}^n$ with $B_1 \cup B_2 = A$ either $B_1 = A$ or $B_2 = A$.

Definition 3.3.4 (Bochnak et al. (1998), Def. 2.2.5, 2.5.12). Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be two semi-algebraic sets. A function $f : A \to B$ is called *semi-algebraic* if its graph $graph(f) = \{(x, f(x)) | x \in A\}$ is a semi-algebraic set in

$\mathbb{R}^{m+n}$. A path (arc) whose graph is semi-algebraic is called a *semi-algebraic path* (*arc*). A semi-algebraic set $A \subseteq \mathbb{R}^n$ is *semi-algebraically path-connected* (*arc-connected*) if, for every pair of (distinct) points $p, q \in A$, there exists a semi-algebraic path (arc) $f: [0,1] \to A$ with $f(0) = p$ and $f(1) = q$.

Proposition 3.3.5. *Let $f: [0,1] \to \mathbb{R}^n$ be a semi-algebraic path with $f(0) \neq f(1)$. Then there is a semi-algebraic arc $g: [0,1] \to \mathbb{R}^n$ with $g(0) = f(0)$, $g(1) = f(0)$ and $g([0,1]) \subseteq f([0,1])$.*

Proof. By the cell stratification theorem, there exists a stratification $\mathcal{C}$ of $|f|$. It follows by (van den Dries, 1998, Chapter 4, Corollary 1.6(ii)) that $\mathcal{C}$ contains 0- and 1-cells only. The natural ordering on $[0,1]$ induces a sequence of the cells of $\mathcal{C}$ which starts with the 0-cell $f(0)$ and ends with the 0-cell $f(1)$. Moreover, in the sequence every 0-cell (except for $f(1)$) is followed by a 1-cell and every 1-cell is followed by a 0-cell. If f is not injective some 0-cells will appear as cross-points more than once in the sequence. If cells appearing between such cross-point are deleted from the sequence, the union of the remaining cells is the locus of some semi-algebraic arc $g: [0,1] \to f([0,1])$. □

Proposition 3.3.6. *Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set. Then the following statements are equivalent.*

1. *A is connected.*
2. *A is semi-algebraically connected.*
3. *A is semi-algebraically path-connected.*
4. *A is semi-algebraically arc-connected.*

Furthermore, A has finitely many components, all of which are semi-algebraic sets.

Proof. By (Bochnak et al., 1998, Theorem 2.4.5), A has finitely many components, all of which are semi-algebraic, and (i) iff (ii). By (Bochnak et al., 1998, Proposition 2.5.13), (ii) iff (iii). By proposition 3.3.5, (iii) iff (iv). □

A further characteristic of semi-algebraic sets is that they have accessible boundaries as the next proposition states.

Proposition 3.3.7 (Bochnak et al. (1998), Theorem 2.5.5). *Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set and $p \in \mathbb{R}^n$ be a boundary point of A. Then there exists a path $f\colon [0,1] \to \mathbb{R}^n$ with $f(0) = p$ and $f((0,1]) \subseteq A$.*

Proposition 3.3.8 (van den Dries (1998), Chapter 6, Lemma 3.5). *Let $A_1, A_2 \subseteq \mathbb{R}^n$ be disjoint closed semi-algebraic sets. Then there exist disjoint semi-algebraic open sets U_1 and U_2 such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$.*

In the remainder of this section, I will restrict my attention to *regular open* semi-algebraic sets.

3.3.2 Regular open semi-algebraic sets and their properties

Topological spaces can be classified according to their separation properties. These are defined as follows.

Definition 3.3.9. A topological space X is said to be *Hausdorff* if for all distinct points $p, q \in X$ there exist disjoint open sets U, V in X such that $p \in U$ and $q \in V$ (T_2 axiom). A topological space X is *regular* if for all distinct points $p, q \in X$ there exists an open set U such that $p \in U$ and $q \notin U$, or $q \in U$ and $p \notin U$ (T_0 axiom) and for every closed set V and every point $p \in X \setminus V$ there exist disjoint open sets U_p and U_V such that $p \in U_p$ and $V \subseteq U_V$ (T_3 axiom). A space X is *normal* if for all two distinct points $p, q \in X$ there exist open sets U_p and U_q containing p and q respectively such that $p \notin U_q$ and $q \notin U_p$ (T_1 axiom) and for all disjoint closed sets A and B there exist disjoint open sets U_A and U_B containing A and B respectively (T_4 axiom).

It is straightforward to show that every normal space is regular and every regular space is Hausdorff. It is easy to see that the Euclidean spaces are normal, and therefore regular and Hausdorff. Let the notion of separation properties be transferred to spatial domains in the following way.

Definition 3.3.10. A spatial domain M over a topological space X is *Hausdorff*, *regular* or *normal* if M satisfies the corresponding separation axioms in the above definition where 'open sets in X' is changed to 'open sets in the spatial domain M', and 'closed set in X' is changed to the 'closure of an element in M'.

Since a spatial domain M over a topological space X may contain only a few open sets, normality does not imply regularity, and regularity does not imply Hausdorffness in general. However, here we have the following result.

Proposition 3.3.11. *The spatial domains* $\widetilde{\mathbf{F}}$*,* $\mathbf{J}^*$ *and* $\widetilde{\mathbf{S}}$ *are Hausdorff, regular and normal, and the spatial domain* $\mathbf{J}$ *is Hausdorff and regular.*

Proof. Let A_1 and A_2 be two disjoint closed sets in $(\mathbb{R}^2)^*$. For each point $p \in A_1$ let U_p be an open ball with centre p such that $[U_p] \cap A_2 = \emptyset$. Let $\mathcal{U}_1 = \{U_p | p \in A_1\}$. For each point $q \in A_2$ let U_q be an open ball with centre q not intersecting $\bigcup \mathcal{U}_1$. Let $\mathcal{U}_2 = \{U_q | q \in A_2\}$. Then $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ is an open cover of $A_1 \cup A_2$. Since $(\mathbb{R}^2)^*$ is compact and $A_1 \cup A_2$ is closed, $A_1 \cup A_2$ is compact. Hence, there exists a finite subcover $\mathcal{U}'$ of $\mathcal{U}$. Then $B_1 = \sum(\mathcal{U}' \cap \mathcal{U}_1)$ and $B_2 = \sum(\mathcal{U}' \cap \mathcal{U}_2)$ are disjoint elements of $\mathbf{S}^*$ such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. Hence, $\mathbf{F}^*$, $\mathbf{J}^*$ and $\mathbf{S}^*$ are Hausdorff, regular and normal.

For $\mathbf{F}$, the proposition follows directly from the normality of $\mathbb{R}^2$ and the fact that for any two disjoint open sets U and V, $[U]^\circ$ and $[V]^\circ$ are regular open, disjoint and contain U and V respectively.

For $\mathbf{J}$, the proposition follows directly from the fact that every open disc and its complement in the Boolean algebra of regular open sets is an element of $\mathbf{J}$.

By proposition 3.3.8, $\mathbf{S}$ is normal. Since points are closed sets in $\mathbb{R}^2$, $\mathbf{S}$ is also regular and Hausdorff. □

Lemma 3.3.12. *Any non-empty element of* $\widetilde{\mathbf{S}}$ *is the sum of finitely many mutually disjoint connected elements of* $\widetilde{\mathbf{S}}$*, all of which are 2-cells.*

Proof. By the cell stratification theorem, there exist finite sets $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ of 0-, 1- and 2-cells respectively, such that $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ is a stratification of $r \in \mathbf{S}$. For each $A \in \mathcal{C}_0 \cup \mathcal{C}_1$, $A \subseteq [s]$ for some $s \in \mathcal{C}_2$. Hence, $[r] = [\bigcup \mathcal{C}_2]$. Thus, $r = [\bigcup \mathcal{C}_2]^\circ = \sum \mathcal{C}_2$.

By proposition 3.2.1, $\mathbf{S}$ and $\mathbf{S}^*$ are isomorphic Boolean algebras such that $r \in \mathbf{S}$ is connected if and only if r^* is connected. Therefore, the lemma holds also for $\mathbf{S}^*$. □

Proposition 3.3.13. *The structure* $(\widetilde{\mathbf{S}}, +, \cdot, -, \emptyset, \mathbb{R}^2)$ *is a Boolean subalgebra of* $(\widetilde{\mathbf{J}}, +, \cdot, -, \emptyset, \mathbb{R}^2)$.

Proof. By definition 3.3.1, the set of semi-algebraic sets is closed under finite unions, intersections and complementation. Furthermore, by proposition 3.3.2, the closure and interior of a semi-algebraic set are semi-algebraic. Hence, for $u, v \in \widetilde{\mathbf{S}}$, $u + v = [u \cup v]^\circ \in \widetilde{\mathbf{S}}$. So are $u \cdot v = u \cap v$ and $-u = \widetilde{\mathbb{R}^2} \cap (\widetilde{\mathbb{R}^2} \setminus u)$. By lemma 3.3.12, every region in $\widetilde{\mathbf{S}}$ is the finite sum of 2-cells. Since 2-cells are elements of $\widetilde{\mathbf{J}}$, $\widetilde{\mathbf{S}} \subseteq \widetilde{\mathbf{J}}$. □

Lemma 3.3.14. *Let $r \in \widetilde{\mathbf{S}}$ and let s be a component of r. Then $s \in \widetilde{\mathbf{S}}$.*

Proof. Since locally Euclidean spaces are locally connected, it follows from lemma 3.1.4 that s is regular open. By proposition 3.3.6, s is a semi-algebraic set. □

Lemma 3.3.15. *A region $r \in \widetilde{\mathbf{S}}$ is connected if and only if for every two non-empty regions $s_1, s_2 \in \widetilde{\mathbf{S}}$ with $r = s_1 + s_2$, $r \cap [s_1] \cap [s_2] \neq \emptyset$.*

Proof. By lemma 3.3.14, the proof proceeds as for lemma 3.1.5. □

Lemma 3.3.16. *If $r \in \widetilde{\mathbf{S}}$ is non-empty, connected and bounded, and has a non-empty connected complement then r is a Jordan region.*

Proof. The proof is the same as for conjecture 3.2.12, but with conjecture 3.2.10 replaced by proposition 3.3.7. □

Theorem 3.3.17 (Newman (1964),Chapter V, Theorem 9.2). *If the common part of the two closed sets F_1 and F_2 in $(\mathbb{R}^2)^*$ is connected, two points which are connected in $(\mathbb{R}^2)^* \setminus F_1$ and $(\mathbb{R}^2)^* \setminus F_1$ are connected in $(\mathbb{R}^2)^* \setminus (F_1 \cup F_2)$.*

Lemma 3.3.18. *Let $r, s \in \mathbf{S}^*$ be two Jordan regions. If $[r] \cap [s]$ is connected then $-(r + s)$ is connected.*

Proof. Let $F_1 = [r]$ and $F_2 = [s]$. Let $p, q \in -(r + s) = (\mathbb{R}^2)^* \setminus (F_1 \cup F_2)$. Then $p, q \in ((\mathbb{R}^2)^* \setminus F_1) = -r$ and $p, q \in ((\mathbb{R}^2)^* \setminus F_2) = -s$. Since $-r$ and $-s$ are connected, p and q are connected in $-r$ and $-s$. Since $[r] \cap [s]$ is connected, it follows from theorem 3.3.17 that p and q are connected in $(\mathbb{R}^2)^* \setminus (F_1 \cup F_2) = -(r + s)$. Hence, $-(r + s)$ is connected. □

Lemma 3.3.19. *Let $r, s \in \mathbf{S}^*$ be two disjoint Jordan regions whose sum is a Jordan region. Then $\partial(r) \cap \partial(s)$ is the locus of a Jordan arc.*

Proof. I use the following three results given in (Newman, 1964).

Result 1, chapter V, theorem 11.5: If the common part of two continua F_1 and F_2 in $(\mathbb{R}^2)^*$ is not connected, there exists a pair of points not separated by F_1, but separated by $F_1 \cup F_2$. (A continuum is a compact connected set with at least two points.)

Result 2, chapter IV, theorem 12.1: A continuum whose connection is destroyed by the removal of two arbitrary points is a simple closed curve.

Result 3, chapter IV, theorem 10.2: A continuum of which all but at most two points are cut-points is a Jordan arc. (A point p of a connected set X is a cut-point if $X \setminus \{p\}$ is disconnected.)

Since r and s are disjoint and $r + s \neq (\mathbb{R}^2)^*$, $\partial(r) \neq \partial(s)$. Since $\partial(r)$ and $\partial(s)$ are simple closed curves, $\partial(s) \setminus \partial(r) \neq \emptyset$. Let $p \in \partial(s) \setminus \partial(r)$. Choose $\epsilon > 0$ such that $B_\epsilon(p) \cap \partial(r) = \emptyset$. Then $\partial(s) \setminus B_\epsilon(p)$ is closed and has a connected complement. It follows from result 1 that $[r] \cap [s] = \partial(r) \cap \partial(s)$ is connected. Since $r + s$ is connected, it follows from lemma 3.3.15 that $\partial(r) \cap \partial(s)$ is not a singleton. Since $\partial(r)$ and $\partial(s)$ are closed, $\partial(r) \cap \partial(s)$ is compact and hence a continuum.

Let $p \in \partial(r) \cap \partial(s) \cap (r+s)$ and $q \in \partial(r) \setminus \partial(s)$. Since $\partial(r)$ is a Jordan curve, it follows from result 2 that $\partial(r) \setminus \{p, q\}$ is disconnected. Hence, $(\partial(r) \cap \partial(s)) \setminus \{p\}$ is disconnected. Lemma 3.1.8 implies that $(\partial(r) \cap \partial(s)) \setminus (r + s)$ contains at most two points. Hence, all but at most two points of $\partial(r) \cap \partial(s)$ are cut-points. By result 3, $\partial(r) \cap \partial(s)$ is (the locus of) a Jordan arc. □

Lemma 3.3.20. *Let $r_1, r_2 \in \widetilde{\mathbf{S}}$ and $p \in [r_1] \cap [r_2]$. Then there exist disjoint bounded connected regions $s_1, s_2 \in \widetilde{\mathbf{S}}$ such that $s_1 \subseteq r_1$, $s_2 \subseteq r_2$, $[s_1] \cap [s_2] = \{p\}$ and $[-(r_1 + r_2)] \cap [s_1] \cap [s_2]$ is a singleton or empty.*

Proof. Since the boundary points of r_1 and r_2 are accessible by lemma 3.3.7, there exist semi-algebraic loops γ_1 at p in r_1 and γ_2 at p in r_2 such that $|\gamma_1|$ and $|\gamma_2|$ are disjoint except for p. Then the loops define connected regions $s_1, s_2 \in \mathbf{S}$ such that $s_1 \subseteq r_1$, $s_2 \subseteq r_2$, $[s_1] \cap [s_2] = \{p\}$ and $[-(r_1 + r_2)] \cap [s_1] \cap [s_2] \subseteq \{p\}$. □

Lemma 3.3.21. *Let $r_1, r_2, r_3, r_4 \in \widetilde{\mathbf{S}}$ be connected and mutually disjoint such that $r_i + r_j$ is connected $(1 \leq i < j \leq 4)$. Then $[r_1] \cap [r_2] \cap [r_3] \cap [r_4] = \emptyset$.*

Proof. Assume $p \in [r_1] \cap [r_2] \cap [r_3] \cap [r_4]$. Let $q_i \in r_i$ $(1 \leq i \leq 4)$. By lemma 3.3.7, the boundaries of r_1, r_2, r_3 and r_4 are accessible. Hence, there exist arcs $\gamma_{i,j}$ from q_i to q_j in $r_i + r_j$ with disjoint interior $(1 \leq i < j \leq 4)$. Moreover, there exist end-cuts ξ_i $(1 \leq i \leq 4)$ from q_i to p in r_i not intersecting $\gamma_{i,j}$ $(1 \leq j \leq 4, j \neq j)$ except in q_i. Then the arcs form a plane embedding of the non-planar graph K_5 which is impossible. □

Lemma 3.3.22. *Let $r_1, r_2, r_3, s_1, s_2 \in \widetilde{\mathbf{S}}$ be connected and mutually disjoint such that $r_i + s_j$ is connected $(1 \leq i \leq 3, 1 \leq j \leq 2)$. Then $[r_1] \cap [r_2] \cap [r_3] = \emptyset$.*

Proof. Assume $p \in [r_1] \cap [r_2] \cap [r_3]$. Let $q_i \in r_i$ $(1 \leq i \leq 3)$ and $p_j \in s_j$ $(j = 1, 2)$. By lemma 3.3.7, the boundaries of r_1, r_2, r_3, s_1 and s_2 are accessible. Hence, there exist mutually disjoint Jordan arcs $\gamma_{i,j}$ from q_i to p_j in $r_i + s_j$ $(1 \leq i \leq 3, 1 \leq j \leq 2)$. Moreover, there exist end-cuts ξ_i $(1 \leq i \leq 3)$ from q_i to p in r_i not intersecting $\gamma_{i,j}$ except in q_i $(1 \leq j \leq 3, j \neq i)$. Then the arcs form a plane embedding of the non-planar graph $K_{3,3}$ which is impossible. □

Lemma 3.3.23. *Let $r_1, r_2, r_3 \in \widetilde{\mathbf{S}}$ be mutually disjoint j-regions such that $r_1 + r_2$ and $r_1 + r_3$ are j-regions, $-(r_2 + r_3)$ is connected and $-(r_1 + r_2 + r_3)$ is non-empty. Then $[r_1] \cap [r_2] \cap [r_3]$ contains at most one point.*

Proof. Suppose for contradiction that p_1 and p_2 are two distinct points in $[r_1] \cap [r_2] \cap [r_3]$. Let $p_3 \in -(r_1 + r_2 + r_3)$ and $q_i \in r_i$ for $i \in \{1, 2, 3\}$. Note that $r_i + -(r_1 + r_2 + r_3)$ for $i \in \{1, 2, 3\}$ is connected. Let $\gamma_{i,j}$ be an end-cut from q_i to p_j in r_i for $i \in \{1, 2, 3\}, j \in \{1, 2\}$ and let $\gamma_{i,3}$ be an arc from q_i to p_3 in $r_i + -(r_1 + r_2 + r_3)$ for $i \in \{1, 2, 3\}$. Certainly, the arcs can be chosen to intersect at endpoints only. Then the loci of these arcs represent a plane embedding of the non-planar graph $K_{3,3}$ which is impossible. □

The partition of a region into mutually disjoint simpler regions will play an important role in later proofs.

Definition 3.3.24. The elements $r_1, \ldots, r_n \in \widetilde{\mathbf{F}}$ are said to form a *partition of* $r \in \widetilde{\mathbf{F}}$ if they are mutually disjoint, non-empty and sum to r. In case of $r = \widetilde{\mathbb{R}^2}$, $r_1, \ldots, r_n$ is simply said to be a *partition.* A partition $r_1, \ldots, r_n$ of r is said to be *connected* if each r_i $(1 \leq i \leq n)$ is connected. A region $r \in \widetilde{\mathbf{F}}$ is called *j-region* if r is non-empty and connected and has a non-empty and connected complement. A connected partition whose elements are j-regions is said to be a *j-partition.* A j-partition $r_1, \ldots, r_n \in \widetilde{\mathbf{F}}$ is *radial about* r_1 if $-(r_1 + r_i)$ is connected for all i with $2 \leq i \leq n$. A *radial partition* is a j-partition which is radial about every element of the partition. The elements $r_1, \ldots, r_n \in \widetilde{\mathbf{F}}$ are said to *refine* the elements $s_1, \ldots, s_k \in \widetilde{\mathbf{F}}$ if every s_i $(1 \leq i \leq k)$ if the sum of various r_j. In this case, $r_1, \ldots, r_n$ is said to be a *refinement* of $s_1, \ldots, s_k$. Given regions $r_1, \ldots, r_n \in \widetilde{\mathbf{F}}$, let $s_1, \ldots, s_k$ be the components of regions of the form $\pm r_1 \cdot \ldots \cdot \pm r_n$. This connected partition $s_1, \ldots, s_k$ is called the *connected partition generated by* $r_1, \ldots, r_n$.

The following lemmas present results concerning radial partitions. Remember that the binary connection graph on connected regions $r_1, \ldots, r_n$ contains an edge $\{r_i, r_j\}$ if and only if $r_i + r_j$ is connected $(1 \leq i < j \leq n)$.

Lemma 3.3.25. *Let $r_1, \ldots, r_n \in \widetilde{\mathbf{S}}$ $(n \geq 4)$ be a radial partition such that for some point p and some k with $3 \leq k < n$, $p \in [r_i]$ if and only if $1 \leq i \leq k$. Then $r_1 + \ldots + r_k$ is a j-region. Moreover, the binary connection graph on $r_1, \ldots, r_k$ is a cycle.*

Proof. Let $r_1, \ldots, r_n \in \mathbf{S}^*$. Assume $p \in [r_1] \cap \ldots \cap [r_k]$ and $p \notin [r_{k+1}] \cup \ldots \cup [r_n]$. I show that $r_1 + \ldots + r_n$ is connected first. Let $s_1, s_2 \in \mathbf{S}^*$ be a separation of $s = r_1 + \ldots + r_k$. If $p \in s_1 \cup s_2$ then for some r_i $(1 \leq i \leq k)$, $r_i \cdot s_1 \neq \emptyset \neq r_i \cdot s_2$. Since r_i is connected, $[r_i \cdot s_1] \cap [r_i \cdot s_2] \cap s_1 + s_2 \neq \emptyset$. Hence $[s_1] \cap [s_2] \cap s_1 + s_2 \neq \emptyset$. If $p \notin s_1 \cup s_2$ then, since $p \notin [-s]$, $p \in \partial(s_1) \cap \partial(s_2) \cap s$. Then $[s_1] \cap [s_2] \cap s \neq \emptyset$. Either way, by lemma 3.3.15, $s = r_1 + \ldots + r_k$ is connected.

Let $t = r_{k+1} + \ldots + r_n$. I show by induction over k, $k \geq 3$, that $r_1 + \ldots + r_k$ is a Jordan region, $r_i + t$ is connected $(1 \leq i \leq k)$, and the binary connection graph on $r_1, \ldots, r_k$ is a cycle.

Assume $k = 3$. Then $r_1 + r_2$, $r_2 + r_3$ and $r_1 + r_3$ are connected. Thus, the binary connection graph on r_1, r_2, r_3 is a cycle. Since the partition is radial, r_i $(1 \leq i \leq 3)$ has a connected sum with every component of t. It follows from lemma 3.3.22, that t is connected. Hence, $r_1 + r_2 + r_3$ is a Jordan region.

Assume that $k > 3$ and the induction hypothesis holds for $k - 1$. Since $r_1 + \ldots + r_k$ is connected, r_1 has a connected sum with some region r_i $(2 \leq i \leq k)$. Assume WLOG that $r_1 + r_2$ is connected. Since the partition is radial, $r_1 + r_2$ is a Jordan region. Suppose that the collection of regions $r_1 + r_2, r_3, \ldots, r_k$ is not radial. Then for some region, r_3 say, $-((r_1 + r_2) + r_3)$ is disconnected. By lemma 3.3.18, $\partial(r_1 + r_2) \cap \partial(r_3)$ is disconnected. Since $p \in \partial(r_1) \cap \partial(r_2) \cap \partial(r_3)$, one component of $\partial(r_1 + r_2) \cap \partial(r_3)$ contains p and another component intersects with $\partial(r_1)$ or $\partial(r_2)$. Assume WLOG that $\partial(r_1) \cap \partial(r_3)$ is disconnected. By lemma 3.3.18, $-(r_1 + r_3)$ is disconnected. However, this is impossible since the partition $r_1, \ldots, r_n$ is radial. Hence, the elements $r_1 + r_2, r_3, \ldots, r_k$ are radial about each other. Therefore, it follows from the induction hypothesis that $r_1 + \ldots + r_k$ is a Jordan region, the binary connection graph on $r_1 + r_2, r_3, \ldots, r_k$ is a cycle and each of $r_1 + r_2$, $r_3, \ldots, r_k$ has a connected sum with $t = r_{k+1} + \ldots + r_n$. Assume WLOG that $r_1 + r_2 + r_3$ and $r_1 + r_2 + r_k$ are connected. By lemma 3.3.22, at most one of r_1 and r_2 has a connected sum with both r_3 and r_k. Suppose $r_1 + r_3$ and $r_1 + r_k$ are connected. Since $r_2 + r_3$ or $r_2 + r_k$ is disconnected and the partition is radial, $r_2 + t$ is connected. Since also $r_3 + t$ and $r_k + t$ are connected, and $p \in [r_2] \cap [r_3] \cap [r_k]$, it follows from lemma 3.3.21 that r_1 cannot have a connected sum with both r_3 and r_k. The same holds for r_2. Assume WLOG that $r_1 + r_k$ and $r_2 + r_3$ are connected. Since $r_1 + r_2$ is connected, the binary connection graph on $r_1, \ldots, r_k$ is a cycle. Since the partition is radial, $r_1 + t$ and $r_2 + t$ are connected. This completes the induction step.

For $r_1, \ldots, r_n \in \mathbf{S}$, the result follows from proposition 3.2.1. □

The following lemmas appeal to a number of results in graph theory. These are:

Theorem 3.3.26 (Bonnington and Little (1995), Theorem 1.25). *A graph Γ with more than one vertex is n-connected if and only if $n < ||V(\Gamma)||$ and $\Gamma \setminus S$ is connected for each $S \subseteq V(\Gamma)$ such that $||S|| < n$.*

Theorem 3.3.27 (Gross and Tucker (1987), Thrm. 1.6.1). *A plane graph Γ is 2-connected if and only if every face of Γ is bounded by a cycle.*

Thus, if the edges of a 2-connected plane graph Γ are semi-algebraic then the faces of Γ form a j-partition in $\widetilde{\mathbf{S}}$. Note, however, that the topological boundary of a j-partition is not necessarily a plane graph since the "graph" may have multiple edges. In case of the open plane, the boundary of a j-partition or radial partition may not even be a "graph" with multiple edges, since several regions might be unbounded.

Theorem 3.3.28 (Whitney 1932, Diestel (1997), Theorem 4.3.2). *Let Γ be a 3-connected planar graph and let I_1 and I_2 be two plane embeddings of Γ in the closed plane such that $\sigma_1 : \Gamma \to I_1$ and $\sigma_2 : \Gamma \to I_2$ are graph isomorphisms. Then there exists a homeomorphism h from the closed plane onto itself such that $h|_{V(I_1)\cup E(I_1)} = \sigma_2 \circ \sigma_1^{-1}$.*

Thus, if I_1 is a plane embedding of a 3-connected planar graph Γ and the cycle $C \subseteq \Gamma$ bounds a face of I_1 then C bounds a face in every plane embedding of Γ. It is easy to see that this is not true for 2-connected planar graphs.

Theorem 3.3.29. *The edges of a plane graph can be continuously deformed into piecewise linear edges without affecting any vertices or any semi-algebraic (or alternatively piecewise linear) edges.*

Proof. Follows directly from a statement in (Bollobás, 1979, p. 16). □

Definition 3.3.30. Given a radial partition $r_1, \ldots, r_n \in \mathbf{S}^*$ $(n \geq 4)$, the *topological boundary graph* on $r_1, \ldots, r_n$ is the plane graph Γ_∂ such that $V(\Gamma_\partial) = \bigcup_{1\leq i<j<k\leq n}([r_i] \cap [r_j] \cap [r_k])$ and $\bigcup E(\Gamma_\partial) = \bigcup_{i=1}^{n} \partial(r_i)$.

Lemma 3.3.31. *The topological boundary graph of a radial partition* $r_1, \ldots, r_n \in \mathbf{S}^*$ $(n \geq 4)$ *is a 3-connected plane graph.*

Proof. Let Γ_∂ be the topological boundary graph of $r_1, \ldots, r_n$. Note that every r_i has at least three neighbours in $r_1, \ldots, r_n$ and, by lemma 3.3.23, $[r_i] \cap [r_j] \cap [r_k]$ contains at most one point $(1 \leq i < j < k \leq n)$. Hence, Γ_∂ is a plane graph. Let v be any vertex of Γ_∂. Then v lies in the closures of at least three regions. By lemma 3.3.25, the sum of these regions is a j-region. Hence, by theorem 3.3.27, $\Gamma_\partial \setminus \{v\}$ is a 2-connected graph. Therefore, by theorem 3.3.26, Γ_∂ is 3-connected. □

Definition 3.3.32. The tuples $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ of a spatial domain over a topological space X are said to be *topologically equivalent*, written $r_1, \ldots, r_n \sim s_1, \ldots, s_n$, if there exists a homeomorphism $h : X \to X$ such that $h(r_i) = s_i$ for $1 \leq i \leq n$. A subset A of $\wp(X)$ is said to be *topologically homogeneous* if, whenever $\bar{a}, \bar{b}$ are n-tuples from A such that $\bar{a} \sim \bar{b}$ and a is an element of A, there exists $b \in A$ such that $\bar{a}, a \sim \bar{b}, b$.

Thus, a topologically homogeneous set $A \subseteq \wp(X)$ need not be closed under arbitrary homeomorphisms of X; however, any two topologically equivalent n-tuples in A must 'look alike' in terms of their topological relations to other elements of A. I will show below, employing radial partitions and 3-connected planar graphs, that the spatial domain $\widetilde{\mathbf{S}}$ is topologically homogeneous. First, I show that any tuple of regions in $\widetilde{\mathbf{S}}$ can be refined by some radial partition and that radial partitions have the following property.

Lemma 3.3.33. *Let* $r_1, \ldots, r_n \in \widetilde{\mathbf{S}}$ *and* $s_1, \ldots, s_n \in \widetilde{\mathbf{S}}$ $(n \geq 4)$ *be radial partitions such that* $r_i + r_j$ *is connected if and only if* $s_i + s_j$ *is connected* $(1 \leq i < j \leq n)$, *and* r_i *is bounded if and only if* s_i *is bounded* $(1 \leq i \leq n)$. *Then* $r_1, \ldots, r_n \sim s_1, \ldots, s_n$.

Proof. I prove the lemma for the closed plane first. Let $r_1, \ldots, r_n, s_1, \ldots, s_n \in \mathbf{S}^*$. By lemma 3.3.31, the topological boundary graphs Γ_r and Γ_s of $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ respectively are 3-connected plane graphs. I show that $[r_i] \cap [r_j] \cap [r_k]$ is a vertex of Γ_r if and only $[s_i] \cap [s_j] \cap [s_k]$ is a vertex of Γ_s $(1 \leq i < j < k \leq n)$, and $[r_i] \cap [r_j]$ is an edge of Γ_r if and only if $[s_i] \cap [s_j]$ is an edge of Γ_s $(1 \leq i < j \leq n)$. Then Γ_r and Γ_s are plane embeddings of the same 3-connected planar graph and the lemma follows by theorem 3.3.28.

Since $r_i + r_j$ is connected iff $s_i + s_j$ is connected, it follows from lemma 3.3.19 that $[r_i] \cap [r_j]$ is an edge of Γ_r iff $[s_i] \cap [s_j]$ is an edge of Γ_s $(1 \leq i < j \leq n)$.

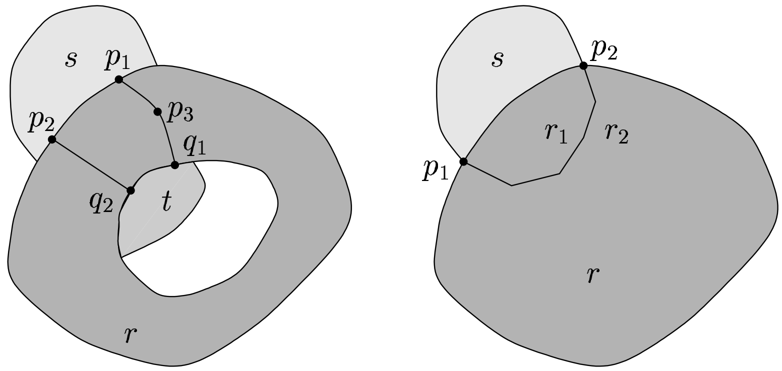
Assume $p \in [r_1] \cap \ldots \cap [r_l]$ and $p \notin [r_{l+1}] \cup \ldots \cup [r_n]$ for some l, $3 \leq l < n$. By lemma 3.3.25, the binary connection graph on $r_1, \ldots, r_l$ is a cycle and $r_1 + \ldots + r_l$ is a Jordan region. Hence, the binary connection graph on $s_1, \ldots, s_l$ is a cycle and $s_1 + \ldots + s_l$ is a Jordan region. WLOG assume that $s_1 + s_2 + s_3$ is connected. By lemma 3.3.23, $[s_1] \cap [s_2] \cap [s_3]$ contains a single point q. Then for any i, j, k with $1 \leq i < j < k \leq l$, $[s_i] \cap [s_j] \cap [s_k] = \{p\}$, since otherwise by lemma 3.3.25 the binary connection graph on $s_1, \ldots, s_l$ could not be a cycle. It is shown analogously that, if $[s_i] \cap [s_j] \cap [s_k]$ is a singleton, then $[r_i] \cap [r_j] \cap [r_k]$ is a singleton.

Now let $r_1, \ldots, r_n, s_1, \ldots, s_n \in \mathbf{S}$ and let Γ_r and Γ_s be the the topological boundary graphs of $r_1^*, \ldots, r_n^*$ and $s_1^*, \ldots, s_n^*$ respectively. By the above proof there exists a homeomorphism $h\colon (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$ taking r_i^* to s_i^* $(1 \leq i \leq n)$. Since r_i is bounded iff s_i is bounded, the point at infinity p_∞ lies on corresponding vertices, edges or faces of Γ_r and Γ_s. Hence, h can be chosen to map p_∞ to itself. Therefore, $h|_{\mathbb{R}^2}$ is a homeomorphism taking r_i to s_i $(1 \leq i \leq n)$. □

The following two lemmas are adapted from (Pratt and Schoop, 1998). The first of the two lemmas will be used to show that every connected partition can be refined to a radial partition. The second lemma will be used in chapter 5. However, both lemmas are technically similar and therefore presented together here.

Lemma 3.3.34. *Let $r, s, t \in \widetilde{\mathbf{S}}$ be connected such that r is non-empty, r and $s+t$ are disjoint and $r+s$ and $r+t$ are connected. Then there are disjoint connected non-empty regions $r_1, r_2 \in \widetilde{\mathbf{S}}$ such that $r = r_1 + r_2$ and $r_1 + s$, $r_1 + t$, $r_2 + s$ and $r_2 + t$ are connected. Furthermore, if $r \in \mathbf{S}$, r is unbounded and $-(r+s)$ is bounded then r_1 can be chosen to be bounded, or r_1 and r_2 can be chosen to be unbounded.*

Proof. Consider the regions r^*, s^* and t^* of $\mathbf{S}^*$. If s^* or t^* is empty an easier or similar proof as the one which follows applies. Assume s^* and t^* to be non-empty. It follows from lemma 3.3.15 that there are two distinct points $p_1, p_2 \in (r^* + s^*) \cap \partial(r^*)$ and two distinct points $q_1, q_2 \in (r^* + t^*) \cap \partial(r^*)$. Let p_3 be a point in r^*. Since by proposition 3.3.7 the boundary of r^* is accessible from r^*, and by proposition 3.3.6, r^* is arc-connected, there exists a semi-algebraic cross-cut γ_1 in r_1^* from p_1 to q_1. Either γ_1 partitions r^* into two connected regions r_1^* and r_2^*, or



(a) (b)

Figure 3.11: The existence of specific regions in $\mathbf{S}$

$r^* \setminus |\gamma_1|$ is connected. In the former case, obviously r_1^* and r_2^* are disjoint elements of $\mathbf{S}^*$ whose sum is r^*. In the later case, let γ_2 be a semi-algebraic cross-cut from p_2 to q_2 in $r^* \setminus |\gamma_1|$ (cf. figure 3.11(a)). Since $p_2 \in [s^*]$, $q_2 \in [r^*]$ and γ_2 connects $[s^*]$ and $[r^*]$, p_2 and q_2 lie in the same component of $(\mathbb{R}^2)^* \setminus (r^* \setminus |\gamma_1|)$. It is a result from (Newman, 1964, Chapter V, Theorem 11.7) that, if γ is a cross-cut in a non-empty open connected set U with endpoints in the same component of $(\mathbb{R}^2)^* \setminus U$ then $U \setminus |\gamma|$ has two components. Hence γ_2 partitions $r^* \setminus |\gamma_1|$ into two connected regions r_1^* and r_2^*. Again, it is easy to see that r_1^* and r_2^* are disjoint elements of $\mathbf{S}^*$ whose sum is r^*. Since p_1, p_2 and q_1, q_2 are points of $(r^*+s^*)\cap\partial(r^*)$ and $(r^*+s^*)\cap\partial(r^*)$ respectively, it follows from lemma 3.3.15 that $r_1^*+s^*$, $r_1^*+t^*$, $r_2^*+s^*$ and $r_2^*+t^*$ are connected. Then by lemma 3.2.1, r_1+s, r_1+t, r_2+s and r_2+t are connected.

If r is unbounded and $-r$ is bounded, i.e. $p_\infty \in r^*$, then choose $|\gamma_1|$ (and $|\gamma_2|$) not to include p_∞ and let r_1^* be the component of $r^* \setminus |\gamma_1|$ $(r^* \setminus (|\gamma_1| \cup |\gamma_2|))$ that does not contain p_∞. Then r_1 is bounded. If γ_1 is chosen to go through p_∞ then r_1 and r_2 are unbounded.

If r and $-r$ are unbounded and $-(r+s)$ is bounded then $p_\infty \in [r^*]\setminus[-(r+s)]$. Then p_1 and p_2 can be chosen not to be p_∞, and r_1 can be chosen to be the bounded region. If p_1 is chosen to be p_∞ then r_1 and r_2 are unbounded. □

Lemma 3.3.35. *Let $r, s \in \widetilde{\mathbf{S}}$ be disjoint j-regions such that $r+s$ is a j-region. Then there exist disjoint connected non-empty regions $r_1, r_2 \in \widetilde{\mathbf{S}}$ such that $r = r_1 + r_2$, $r_1 + s$ and $r_2 + -(r+s)$ are connected and $r_2 + s$ and $r_1 + -(r+s)$ are disconnected. Furthermore, if $r \in \mathbf{S}$, r is unbounded and $-r$ is bounded then* *one of r_1 and r_2 can be chosen to be bounded, or r_1 and r_2 can be chosen to be unbounded.*

Proof. Consider the regions $r^*, s^* \in \mathbf{S}^*$. By lemma 3.3.19, $\partial(r^*) \cap \partial(s^*)$ is a Jordan arc α. Let p_1 and p_2 be its endpoints. By the accessibility of the boundary of r, there exists a semi-algebraic cross-cut γ from p_1 to p_2 in r^* (cf. figure 3.11(b)). The arc γ separates r^* into two disjoint Jordan regions r_1^* and r_2^* whose boundaries are $|\alpha| \cup |\gamma|$ and $(\partial(r^*) \setminus |\alpha|) \cup |\gamma|$ respectively. By lemma 3.3.15, $r_1^* + s^*$ and $r_2^* + -(r^* + s^*)$ are connected, and $r_2^* + s^*$ and $r_1^* + -(r^* + s^*)$ are disconnected. By proposition 3.2.1, $r_1 + s$ and $r_2 + -(r + s)$ are connected, and $r_2 + s$ and $r_1 + -(r + s)$ are disconnected.

If r is unbounded and $-r$ is bounded, i.e. $p_\infty \in r^*$, then $|\gamma|$ can be chosen to include p_∞. Then r_1 and r_2 are unbounded. On the other hand, γ can be chosen to pass p_∞ "on the left" or "on the right". Then either r_1 or r_2 is unbounded. □

Lemma 3.3.36. *(i) There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that any j-partition in $\widetilde{\mathbf{S}}$ with n elements has a radial partition with $f(n)$ elements as refinement.*

(ii) There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that any connected partition in $\widetilde{\mathbf{S}}$ with n elements has a j-partition with $g(n)$ elements as refinement.

Proof. (i) Let $r_1, \ldots, r_n \in \widetilde{\mathbf{S}}$ be a j-partition. Suppose s_1 and s_2 are two components of $-(r_1 + r_2)$. Then $r_1 + s_1$ and $r_1 + s_2$ are connected. By lemma 3.3.34, there exist disjoint connected regions $t_1, t_2 \in \widetilde{\mathbf{S}}$ such that $r_1 = t_1 + t_2$ and $s_1 + t_1$, $s_1 + t_2$, $s_2 + t_1$ and $s_2 + t_2$ are connected. Hence, t_1 and t_2 are j-regions. Moreover, the number of components of $-(t_1 + r_2)$ and $-(t_2 + r_2)$ together is equal to the number of components of $-(r_1 + r_2)$. Therefore, proceeding with the application of lemma 3.3.34 in this way, in a finite number of steps the j-partition $r_1, \ldots, r_n$ is refined to a radial partition. It is easy to see that at most $n - 2$ such steps are necessary. Hence, the new partition has at most $2n - 2$ elements. Let $f(n) = 2n - 2$. If $s_1, \ldots, s_k$ is a radial refinement of $s_1, \ldots, s_n$ with $k < f(n)$ then lemma 3.3.34 guarantees that $s_1, \ldots, s_k$ can be further refined to an $f(n)$-element radial partition.

(ii) Let $r_1, \ldots, r_n \in \widetilde{\mathbf{S}}$ be a connected partition. Suppose $s_1, s_2 \in \widetilde{\mathbf{S}}$ are two components of $-r_1$. Then $r_1 + s_1$ and $r_1 + s_2$ are connected. By lemma 3.3.34, there exist disjoint connected regions $t_1, t_2 \in \widetilde{\mathbf{S}}$ such that $r_1 = t_1 + t_2$, and $t_1 + s_1$, $t_1 + s_2$, $t_2 + s_1$ and $t_2 + s_2$ are connected. Then the number of components of $-t_1$ plus the number of components of $-t_2$ is smaller or equal to the number of components of $-r_1$. Therefore, a repeated but finite application of lemma 3.3.34 results in a j-partition that refines $r_1, \ldots, r_n$. It is easy to see that at most $n-1$ applications are required. Hence, let $g(n) = 2n - 1$. If $s_1, \ldots, s_k$ is a radial refinement of $s_1, \ldots, s_n$ with $k < g(n)$ then lemma 3.3.34 guarantees that $s_1, \ldots, s_k$ can be further refined to an $g(n)$-element radial partition. □

Proposition 3.3.37. *There exist only finitely many n-element radial partitions in $\widetilde{\mathbf{S}}$ up to topological equivalence.*

Proof. Since there are only finitely many 3-connected planar graphs with n vertices, it follows from lemma 3.3.33 that there are only finitely many n-element radial partitions up to topological equivalence. □

The following proposition is a direct consequence of proposition 3.3.37 and lemma 3.3.36.

Proposition 3.3.38. *There exist only finitely many n-element connected partitions up to topological equivalence in $\widetilde{\mathbf{S}}$.*

Note that the proposition neither holds for the spatial domain $\widetilde{\mathbf{F}}$ nor the spatial domain of regular open semi-algebraic sets in $\mathbb{R}^3$. A consequence of the proposition is that there are only countable many regions in $\widetilde{\mathbf{S}}$ up to topological equivalence. This is not true for $\widetilde{\mathbf{F}}$ or $\widetilde{\mathbf{J}}$:

Lemma 3.3.39. *There exist uncountably many regions up to topological equivalence in $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{J}}$.*

Proof. Consider two Jordan regions r and s of $\mathbf{J}$ such that $r \cdot s$ has infinitely many components as depicted in figure 3.7 on page 62. Certainly for any subset C of the components of $r \cdot s$ there exist two Jordan regions r' and s' such that $r' \cdot s' \subseteq \bigcup C$, for all $c \in C$, $c \cdot r' \cdot s' \neq \emptyset$, and $[r' \cdot s'] \cap [-(r \cdot s)] = \emptyset$. Thus, there is a region t in $\widetilde{\mathbf{J}}$ with denumerable many components that are ordered through their contact-relation. Moreover, for any subset C of the components of t there is a region t_C that is identical to t except that all and only those components of t_C which are covered by a region in C have a hole. Therefore, there are uncountably many regions up to homeomorphism in $\widetilde{\mathbf{J}} \subset \widetilde{\mathbf{F}}$. □

Proposition 3.3.40. *The spatial domain $\widetilde{\mathbf{S}}$ is topologically homogeneous.*

Proof. Let $r_1, \ldots, r_n, r, s_1, \ldots, s_n \in \mathbf{S}^*$ such that $r_1, \ldots, r_n \sim s_1, \ldots, s_n$. By lemma 3.3.36, there is a radial partition $u_1, \ldots, u_k \in \mathbf{S}^*$ $(k \geq 4)$ refining the connected partition generated by $r_1, \ldots, r_n, r$. Since $r_1, \ldots, r_n \sim s_1, \ldots, s_n$, there exists a homeomorphism $h : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$ taking r_i to s_i $(1 \leq i \leq n)$. Let $v_i = h(u_i)$ $(1 \leq i \leq k)$. Then $v_1, \ldots, v_k$ is a radial refinement of $s_1, \ldots, s_n$. However, note that the sets $v_1, \ldots, v_n$ are not necessarily elements of $\mathbf{S}^*$. Let Γ be the topological boundary graph of $v_1, \ldots, v_k$. By theorem 3.3.29, Γ can be continuously deformed into a plane graph Γ' where all non-semi-algebraic edges are replaced by piecewise liner ones. Let v_i' be the face of Γ' whose boundary is the deformed boundary of v_i $(1 \leq i \leq k)$. Then $v_1', \ldots, v_k'$ are elements of $\mathbf{S}^*$ refining $s_1, \ldots, s_n$ such that $v_i' + v_j'$ is connected if and only if $v_i + v_j$ is connected $(1 \leq i \leq k)$. By lemma 3.3.33, there exists a homeomorphism $g : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$ taking u_i to v_i' $(1 \leq i \leq k)$. Let $s = g(r)$. Then $r_1, \ldots, r_n, r \sim s_1, \ldots, s_n, s$.

Let $r_1, \ldots, r_n, r, s_1, \ldots, s_n \in \mathbf{S}$ such that $r_1, \ldots, r_n \sim s_1, \ldots, s_n$. Hence, there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ taking r_i to s_i $(1 \leq i \leq n)$. By lemma 3.2.2, h extends to an homeomorphism $h^* : (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$ such that $h^*(r_i^*) = s_i^*$ $(1 \leq i \leq n)$ and $h^*(p_\infty) = p_\infty$. By the above argumentation there are radial refinements $u_1^*, \ldots, u_k^*$ and $v_1^*, \ldots, v_k^*$ in $\mathbf{S}^*$ of $r_1^*, \ldots, r_n^*, r^*$ and $s_1^*, \ldots, s_n^*$ respectively such that $u_1^*, \ldots, u_k^* \sim v_1^*, \ldots, v_k^*$. Moreover, the graph Γ' in the above argumentation can be chosen such that $p_\infty \in [u_i^*]$ iff $p_\infty \in [v_i^*]$. It follows from lemma 3.3.33 that there exists a homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$ taking u_i to v_i $(1 \leq i \leq k)$. Let $s = g(r)$. Then $r_1, \ldots, r_n, r \sim s_1, \ldots, s_n, s$. $\square$

In this section, we have seen that the set $\widetilde{\mathbf{S}}$ of regular open semi-algebraic sets in $\widetilde{\mathbb{R}^2}$ is very well-behaved: every region in $\widetilde{\mathbf{S}}$ has an accessible boundary and finitely many components, $\widetilde{\mathbf{S}}$ forms a Boolean algebra and $\widetilde{\mathbf{S}}$ is topologically homogeneous. Such pleasant properties have been observed not only for semi-algebraic sets but also for other structures, called o-minimal structures.

3.4 O-minimal structures

The set of semi-algebraic sets is one example for an o-minimal structure in the sense given below. O-minimal structures have recently become the centre of intensive research in model theory. The term ‘o-minimal’, standing for ‘order-minimal’, was introduced in model theory to describe a class of “nice” structures (Pillay and Steinhorn, 1986; Knight et al., 1986). However, o-minimal structures are also of interest to topologists, since the definable sets of an o-minimal (model-theoretic) structure can be interpreted as sets over a topological space. As it turns out, these definable sets are well-behaved and satisfy certain finiteness properties that make them interesting for the development of a "tame topology" as envisaged by Grothendieck (see Schneps and Lochak, 1997).

O-minimal structures can be defined on any dense linear order without endpoints. Since this thesis is restricted to the Euclidean setting, I give a definition of o-minimal structure on $\mathbb{R}$.

Definition 3.4.1 (van den Dries (1998)). An *o-minimal structure on* $\mathbb{R}$ is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$:

(1) $\mathcal{S}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$, that is, $\emptyset \in \mathcal{S}_n$ and if $A, B \in \mathcal{S}_n$, then $A \cup B \in \mathcal{S}$ and $\mathbb{R}^n \setminus A \in \mathcal{S}_n$;

(2) if $A \in \mathcal{S}_n$ then $A \times \mathbb{R} \in \mathcal{S}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{S}_{n+1}$;

(3) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_i = x_j\} \in \mathcal{S}_n$ for some i, j with $1 \le i < j \le n$;

(4) if $A \in \mathcal{S}_{n+1}$ then $\pi(A) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the usual projection map, i.e. $\pi(A) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | (x_1, \ldots, x_n, x_{n+1}) \in A\}$;

(5) for each $r \in \mathbb{R}$, $\{r\} \in \mathcal{S}_1$, and $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 < x_2\} \in \mathcal{S}_2$;

(6) the only sets in $\mathcal{S}_1$ are the finite unions of sets of the form $\{r\}$ or (r_1, r_2) where $r \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R} \cup \{-\infty, +\infty\}$.

From a model-theoretic point of view, an o-minimal structure can be defined as a model-theoretic $\mathcal{L}(\le, \ldots)$-structure $\mathfrak{A}$ whose domain A is a dense linear order without endpoints such that every subset of A which is definable with parameters in $\mathfrak{A}$ is the finite union of points and open intervals of A. Given such o-minimal (model-theoretic) structure $\mathfrak{A}$, the sets definable with parameters in $\mathfrak{A}$ form an o-minimal structure in the first sense.

Examples of o-minimal structures are (van den Dries, 1996):

1. The class of semi-algebraic sets
   They were already introduced in the previous section. The semi-algebraic sets are exactly the sets definable with parameters in the model-theoretic structure with domain $\mathbb{R}$ and language $\mathcal{L}(<, 0, 1, +, -, \cdot)$ with its usual interpretation.

2. The class of semi-linear sets
   A semi-linear subset of $\mathbb{R}^n$ is the finite union of sets of the form

$$\{x \in \mathbb{R}^n | f_1(x) = 0, \ldots, f_k(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0\}$$

   where f_i $(1 \leq i \leq k)$ and g_j $(1 \leq j \leq l)$ are polynomials of degree smaller or equal to 1. The semi-linear sets are exactly the sets definable with parameters in the model-theoretic structure with domain $\mathbb{R}$ and language $\mathcal{L}(<, 0, +, -, (\lambda_r)_{r \in \mathbb{R}})$ where $<$, $+$ and $-$ have their usual interpretation and λ_r is the scalar multiplication by r. Obviously, every semi-linear set is a semi-algebraic set.

3. It has been shown in (Peterzil, 1992) that there exists one o-minimal structure which lies exactly between the semi-linear and the semi-algebraic sets. This structure contains all semi-linear set and all *bounded* semi-algebraic sets.

4. The class of sets definable with parameters in the model-theoretic structure with domain $\mathbb{R}$ and language $\mathcal{L}(<, 0, 1, +, -, \cdot, \exp)$ where $\exp(x)$ is interpreted as the exponential function e^x.

5. The class of sets definable with parameters in the model-theoretic structure with domain $\mathbb{R}$ and language $\mathcal{L}(<, 0, 1, +, -, \cdot, (f))$ where (f) ranges over all restricted analytic functions, i.e. over all functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{[-1,1]^n}$ is analytic and f is identically 0 outside $[-1, 1]^n$.

3.5 The spatial domain of regular open semi-linear sets

I denote the *spatial domain of regular open semi-linear sets* in $\mathbb{R}^2$ by $\mathbf{P}$ (standing for "polygonal"). Note that $\mathbf{P} \subseteq \mathbf{S}$ and, moreover, $(\mathbf{P}, +, \cdot, -, \emptyset, \mathbb{R}^2)$ is a Boolean subalgebra of $(\mathbf{S}, +, \cdot, -, \emptyset, \mathbb{R}^2)$.

So far, only uncountable spatial domains have been introduced. If the affine functions in the definition of semi-linear sets are restricted to take only rational parameters then the set of such restricted semi-linear sets is a countable subset of all semi-linear set. I denote the regular open set of this restricted set by

$\mathbf{Q}$. Then $\mathbf{Q} \subseteq \mathbf{P}$ and, moreover, $(\mathbf{Q}, +, \cdot, -, \emptyset, \mathbb{R}^2)$ is a Boolean subalgebra of $(\mathbf{P}, +, \cdot, -, \emptyset, \mathbb{R}^2)$.

The following two lemmas will be of importance later on.

Lemma 3.5.1. *If $r_1, \ldots, r_n \in \widetilde{\mathbf{P}}$ and $r \in \widetilde{\mathbf{S}}$ then there exists $s \in \widetilde{\mathbf{P}}$ such that $r_1, \ldots, r_n, r \sim r_1, \ldots, r_n, s$.*

Proof. Assume $r_1, \ldots, r_n \in \mathbf{P}^*$. By lemma 3.3.36, there exists a radial partition $u_1, \ldots, u_k \in \mathbf{S}^*$ refining the connected partition generated by $r_1, \ldots, r_n, r$. Hence, $u_1, \ldots, u_k$ refines $r_1, \ldots, r_n, r$. By theorem 3.3.29, the edges of the topological boundary graph Γ of $u_1, \ldots, u_k$ can be continuously deformed into a graph Γ' with piecewise linear edges without affecting any edges which are already piecewise linear. Hence, the faces of Γ' are elements of $\mathbf{P}^*$. Moreover, Γ and Γ' are plane embeddings of the same 3-connected planar graph, whence by theorem 3.3.28 there is a homeomorphism h of the closed plane onto itself taking each u_i to a face v_i of Γ' $(1 \leq i \leq k)$. Hence, $u_1, \ldots, u_k \sim v_1, \ldots, v_k$. Since the boundaries of r_i are left unaffected by h, $h(r_i) = r_i$ $(1 \leq i \leq n)$. Let $s = h(r)$. Then $r_1, \ldots, r_n, r \sim r_1, \ldots, r_n, s$.

Assume $r_1, \ldots, r_n \in \mathbf{P}$. Then $r_1^*, \ldots, r_n^* \in \mathbf{P}^*$. Certainly, there is a refinement of the connected partition generated by $r_1^*, \ldots, r_n^*$ such that p_∞ is a vertex of the topological boundary graph. Then the proof proceeds as before. Hence, $r_1^*, \ldots, r_n^*, r^* \sim r_1^*, \ldots, r_n^*, s^*$ and r is bounded if and only if s is bounded. Then $r_1, \ldots, r_n, r \sim r_1, \ldots, r_n, s$. □

Lemma 3.5.2. *If $r_1, \ldots, r_n \in \widetilde{\mathbf{Q}}$ and $s \in \widetilde{\mathbf{S}}$ then there exists $r \in \widetilde{\mathbf{Q}}$ such that $r_1, \ldots, r_n, s \sim r_1, \ldots, r_n, r$.*

Proof. The proof proceeds as for lemma 3.5.1 except that the piece-wise linear edges are chosen to have rational vertices. □

The topological homogeneity of $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ follows immediately from the above two lemmas and the homogeneity of $\widetilde{\mathbf{S}}$ (proposition 3.3.40).

Corollary 3.5.3. *The spatial domains $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ are topologically homogeneous.*

3.6 Conclusion

This chapter introduced the five spatial domains $\mathbf{F}$, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{Q}$ over the topological space $\mathbb{R}^2$ and their counterparts over $(\mathbb{R}^2)^*$. The spatial domain $\mathbf{F}$ of regular open sets in $\mathbb{R}^2$ was shown to contain regions that are not appropriate for a common-sense representation of everyday objects in space. The spatial domain **J** is a strict subset of **F** and was only conjectured to be better-behaved than **F**. The set **S** of regular open semi-algebraic sets was shown to be extremely well-behaved. Further refinements of **S** were given by the polynomial spatial domain **P** and its refinement **Q**, the spatial domain of rational polygons.

The properties that make **S** and therefore also **P** and **Q** so well-behaved are:

1. **S** forms a Boolean algebra with respect to the subset relation.
2. **S** obeys the separation property of normality and is topologically homogeneous.
3. There are only finitely many n-element connected partitions in **S** up to topological equivalence.
4. Every region in **S** has finitely many components.
5. Every region in **S** has an accessible boundary.

The following chapters make use of these properties of **S**.

The spatial domain **S** is a subset of the o-minimal structure of semi-algebraic sets over the real line. It is an open problem whether the two-dimensional regular open sets of some other o-minimal structure over the real line provide an equally well-behaved spatial domain. The investigation of spatial domains over $\mathbb{R}^3$ remains for future work.

Chapter 4

Planar mereotopologies and their properties

The introduction of spatial domains in the previous chapter settled the question of which regions are taken to exist. In this chapter, mereological and topological primitives will be introduced and represented by non-logical symbols in first-order languages. These first-order languages will be interpreted over the spatial domains and so model-theoretic structures will be introduced. The properties of these structures as well as the expressivity of the first order-languages will be investigated.

Definition 4.0.1. Let X be a topological space and $\mathcal{L}$ a first-order language with equality. An $\mathcal{L}$-structure $\mathfrak{M}$ with domain $M \subseteq \wp(X)$ is a *mereotopology* (over X) if

(i) the set $\{(r, s) \subseteq M^2 | r \subseteq s\}$ is $\mathcal{L}$-definable in $\mathfrak{M}$ (mereological criterion)

(ii) for all $\bar{r}, \bar{s} \in M^k$ $(k \geq 1)$ and every atomic formula $\phi(x_1, \ldots, x_k)$ in $\mathcal{L}$, $\bar{r} \sim \bar{s}$ implies $\mathfrak{M} \models \phi[\bar{r}] \leftrightarrow \phi[\bar{s}]$ (topological criterion).

All mereotopologies considered here will be $\mathcal{L}$-structures where $\mathcal{L}$ has only a small number of non-logical symbols with mereological or topological interpretation. These symbols are:

(i) the binary predicate symbol $\leq$ where $x \leq y$ is read as 'region x is part of region y',

(ii) the binary predicate symbol C where $\mathrm{C}(x, y)$ is read as 'regions x and y are in contact',

(iii) the unary predicate symbol c where $\mathrm{c}(x)$ is read as 'region x is connected', and

(iv) the unary predicate symbol b where $\mathrm{b}(x)$ is read as 'region x is bounded'.

Different combinations of these symbols will be employed. I write $\mathcal{L}(\Sigma)$ for the usual first-order language with equality and signature Σ. Then $\mathcal{L}(\{\leq, \mathrm{c}\})$, or $\mathcal{L}(\leq, \mathrm{c})$ for short, represents the first-order language with equality and predicate symbols $\leq$ and c. The languages $\mathcal{L}(\leq, \mathrm{c})$, $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$, $\mathcal{L}(\mathrm{C})$ and $\mathcal{L}(\leq, \mathrm{C})$ will feature in this thesis. These four languages will be called *mereotopological languages.* I use the symbol $\mathcal{L}_{mt}$ to refer to any of these four mereotopological languages.

The predicate symbols $\leq, \mathrm{C}$, c and b have the following standard interpretation which will be observed throughout the thesis. Given a topological space X and an $\mathcal{L}_{mt}$-structure $\mathfrak{M}$ with domain $M \subseteq \wp(X)$, I define

$$\begin{array}{lcl} [\leq]^{\mathfrak{M}} & = & \{(r, s) \in M^2 | r \subseteq s\} \\ [\,\mathrm{C}]^{\mathfrak{M}} & = & \{(r, s) \in M^2 | [r] \cap [s] \neq 0\} \\ [\,\mathrm{c}]^{\mathfrak{M}} & = & \{r \in M | r \text{ is connected}\} \\ [\,\mathrm{b}]^{\mathfrak{M}} & = & \begin{cases} \{r \in M | r \text{ is bounded}\} & \text{if } X \text{ is a metric space} \\ M & \text{otherwise} \end{cases} \end{array}$$

The spatial domains introduced in the previous chapter are now interpreted over the mereotopological languages. The various $\mathcal{L}_{mt}$-structures are symbolised by the Gothic letters corresponding to the letters of the spatial domains. Thus, $\mathfrak{F}$ is an $\mathcal{L}_{mt}$-structure with domain $\mathbf{F}$, and the $\mathcal{L}_{mt}$-structures $\mathfrak{J}$, $\mathfrak{S}$, $\mathfrak{P}$ and $\mathfrak{Q}$ have the domains $\mathbf{J}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{Q}$ respectively. The one-point compactifications will be treated analogously. Thus, the $\mathcal{L}_{mt}$-structure $\mathfrak{F}^*$ has the domain $\mathbf{F}^*$ etc. An $\mathcal{L}_{mt}$-structure in a specific mereotopological language is denoted by the Gothic symbol followed by the signature of the language. For example, $\mathfrak{S}^*(\leq, \mathrm{c})$ denotes the model-theoretic structure which interprets the mereotopological language $\mathcal{L}(\leq, \mathrm{c})$ over the spatial domain $\mathbf{S}^*$. Sometimes it will be convenient to refer indifferently to an $\mathcal{L}_{mt}$-structure over the open or the closed plane. Therefore, I will transfer the $\tilde{\ }$-notation, which was used in the previous chapter for spatial domains, to mereotopologies. For example, I take $\tilde{\mathfrak{S}}$ to refer indifferently to $\mathfrak{S}$ or $\mathfrak{S}^*$. I write $\mathfrak{R}$ to refer indifferently to $\mathfrak{F}$, $\mathfrak{J}$ or $\mathfrak{S}$. The domain of $\mathfrak{R}$ is denoted R.

The signature-notation will be used in combination with the $^{\sim}$- or *-notation. Note, however, that a multiple occurrence of one of the $^{\sim}$-notations in a lemma, theorem or proof etc. is understood to refer to the same model-theoretic structure and its spatial domain. Moreover, the occurrence of the $^{\sim}$-notation for several models in the same lemma, theorem or proof etc. is to be resolved either in the *-or non-*-notation for *all* domains and mereotopologies in the respective lemma etc. Thus, if a theorem refers to $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{S}}$ then the theorem holds for $\mathfrak{F}$ and $\mathfrak{S}$, and for $\mathfrak{F}^*$ and $\mathfrak{S}^*$, but not necessarily for $\mathfrak{F}$ and $\mathfrak{S}^*$.

Given the above standard interpretation of the predicate symbols it is easy to see that $\tilde{\mathfrak{R}}(\leq, \mathrm{c})$, $\tilde{\mathfrak{R}}(\leq, \mathrm{c}, \mathrm{b})$ and $\tilde{\mathfrak{R}}(\leq, \mathrm{C})$ are mereotopologies. It will be shown in lemma 4.1.2 that $\tilde{\mathfrak{R}}(\mathrm{C})$ is a mereotopology as well.

The following lemma shows $\tilde{\mathfrak{R}}$ be a mereotopology in the stronger sense of requiring two topologically equivalent tuples of regions to satisfy the same formulae and not only the same atomic formulae. The lemma shows the importance of the notion of topological homogeneity.

Lemma 4.0.2. *Let $\mathfrak{M}$ be a mereotopology in the language $\mathcal{L}_{mt}$ such that M is topologically homogeneous. Let $\bar{r}, \bar{s} \in M^n$ be topologically equivalent. Then $\bar{r}$ and $\bar{s}$ have the same type in $\mathfrak{M}$.*

Proof. By induction on the complexity of ϕ. If ϕ is an atomic formula, then, by assumption, $\mathfrak{M} \models \phi[\bar{r}]$ iff $\mathfrak{M} \models \phi[\bar{s}]$. The only non-trivial recursive case is where ϕ is $\exists x \psi(\bar{x}, x)$. Suppose, then $\mathfrak{M} \models \phi[\bar{a}]$. Let $a \in M$ be such that $\mathfrak{M} \models \psi[\bar{a}, a]$. Since M is topologically homogeneous, let $b \in M$ satisfy $\bar{a}, a \sim \bar{b}, b$. By inductive hypothesis, $\mathfrak{M} \models \psi[\bar{b}, b]$, and hence $\mathfrak{M} \models \phi[\bar{b}]$. □

4.1 The relative expressivity of the mereotopological languages

This section extends the work presented in (Pratt and Schoop, 1999) on the relative expressivity of mereotopological languages in plane mereotopology. Some of the lemmas have been taken directly from the cited work.

Given a set A, a subset $\mathcal{C}$ of A^n for some $n \geq 1$ is said to be a *relation* over A. The main result of this section is:

Theorem 4.1.1. *(i) All relations over* $\mathbf{R}$ *that are* $\mathcal{L}(\leq, \mathrm{c})$*-definable in* $\mathfrak{R}(\leq, \mathrm{c})$ *are also* $\mathcal{L}(\mathrm{C})$*-definable in* $\mathfrak{R}(\mathrm{C})$*, but not vice versa.*

(ii) *All relations over* $\mathbf{S}^*$ *that are* $\mathcal{L}(\mathrm{C})$*-definable in* $\mathfrak{S}^*(\mathrm{C})$ *are also* $\mathcal{L}(\leq, \mathrm{c})$*-definable in* $\mathfrak{S}^*(\leq, \mathrm{c})$*, and vice versa.*

(iii) *All relations over* $\widetilde{\mathbf{S}}$ *that are* $\mathcal{L}(\mathrm{C})$*-definable in* $\mathfrak{S}(\mathrm{C})$ *are also* $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$*-definable in* $\widetilde{\mathbf{S}}(\leq, \mathrm{c}, \mathrm{b})$*, and vice versa.*

Thus, the language $\mathcal{L}(\mathrm{C})$ is strictly more expressive in the open plane than $\mathcal{L}(\leq, \mathrm{c})$; but this advantage is lost when we move to the closed plane. The language $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$, however, has the same expressivity as $\mathcal{L}(\mathrm{C})$ in the open and in the closed plane with respect to the semi-algebraic spatial domains.

The above theorem will be proved in separate lemmas. First I prove that $\mathcal{L}(\mathrm{C})$ is at least as expressive as $\mathcal{L}(\leq, \mathrm{c})$. Specifically, I prove that the interpretations of the primitives $\leq$ and c in $\mathcal{L}(\leq, \mathrm{c})$ are $\mathcal{L}(\mathrm{C})$-definable in $\widetilde{\mathfrak{R}}(\mathrm{C})$.

Lemma 4.1.2. *Let* $r_1, r_2 \in \widetilde{\mathbf{R}}$*. Then* $r_1 \subseteq r_2$ *if and only if* $\widetilde{\mathfrak{R}}(\mathrm{C}) \models \phi_{\leq}[r_1, r_2]$*, where* $\phi_{\leq}(x, y)$ *is the* $\mathcal{L}(\mathrm{C})$*-formula* $\forall z(\mathrm{C}(x, z) \rightarrow \mathrm{C}(y, z))$*.*

Proof. If $r_1 \subseteq r_2$ then $[r_1] \subseteq [r_2]$, so $[t] \cap [r_1] \neq \emptyset$ implies $[t] \cap [r_2] \neq \emptyset$ for any set t. Conversely, if $r_1 \cdot (-r_2)$ is non-empty, by the regularity of $\widetilde{\mathbf{R}}$ (proposition 3.3.11), a region t can be found lying in the interior of $r_1 \cdot (-r_2)$, so that $[t] \cap [r_1] \neq \emptyset$, but $[t] \cap [r_2] = \emptyset$. □

Since the subset relation $\subseteq$ on $\widetilde{\mathbf{R}}$ is $\mathcal{L}(\mathrm{C})$-definable in $\widetilde{\mathfrak{R}}(\mathrm{C})$, the symbol $\leq$ will be used in $\mathcal{L}(\mathrm{C})$-formulae as a shorthand for $\phi_{\leq}$. It follows that the Boolean functions $+$, $\cdot$ and $-$, as well as the constants 0 and 1 are also $\mathcal{L}(\mathrm{C})$-definable; so again, these symbols will be used in $\mathcal{L}(\mathrm{C})$-formulae as a shorthand for their definitions.

Lemma 4.1.3. *Let* $r \in \widetilde{\mathbf{R}}$*. Then* r *is connected if and only if* $\widetilde{\mathfrak{R}}(\mathrm{C}) \models \phi_{\mathrm{c}}[r]$*, where* $\phi_{\mathrm{c}}(x)$ *is the* $\mathcal{L}(\mathrm{C})$*-formula*

$$\begin{aligned}&\forall x_1 \forall x_2 (x = x_1 + x_2 \wedge x_1 \cdot x_2 = 0 \wedge x_1 \neq 0 \wedge x_2 \neq 0 \\ &\qquad \rightarrow \exists x_1' \exists x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge \mathrm{C}(x_1', x_2') \wedge \neg\, \mathrm{C}(x_1' + x_2', -x)))\,.\end{aligned}$$

Proof. It follows from the regularity of $\widetilde{\mathbf{R}}$ (proposition 3.3.11) that for non-empty and disjoint regions r_1 and r_2, $\partial(r_1) \cap \partial(r_2) \cap (r_1 + r_2) \neq \emptyset$ if and only if r_1 and r_2 satisfy the formula

$$\exists x_1' \exists x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge \mathrm{C}(x_1', x_2') \wedge \neg\, \mathrm{C}(x_1' + x_2', -(x_1 + x_2)))$$

in $\widetilde{\mathfrak{R}}(\mathrm{C})$. The result then follows from lemmas 3.1.5, 3.2.14 and 3.3.15. □

I henceforth write the symbol c in $\mathcal{L}(\mathrm{C})$-formulae, understanding it as a shorthand for ϕ_{c}. The next task is to show that not every relation over $\mathbf{R}$ is $\mathcal{L}(\leq, \mathrm{c})$-definable in $\mathfrak{R}$.

Lemma 4.1.4. *There is no formula in $\mathcal{L}(\leq, \mathrm{c})$ defining the contact-relation in $\mathfrak{R}$.*

Proof. Let $r_1^*, r_2^*, s_1^*, s_2^* \in \mathbf{S}^*$ be disjoint Jordan regions such that $[r_1^*] \cap [r_2^*] = \{p_\infty\}$ and $[s_2^*] \cap [s_2^*] = \{p\}$ for some point $p \neq p_\infty$. It follows from standard results in topology that there exists a homeomorphism $h\colon (\mathbb{R}^2)^* \to (\mathbb{R}^2)^*$ that maps r_i to s_i $(i = 1, 2)$. By lemma 4.0.2, r_1^*, r_2^* and s_1^*, s_2^* satisfy the same formulae in $\mathfrak{R}^*(\leq, \mathrm{c})$, and it follows from lemma 3.2.1 on page 60, r_1, r_2 and s_1, s_2 satisfy the same formulae in $\mathfrak{R}(\leq, \mathrm{c})$. However, $[r_1] \cap [r_2] = \emptyset$ and $[s_1] \cap [s_2] \neq \emptyset$. □

The next task is to show that $\mathcal{L}(\leq, \mathrm{c})$ is as expressive as $\mathcal{L}(\mathrm{C})$ in $\mathfrak{S}^*$. Specifically, I prove that the interpretation of the primitive C is $\mathcal{L}(\leq, \mathrm{c})$-definable in $\mathfrak{S}^*$.

Lemma 4.1.5. *Let $s_1, s_2, t \in \mathbf{R}^*$ with $-(s_1 + t)$, $-(s_2 + t)$ and t all connected, and $[s_1] \cap [s_2] = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.*

Proof. I appeal to theorem 3.3.17 on page 71. Let $F_1 = s_1 + t$ and $F_2 = s_2 + t$. Since t is connected, $F_1 \cap F_2 = [t]$ is connected. Let $p, q \in -(s_1 + s_2 + t) = (\mathbb{R}^2)^* \setminus (F_1 \cup F_2)$. Then $p, q \in -(s_1 + t) = (\mathbb{R}^2)^* \setminus F_1$ and $p, q \in -(s_2 + t) = (\mathbb{R}^2)^* \setminus F_2$. Since $-(s_1 + t)$ and $-(s_2 + t)$ are connected, p and q are connected in these sets. Then p and q are connected in $-(s_1 + s_2 + t)$. Hence, $-(s_1 + s_2 + t)$ is connected. □

In the sequel, I write $\pi(y_1, y_2)$ to abbreviate the formula

$$\exists z(\mathrm{c}(-(y_1 + z)) \wedge \mathrm{c}(-(y_2 + z)) \wedge \mathrm{c}(z) \wedge \neg\, \mathrm{c}(-(y_1 + y_2 + z))).$$

Lemma 4.1.6. *Let $r_1, r_2 \in \mathbf{S}^*$. Then $[r_1] \cap [r_2] \neq \emptyset$ if and only if $\mathfrak{S}^*(\leq, \mathrm{c}) \models \phi_{\mathrm{C}}[r_1, r_2]$, where $\phi_{\mathrm{C}}^*(x_1, x_2)$ is the $\mathcal{L}(\leq, \mathrm{c})$-formula:*

$$\exists y_1 \exists y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge \pi(y_1, y_2)).$$

Proof. The if-direction is immediate given lemma 4.1.5. For the only-if direction, it is easy to see by the accessibility of boundaries that if $[r_1] \cap [r_2] \neq \emptyset$ then we can find $s_1 \leq r_1$, $s_2 \leq r_2$ such that s_1 and s_2 satisfy π, whence $\mathfrak{S}^*(\leq, \mathrm{c}) \models \phi_{\mathrm{C}}^*[r_1, r_2]$. □

Hence, the language $\mathcal{L}(\mathrm{C})$ is expressive enough to distinguish between the open- and the closed-plane mereotopologies $\mathfrak{S}$ and $\mathfrak{S}^*$ while the language $\mathcal{L}(\leq, \mathrm{c})$ cannot tell them apart. The following lemma will be used to show that boundedness is $\mathcal{L}(\mathrm{C})$-definable in $\widetilde{\mathfrak{S}}$.

Lemma 4.1.7. *Let $s_1, s_2, t \in \mathbf{R}^*$ with $-(s_1 + t)$, $-(s_2 + t)$ and t all connected, $-(s_1 + s_2 + t)$ disconnected, and $[s_1] \cap [s_2] = \emptyset$. Then s_1 and s_2 are unbounded.*

Proof. Let $^* : \mathfrak{R}(\leq, \mathrm{c}) \to \mathfrak{R}^*(\leq, \mathrm{c})$ be the model isomorphism given by proposition 3.2.1 on page 60. Then $(-(s_1 + t))^* = -(s_1^* + t^*)$, $(-(s_2 + t))^* = -(s_2^* + t^*)$, and t^* are all connected, while $(-(s_1 + s_2 + t))^* = -(s_1^* + s_2^* + t^*)$ is disconnected. It follows from lemma 4.1.5 that $[s_1^*] \cap [s_2^*] \neq \emptyset$. Since $[s_1] \cap [s_2] = \emptyset$, it is clear from the definition of * that $[s_1^*]$ and $[s_2^*]$ both contain the point at infinity, whence s_1 and s_2 are unbounded. □

Lemma 4.1.8. *Let $r \in \mathbf{S}$. Then r is bounded if and only if $\mathfrak{S}(\mathrm{C}) \models \phi_{\mathrm{b}}[r]$, where $\phi_{\mathrm{b}}(x)$ is the $\mathcal{L}(\mathrm{C})$-formula:*

$$\neg \exists y_1 \exists y_2 (y_1 \leq x \wedge y_2 \leq x \wedge \pi(y_1, y_2) \wedge \neg\, \mathrm{C}(y_1, y_2)).$$

Proof. If r satisfies $\neg \phi_{\mathrm{b}}(x)$ in $\mathfrak{S}(\mathrm{C})$, then, by lemma 4.1.7, r contains two unbounded regions, so is certainly itself unbounded. Conversely, if r is unbounded, by the accessibility of boundaries, it is simple to construct regions $s_1, s_2 \in \mathbf{S}$ such that, $s_1 \leq r$, $s_2 \leq r$ and $[s_1] \cap [s_2] = \emptyset$, and satisfying the $\mathcal{L}(\mathrm{C})$-formula $\pi(y_1, y_2)$. □

Since boundedness is trivially definable in $\mathfrak{S}^*(\mathrm{C})$, it follows from lemmas 4.1.2 and 4.1.3 that $\mathcal{L}(\mathrm{C})$ is as expressive as $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$ in $\mathfrak{S}^*$. Since parthood, connection and boundedness are $\mathcal{L}(\mathrm{C})$-definable in $\mathfrak{S}$, it remains to show that contact is $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$-definable in $\mathfrak{S}$.

Lemma 4.1.9. *Let $r_1, r_2 \in \mathbf{S}$. Then $[r_1] \cap [r_2] \neq \emptyset$ if and only if $\mathfrak{S} \models \phi_{\mathrm{C}}[r_1, r_2]$ where $\phi_{\mathrm{C}}(x_1, x_2)$ is the formula*

$$\exists y_1 \exists y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge \pi(y_1, y_2) \wedge (\mathrm{b}(y_1) \vee \mathrm{b}(y_2)))\,.$$

Proof. Assume $(r_1, r_2) \in \mathbf{S}^2$ satisfies $\phi_{\mathrm{C}}(x_1, x_2)$ in $\mathfrak{S}$. Then there are regions $s_1, s_2 \in \mathbf{S}$ at least one of which is bounded such that $s_1 \subseteq r_1$, $s_2 \subseteq r_2$ and s_1, s_2 satisfies $\pi(y_1, y_2)$ in $\mathfrak{S}$. Then the pair (s_1^*, s_2^*) satisfies $\phi_{\mathrm{C}}^*(x_1, x_2)$ in $\mathfrak{S}^*$. Hence,

$[s_1^*] \cap [s_2^*] \neq \emptyset$. Since $p_\infty \notin [s_1^*] \cap [s_2^*]$, $([s_1^*] \cap [s_2^*]) \setminus \{p_\infty\} \neq \emptyset$. Thus, $[s_1] \cap [s_2] \neq \emptyset$ and therefore $[r_1] \cap [r_2] \neq \emptyset$.

For the converse direction, assume that the closures of $r_1, r_2 \in \mathbf{S}$ have a point p in common. By lemma 3.3.20, there exist bounded regions $s_1, s_2 \in \mathbf{S}$ such that $s_1 \subseteq r_1$, $s_2 \subseteq r_2$ and $p \in [s_1] \cap [s_2]$. Then by lemma 4.1.9 the pair (s_1^*, s_2^*) satisfies $\phi_C^*(x_1, x_2)$ in $\mathfrak{S}^*$. Hence, there are regions $t_1^*, t_2^* \in \mathbf{S}^*$ such that $t_1^* \subseteq s_1^*$, $t_2^* \subseteq s_2^*$ and $\mathfrak{S}^* \models \pi[t_1^*, t_2^*]$. Since $p_\infty \notin [s_1^*] \cup [s_2^*]$, it follows that $p_\infty \notin [t_1^*] \cup [t_2^*]$. Hence t_1 and t_2 are bounded and $\mathfrak{S} \models \pi[t_1, t_2]$. Hence, $\mathfrak{S} \models \phi_C[r_1, r_2]$. □

The last lemma completes the proof of theorem 4.1.1.

4.2 Homeomorphisms from automorphisms

Before I go on to investigate the model-theoretic properties of the mereotopologies and provide further expressivity results, I prove a technical result that will be used later on.

Definition 4.2.1. Let X be a topological space and $\mathfrak{M}$ be a mereotopology over X. The mereotopology $\mathfrak{M}$ will be called

(i) Hausdorff if the spatial domain M is Hausdorff,

(ii) regular if the spatial domain M is regular and

(iii) normal if the spatial domain M is normal.

Theorem 4.2.2. *Let $\mathfrak{M}$ be a mereotopology over a topological space X satisfying the following conditions:*

(i) $\mathfrak{M}$ is a Boolean subalgebra of $RO(X)$;

(ii) $\mathfrak{M}$ is Hausdorff;

(iii) $\mathfrak{M}$ is regular;

(iv) if $p \in X$ then there is $a \in M$ such that $p \in a$ and $[a]$ is compact (local compactness);

(v) the relations $\{(a, b) \in M^2 | [a] \cap [b] \neq \emptyset\}$ and $\{a \in M | [a] \text{ compact}\}$ are $\mathcal{L}$-definable in $\mathfrak{M}$.

Then, if α is an $\mathfrak{M}$-automorphism, there exists a homeomorphism $h : X \to X$ such that, for all $a \in M$, $\alpha(a) = h(a) =_{df} \{h(p) | p \in a\}$.

The proof of this theorem is taken nearly unchanged from (Pratt and Schoop, 1999). It employs a technique of Roeper (1997), to reconstruct points as equivalence classes of ultrafilters on $\mathfrak{M}$ as defined below. By showing that automorphisms of $\mathfrak{M}$ map equivalent ultrafilters to equivalent ultrafilters, corresponding homeomorphisms of the space X onto itself are obtained. For the remainder of this section, I assume that $\mathfrak{M}$ satisfies the conditions of theorem 4.2.2.

Definition 4.2.3. An ultrafilter on $\mathfrak{M}$ is a a set $U \subseteq M$ such that

(i) $X \in U$,

(ii) if $a \in U$, $b \in U$ and $a \subseteq b$, then $b \in U$,

(iii) if $a \in U$ and $b \in U$, then $a \cap b \in U$,

(iv) for each $a \in M$, either a or its complement is in U, but not both.

An ultrafilter U on $\mathfrak{M}$ is said to be a *compact* if U contains some u such that $[u]$ is compact.

Lemma 4.2.4. *Let U be a compact ultrafilter on $\mathfrak{M}$. Then the set $\bigcap\{[u]|u \in U\}$ is a singleton.*

Proof. This is an adaptation of a standard result (Koppelberg, 1989, Chapter 1, Exercise 2). I first show that $\bigcap\{[u]|u \in U\}$ contains at least one point. Choose $u_0 \in U$ such that $[u_0]$ is compact. Then $\bigcap\{[u]|u \in U\} = \emptyset$ implies $\bigcup\{X \setminus [u]|u \in U\} = X$, whence $\{-u|u \in U\}$ covers X and hence $[u_0]$. By compactness of $[u_0]$, let $u_1, \ldots, u_n \in U$ be such that $u_0 \subseteq [u_0] \subseteq -u_1 \cup \ldots \cup -u_n \subseteq -u_1 + \ldots + -u_n$, whence $u_0 \cdot u_1 \cdot \ldots \cdot u_n = \emptyset$ contradicting the fact that U is a proper filter. Next we show that $\bigcap\{[u]|u \in U\}$ contains at most one point. Suppose that $p, q \in \bigcap\{[u]|u \in U\}$ with $p \neq q$. By assumption (Hausdorffness), we can find disjoint $a, b \in A$ such that $p \in a$ and $q \in b$. Hence $p \notin [-a]$ and $q \notin [a]$. Since U is maximal, either a or $-a$ is in U, so that either p or q is not in $\bigcap\{[u]|u \in U\}$. □

Given a compact ultrafilter U, the element of $\bigcap\{[u]|u \in U\}$ will be denoted by p_U and U is said to *converge to* p_U.

Lemma 4.2.5. *Let U be a compact ultrafilter on $\mathfrak{M}$ converging to p_U. If $p_U \in a \in M$ then $a \in U$. Furthermore, there exists $b \in U$ such that $p_U \in b$ and $[b] \subseteq a$.*

Proof. Suppose $p_U \in a \in M$. Then $p_U \notin [-a]$. Since U is an ultrafilter converging to p_U, $p_U \in [u]$ for every $u \in U$, so $-a \notin U$, whence $a \in U$. For the second part of the lemma, observe that $X \setminus a$ is closed. By assumption (regularity), there exist disjoint $b, b' \in A$ such that $p_U \in b$ and $X \setminus a \subseteq b'$. Thus $[b] \subseteq a$; and by the first part of the lemma, $b \in U$. □

Definition 4.2.6. If U and V are ultrafilters on $\mathfrak{M}$, we say U and V are *equivalent* if $[u] \cap [v] \neq \emptyset$ for all $u \in U, v \in V$.

Lemma 4.2.7. *If U and V are compact ultrafilters on $\mathfrak{M}$, then $p_U = p_V$ iff U and V are equivalent.*

Proof. The only-if direction is trivial. For the if-direction suppose that $p_U \neq p_V$. By assumption (Hausdorffness), there exist disjoint $a, b \in M$ such that $p_U \in a$, and $p_V \in b$. By lemma 4.2.5, $b \in V$ and, furthermore, there exists $u \in U$ such that $[u] \subseteq a$. Hence, $[u] \cap [b] = \emptyset$ contradicting the equivalence of U and V. □

Lemma 4.2.8. *Let α be an $\mathfrak{M}$-automorphism and U and V equivalent compact ultrafilters on $\mathfrak{M}$. Then $\alpha(U)$ and $\alpha(V)$ are equivalent compact ultrafilters on $\mathfrak{M}$.*

Proof. It is straightforward to show that α maps ultrafilters to ultrafilters. The result then follows because the relations $\{a \in M | [a] \text{ compact}\}$ and $\{(a, b) \in M^2 | [a] \cap [b] \neq \emptyset\}$ are, by assumption, definable in $\mathfrak{M}$. □

Lemma 4.2.9. *Let α be an $\mathfrak{M}$-automorphism, $a \in M$, and U a compact ultrafilter with $p_U \in a$. Then $p_{\alpha(U)} \in \alpha(a)$.*

Proof. By lemma 4.2.5, $a \in U$, and there exists $b \in U$ such that $p_U \in [b]$ and $[b] \subseteq a$, so that $[b] \cap [-a] = \emptyset$. By assumption, $\{(a, b) \in M^2 | [a] \cap [b] = \emptyset\}$ is $\mathcal{L}$-definable in $\mathfrak{M}$. Then, since α is an automorphism, $[\alpha(b)] \cap [-\alpha(a)] = \emptyset$, i.e. $[\alpha(b)] \subseteq \alpha(a)$. Since $\alpha(b) \in \alpha(U)$, $p_{\alpha(U)} \in [\alpha(b)] \subseteq \alpha(a)$. □

It remains to show theorem 4.2.2.

Proof of theorem 4.2.2. Suppose that α is an $\mathfrak{M}$-automorphism. Let for a compact ultrafilter U on $\mathfrak{M}$ the function h be defined by $h(p_U) = p_{\alpha(U)}$. I show: (i) h is well-defined and injective, (ii) both the domain and range of h are the set X, (iii) for all $a \in A$, $\alpha(a) = h(a) =_{df} \{h(p) | p \in a\}$ and $\alpha^{-1}(a) = h^{-1}(a)$, and (iv) h and h^{-1} are continuous.

(i) Let U and V be compact ultrafilters on $\mathfrak{M}$ both converging to p. By lemma

4.2.8, the automorphism α maps equivalent ultrafilters to equivalent ultrafilters. Hence, h is well defined. Applying the same reasoning to α^{-1}, h is injective.
(ii) Let $p \in X$. Then $\{a \in M | p \in a\}$ is a filter on $\mathfrak{M}$ and by assumption (local compactness) contains some $a \in M$ with $[a]$ compact. By the prime ideal theorem (Koppelberg, 1989, Chapter 1, 2.16), this filter can be extended to a (compact) ultrafilter U on $\mathfrak{M}$. By lemma 4.2.4, U converges to some point p_U. By the assumption of Hausdorffness of M and lemma 4.2.5, $p = p_U$. Thus, the domain of h is X. Applying the same reasoning to α^{-1}, the range of h is X.
(iii) Let $p_U \in \alpha(a)$ and U be some compact ultrafilter on $\mathfrak{M}$ converging to p_U. By lemma 4.2.9, $p_{\alpha^{-1}(U)} \in a$. Hence, $p_U = h(p_{\alpha^{-1}(U)}) \in h(a)$. Conversely, let $p_U \in h(a)$. By the definition of h, $p_{\alpha^{-1}(U)} \in a$ and by lemma 4.2.9, $p_U \in \alpha(a)$. Hence $\alpha(a) = h(a)$. Since $\alpha^{-1}(a) \in M$, $\alpha^{-1}(a) = h^{-1}(h(\alpha^{-1}(a)) = h^{-1}(\alpha(\alpha^{-1}(a))) = h^{-1}(a)$.
(vi) Let $u \subseteq X$ be an open set. By the regularity of M, there is for each point $p \in u$ an open set $a_p \in M$ with $p \in a_p \subseteq u$. Thus the set $\mathcal{U} = \{a_p \in M | p \in u\}$ satisfies $\bigcup \mathcal{U} = u$. Then $h(u) = h(\bigcup \mathcal{U}) = \bigcup_{a \in \mathcal{U}} h(a) = \bigcup_{a \in \mathcal{U}} \alpha(a)$ is a union of open sets and hence is itself an open set. Therefore, h^{-1} is continuous. By substituting h^{-1} and α^{-1} for h and a, respectively, h is continuous. □

4.3 Model-theoretic properties of planar mereotopologies

In this section, I will investigate the model-theoretic properties of the mereotopologies defined at the beginning of this chapter. Some of the results in this section have been presented in a similar form in (Pratt and Lemon, 1997).

Theorem 4.3.1. *The mereotopologies* $\mathfrak{R}(\leq, \mathrm{c})$ *and* $\mathfrak{R}^*(\leq, \mathrm{c})$ *are isomorphic.*

Proof. The theorem is a direct consequence of proposition 3.2.1 on page 60. □

Theorem 4.3.2. *The mereotopologies* $\mathfrak{R}(\mathrm{C})$ *and* $\mathfrak{R}^*(\mathrm{C})$ *are not elementarily equivalent.*

Proof. Consider the formula $\pi(y_1, y_2)$ given on page 90. It follows from lemma 4.1.5 that, given $r_1, r_2 \in \mathbf{R}^*$, $\mathfrak{R}^* \models \pi[r_1, r_2]$ implies $[r_1] \cap [r_2] \neq 0$. Hence, the formula $\forall x_1 \forall x_2 (\pi(x_1, x_2) \rightarrow \mathrm{C}(x_1, x_2))$ holds in $\mathfrak{R}^*$. However, it is easy to see that the formula does not hold for all unbounded regions in the open plane mereotopology $\mathfrak{R}$. □

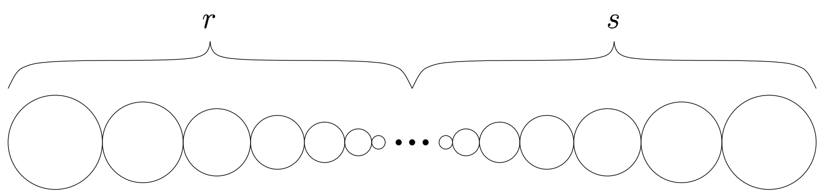


Figure 4.1: Two elements of $\widetilde{\mathbf{J}}$ in contact

Theorem 4.3.3. *(i)* $\widetilde{\mathfrak{J}}(\leq, \mathrm{c}) \not\equiv \widetilde{\mathfrak{S}}(\leq, \mathrm{c})$ *(ii)* $\widetilde{\mathfrak{J}}(\mathrm{C}) \not\equiv \widetilde{\mathfrak{S}}(\mathrm{C})$
(iii) $\widetilde{\mathfrak{F}}(\leq, \mathrm{c}) \not\equiv \widetilde{\mathfrak{S}}(\leq, \mathrm{c})$ *(iv)* $\widetilde{\mathfrak{F}}(\mathrm{C}) \not\equiv \widetilde{\mathfrak{S}}(\mathrm{C})$

Proof. (i) Remember that in lemma 4.1.6 the formula $\phi^*_{\mathrm{C}}(x, y)$ was shown to define the set $\{(r, s) \in (\mathbf{S}^*)^2 | [r] \cap [s] \neq \emptyset\}$ in $\mathfrak{S}^*(\leq, \mathrm{c})$. Let σ stand for the sentence

$$\forall x \forall y (\phi^*_{\mathrm{C}}(x, y) \rightarrow \exists x' \exists y' (x' \leq x \wedge \mathrm{c}(x') \wedge y' \leq y \wedge \mathrm{c}(y') \wedge \phi^*_{\mathrm{C}}(x', y'))) \,.$$

Assume $(r, s) \in (\mathbf{S}^*)^2$ satisfies $\phi^*_{\mathrm{C}}(x, y)$ in $\mathfrak{S}^*(\leq, \mathrm{c})$. Then $[r] \cap [s] \neq \emptyset$. Since r and s have finitely many components, there exist components r' and s' of r and s respectively such that $[r'] \cap [s'] \neq \emptyset$. Hence, (r', s') satisfies $\phi^*_{\mathrm{C}}(x, y)$ in $\mathfrak{S}^*(\leq, \mathrm{c})$, whence $\mathfrak{S}^*(\leq, \mathrm{c}) \models \sigma$.

The regions $r, s \in \mathbf{J}^*$ which are depicted in figure 4.1 visibly satisfy $\phi^*_{\mathrm{C}}(x, y)$ in $\mathfrak{J}^*$. However, no two connected parts r' and s' of r and s respectively are in contact. By lemma 4.1.5, (r', s') does not satisfy $\phi^*_{\mathrm{C}}(x, y)$ in $\mathfrak{J}^*(\leq, \mathrm{c})$. Hence, $\mathfrak{J}^*(\leq, \mathrm{c}) \not\models \sigma$, whence $\mathfrak{J}(\leq, \mathrm{c}) \equiv \mathfrak{J}^*(\leq, \mathrm{c}) \not\equiv \mathfrak{S}^*(\leq, \mathrm{c}) \equiv \mathfrak{S}(\leq, \mathrm{c})$.

(ii) Follows directly from (i) and lemmas 4.1.2 and 4.1.3.

(iii) Follows directly from (i) and the fact that $\widetilde{\mathbf{J}} \subseteq \widetilde{\mathbf{F}}$.

(iv) Follows directly from (iii) and lemmas 4.1.2 and 4.1.3. □

Theorem 4.3.4. $\mathfrak{Q} \preceq \mathfrak{P} \preceq \mathfrak{S}$ *and* $\mathfrak{Q}^* \preceq \mathfrak{P}^* \preceq \mathfrak{S}^*$.

Proof. I show $\mathfrak{P} \preceq \mathfrak{S}$ first. By construction, $\mathfrak{P} \subseteq \mathfrak{S}$. By the Tarski-Vaught-Lemma (Mendelson, 1997, Proposition 2.37), it is sufficient to show that for every formula $\phi(x_0, \ldots, x_n)$ in $\mathcal{L}$, $r \in \mathbf{S}$ and $a_1, \ldots, a_n \in \mathbf{P}$, if $\mathfrak{S} \models \phi[r, a_1, \ldots, a_n]$ then there exists $b \in \mathbf{P}$ such that $\mathfrak{S} \models \phi[b, a_1, \ldots, a_n]$. By lemma 3.5.1, for any $r \in \mathbf{S}$, $a_1, \ldots, a_n \in \mathbf{P}$ there exists $b \in \mathbf{P}$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$. Then by lemma 4.0.2, if $\mathfrak{S} \models \phi[r, a_1, \ldots, a_n]$ then $\mathfrak{S} \models \phi[b, a_1, \ldots, a_n]$. The other cases can be proved analogously appealing to lemma 3.5.2 where necessary. □

Having investigated the model-theoretic relations between the mereotopologies, now I will investigate the model-theoretic properties of the mereotopologies and their theories. This requires the introduction of further model-theoretic concepts.

Definition 4.3.5. Let $\Sigma(\bar{x})$ be a set of formulae in the free variables $\bar{x} = x_1, \ldots, x_n$ and let $\mathfrak{A}$ be an $\mathcal{L}$-structure. The structure $\mathfrak{A}$ is said to *realize* $\Sigma(\bar{x})$ if some n-tuple of elements of A satisfies $\Sigma(\bar{x})$. The structure $\mathfrak{A}$ is said to *omit* $\Sigma(\bar{x})$ if $\mathfrak{A}$ does not realize $\Sigma(\bar{x})$.

Given a model $\mathfrak{A}$ and a subset $X \subseteq A$, the model $\mathfrak{A}$ expanded by new constant symbols taking the elements of X as interpretation will be denoted by $\mathfrak{A}_X$, and its language by $\mathcal{L}_X$. Then $\mathfrak{A}_X$ is called an *extension* of $\mathfrak{A}$. A model $\mathfrak{A}$ is said to be *ω-saturated* if for every finite set $X \subseteq A$, every set of formulae $\Sigma(x)$ of $\mathcal{L}_X$ consistent with $Th(\mathfrak{A}_X)$ is realized in $\mathfrak{A}_X$.

A theory T is said to be *categorical in cardinality* κ, or *κ-categorical*, if T has a model of cardinality κ and every two models of T of cardinality κ are isomorphic. A theory T is *α-stable* if for every model $\mathfrak{A}$ of T and any set $X \subseteq A$ of cardinality α, the extension $\mathfrak{A}_X$ realizes exactly α types in a single variable.

Let $\delta_n(x)$ $(n \geq 1)$ stand for the formula

$$\exists x_1 \ldots \exists x_n \Big(x = x_1 + \ldots + x_n \wedge \bigwedge_{i=1}^{n} \mathrm{c}(x_i) \Big) .$$

Then given a mereotopology $\mathfrak{M}(\leq, \mathrm{c})$ and a region $r \in M$, r will satisfy $\delta_n(x)$ if and only if r is the sum of n connected regions. Let $\Delta(x) = \{\neg\delta_n(x) | n \geq 1\}$. Then all regions in M have finitely many components if and only if $\mathfrak{M}(\leq, \mathrm{c})$ omits $\Delta(x)$. For example, $\widetilde{\mathfrak{S}}(\leq, \mathrm{c})$ omits $\Delta(x)$.

In the sequel, I make use of the fact that by virtue of lemmas 4.1.2 and 4.1.3 it is admissible to use with respect to $\widetilde{\mathfrak{S}}$ the symbols $\leq$ and c as abbreviations in any of the mereotopological languages.

Proposition 4.3.6. *There exists a countable model of the $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{S}}$ realizing $\Delta(x)$.*

Proof. Let $\mathcal{L}^c_{mt}$ be the language $\mathcal{L}_{mt}$ enriched with the infinite set $\{c_i | i \in \mathbb{N}\}$ of constant symbols. Let T^c be the theory $Th(\widetilde{\mathfrak{S}}) \cup \{\mathrm{comp}(c_i, c_0) | 1 \leq i\} \cup \{c_i \neq c_j | 1 \leq i, j, i \neq j\}$ where $\mathrm{comp}(x, y)$ stands for the formula $\forall z(x \leq z \wedge z \leq y \wedge \mathrm{c}(z) \rightarrow x = z)$ expressing that x is a component of y. It is easy to see that any finite subtheory of T^c is consistent. Hence, by the compactness theorem, T^c

is consistent. Therefore, there exists a model $\mathfrak{M}$ of T^c not omitting $\Delta(x)$, which, by the Downwards Löwenheim-Skølem theorem, can be chosen to be countable. The reduct of $\mathfrak{M}$ to $\mathcal{L}_{mt}$, i.e. the model $\mathfrak{M}$ where the interpretation is restricted to the language $\mathcal{L}_{mt}$, is the required model. □

Corollary 4.3.7. *The model $\widetilde{\mathfrak{S}}$ is not ω-saturated.*

Corollary 4.3.8. *The $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{S}}$ is not ω-categorical.*

Proof. By proposition 4.3.6 there exists a countable model $\mathfrak{M}$ of $Th(\widetilde{\mathfrak{S}}) = Th(\widetilde{\mathfrak{Q}})$ which realizes $\Delta(x)$. Hence, $\widetilde{\mathfrak{Q}} \not\cong \mathfrak{M}$. □

However, I show below that the $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{S}}$ satisfies the following weaker categoricity result.

Proposition 4.3.9. *Any two countable models of the $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{S}}$ which omit $\Delta(x)$ are isomorphic.*

By Scott's isomorphism theorem (Keisler, 1971), there exists a formula ϕ in the infinitary language $\mathcal{L}_{\omega_1\omega}$ such that any two countable models of $Th(\widetilde{\mathfrak{S}}(\mathcal{L}_{mt}))$ are isomorphic if and only if they satisfy ϕ. Obviously, in this case, the formula ϕ is

$$\bigvee_{n\geq 1}\left(\exists x_1 \ldots \exists x_n (x = x_1 + \ldots + x_n \wedge \bigwedge_{i=1}^{n} \mathrm{c}(x_i))\right).$$

Theorem 4.3.10. *The $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{R}}$ is not ω-stable.*

Proof. Let $a_0 \in \widetilde{\mathbf{R}}$ be non-empty. By proposition 3.3.11, $\widetilde{\mathfrak{R}}$ is regular. Hence, there exists a countable sequence $a_1, a_2, \ldots$ in $\widetilde{\mathbf{R}}$ such that $a_0 \supset a_1 \supset a_2 \supset \ldots$. Again since $\widetilde{\mathbf{R}}$ is regular, for each $a_i \supset a_j$ there is a'such that $a_i \supset a' \supset a_j$. Hence, there is a countable subset $X \subseteq \widetilde{\mathbf{R}}$ on which $\subset$ defines a dense linear order. For each initial segment $Y \subseteq X$, the set of formulae

$$\{c_y \leq v | y \in Y\} \cup \{v \leq c_x | x \in X \setminus Y\}$$

is consistent with $Th(\widetilde{\mathfrak{R}}_X)$ and can be extended to a unique type $\theta_Y(v)$ consistent with $Th(\widetilde{\mathfrak{R}}_X)$. Hence, there is a model $\mathfrak{M}$ of $Th(\widetilde{\mathfrak{R}}_X)$ realizing each type $\theta_Y(v)$. Since X has 2^ω initial segments, $\mathfrak{M}$ realizes at least 2^ω types in one variable. □

Theorem 4.3.11. *The $\mathcal{L}_{mt}$-theory of $\widetilde{\mathfrak{S}}$ is not categorical in any cardinality.*

Proof. By (Chang and Keisler, 1990, Lemma 7.1.4), if a theory T is categorical in some uncountable cardinality, then T is ω-stable. It follows from corollary 4.3.8 and theorem 4.3.10 that $Th(\widetilde{\mathfrak{S}})$ is not categorical in any cardinality. □

Definition 4.3.12. A formula $\phi(\bar{x})$ in $\mathcal{L}$ is *complete* in a theory T if, for all formulae $\psi(\bar{x})$, either $T \models \phi(\bar{x}) \to \psi(\bar{x})$ or $T \models \phi(\bar{x}) \to \neg\psi(\bar{x})$ holds. A model $\mathfrak{A}$ is *atomic* if every n-tuple $\bar{a}$ in A satisfies a formula $\phi(\bar{x})$ in $\mathfrak{A}$ which is complete in $Th(\mathfrak{A})$. A formula $\phi(x_1, \dots x_n)$ is said to be *topologically complete* in a mereotopology $\mathfrak{M}$ if any two tuples $\bar{r}, \bar{s} \in M^n$ which satisfy $\phi(\bar{x})$ in $\mathfrak{M}$ are topologically equivalent.

In the remainder of this chapter, let $\theta_k(x_1, \dots, x_n)$ stand for an $\mathcal{L}_{mt}$-formula of the form

$$\exists y_1 \dots \exists y_k(\rho_{\Gamma,I}(y_1, \dots, y_k) \wedge \sigma(x_1, \dots, x_n, y_1, \dots, y_k))$$

where $\rho_{\Gamma,I}(y_1, \dots, y_k)$ defines $y_1, \dots, y_k$ to be a radial partition with binary connection graph Γ and defines y_i to be bounded if and only if $y_i \in I$, and $\sigma(\bar{x}, \bar{y})$ defines each x_i be the sum of specific y_j's. In case of $\mathcal{L}_{mt} = \mathcal{L}(\leq, \mathrm{c})$, a formula $\rho_{\Gamma,I}(y_1, \dots, y_k)$ defines $y_1, \dots, y_k$ to be a radial partition with binary connection graph Γ. Such formulae $\rho_{\Gamma,I}(\bar{y})$ and $\sigma(\bar{x}, \bar{y})$ can certainly be constructed.

Theorem 4.3.13. *A formula $\theta_k(x_1, \dots, x_n)$ is topologically complete in $\widetilde{\mathfrak{S}}(\mathrm{C})$, $\widetilde{\mathfrak{S}}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathfrak{S}^*(\leq, \mathrm{c})$. Moreover, $\theta_k(x_1, \dots, x_n)$ is complete in $Th(\widetilde{\mathfrak{S}})$, and every n-tuple in $\widetilde{\mathbf{S}}$ satisfies such formula.*

Proof. Let for this proof $\widetilde{\mathfrak{S}}$ only stand for $\widetilde{\mathfrak{S}}(\mathrm{C})$, $\widetilde{\mathfrak{S}}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathfrak{S}^*(\leq, \mathrm{c})$. Assume that the n-tuples $\bar{r}$ and $\bar{s}$ in $\widetilde{\mathbf{S}}$ satisfy $\theta_k(\bar{x})$. Then there are k-element radial partitions $\bar{u}$ and $\bar{v}$ refining $\bar{r}$ and $\bar{s}$ respectively such that $u_i + u_j$ is connected iff $v_i + v_j$ is connected $(1 \leq i < j \leq k)$, and u_i is bounded iff v_i is bounded $(1 \leq i \leq k)$. By lemma 3.3.33 on page 76, $\bar{u} \sim \bar{v}$ and hence $\bar{r} \sim \bar{s}$. Thus, $\theta_k(\bar{x})$ is topologically complete.

It follows from lemma 4.0.2 that all tuples satisfying $\theta_k(\bar{x})$ have the same type. Then $\widetilde{\mathfrak{S}} \models \theta_k(\bar{x}) \to \phi(\bar{x})$ or $\widetilde{\mathfrak{S}} \models \theta_k(\bar{x}) \to \neg\phi(\bar{x})$ for all formulae $\phi(\bar{x})$. Hence, $Th(\widetilde{\mathfrak{S}}) \models \theta_k(\bar{x}) \to \phi(\bar{x})$ or $Th(\widetilde{\mathfrak{S}}) \models \theta_k(\bar{x}) \to \neg\phi(\bar{x})$. Thus, $\theta_k(\bar{x})$ is complete.

Let $r_1, \dots, r_n \in \widetilde{\mathbf{S}}$. By lemma 3.3.36 on page 79 there exists a radial partition $s_1, \dots, s_k$ refining the connected partition generated by $r_1, \dots, r_n$. Hence, $s_1, \dots, s_k$ refines $r_1, \dots, r_n$. Then $s_1, \dots, s_k$ satisfies some formula $\rho_{\Gamma,I}(\bar{y})$ and $r_1, \dots, r_n, s_1, \dots, s_k$ satisfies some formula $\sigma(\bar{x}, \bar{y})$. Hence, $r_1, \dots, r_n$ satisfies some formula $\theta_k(\bar{x})$.

Since $\mathfrak{S}(\leq, \mathrm{c})$ and $\mathfrak{S}^*(\leq, \mathrm{c})$ are isomorphic by theorem 4.3.1, any n-tuple of $\mathbf{S}$ satisfies some formula $\theta_k(\bar{x})$, which is complete in $Th(\mathfrak{S}^*(\leq, \mathrm{c})) = Th(\mathfrak{S}(\leq, \mathrm{c}))$. □

Thus, $\widetilde{\mathfrak{S}}$ and therefore also $\widetilde{\mathfrak{P}}$ and $\widetilde{\mathfrak{Q}}$ are atomic models. However, to prove the promised categoricity result I need the following more general theorem.

Theorem 4.3.14. *A model of $Th(\mathfrak{S})$ which omits $\Delta(x)$ is atomic.*

Proof. Let $a_1, \dots, a_n \in A$. Let $b_1, \dots, b_m \in A$ be the (necessarily disjoint) non-zero elements of the form $\pm a_1 \cdot \ldots \cdot \pm a_n$. Since $\mathfrak{A}$ omits $\Delta(x)$, each b_i is the sum of connected elements $b_{i,1}, \dots, b_{i,k_i} \in A$. It follows from lemma 3.1.6 that $Th(\widetilde{\mathfrak{S}}) = Th(\mathfrak{A}) \models \forall x \forall y (\mathrm{c}(x) \wedge \mathrm{c}(y) \wedge x \cdot y \neq \emptyset \rightarrow \mathrm{c}(x + y))$. Summing together those $b_{i,j}$'s which overlap, we get a connected partition $c_1, \dots, c_k \in A$ refining $b_1, \dots, b_m$ and $a_1, \dots, a_n$. By lemma 3.3.36 on page 79, there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that any k-element connected partition in $\widetilde{\mathbf{S}}$ has an $f(k)$-element radial partition as refinement. Since this result can be expressed in a formula for each $k \geq 1$, there exists an $f(k)$-element radial partition refining $a_1, \dots, a_n$. Hence, $a_1, \dots, a_n$ satisfies a complete formula $\theta_{f(k)}(\bar{x})$. □

Since $\widetilde{\mathfrak{Q}}$ is a countable atomic model, proposition 4.3.9, whose proof was omitted above, follows directly from theorem 4.3.14 and the following theorem.

Theorem 4.3.15 (Chang and Keisler (1990), Theorem 2.3.3). *If $\mathfrak{A}$ and $\mathfrak{B}$ are countable atomic models and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

The mereotopology $\widetilde{\mathfrak{Q}}$ has a special status: it is the smallest model of $Th(\widetilde{\mathfrak{S}})$ in the following sense.

Definition 4.3.16. A model $\mathfrak{A}$ is said to be *prime* if, for any model $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \preceq \mathfrak{B}$.

Theorem 4.3.17 (Chang and Keisler (1990), Theorem 2.3.4). *A model is countable atomic if and only if it is prime.*

Then directly from theorem 4.3.14, we have the following result.

Theorem 4.3.18. *The mereotopology $\widetilde{\mathfrak{Q}}$ is a prime model.*

In the sequel, I will appeal to the following two standard results in model theory.

Theorem 4.3.19 (Chang and Keisler (1990), Theorem 3.1.6). *Let $\mathfrak{A}$ be a model of cardinality α and let $||\mathcal{L}|| \leq \beta \leq \alpha$. Then $\mathfrak{A}$ has an elementary submodel of cardinality β. Furthermore, given any set $X \subseteq A$ of cardinality smaller than or equal to β, $\mathfrak{A}$ has an elementary submodel of cardinality β which contains X.*

Theorem 4.3.20 (Chang and Keisler (1990), Prop. 2.4.4, Ex. 2.4.5). *Let $\mathfrak{A}$ be a countably atomic model and let $\bar{a}, \bar{b} \in A^n$ $(n \geq 1)$ have the same type in $\mathfrak{A}$. Then there is an automorphism of $\mathfrak{A}$ taking $\bar{a}$ to $\bar{b}$.*

Theorem 4.3.21. *The mereotopologies $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{J}}$ are not atomic.*

Proof. The proof follows an idea by Dr. Ian Pratt and is identical for $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{J}}$. Suppose for contradiction that $\widetilde{\mathfrak{F}}$ is atomic. Let $\bar{r}, \bar{s} \in (\widetilde{\mathbf{F}})^n$ satisfy the same complete formula $\phi(\bar{x})$. Certainly, $\widetilde{\mathbf{Q}}$ is a countable subset of $\widetilde{\mathbf{F}}$. By theorem 4.3.19, there exists a countable elementary submodel $\mathfrak{M}$ of $\widetilde{\mathfrak{F}}$ containing $\widetilde{\mathbf{Q}}$ and the regions of the tuples $\bar{r}$ and $\bar{s}$. Then $\mathfrak{M}$ is atomic and $\bar{r}$ and $\bar{s}$ satisfy the same complete formula $\phi(\bar{x})$ in $\mathfrak{M}$. By theorem 4.3.20, there exists an automorphism α on $\mathfrak{M}$ taking $\bar{r}$ to $\bar{s}$. Hence, by theorem 4.2.2, there exists a homeomorphism $h : \widetilde{\mathbb{R}^2} \to \widetilde{\mathbb{R}^2}$ taking $\bar{r}$ to $\bar{s}$, i.e. $\bar{r}$ and $\bar{s}$ are topologically equivalent. Therefore, $\phi(\bar{x})$ is topological complete. Thus, every n-tuple satisfies a topologically complete formula. However, $\mathcal{L}_{mt}$ is only countable and by lemma 3.3.39 on page 80, there exist uncountably many elements up to topological equivalence in $\widetilde{\mathbf{F}}$. □

4.4 The absolute expressivity of the mereotopological languages

The relative expressivity of the mereotopological languages was investigated in section 4.1. This section investigates how far the mereotopological languages can distinguish between various arrangements of regions. Since the languages are mereotopological, by lemma 4.0.2 they cannot distinguish between topologically equivalent n-tuples of regions. This gives an upper bound of the expressivity of the mereotopological languages. The lower bound of their expressivity was provided by theorem 4.3.13: every n-tuple of $\widetilde{\mathbf{S}}$ satisfies a topologically complete formula in $\widetilde{\mathfrak{S}}(\mathrm{C})$, $\widetilde{\mathfrak{S}}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathfrak{S}^*(\leq, \mathrm{c})$. Hence, the mereotopological languages $\mathcal{L}(\mathrm{C})$ and $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$ distinguish n-tuples in $\widetilde{\mathbf{S}}$ up to topological equivalence. The language $\mathcal{L}(\leq, \mathrm{c})$ is capable to distinguish n-tuples in the closed plane. However, this is not true for the open plane as lemma 4.1.4 shows.

These expressivity results can be used to show that the signatures $\{\mathrm{C}\}$ and $\{\leq, \mathrm{c}, \mathrm{b}\}$, and, in the case of the closed plane, the signature $\{\leq, \mathrm{c}\}$ as well, are *topologically adequate* over the semi-algebraic domains in the following sense. Suppose we construct the infinitary languages $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$, $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c})$ in exactly the same way as $\mathcal{L}(\mathrm{C})$, $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathcal{L}(\leq, \mathrm{c})$, except that, if $\Phi(x_1, \ldots, x_n)$ is a countable set of formulae in the variables $x_1, \ldots, x_n$, then $\bigwedge \Phi(x_1, \ldots, x_n)$ and $\bigvee \Phi(x_1, \ldots, x_n)$ are also formulae. (Thus, formulae of $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$, $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c}, \mathrm{b})$ and $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c})$, although infinitary, may contain only finitely many variables.) Let a relation $\mathcal{C}$ over a subset A of $\wp(X)$ be called *topological* if for all $\bar{a} \in \mathcal{C}$ and $\bar{b}$ in A^n (with n the length of $\bar{a}$) $\bar{a} \sim \bar{b}$ implies $\bar{b} \in \mathcal{C}$. Then we have:

Theorem 4.4.1. *Let $\mathcal{C}$ be a relation over the open-plane spatial domain* $\mathbf{S}$. *Then the following are equivalent:*

(i) $\mathcal{C}$ *is topological;*
(ii) $\mathcal{C}$ *is* $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$*-definable in the structure* $\mathfrak{S}(\mathrm{C})$*;*
(iii) $\mathcal{C}$ *is* $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c}, \mathrm{b})$*-definable in the structure* $\mathfrak{S}(\leq, \mathrm{c}, \mathrm{b})$.

Let $\mathcal{C}$ be a relation over the closed-plane spatial domain $\mathbf{S}^*$. *Then the following are equivalent:*

(i) $\mathcal{C}$ *is topological;*
(ii) $\mathcal{C}$ *is* $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$*-definable in the structure* $\mathfrak{S}^*(\mathrm{C})$*;*
(iii) $\mathcal{C}$ *is* $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c}, \mathrm{b})$*-definable in the structure* $\mathfrak{S}^*(\leq, \mathrm{c}, \mathrm{b})$*;*
(iv) $\mathcal{C}$ *is* $\mathcal{L}_{\omega_1\omega}(\leq, \mathrm{c})$*-definable in the structure* $\mathfrak{S}^*(\leq, \mathrm{c})$.

Proof. I give the proof for the mereotopology $\mathfrak{S}(\mathrm{C})$. Corresponding remarks apply to $\mathfrak{S}^*(\leq, \mathrm{c})$, $\mathfrak{S}^*(\mathrm{C})$ and $\widetilde{\mathfrak{S}}(\leq, \mathrm{c}, \mathrm{b})$. That all $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$-definable relations in $\mathfrak{S}(\mathrm{C})$ are topological follows using the same proof strategy as for lemma 4.0.2; the details are routine. Conversely, if $\mathcal{C}$ is a topological relation over $\widetilde{\mathbf{S}}$, then

$$\bigvee \left\{ \phi(\bar{x}) \in \mathcal{L}(\mathrm{C}) \;\middle|\; \begin{array}{l} \phi \text{ is a topologically complete } \mathcal{L}(\mathrm{C})\text{-formula} \\ \text{s.t. } \mathfrak{S}(\mathrm{C}) \models \phi[\bar{a}] \text{ for some } \bar{a} \in \mathcal{C} \end{array} \right\}$$

is a formula of $\mathcal{L}_{\omega_1\omega}(\mathrm{C})$ (by the countability of $\mathcal{L}(\mathrm{C})$), and is clearly satisfied in $\mathfrak{S}(\mathrm{C})$ by all and only those n-tuples in $\mathcal{C}$. □

It is obvious, by a simple counting argument, that no such result as theorem 4.4.1 could hold for the finitary versions of the mereotopological languages.

4.5 Conclusion

In this chapter, I introduced a number of mereotopological languages that employ predicate symbols to express the notions of parthood, contact, connectedness and boundedness. The mereotopological languages were interpreted over the spatial domains $\widetilde{\mathbf{F}}$, $\widetilde{\mathbf{J}}$, $\widetilde{\mathbf{S}}$, $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$, and thereby the mereotopologies $\widetilde{\mathfrak{F}}$, $\widetilde{\mathfrak{J}}$, $\widetilde{\mathfrak{S}}$, $\widetilde{\mathfrak{P}}$ and $\widetilde{\mathfrak{Q}}$, called mereotopologies, were defined.

The relative expressivity of the mereotopological languages with respect to these mereotopologies was investigated. It was shown that the mereotopological languages cannot distinguish between the mereotopologies $\widetilde{\mathfrak{S}}$, $\widetilde{\mathfrak{P}}$ and $\widetilde{\mathfrak{Q}}$ but can tell $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathfrak{F}}$ apart. The topological notion of contact was shown to be more expressive than the notions of parthood and connectedness taken together. However, parthood and connectedness are sufficient to define contact in the mereotopology $\mathfrak{S}^*$. The topological notion of contact was shown to have the same expressivity with respect to the mereotopology $\widetilde{\mathfrak{S}}$ as the notions of parthood, connectedness and boundedness taken together. The relative expressivity results were extended to absolute results. The mereotopological languages $\mathcal{L}(\mathrm{C})$ and $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$ were shown to be topologically adequate in the sense that these languages distinguish all n-tuples of regions in $\widetilde{\mathbf{S}}$ up to topological equivalence, and that the infinitary versions of these languages distinguish between all topological relations over $\widetilde{\mathbf{S}}$. For $\mathcal{L}(\leq, \mathrm{c})$ these results only hold in the closed plane mereotopology $\mathfrak{S}^*$.

The theory of $\widetilde{\mathfrak{S}}$ omits the set of formulae $\Delta(x)$, since every region in $\widetilde{\mathbf{S}}$ has only finitely many components. However, it was shown that there are models of $Th(\widetilde{\mathfrak{S}})$ which do not omit $\Delta(x)$. This results shows that the finitary mereotopological languages are not capable to capture all properties of $\widetilde{\mathbf{S}}$. It was shown that $\widetilde{\mathfrak{Q}}$ is a prime model. Thus, since $\widetilde{\mathfrak{Q}}$ and $\widetilde{\mathfrak{S}}$ are elementarily equivalent, any model of $Th(\widetilde{\mathfrak{S}})$ contains a copy of $\widetilde{\mathfrak{Q}}$. Therefore, if we accept the spatial domain $\widetilde{\mathbf{S}}$ to be adequate for the common-sense representation of space, then the above result shows that the much simpler spatial domain $\widetilde{\mathbf{Q}}$ is adequate as well. Moreover, since prime models of $Th(\widetilde{\mathfrak{S}})$ are isomorphic, there is no ontologically more parsimonious spatial domain which is adequate as well. This result becomes especially interesting since the spatial domain $\mathbf{Q}$ is employed in computer science for the representation of spatial data (e.g. in the areas of Computer Aided Design (CAD) and Geographical Information Systems (GIS)). Bearing in mind the importance of the mereotopology $\mathfrak{Q}(\mathrm{C})$, the next chapter establishes which formulae of $\mathcal{L}(\mathrm{C})$

hold in $\mathfrak{S}$(C), and hence in $\mathfrak{Q}$(C).

Chapter 5

A complete axiomatisation for the mereotopology $\mathfrak{S}(\mathrm{C})$

In this chapter, I present a complete axiomatisation of the $\mathcal{L}(\mathrm{C})$-theory of $\mathfrak{S}$ in the predicate calculus extended by an infinitary rule of inference. The following section introduces this extension. Section 2 introduces the axiom system, and section 3 presents the consistency and completeness proofs.

5.1 The predicate calculus with an infinitary rule of inference: the Δ-calculus

In order to approach a complete axiomatisation of $Th(\mathfrak{S})$, an extension to the predicate calculus will be introduced in this section. The extension, which will be called Δ-*calculus*, has close relations to ω-logic, which has been used to investigate the standard model of arithmetic (cf. Chang and Keisler, 1990).

Let $\Delta = \Delta(\bar{x})$ be a consistent infinite set of formulae in some countable first-order language $\mathcal{L}$. Then the Δ-calculus is formed by adding to the usual axioms and rules of inference of the predicate calculus the Δ-*rule*:

$$\frac{\{\forall\bar{x}(\delta(\bar{x}) \vee \phi(\bar{x}))|\delta(\bar{x}) \in \Delta(\bar{x})\}}{\forall\bar{x}(\phi(\bar{x}))}$$

Note that proofs in the Δ-calculus can have infinite length, since $\Delta(\bar{x})$ is infinite. The terminology of usual first-order logic is transferred to the Δ-calculus: Given a theory $\Sigma \cup \{\sigma\}$, σ is said to be *deducible from* Σ *in the* Δ-*calculus*, in symbols $\Sigma \vdash_\Delta \sigma$, if there is a proof in the Δ-calculus of σ from Σ. The set of all sentences deducible in the Δ-calculus from Σ is the *deductive closure of* Σ *in the* Δ*-calculus.* A theory is called *inconsistent in the* Δ*-calculus* if for every sentence σ in $\mathcal{L}$, $\Sigma \vdash_\Delta \sigma$, and *consistent in the* Δ*-calculus* if it is not inconsistent in the Δ-calculus. A theory is called *complete in the* Δ*-calculus* if its deductive closure in the Δ-calculus is maximal consistent in the Δ-calculus. Unsurprisingly, there are deduction, consistency, soundness and completeness theorems for the Δ-calculus.

Deduction theorem for the Δ-calculus. *Let Σ be a set of sentences, and σ and τ be two sentences. Then $\Sigma \vdash_\Delta \sigma \to \tau$ if and only if $\Sigma \cup \{\sigma\} \vdash_\Delta \tau$.*

Proof. If $\Sigma \vdash_\Delta \sigma \to \tau$ then a proof of τ from $\Sigma \cup \{\sigma\}$ is obtained by a proof of $\sigma \to \tau$ from Σ with application of modus ponens.

Conversely, let $\{\tau_\alpha\}_{0 \leq \alpha \leq \beta}$ be a proof of $\tau_\beta = \tau$ from $\Sigma \cup \{\sigma\}$ where for each $\tau_{\alpha'}$ $(0 \leq \alpha' \leq \beta)$, $\{\tau_\alpha\}_{0 \leq \alpha \leq \alpha'}$ is a proof of $\tau_{\alpha'}$. I show by transfinite induction on α' that $\Sigma \vdash_\Delta \sigma \to \tau_{\alpha'}$.

If $\tau_{\alpha'}$ is an axiom, $\tau_{\alpha'} \in \Sigma \cup \{\sigma\} \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'}$, or $\tau_{\alpha'}$ was deduced from $\Sigma \cup \{\sigma\} \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'}$ by modus ponens or the rule of generalisation then $\Sigma \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'} \vdash_\Delta \sigma \to \tau_{\alpha'}$ by the deduction theorem of the predicate calculus. If $\alpha' = 0$ then $\Sigma \vdash_\Delta \sigma \to \tau_{\alpha'}$. If $\alpha' > 0$ then it follows from the induction hypothesis that $\Sigma \vdash_\Delta \sigma \to \tau_{\alpha'}$. If $\tau_{\alpha'} = \forall x(\phi(\bar{x}))$ for some ϕ and $\forall \bar{x}(\phi(\bar{x}))$ is deduced from $\Sigma \cup \{\sigma\} \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'}$ by the application of the Δ-rule then $\forall x(\delta(\bar{x}) \vee \phi(\bar{x})) \in \Sigma \cup \{\sigma\} \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'}$ for all $\delta(\bar{x}) \in \Delta(\bar{x})$. If $\alpha' = 0$ then $\forall x(\delta(\bar{x}) \vee \phi(\bar{x})) \in \Sigma \cup \{\sigma\}$ and hence $\Sigma \vdash_\Delta \sigma \to \forall x(\delta(\bar{x}) \vee \phi(\bar{x}))$ for all $\delta(\bar{x}) \in \Delta(\bar{x})$. If $\alpha' > 0$ then $\forall x(\delta(\bar{x}) \vee \phi(\bar{x})) \in \Sigma \cup \{\sigma\} \cup \{\tau_\alpha\}_{0 \leq \alpha < \alpha'}$ whence by induction hypothesis $\Sigma \vdash_\Delta \sigma \to \forall x(\delta(\bar{x}) \vee \phi(\bar{x}))$ for all $\delta(\bar{x}) \in \Delta(\bar{x})$. Either way, since $\bar{x}$ is not free in the sentence σ, $\Sigma \vdash_\Delta \forall \bar{x}(\delta(\bar{x}) \vee (\sigma \to \phi(\bar{x})))$. Then by the Δ-rule, $\Sigma \vdash_\Delta \forall \bar{x}(\sigma \to \phi(\bar{x}))$. Again, since $\bar{x}$ is not free in σ, $\Sigma \vdash_\Delta \sigma \to \forall \bar{x}(\phi(\bar{x}))$. Hence, $\Sigma \vdash_\Delta \sigma \to \tau_{\alpha'}$. □

Corollary 5.1.1. *A set of sentences $\Sigma \cup \{\sigma\}$ is consistent in the Δ-calculus if and only if $\Sigma \nvdash_\Delta \neg\sigma$.*

Definition 5.1.2. Let $\Sigma(\bar{x})$ be a set of formulae in the free variables $\bar{x} = x_1, \ldots, x_n$ and let $\mathfrak{A}$ be an $\mathcal{L}$-structure. The structure $\mathfrak{A}$ is said to *realize* $\Sigma(\bar{x})$ if some n-tuple of elements of A satisfies $\Sigma(\bar{x})$. The structure $\mathfrak{A}$ is said to *omit* $\Sigma(\bar{x})$ if $\mathfrak{A}$ does not realize $\Sigma(\bar{x})$. A theory T in $\mathcal{L}$ is said to *locally omit* $\Sigma(\bar{x})$ if for every formula $\phi(\bar{x})$ in $\mathcal{L}$ consistent with T, there is $\sigma(\bar{x}) \in \Sigma(\bar{x})$ such that $\phi(\bar{x}) \wedge \neg\sigma(\bar{x})$ is consistent with T.

Omitting types theorem (Chang and Keisler (1990), Theorem 2.2.9). *Let T be a consistent theory in a countable language $\mathcal{L}$, and let $\Sigma(\bar{x})$ be a set of formulae. If T locally omits $\Sigma(\bar{x})$, then T has a countable model which omits $\Sigma(\bar{x})$.*

In the following sections, $\mathcal{L}$-structures omitting some set $\Delta(\bar{x})$ of formulae will be of specific interest. They will be called Δ-*models*.

Soundness theorem for the Δ-calculus. *Let Σ be a set of sentences and σ be a sentence. If $\Sigma \vdash_\Delta \sigma$ then all Δ-models of Σ are Δ-models of σ.*

Proof. Let $\mathfrak{A}$ be a Δ-model of Σ and $\{\sigma_\alpha\}_{0\leq\alpha\leq\beta}$ a proof of $\sigma_\beta = \sigma$ from Σ in the Δ-calculus such that $\{\sigma_\alpha\}_{0\leq\alpha\leq\alpha'}$ is a proof of $\sigma_{\alpha'}$. I argue by induction over α. Assume $\sigma_{\alpha'}$ is deduced from Σ, but is not derived by the Δ-rule. Then $\sigma_{\alpha'}$ is an axiom, $\sigma_{\alpha'} \in \Sigma \cup \{\sigma_\alpha\}_{0\leq\alpha<\alpha'}$, or σ'_α is deduced from $\Sigma \cup \{\sigma_\alpha\}_{0\leq\alpha<\alpha'}$ by modus ponens or rule of generalisation. It follows from the soundness theorem of the predicate calculus and for $\alpha' > 0$ also by the induction hypothesis that $\mathfrak{A} \models \sigma_{\alpha'}$. Assume $\sigma_{\alpha'} = \forall\bar{x}(\phi(\bar{x}))$ for some ϕ and $\forall\bar{x}(\phi(\bar{x}))$ is deduced from $\Sigma \cup \{\sigma_\alpha\}_{0\leq\alpha<\alpha'}$ by an application of the Δ-rule. If $\alpha' = 0$ then $\forall\bar{x}(\delta(\bar{x}) \vee \phi(\bar{x})) \in \Sigma$ for all $\delta(\bar{x}) \in \Delta$. Hence, $\mathfrak{A} \models \forall\bar{x}(\delta(\bar{x}) \vee \phi(\bar{x}))$. If $\alpha' > 0$ then by induction hypothesis, $\mathfrak{A} \models \forall\bar{x}(\delta(\bar{x}) \vee \phi(\bar{x}))$ for all $\delta(\bar{x}) \in \Delta$. Either way, since $\mathfrak{A}$ omits Δ, for all $\bar{a}$ in A, for some $\delta(\bar{x}) \in \Delta(\bar{x})$, $\mathfrak{A} \not\models \delta[\bar{a}]$. Hence, $\mathfrak{A} \models \forall\bar{x}(\phi(\bar{x}))$. □

Consistency theorem for the Δ-calculus. *A set of sentences Σ is consistent in the Δ-calculus if and only if Σ has a Δ-model.*

Proof. Let $\overline{\Sigma}$ be the deductive closure of Σ in the Δ-calculus. Assume Σ is consistent in the Δ-calculus. Then $\overline{\Sigma}$ is consistent. Suppose $\phi(\bar{x})$ is a formula consistent with $\overline{\Sigma}$. Then $\forall\bar{x}(\neg\phi(\bar{x})) \notin \overline{\Sigma}$. Then $\{\forall\bar{x}(\delta(\bar{x})\vee\neg\phi(\bar{x}))|\delta(\bar{x}) \in \Delta(\bar{x})\} \not\subseteq \overline{\Sigma}$ and hence $\forall\bar{x}(\delta(\bar{x})\vee\neg\phi(\bar{x})) \notin \overline{\Sigma}$ for some $\delta(\bar{x}) \in \Delta(\bar{x})$. Therefore, $\neg\delta(\bar{x}) \wedge \phi(\bar{x})$ is consistent with $\overline{\Sigma}$, i.e. $\overline{\Sigma}$ locally omits $\Delta(\bar{x})$. Therefore, by the omitting types theorem, there is a model of $\Sigma \subseteq \overline{\Sigma}$ omitting $\Delta(\bar{x})$.

For the converse direction, assume that Σ is inconsistent in the Δ-calculus. Then for some $\sigma \in \Sigma$, $\Sigma \vdash_\Delta \neg\sigma$ and hence Σ has no (Δ-)model. □

The completeness theorem for the Δ-calculus is a corollary of the deduction theorem and the consistency theorem for the Δ-calculus:

Completeness theorem for the Δ-calculus. *Let Σ be a set of sentences and σ be a sentence. If all Δ-models of Σ are Δ-models of σ then $\Sigma \vdash_\Delta \sigma$.*

The completeness of a theory Σ in first-order logic is characterised by the elementary equivalence of all models of Σ. The proposition below shows that the completeness of Σ in the Δ-calculus is characterised by the elementary equivalence of all Δ-models of Σ.

Proposition 5.1.3. *A theory Σ which is consistent in the Δ-calculus is complete in the Δ-calculus if and only if any two Δ-models of Σ are elementarily equivalent.*

Proof. Straightforward. □

5.2 The axiom system $\mathcal{P}$

In this section, I present a complete axiom system $\mathcal{P}$. To achieve readability of the axiom system $\mathcal{P}$ and to make the completeness proof more transparent, the axiom system $\mathcal{P}$ will be stated in the formal language $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$ where c and b are unary predicate symbols, $+$ and $\cdot$ are binary function symbols, $-$ is a unary function symbol, and 0 and 1 are constant symbols. The mereotopology $\mathfrak{S}$ will be considered as an $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$-structure where the function and constant symbols are interpreted as join, meet, complement and bottom- and top-element respectively of the Boolean algebra $(\mathbf{S}, +, \cdot, -, 0, 1)$ and c and b define the connected and bounded regions respectively (cf. chapter 4). It follows from theorem 4.1.1 that the subset-, connectedness- and boundedness-relation and the Boolean operations and constants are $\mathcal{L}(\mathrm{C})$-definable in $\mathfrak{S}$. Therefore, the axiom system $\mathcal{P}$ in the language $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$ can be translated into an axiom system in the language $\mathcal{L}(\mathrm{C})$.

The following abbreviations will be used in the axiom system.

1. Let $x \leq y$ stand for $x \cdot y = x$. In $\mathfrak{S}$, $x \leq y$ defines the subset-relation.

2. For $n > 1$ let $x = x_1 \oplus \ldots \oplus x_n$ stand for

$$x = x_1 + \ldots + x_n \wedge \bigwedge_{1 \leq i \leq n} (\mathrm{c}(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq n} x_i \cdot x_j = 0\,.$$

   In $\mathfrak{S}$, $x = x_1 \oplus \ldots \oplus x_n$ defines the sets of regions which have at most n components.

3. Let $\mathrm{j}(x)$ stand for $\mathrm{c}(x) \wedge x \neq 0 \wedge \mathrm{c}(-x) \wedge -x \neq 0$. By lemma 3.2.12, $\mathrm{j}(x)$ defines the set of j-regions in $\mathfrak{S}$.

The axiom system $\mathcal{P}$ consists of the following axioms and axiom schemata.

A1 The usual axioms of non-trivial Boolean algebra with Boolean operations $+$, $\cdot$ and $-$, and (distinct) top- and bottom-elements 1 and 0.

A2 $\forall x \forall y \forall z \Big((\mathrm{c}(x+y) \wedge \mathrm{c}(y+z) \wedge y \neq 0) \rightarrow \mathrm{c}(x+y+z) \Big)$

A3 Where $n > 1$, the axioms

$$\forall x_1 \ldots \forall x_n \Big((\mathrm{c}\Big(\sum_{1 \leq i \leq n} x_i \Big) \wedge \bigwedge_{1 \leq i \leq n} \mathrm{c}(x_i)) \rightarrow \bigvee_{2 \leq i \leq n} \mathrm{c}(x_1 + x_i) \Big)$$

A4 Where $n > 1$, the axioms

$$\forall x_1 \ldots \forall x_n \Big((\mathrm{c}\Big(\sum_{1 \leq i \leq n} x_i \Big) \wedge \bigwedge_{1 \leq i \leq n} \mathrm{c}(x_i)) \rightarrow \bigvee_{1 \leq i \leq n} \mathrm{c}\Big(\sum_{\substack{1 \leq j \leq n \\ j \neq i}} x_j \Big) \Big)$$

A5 $\neg \exists x_1 \ldots \exists x_6 \Big(\bigwedge_{1 \leq i \leq 6} (\mathrm{c}(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 6} x_i \cdot x_j = 0 \wedge \bigwedge_{\substack{1 \leq i \leq 3 \\ 4 \leq j \leq 6}} \mathrm{c}(x_i + x_j) \Big)$

A6 $\mathrm{c}(1)$

A7 $\neg\, \mathrm{b}(1)$

A8 Where $n > 1$, the axioms $\forall x_1 \ldots \forall x_n (\mathrm{b}(\sum_{i=1}^{n} x_i) \leftrightarrow \bigwedge_{i=1}^{n} \mathrm{b}(x_i))$

A9 Where $n \geq 4$ and $3 \leq k < n$, the axioms

$$\forall x_1 \ldots \forall x_n \Big(\Big(1 = x_1 \oplus \ldots \oplus x_n \wedge \bigwedge_{1 \leq i \leq j \leq n} (\mathrm{j}(x_i) \wedge \mathrm{c}(-(x_i + x_j)))$$

$$\wedge \bigwedge_{i=1}^{k} \neg\, \mathrm{b}(x_i) \wedge \bigwedge_{i=k+1}^{n} \mathrm{b}(x_i) \Big)$$

$$\rightarrow \mathrm{j}\Big(\sum_{i=1}^{k} x_i \Big) \wedge \pi(x_1, \ldots, x_k) \Big)$$

where $\pi(x_1, \ldots, x_k)$ stands for the formula which expresses that the binary connection graph on $x_1, \ldots, x_k$ is a cycle with edges $\{x_i, x_{i+1}\}$ $(1 \leq i < k)$ and $\{x_1, x_k\}$. Such formula $\pi(x_1, \ldots, x_k)$ can certainly be constructed. However, I omit the details of this construction.

A10 $\forall x \forall y \forall z \Big((\mathrm{c}(x) \wedge \mathrm{c}(y) \wedge \mathrm{c}(z) \wedge \mathrm{c}(x+y) \wedge \mathrm{c}(x+z)$
$\wedge x \cdot y = 0 \wedge x \cdot z = 0 \wedge x \neq 0 \wedge \phi(x,y)) \rightarrow$
$\exists x_1 \exists x_2 (x = x_1 \oplus x_2 \wedge \bigwedge_{1 \leq i \leq 2} (\mathrm{c}(x_i + y) \wedge \mathrm{c}(x_i + z)) \wedge \psi(x_1, x_2))\Big)$

where $\phi(x,y)$ and $\psi(x_1,x_2)$ are one of the following pairs:

$\phi(x,y)$	$\psi(x_1,x_2)$
$x = x$	$x_1 = x_1$
$\neg\,\mathrm{b}(x) \wedge \mathrm{b}(-(x+y))$	$\mathrm{b}(x_1)$
$\neg\,\mathrm{b}(x) \wedge \mathrm{b}(-(x+y))$	$\neg\,\mathrm{b}(x_1) \wedge \neg\,\mathrm{b}(x_2)$

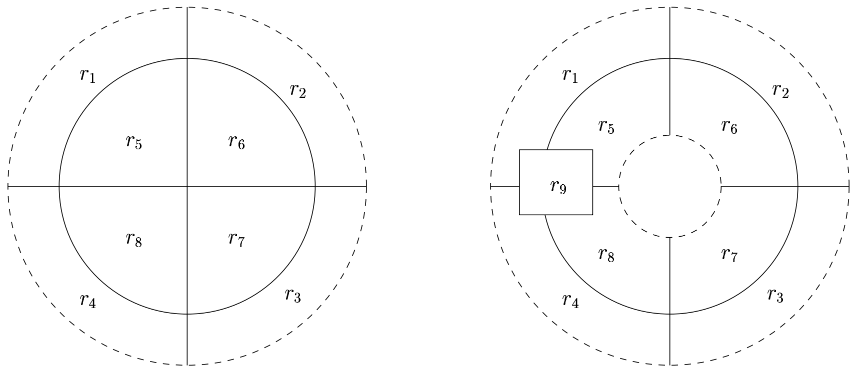
A11 $\forall x \forall y \Big((\mathrm{j}(x) \wedge \mathrm{j}(y) \wedge \mathrm{j}(x+y) \wedge x \cdot y = 0 \wedge \phi(x,y)) \rightarrow$
$\exists x_1 \exists x_2 (x = x_1 \oplus x_2 \wedge \mathrm{c}(x_1 + y) \wedge \neg\,\mathrm{c}(x_1 + (-x)\cdot(-y)) \wedge$
$\mathrm{c}(x_2 + (-x)\cdot(-y)) \wedge \neg\,\mathrm{c}(x_2 + y) \wedge \psi(x_1,x_2))\Big)$

where $\phi(x,y)$ and $\psi(x_1,x_2)$ are one of the following pairs:

$\phi(x,y)$	$\psi(x_1,x_2)$
$x = x$	$x_1 = x_1$
$\neg\,\mathrm{b}(x) \wedge \mathrm{b}(-x)$	$\mathrm{b}(x_1)$
$\neg\,\mathrm{b}(x) \wedge \mathrm{b}(-x)$	$\mathrm{b}(x_2)$
$\neg\,\mathrm{b}(x) \wedge \mathrm{b}(-x)$	$\neg\,\mathrm{b}(x_1) \wedge \neg\,\mathrm{b}(x_2)$

The respective task of the individual axioms are as follows. Axiom 1 reflects the structure of the spatial domain **S**. Axiom 2 ensures that two connected regions with a non-empty intersection have a connected sum. Axiom schemata 3 and 4 impose restrictions on n-tuples of connected regions whose sum is connected. Given a connected sum of a finite collection of connected regions, firstly, each of these regions has a connected sum with at least one other region of the collection, and secondly, one of the regions can be removed from the collection without destroying the connectedness of the sum of the remaining regions. Both axiom schemata reflect a correspondence between sums of connected regions and the binary connection graph on these regions.

Axiom 5 characterises planarity by forbidding the non-planar graph $K_{3,3}$ to be the binary connection graph of some collection of mutually disjoint connected regions. It is shown by lemma 5.3.4 below that axiom 5 is sufficient to exclude the non-planar graph K_5 as binary-connection graph as well.



(a) A j-partition of the open plane

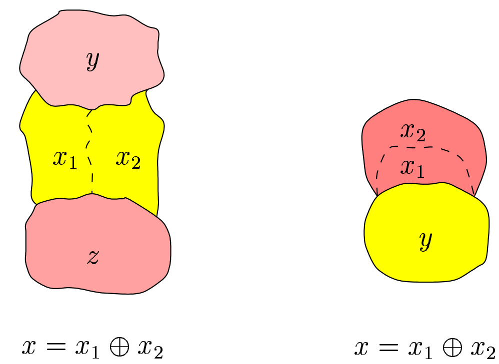
(b) A j-partition of the open annulus

Figure 5.1: Explanation for axiom 9

Axioms 6 and 7 simply state the connectedness and unboundedness of the entire space. Axiom schema 8 ensures that a region is bounded if and only if all of its parts are bounded. Therefore, given a partition of the whole space, it follows from axiom 7 that at least one of the regions of the partition must be unbounded.

Axiom schema 9 ensures that the entire space has the (un)boundedness properties of the open plane and not those of, for example, an open annulus. Figure 5.1(a) shows an instantiation of the axiom schema for $n = 8$ and $k = 4$ for the open plane that is depicted in the figure by an open disc. The regions $r_1, \ldots, r_8$ form a radial partition of the entire space as required by the condition $1 = x_1 \oplus \ldots \oplus x_8 \wedge \bigwedge_{1 \leq i \leq j \leq n} (\mathrm{j}(x_i) \wedge \mathrm{c}(-(x_i + x_j)))$ of the axiom. Regions $r_1, \ldots, r_4$ are unbounded and regions $r_5, \ldots, r_8$ are bounded as required by the condition $\bigwedge_{i=1}^{k} \neg\, \mathrm{b}(x_i) \wedge \bigwedge_{i=k+1}^{n} \mathrm{b}(x_i)$. By the axiom, the sum of unbounded regions has to be a j-region. The regions $r_1, \ldots, r_4$ satisfy this condition. Furthermore, the binary connection graph of the unbounded regions must be a cycle ($\pi(x_1, \ldots, x_4)$) which is correct for $r_1, \ldots, r_4$. Now consider the annulus depicted in figure 5.1(b). The regions $r_1, \ldots, r_9$ instantiate the antecedent of the axiom for $k = 8$. Furthermore, the sum of the unbounded regions $r_1, \ldots, r_8$ is a j-region. However, the binary connection graph on $r_1, \ldots, r_8$ is not a cycle. Thus, the axiom does not hold for these nine regions partitioning the annulus.

Finally, axioms 10 and 11 ensure the existence of sufficiently many regions of various shapes, and the existence of bounded and unbounded regions. Figure 5.2



(a) Axiom 10 (b) Axiom 11

Figure 5.2: Instantiations of axioms 10 and 11

shows two simple instantiations of the two axioms.

5.3 Consistency and Completeness of $\mathcal{P}$

In this section, I will show that the axiom system $\mathcal{P}$ is consistent and complete in the Δ-calculus where

$$\Delta(x) = \left\{ \neg\exists x_1 \ldots \exists x_n (x = x_1 + \ldots + x_n \wedge \bigwedge_{i=1}^{n} \mathrm{c}(x_i)) \middle| n \geq 1 \right\}.$$

Therefore, a mereotopology $\mathfrak{M}$ is a Δ-model, i.e. a model omitting the set of formulae $\Delta(x)$, if and only if every region of $\mathfrak{M}$ has only finitely many components.

Theorem 5.3.1 (Consistency). *The axiom system $\mathcal{P}$ is consistent in the Δ-calculus.*

Proof. By lemma 3.3.12 on page 70, $\mathfrak{S}$ omits $\Delta(x)$ and hence is a Δ-model. Moreover, the following list shows that every axiom (schema) holds in $\mathfrak{S}$.

A1 By proposition 3.3.13 on page 70.

A2 By lemma 3.1.6 on page 54.

A3 By lemma 3.1.10 on page 57 and proposition 3.3.7 on page 69.

A4 By lemma 3.1.14 on page 59 and proposition 3.3.7 on page 69.

A5 Suppose there are mutually disjoint connected regions $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbf{S}$ such that $r_i + s_j$ for $1 \leq i \leq j \leq 3$ is connected. For each $i \in \{1, 2, 3\}$ let p_i be some point in r_i and let q_i be some point in s_i. By lemma 3.3.15, $\partial(r_i) \cap \partial(s_i) \cap (r_i + s_j) \neq \emptyset$ $(1 \leq i \leq j \leq 3)$, and by proposition 3.3.7, the boundaries of the regions are accessible. Hence, there exist arcs $\gamma_{i,j}$ from p_i to q_j in $r_i + s_j$ $(1 \leq i \leq j \leq 3)$ such that the arcs have mutually disjoint interiors. It is easy to see that the arcs form a planar embedding of the non-planar graph $K_{3,3}$ in the plane which is impossible.

A6 $\mathbb{R}^2$ is connected.

A7 $\mathbb{R}^2$ is unbounded.

A8 The union of any finite number of bounded subsets is bounded (see e.g. Sutherland, 1975, Proposition 2.2.15). Thus, if $r_1, \ldots, r_n \in \mathbf{S}$ are bounded so is $\sum_{i=1}^n r_i \subseteq \bigcup_{i=1}^n [r_i]$. The other direction is trivial.

A9 By lemma 3.3.25 on page 74 with $p = p_\infty$.

A10 By lemma 3.3.34 on page 77.

A11 By lemma 3.3.35 on page 79.

Hence, by the consistency theorem of the Δ-calculus, $\mathcal{P}$ is consistent in the Δ-calculus. □

To show the completeness of $\mathcal{P}$ in the Δ-calculus, considerable more effort is necessary. The notations and lemmas presented in the following subsection provide some help.

5.3.1 Properties of models of $\mathcal{P}$

For this subsection, let the $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$-structure $\mathfrak{A}$ with domain A be some Δ-model of $\mathcal{P}$. Although $\mathfrak{A}$ is not necessarily a mereotopology, the terminology introduced for mereotopologies will be use for elements of A. An element $a \in A$ is said to be "connected" if $\mathfrak{A} \models \mathrm{c}[a]$ and "bounded" if $\mathfrak{A} \models \mathrm{b}[a]$. To simplify notation, I will write the constant symbols 0 and 1 in place of the elements $0^{\mathfrak{A}}$ and $1^{\mathfrak{A}}$ and will treat the function symbols similarly, i.e. I write $a_1 + a_2$ instead of $+^{\mathfrak{A}}(a_1, a_2)$ and refer to "the sum of a_1 and a_2" etc. An element $a \in A \setminus \{0\}$ is said to be "non-zero" and two elements $a_1, a_2 \in A$ are "disjoint" if $a_1 \cdot a_2 = 0$. If $a_1, \ldots, a_n \in A$ are non-zero, connected, mutually disjoint and sum to $a \in A$ then I write $a = a_1 \oplus \ldots \oplus a_n$. If in this case $a_1 + a_2$ is connected then a_1 is said to be a *neighbour* of a_2. The element a_1 is said to be *a part of* a_2, written $a_1 \leq a_2$, if $a_1 \cdot a_2 = a_1$. As in topological spaces, an element a_1 is a *component* of an element a_2 if a_1 is a maximal connected part of a_2. With this terminology the definitions of the various kinds of partitions and refinements are directly transfered from those for spatial domains (see definition 3.3.24 on page 73).

Lemma 5.3.2 (Pratt and Schoop (1998)). *The formula* $\mathrm{c}(0)$ *is a theorem of* $\mathcal{P}$ *in the* Δ*-calculus.*

Proof. By the axioms for a non-trivial Boolean algebra, we have, for all $n \geq 1$:

$$\mathcal{P} \vdash \forall x \forall x_1 \ldots \forall x_n \Big((\bigwedge_{1 \leq i \leq n} \mathrm{c}(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i) \wedge \bigwedge_{1 \leq i \leq n} (x_i = 0)) \rightarrow \mathrm{c}(x) \Big)$$

$$\mathcal{P} \vdash \forall x \forall x_1 \ldots \forall x_n \Big((\bigwedge_{1 \leq i \leq n} \mathrm{c}(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i) \wedge \neg \bigwedge_{1 \leq i \leq n} (x_i = 0)) \rightarrow x \neq 0 \Big)$$

and, therefore,

$$\mathcal{P} \vdash \forall x \exists x_1 \ldots \exists x_n \Big((\bigwedge_{1 \leq i \leq n} \mathrm{c}(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i)) \rightarrow (\mathrm{c}(x) \vee x \neq 0) \Big).$$

Hence, for every $\delta(x) \in \Delta(x)$, $\mathcal{P} \vdash \forall x \Big(\delta(x) \rightarrow (\mathrm{c}(x) \vee x \neq 0) \Big)$. By the infinitary rule of inference in the Δ-calculus, $\forall x (\mathrm{c}(x) \vee x \neq 0)$ and therefore $\mathrm{c}(0)$ are theorems of $\mathcal{P}$ in the Δ-calculus. □

Lemma 5.3.3. *The formula* $\mathrm{b}(0)$ *is a theorem of* $\mathcal{P}$.

Proof. Let $\mathfrak{B}$ be a model of $\mathcal{P}$. By axiom 6 and lemma 5.3.2, 1 and 0 are connected. It follows from axiom 10 that there exists a bounded element $b \in B$. Since $b = b + 0$ is bounded, it follows from axiom 8 that 0 is bounded. By the completeness theorem, $\mathrm{b}(0)$ is a theorem of $\mathcal{P}$. □

Lemma 5.3.4. *The formula*

$$\neg \exists x_1 \ldots \exists x_5 \Big(\bigwedge_{1 \leq i \leq 5} (\mathrm{c}(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 5} (\mathrm{c}(x_i + x_j) \wedge x_i \cdot x_j = 0) \Big)$$

is a theorem of $\mathcal{P}$.

Proof. Let $\phi_{\neg K5}$ stand for the above formula. Let $\mathfrak{B}$ be a model of $\mathcal{P}$. Suppose for a proof by contradiction that there are mutually disjoint non-zero elements $r_1, \dots, r_5 \in B$ such that $r_i + r_j$ is connected for $1 \le i \le j \le n$, i.e. $\mathfrak{B} \models \neg\phi_{\neg K5}$. By axiom 10, there exist disjoint connected non-zero elements $r_{11}, r_{12} \in B$ such that $r_1 = r_{11} + r_{12}$ and $r_{11} + r_2$, $r_{11} + r_3$, $r_{12} + r_2$ and $r_{12} + r_3$ are connected. By axiom 3 and the fact that $r_4 + r_{11} + r_{12}$ and $r_5 + r_{11} + r_{12}$ are connected, it can be assumed that $r_4 + r_{11}$ and $r_5 + r_{11}$ are connected or that $r_4 + r_{11}$ and $r_5 + r_{12}$ are connected. If $r_4 + r_{11}$ and $r_5 + r_{11}$ are connected then each of r_{12}, r_4 and r_5 has a connected sum with each of r_{11}, r_2 and r_3. If $r_4 + r_{11}$ and $r_5 + r_{12}$ are connected then each of r_{12}, r_2 and r_4 has a connected sum with each of r_{11}, r_3 and r_5. However, this is impossible, since by assumption $\mathfrak{B}$ satisfies axiom 5, whence $\phi_{\neg K5}$ is a theorem of $\mathcal{P}$. □

The following lemma will be used in the sequel without further mention.

Lemma 5.3.5. *An element $a \in A$ is the sum of finitely many components. Moreover, these components are unique.*

Proof. Since $\mathfrak{A}$ omits $\Delta(x)$, there is a finite set π of connected elements $a_1, \dots, a_n \in A$ that sum to a. If every pair a_i, a_j in π is replaced with $a_i + a_j$ if $a_i + a_j$ is connected and this step is repeated as often as possible then eventually π is a finite set of connected elements $b_1, \dots, b_k \in A$ which sum to a. Furthermore, every two elements b_i, b_j $(1 \le i < j \le k)$ have a disconnected sum and by axiom 2 are disjoint. If $k = 1$ then b_1 is the only component of a. Suppose $k > 1$ and b_1 is not a component of a. Then there is a connected $b' \in A$ with $b_1 < b' \le a$. Hence, for some j $(2 \le j \le k)$ $b' \cdot b_j \neq 0$ and therefore by axiom 2, $b_1 + b_j$ is connected contradicting the construction of $b_1, \dots, b_k$. Hence, $b_1, \dots, b_k$ are maximally connected and thus components of a. It follows from axiom 2 that components are unique. Hence, $b_1, \dots, b_k$ are *the* components of a. □

Lemma 5.3.6. *Let $a \in A$ be non-zero with $a \neq 1$. Then there exists a j-partition $c_1, \dots, c_k \in A$ of a.*

Proof. It is sufficient to show that there exists a j-partition for every component of a. Therefore, assume a is connected. If a is a j-element then the lemma holds trivially. Otherwise, let $d_1, \dots, d_n \in A$ $(n \ge 2)$ be the components of $-a$. Since by axiom 6, $1 = a + d_1 + \dots + d_n$ is connected, by axiom 3, $a + d_i$ $(1 \le i \le n)$ is connected. By axiom 10, there exist disjoint connected non-zero elements

$a_1, a_2 \in A$ such that $a = a_1 + a_2$, and $d_1 + a_1$, $d_1 + a_2$, $d_2 + a_1$ and $d_2 + a_2$ are all connected. I show by induction over n that there exists a j-partition of a.

Assume $n = 2$. By axiom 2, $-a_1 = a_2 + d_1 + d_2$ and $-a_2 = a_1 + d_1 + d_2$ are connected. Hence, a_1 and a_2 are j-regions and a_1, a_2 is a j-partition of a.

Assume $n > 2$. Then $-a_1 = a_2 + d_1 + \ldots + d_n$ and $-a_2 = a_1 + d_1 + \ldots + d_n$. Since by axiom 2, $a_1 + d_1 + d_2$ and $a_2 + d_1 + d_2$ are connected, both $-a_1$ and $-a_2$ have fewer than n components. By induction hypothesis, there exist j-partitions $c_1, \ldots, c_l$ and $c_{l+1}, \ldots, c_k$ of a_1 and a_2 respectively. Hence, $c_1, \ldots, c_k$ is a j-partition of a. □

Lemma 5.3.7. *Let $a_1, \ldots, a_n \in A$ be non-zero with $a_1 \neq 1$. Then there exists a j-partition $c_1, \ldots, c_k \in A$ of $a = a_1 + \ldots + a_n$ which refines $a_1, \ldots, a_n$.*

Proof. Let $b_1, \ldots, b_m \in A$ be the non-zero elements of the form $a \cdot \pm a_1 \cdot \ldots \cdot \pm a_n$. Then $b_1, \ldots, b_m$ is a partition of a refining $a_1, \ldots, a_n$. Then with lemma 5.3.6 applied to each b_i there exists a j-partition $c_1, \ldots, c_k \in A$ of a refining $a_1, \ldots, a_n$. □

Lemma 5.3.8. *Let $a_1, \ldots, a_n \in A$ be a j-partition. Then there exist $c_1, \ldots, c_k \in A$ such that $a_1, c_1, \ldots, c_k$ is a radial partition refining $a_1, \ldots, a_n$, and a_1 has at least three neighbours in $c_1, \ldots, c_k$.*

Proof. If $a_1, \ldots, a_n$ is not radial about a_1 then for some a_i $(2 \leq i \leq n)$, a_2 say, $-(a_1 + a_2)$ is disconnected. Let $b_1, \ldots, b_n$ be the components of $-(a_1 + a_2)$. Since a_1 is a j-element, it follows from axiom 3 that $a_2 + b_1$ and $a_2 + b_2$ are connected. By axiom 10, there exist elements $d_1, d_2 \in A$ such that $a_2 = d_1 \oplus d_2$ and $b_1 + d_1$, $b_1 + d_2$, $b_2 + d_1$ and $b_2 + d_2$ are connected. Hence, d_1 and d_2 are j-elements and both $-(a_1 + d_1)$ and $-(a_1 + d_2)$ have fewer than m components. Hence, $a_1, d_1, d_2, a_3, \ldots, a_n$ is a j-partition refining $a_1, \ldots, a_n$. The above argument can be repeated until eventually there is some j-partition $c'_1, \ldots, c'_{k'} \in A$ of $-a_1$ which refines $a_2, \ldots, a_n$ such that $a_1, c'_1, \ldots, c'_{k'}$ is radial about a_1. However, possibly $a_1, c'_1, \ldots, c'_{k'}$ is not radial about c'_1. It has to be shown that $a_1, c'_1, \ldots, c'_{k'}$ can be refined into a radial partition without splitting a_1.

Assume c'_1 is a neighbour of a_1. Then $-(a_1 + c'_1)$ is connected. Assume $-(c'_1 + c'_2)$ is disconnected. Let e_1 and e_2 be two components of $-(c'_1 + c'_2)$ and let $f_1, f_2 \in A$ such that $c'_1 = f_1 \oplus f_2$ and $f_1 + e_1$, $f_1 + e_2$, $f_2 + e_1$ and $f_2 + e_2$ are connected. Since either $e_1 \leq -(a_1 + c_1)$ or $e_2 \leq -(a_1 + c'_1)$, $-(a_1 + f_1) = -(a_1 + c'_1) + f_2$ and $-(a_1 + f_2) = -(a_1 + c'_1) + f_1$ are connected. Therefore, by the same argumentation as above there exist elements $c''_1, \ldots, c''_{k''} \in A$ which refine $c'_2, \ldots, c'_{k'}$ such that $a_1, c'_1, c''_1, \ldots, c''_{k''}$ is a j-partition radial about a_1 and c'_1. Hence, eventually there are $c_1, \ldots, c_k \in A$ such that $a_1, c_1, \ldots, c_k$ is a radial partition.

Possibly, a_1 has only one or two neighbours in $c_1, \ldots, c_k$. If a_1 has only one neighbour in $c_1, \ldots, c_k$ then $k = 1$, $c_1 = a_2$ and $n = 2$. Then by axiom 10, there exist connected elements $e_1, e_2 \in A$ such that $c_1 = e_1 \oplus e_2$, and $a_1 + e_1$ and $a_1 + e_2$ are connected. Then a_1, e_1, e_2 is a j-partition radial about a_1 and the situation is similar to the following case.

If a_1 has only two neighbours in $c_1, \ldots, c_k$, c_1 and c_2 say, then by axiom 10, there exist connected elements $e_1, e_2 \in A$ such that $c_1 = e_1 \oplus e_2$ and $a_1 + e_1$, $a_1 + e_2$, $-(a_1 + c_1) + e_1 = -(a_1 + e_2)$ and $-(a_1 + c_1) + e_2 = -(a_1 + e_1)$ are connected. Then $a_1, e_1, e_2, c_2, \ldots, c_k$ is a j-partition radial about a_1 such that a_1 has three neighbours. It remains to show that $a_1, e_1, e_2, c_2, \ldots, c_k$ is a radial partition. Since $-(c_1 + c_i)$ and $a_1 + e_2$ are connected and $a_1 \leq -(c_1 + c_i)$, it follows from axiom 2 that $-(c_i + e_1) = -(c_1 + c_i) + e_2$ is connected $(2 \leq i \leq k)$. So is $-(c_i + e_2)$ $(2 \leq i \leq k)$. Hence, $a_1, e_1, e_2, c_2, \ldots, c_k$ is a radial partition such that a_1 has three neighbours. □

Lemma 5.3.9. *If $a, b \in A$, $-a$ is connected and b is a component of a, then $-b$ and $b + -a$ are connected.*

Proof. Assume $b_1, b_2, \ldots, b_n$ are the components of a. Since $b_i + b_j$ is disconnected $(1 \leq i < j \leq n)$, by axioms 3 and 6, $b_i + -a$ is connected $(1 \leq i \leq n)$. Then by axiom 2, $-b_1 = -a + b_2 + \ldots + b_n$ is connected. □

On the analogue of the binary connection graph on non-empty connected elements of the domain **S**, the binary connection graph of non-zero connected elements $a_1, \ldots, a_n \in A$ is defined by the vertex set $V = \{a_1, \ldots, a_n\}$ and the edge set $E = \{\{a_i, a_j\} \subseteq V | a_i \neq a_j \text{ and } a_i + a_j \text{ is connected}\}$.

Lemma 5.3.10. *Let $a_1, \ldots, a_n \in A$ be non-zero and connected. Then $a_1 + \ldots + a_n$ is connected if and only if the binary connection graph on $a_1, \ldots, a_n$ is connected.*

Proof. The lemma is shown by induction over n. If $n = 1$ then the lemma holds trivially. Let $n > 1$.

Assume that the binary connection graph Γ on $a_1, \ldots, a_n$ is connected. By lemma 3.1.12 on page 58, there is a_i $(1 \leq i \leq n)$, a_1 say, such that $\Gamma \setminus \{a_1\}$ is connected. Then by induction hypothesis, $a_2 + \ldots + a_n$ is connected. Since Γ is connected, there is some j $(2 \leq j \leq n)$ such that $\{a_1, a_j\}$ is an edge of Γ. Hence, $a_1 + a_j$ is connected. Then by axiom 2, $a_1 + \ldots + a_n$ is connected.

Assume $a_1 + \ldots + a_n$ is connected. Then by axiom 4 there is a_i $(1 \leq i \leq n)$, a_1 say, such that $a_2 + \ldots + a_n$ is connected. By induction hypothesis, the binary connection graph on $a_2, \ldots, a_n$ is connected. By axiom 3, $a_1 + a_j$ is connected for some j with $2 \leq j \leq n$. Hence, the binary connection graph on $a_1, \ldots, a_n$ is connected. □

Lemma 5.3.11. *Let $a_1, \ldots, a_n \in A$ be a connected partition. Then the binary connection graph Γ on $a_1, \ldots, a_n$ is planar. Furthermore, if $a_1, \ldots, a_n$ is a radial partition with at least four elements then Γ is a 3-connected graph.*

Proof. Suppose for proof by contradiction that Γ is non-planar. Since, K_5 or $K_{3,3}$ is a minor of Γ, some subgraph Γ' of Γ is contractible to K_5 or $K_{3,3}$. If Γ'' is the result of the contraction of an edge of Γ', $\{a_1, a_2\}$ say, then Γ'' is a subgraph of the binary connection graph on the connected partition $a_1 + a_2, a_3, \ldots, a_n$. Therefore, K_5 or $K_{3,3}$ is the subgraph of the binary connection graph of some connected partition. However, by axiom 5 and lemma 5.3.4 this is impossible.

For the 3-connectedness of Γ, by theorem 3.3.26 on page 75 it is sufficient to show that for any i, j with $1 \leq i \leq j \leq n$, $\Gamma \backslash \{a_i, a_j\}$ is connected. Since $a_1, \ldots, a_n$ is radial, $-(a_i + a_j)$ is connected. Then by lemma 5.3.10, the binary connection graph on $\{a_1, \ldots, a_n\} \setminus \{a_i, a_j\}$, i.e. the graph $\Gamma \setminus \{a_i, a_j\}$, is connected. □

Lemma 5.3.12. *Let $a_1, \ldots, a_n \in A$ $(n \geq 4)$ be a radial partition such that $a_1, \ldots, a_k$ $(k \geq 3)$ are the unbounded elements of the partition. Let Γ be the binary connection graph on $a_1, \ldots, a_n$. Then $\Gamma \setminus \{a_{k+1}, \ldots, a_n\}$ bounds a face in every plane embedding of Γ.*

Proof. By lemma 5.3.11, Γ is a planar graph. By axiom 9, the graph $C = \Gamma \backslash \{a_{k+1}, \ldots, a_n\}$ is a cycle and $a_1 + \ldots + a_k$ is a j-element. Hence, by lemma 5.3.10, $\Gamma \setminus \{a_1, \ldots, a_k\}$ is connected. Since C is a cycle and $\Gamma \setminus \{a_1, \ldots, a_k\}$ is connected, C bounds a face in any plane embedding of Γ. □

Now I am sufficiently equipped to show the completeness of the axiom system $\mathcal{P}$ in the Δ-calculus.

5.3.2 Completeness of $\mathcal{P}$ in the Δ-calculus

Theorem 5.3.13 (Completeness). *The axiom system $\mathcal{P}$ is complete in the Δ-calculus.*

Proof. Since by theorem 5.3.1, $\mathcal{P}$ is consistent in the Δ-calculus, it follows from proposition 5.1.3 that it is sufficient to show that any two Δ-models of $\mathcal{P}$ are elementarily equivalent. It will be shown that any Δ-model of $\mathcal{P}$ is elementarily equivalent to the Δ-model $\mathfrak{S}$ of $\mathcal{P}$. The theorem then follows immediately.

Let $\mathfrak{A}'$ be any Δ-model of $\mathcal{P}$. By the Downwards-Löwenheim-Skølem theorem, there exists a countable elementary submodel $\mathfrak{A}$ of $\mathfrak{A}'$. Therefore, $\mathfrak{A}$ is a countable Δ-model of $\mathcal{P}$. The elementary equivalence of $\mathfrak{A}$ and $\mathfrak{S}$ will be shown in two stages. In the first stage, subsets of A of increasing but finite cardinality will be embedded into $\mathbf{S}$ such that $\mathfrak{A}$ is isomorphic to a submodel of $\mathfrak{S}$. In the second stage, it will be shown that $\mathfrak{A}$ is isomorphic to an elementary submodel of $\mathfrak{S}$.

Stage 1:

Assume $a_1, a_2, a_3, \ldots$ is an enumeration of the domain A of $\mathfrak{A}$. It will be shown that, for each initial segment $a_1, \ldots, a_n$, some embedding of $a_1, \ldots, a_n$ refines some embedding of $a_1, \ldots, a_m$ for all $m < n$.

Axiom 10 shows that A has more than two elements. Therefore, assume WLOG that $a_1 \notin \{0, 1\}$. I show by induction over n that for every initial segment $a_1, \ldots, a_n$ there is a radial partition $c_1^{(n)}, \ldots, c_{k_n}^{(n)} \in A$ ($k_n \geq 4$) with at least three unbounded elements which refines $a_1, \ldots, a_n$. Moreover, if $n > 1$, $c_1^{(n)}, \ldots, c_{k_n}^{(n)}$ also refines the radial partition $c_1^{(n-1)}, \ldots, c_{k_{n-1}}^{(n-1)}$.

By lemma 5.3.5, a_1 has finitely many components $b_1, \ldots, b_l$ and $-a_1$ has finitely many components $b_{l+1}, \ldots, b_k$. Then $b_1, \ldots, b_k$ is a connected partition. By lemmas 5.3.7 and 5.3.8, there exists a radial partition $c_1^{(1)}, \ldots, c_{k_1}^{(1)} \in A$ ($k_1 \geq 4$) which refines $b_1, \ldots, b_k$ and hence a_1. By axiom 10, the radial partition $c_1^{(1)}, \ldots, c_{k_1}^{(1)}$ can be chosen to have at least three unbounded elements. Now let $n > 1$ and $c_1^{(n-1)}, \ldots, c_{k_{n-1}}^{(n-1)}$ be a radial partition with at least three unbounded elements which refines $a_1, \ldots, a_{n-1}$. By lemma 5.3.5, for each i ($1 \leq i \leq k_{n-1}$), $a_n \cdot c_i^{(n-1)}$ has finitely many components $d_{i,1}, \ldots, d_{i,l_i}$ and $a_n \cdot (-c_i^{(n-1)})$ has finitely many components $d_{i,l_i+1}, \ldots, d_{i,m_i}$. Then $d_{1,1}, \ldots, d_{k_{n-1},m_{k_{n-1}}}$ is a connected partition, that refines $c_1^{(n-1)}, \ldots, c_{k_{n-1}}^{(n-1)}$ and $a_1, \ldots, a_n$. By lemmas 5.3.7 and 5.3.8, there is a radial partition $c_1^{(n)}, \ldots, c_{k_n}^{(n)} \in A$ ($k_{n-1} \geq 4$) with at least three unbounded regions that is a refinement of $d_{1,1}, \ldots, d_{1,m_1}, \ldots, d_{k_{n-1},1}, \ldots, d_{k_{n-1},m_{k_{n-1}}}$ and hence of $c_1^{(n-1)}, \ldots, c_{k_{n-1}}^{(n-1)}$ and $a_1, \ldots, a_n$.

Each radial partition $c_1^{(n)}, \ldots, c_{k_n}^{(n)}$ and hence each initial segment $a_1, \ldots, a_n$ will now be mapped into the domain $\mathbf{S}$.

For $n \geq 1$, let $G^{(n)}$ be the set of all functions $g^{(n)} : \{c_1^{(n)}, \ldots, c_{k_n}^{(n)}\} \to \mathbf{S}$ such that the following three conditions are satisfied:

B1: $g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_{k_n}^{(n)})$ is a radial partition,

B2: For $1 \leq i \leq j \leq k_n$, $c_i^{(n)} + c_j^{(n)}$ is connected if and only if $g^{(n)}(c_i^{(n)}) + g^{(n)}(c_j^{(n)})$ is connected,

B3: For $1 \leq i \leq n$, $c_i^{(n)}$ is bounded if and only if $g^{(n)}(c_i^{(n)})$ is bounded.

Claim 1. *For $n \geq 1$, $G^{(n)} \neq \emptyset$.*

Proof. In this proof, let $c_1, \ldots, c_k$ stand for $c_1^{(n)}, \ldots, c_{k_n}^{(n)}$. Since $c_1, \ldots, c_k$ is a radial partition, it follows from lemma 5.3.11 that the binary connection graph Γ on $c_1, \ldots, c_k$ is a 3-connected planar graph. Let $c_1, \ldots, c_m$ be the unbounded elements of $c_1, \ldots, c_k$. Since $m \geq 3$, it follows from lemma 5.3.12 that $c_1, \ldots, c_m$ are the vertices of a face in any plane embedding of Γ. Let $\Gamma_t = (V = \{v_1, \ldots, v_k\}, E)$ be a plane embedding of Γ in the closed plane such that the vertices $v_1, \ldots, v_m$ are the vertices of the boundary of the unbounded face f of Γ_t, i.e. $p_\infty \in f$. In the following, I construct a plane graph $\Gamma_t' = (V', E')$ such that the faces $r_1^*, \ldots, r_k^*$ of Γ_t' are elements of $\mathbf{S}^*$ which form a radial partition, $r_i^* + r_j^*$ is connected if and only if $c_i + c_j$ is connected and r_i is bounded if and only if c_i is bounded $(1 \leq i \leq j \leq k)$. Moreover, Γ_t' will be constructed such that there is a bijection $\phi : V \cup E \cup F \to V' \cup E' \cup F'$ where F and F' are the sets of faces of Γ_t and Γ_t' respectively and $\phi(V) = F'$, $\phi(E) = E'$, $\phi(F) = V'$, $v \in \phi(v)$ for all $v \in V$ and $\phi(f) \in f$ for all $f \in F$.

Choose, in every face f of Γ_t, a point p_f such that, if $p_\infty \in f$ then $p_f = p_\infty$. Let V' be the set of these points and let $\phi(f) = p_f$. Since Γ_t is 3-connected, it follows from theorem 3.3.27 that the faces of Γ_t are Jordan regions. Hence, any edge of Γ_t is shared by exactly two faces. For any edge γ of Γ_t construct an edge γ' of Γ_t' as follows. Let f_1 and f_2 be the faces sharing the edge γ. Let γ' be a semi-algebraic arc from p_{f_1} to p_{f_2} in $f_1 + f_2$ such that the interior of γ' does not intersect any edge of Γ_t' which has already been defined. Let $\phi(\gamma) = \gamma'$. After the construction of all edges, each vertex v_i of Γ_t lies in a face $r_i^* \in \mathbf{S}^*$ of Γ_t' and in each face of Γ_t' lies exactly one vertex of Γ_t. Then Γ_t' is a so-called *geometric dual* of Γ_t with $\phi(v_i) = r_i^*$ $(1 \leq i \leq k)$. Since Γ_t is 3-connected it follows from

(Diestel, 1997, Chapter 4, Section 6) that Γ'_t is 3-connected. By theorem 3.3.27, $r_1^*, \ldots, r_k^* \in \mathbf{S}^*$ is a j-partition of the closed plane.

If $c_i + c_j$ is connected for some distinct $i, j \in \{1, \ldots, k\}$, then v_i and v_j are the endpoints of some edge γ in Γ_t. By construction, $\phi(\gamma)$ is the common edge of the faces $\phi(v_i) = r_i^*$ and $\phi(v_j) = r_j^*$. Hence, $r_i^* + r_j^*$ is connected. If $r_i^* + r_j^*$ is connected for some distinct $i, j \in \{1, \ldots, k\}$, then r_i^* and r_j^* share a common edge γ'. By construction, v_i and v_j are the endpoints of the edge $\phi^{-1}(\gamma)$. Hence, $c_i + c_j$ is connected.

If c_i $(1 \leq i \leq k)$ is unbounded then v_i is a vertex of an edge γ of the face f with $p_\infty \in f$. Hence, $p_\infty \in \phi(\gamma)$. Since $\phi(\gamma)$ is part of the boundary of $\phi(v_i) = r_i^*$, $p_\infty \in [r_i^*]$, whence r_i is unbounded. If $p_\infty \in [r_i^*]$ then p_∞ is a point in some edge γ' of r_i^*. Hence, the vertices of $\phi^{-1}(\gamma')$ belong to the unbounded vertices, and one of the endpoints of $\phi^{-1}(\gamma')$ is $\phi^{-1}(r_i^*) = v_i$. Hence, c_i is an unbounded element.

Hence, $r_1, \ldots, r_k \in \mathbf{S}$ is a radial partition such that $r_i + r_j$ is connected iff $c_i + c_j$ is connected and r_i is bounded if c_i is bounded. □

Claim 2. *Let $g_1, g_2 \in G^{(n)}$. Then $g_1(c_1^{(n)}), \ldots, g_1(c_{k_n}^{(n)}) \sim g_2(c_1^{(n)}), \ldots, g_2(c_{k_n}^{(n)})$.*

Proof. Since $g_1(c_i^{(n)})$ is bounded if and only if $g_2(c_i^{(n)})$ is bounded, and $g_1(c_i^{(n)}) + g_1(c_j^{(n)})$ is connected if and only if $g_2(c_i^{(n)}) + g_2(c_j^{(n)})$ is connected $(1 \leq i \leq j \leq k_n)$, the claim follows by lemma 3.3.33 on page 76. □

In the following, given a non-empty set $C \subseteq \{c_1^{(n)}, \ldots, c_k^{(n)}\}$, I write $g^{(n)}(C)$ for the set $\{g^{(n)}(c_i^{(n)}) | c_i^{(n)} \in C\}$.

Claim 3. *Let $C \subseteq \{c_1^{(n)}, \ldots, c_{k_n}^{(n)}\}$ be non-empty. Then*
(i) $\sum C$ is connected iff $\sum g^{(n)}(C)$ is connected and
(ii) $\sum C$ is bounded iff $\sum g^{(n)}(C)$ is bounded.

Proof. Since $\mathfrak{S}$ is a model of $\mathcal{P}$, (i) follows from lemma 5.3.10 and condition B2 of $g^{(n)}$, and (ii) follows from axiom 8 and condition B3 of $g(n)$. □

By construction, the radial partition $c_1^{(n)}, \ldots, c_{k_n}^{(n)}$ refines the radial partition $c_1^{(m)}, \ldots, c_{k_m}^{(m)}$ for each n, m with $1 \leq m \leq n$. Let for each i with $1 \leq i \leq k_n$, $c_{i_1}^{(n)}, \ldots, c_{i_l}^{(n)}$ be the elements of $c_1^{(n)}, \ldots, c_{k_n}^{(n)}$ that sum to $c_i^{(m)}$. Then for $g^{(n)} \in G^{(n)}$, the *restriction* of $g^{(n)}$ to $c_1^{(m)}, \ldots, c_{k_m}^{(m)}$ $(1 \leq m \leq n)$, written $g^{(n)}|_m$ is defined by

$$g^{(n)}|_m(c_i^{(m)}) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_l}^{(n)}).$$

Claim 4. *Let $g^{(n)} \in G^{(n)}$ $(n > 1)$. Then $g^{(n)}|_{n-1} \in G^{(n-1)}$.*

Proof. It has to be shown that the conditions B1, B2 and B3 hold for $g^{(n)}|_{n-1}$. For B2, note that by construction of $g^{(n)}|_{n-1}$,

$$\begin{aligned} & g^{(n)}|_{n-1}(c_i^{(n-1)}) + g^{(n)}|_{n-1}(c_j^{(n-1)}) \\ & \quad = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_{l_i}}^{(n)}) + g^{(n)}(c_{j_1}^{(n)}) + \ldots + g^{(n)}(c_{j_{l_j}}^{(n)}) \end{aligned}$$

which is by claim 3(i) connected if and only if $c_{i_1}^{(n)} + \ldots + c_{i_{l_i}}^{(n)} + c_{j_1}^{(n)} + \ldots + c_{j_{l_j}}^{(n)} = c_i^{(m)} + c_j^{(m)}$ is connected. Consequently, condition B1 holds as well.

Condition B3 holds, since by claim 3(ii), $g^{(n)}|_{n-1}(c_i^{(n-1)}) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_{l_i}}^{(n)})$ is bounded if and only if $c_{i_1}^{(n)} + \ldots + c_{i_{l_i}}^{(n)} = c_i^{(n-1)}$ is bounded. □

Claim 5. *Let $g^{(n)} \in G^{(n)}$. Then there exists $g^{(n+1)} \in G^{(n+1)}$ such that $g^{(n+1)}|_n = g^{(n)}$.*

Proof. Let $g^{(n+1)} \in G^{(n+1)}$. It follows from claim 4 that $g^{(n+1)}|_n \in G^{(n)}$, and from claim 2 that $g^{(n+1)}|_n(c_1^{(n)}), \ldots, g^{(n+1)}|_n(c_{k_n}^{(n)}) \sim g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_{k_n}^{(n)})$. Since $\mathbf{S}$ is topologically homogeneous as shown by proposition 3.3.40 on page 80, there exist $s_1, \ldots, s_{k_{n+1}} \in \mathbf{S}$ $(1 \leq i \leq k_n)$ such that

$$\begin{aligned} & g^{(n+1)}|_n(c_1^{(n)}), \ldots, g^{(n+1)}|_n(c_{k_n}^{(n)}), g^{(n+1)}(c_1^{(n+1)}), \ldots, g^{(n+1)}(c_{k_{n+1}}^{(n+1)}) \\ & \quad \sim g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_{k_n}^{(n)}), s_1, \ldots, s_{k_{n+1}} . \end{aligned}$$

Hence, there exists a homeomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(g^{(n+1)}(c_i^{(n+1)})) = s_i$ $(1 \leq i \leq k_{n+1})$. Then $h \circ g^{(n+1)}$ is the required function. □

Claim 6. *The model $\mathfrak{A}$ is isomorphically embedded into $\mathfrak{S}$.*

Proof. Choose a function $g^{(1)} \in G^{(1)}$. It follows from claim 5 that for every $n > 1$ a function $g^{(n)} \in G^{(n)}$ can be chosen such that $g^{(n)}|_{n-1} = g^{(n-1)}$. Any non-zero $a \in A$, is the sum of some $c_{j_1}^{(n)}, \ldots, c_{j_l}^{(n)}$ for some $n \geq 1$. Let the embedding function $g\colon A \to \mathbf{S}$ be defined by $g(0) = \emptyset$ and $g(a) = g^{(n)}(c_{j_1}^{(n)}) + \ldots + g^{(n)}(c_{j_l}^{(n)})$ for $a \neq 0$. Then g is a Boolean algebra isomorphism such that $a \in A$ is connected iff $g(a)$ is connected and $a \in A$ is bounded iff $g(a)$ is bounded. For $a = 0$ the equivalences are ensured by lemmas 5.3.2 and 5.3.3. □

Stage 2:

By the last claim, A can be embedded into $\mathbf{S}$ such that some model $\mathfrak{C}$ which is isomorphic to $\mathfrak{A}$ is a submodel of $\mathfrak{S}$. For simplicity I regard A as the image of the embedding and therefore consider from now on the elements of A as regular open semi-algebraic sets in the plane. It remains to show that $\mathfrak{A} \preceq \mathfrak{S}$.

Claim 7. *Let $a_1, \ldots, a_n \in A$ be a j-partition radial about a_1 such that a_1 has at least three neighbours. Let $r_1, r_2 \in \mathbf{S}$ be j-regions with $a_1 = r_1 \oplus r_2$. Then there exist $b_1, b_2 \in A$ such that $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2$.*

Proof. Consider the elements $a_1^*, \ldots, a_n^*, r_1^*, r_2^* \in \mathbf{S}^*$. Remember that by proposition 3.2.1 for all $s_1, s_2 \in \mathbf{S}$, $s_1 + s_2$ is connected if and only if $s_1^* + s_2^*$ is connected, and s_1 is bounded if and only if $p_\infty \notin [s_1^*]$. Then it is sufficient to show that there exist $b_1, b_2 \in A$ such that $a_1^*, \ldots, a_n^*, r_1^*, r_2^* \sim a_1^*, \ldots, a_n^*, b_1^*, b_2^*$ and for $i \in \{1, 2\}$, $p_\infty \in [b_i^*]$ is bounded iff $p_\infty \in [r_i^*]$ is bounded.

It follows from planarity considerations that r_1 and r_2 have at most two neighbours in $a_2, \ldots, a_n$ in common. I will consider the number of common neighbours of r_1 and r_2 as separate cases.

Case 1: Assume the regions r_1 and r_2 have no neighbour in $a_2, \ldots, a_n$ in common.

Since a_1, r_1 and r_2 are j-regions, a_1^*, r_1^* and r_2^* are Jordan regions and $[r_1^*] \cap [r_2^*]$ is the locus of a cross-cut γ_r in a_1^*. Therefore, to show that $a_1^*, \ldots, a_n^*, r_1^*, r_2^* \sim a_1^*, \ldots, a_n^*, b_1^*, b_2^*$ for some $b_1, b_2 \in A$ it is sufficient to show that $[b_1^*] \cap [b_2^*]$ is the locus of a cross-cut γ_b in a_1^* that has the same endpoints as γ_r.

Assume $a_2, \ldots, a_k$ are the elements of $\{a_2, \ldots, a_n\}$ such that $\partial(a_2^*) \cap (\partial(r_1^*) \setminus \partial(r_2^*)) \neq \emptyset$. It is easy to see that no a_i $(2 \leq i \leq k)$ is a neighbour of r_2, and that $a_2^* + \ldots + a_k^*$ is connected. Let d^* be the components of $-(a_2^* + \ldots + a_k^*)$ such that $a_1^* \subseteq d^*$. Let $c^* = d^* \cdot -a_1^*$. Since d^* and a_1^* are Jordan regions, c^* is a Jordan region. By axiom 11, there exist $b_1, b_2 \in A$ such that $a_1 = b_1 \oplus b_2$, $b_1 + c$ and $b_2 + -(a_1 + c)$ are connected, and $b_1 + -(a_1 + c)$ and $b_2 + c$ are disconnected. Hence, for $i \in \{1, 2\}$, b_i has the same neighbours as r_i. Then the cross-cut γ_b defined by b_1^* and b_2^* has the same endpoints as γ_r. Hence, $a_1^*, \ldots, a_n^*, r_1^*, r_2^* \sim a_1^*, \ldots, a_n^*, b_1^*, b_2^*$.

It remains to show that b_i is bounded iff a_i is bounded $(i \in \{1, 2\})$. Note that the closed plane is the disjoint union of the sets $-a_1^*$, a_1^*, $\partial(a_1^*) \cap \partial(c^*) \cap (a_1 + c)^*$, $(\partial(a_1^*) \cap \partial(c^*)) \setminus (a_1 + c)^*$ and $\partial(a_1^*) \setminus \partial(c^*)$. Therefore, the point at infinity p_∞ lies in one of these sets. If $p_\infty \in -a_1^*$ then a_1 is bounded and so are r_1, r_2, b_1 and b_2. If $p_\infty \in a_1^*$ then a_1 is unbounded and $-a_1$ is bounded. Then r_1 or r_2 is unbounded and by axioms 8 and 11, b_1 and b_2 can be chosen as required. If $p_\infty \in \partial(a_1^*) \cap \partial(c^*) \cap (a_1 + c)^*$ then a_1 and c are unbounded but $-(a_1 + c)$ is bounded. Then r_1 is unbounded and r_2 bounded. Since $\partial(a_1^*) \cap \partial(c^*) \cap (a_1 + c)^* \subseteq \partial(b_1^*)$, b_1 is unbounded and b_2 is bounded. If $p_\infty \in (\partial(a_1^*) \cap \partial(c^*)) \setminus (a_1 + c)^*$ then a_1, c, r_1, r_2 and $-(a_1 + c)$ are unbounded. So are the regions b_1 and b_2. And finally,

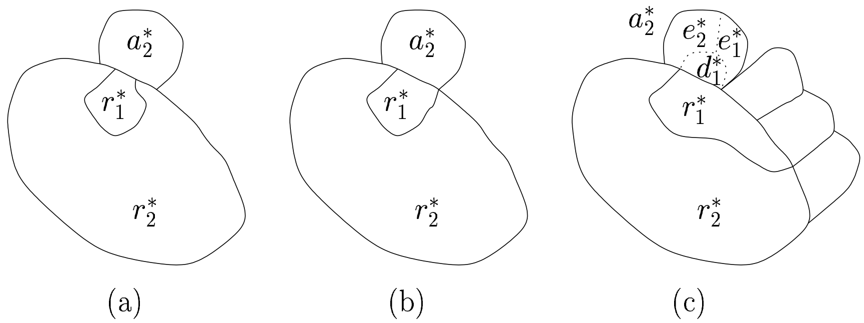


Figure 5.3: Regions r_1 and r_2 having one neighbour in $a_2, \dots, a_n$ in common

if $p_\infty \in \partial(a_1^*) \setminus \partial(c^*)$ then r_1 and c are bounded and r_2 and $-a_1$ are unbounded. Since $\partial(a_1^*) \setminus \partial(c^*) \subseteq \partial(b_2^*)$, b_1 is bounded and b_2 is unbounded.

Case 2: The regions r_1 and r_2 have two neighbours in common.

Assume WLOG that a_2 and a_3 are the common neighbours of r_1 and r_2. Then $[r_1^*] \cap [r_2^*]$ is the locus of a cross-cut γ_r in a_1^* which has one end-point in $a_1^* + a_2^*$ and one end-point in $a_1^* + a_3^*$. It is sufficient to show that there exist disjoint j-regions $b_1, b_2 \in A$ such that $a = b_1 \oplus b_2$, the cross-cut γ_b in a_1^* defined by $[b_1^*] \cap [b_2^*]$ has one end-point in $a_1^* + a_3^*$ and one end-point in $a_1^* + a_3^*$ and the regions b_1 and b_2 can be chosen such that for $i \in \{1, 2\}$, b_i has the same neighbours as r_i in $a_2, \dots, a_n$ and b_i is bounded if and only if r_i is bounded. By axiom 10, there are $b_1, b_2 \in A$ such that $a_2 + b_1$, $a_2 + b_2$, $a_3 + b_1$ and $a_3 + b_2$ are connected. Hence, $[b_1^*] \cap [b_2^*]$ is the locus of a cross-cut γ_b that has one end-point in $a_1^* + a_3^*$ and one end-point in $a_1^* + a_3^*$. WLOG let b_1 have the same neighbours as r_1.

The closed plane is the union of the mutually disjoint sets $-a_1^*, a_1^*, \partial(a_1^*) \cap \partial(a_2^*) \cap (a_1 + a_2)^*$, $\partial(a_1^*) \cap \partial(a_3^*) \cap (a_1 + a_3)^*$ and $\partial(a_1^*) \cap \partial((a_1 + a_2 + a_3)^*)$. If $p_\infty \in -a_1^*$ then a_1, r_1 and r_2 are bounded and so are b_1 and b_2. If $p_\infty \in a_1^*$ then a_1 and r_1 or r_2 are unbounded. By axioms 8 and 10, the elements b_1 and b_2 can be chosen as required. If $p_\infty \in \partial(a_1^*) \cap \partial(a_2^*) \cap (a_1 + a_2)^*$ then a_1, a_2 and r_1 or r_2 are unbounded and $-(a_1 + a_2)$ is bounded. Then by axioms 8 and 10, b_1 and b_2 can be chosen as required. The case $p_\infty \in \partial(a_1^*) \cap \partial(a_3^*) \cap (a_1 + a_3)^*$ is equivalent to the previous one. Finally, if $p_\infty \in \partial(a_1^*) \cap \partial((a_1 + a_2 + a_3)^*)$ then a_1, $-(a_1 + a_2)$ and only one of r_1 and r_2 is unbounded. Then only one of b_1 and b_2 is unbounded, and by axioms 8 and 10, b_1 and b_2 can be chosen as required.

Case 3: The regions r_1 and r_2 have only one neighbour in $a_2, \dots, a_n$ in common.

I distinguish two subcases.

Case 3.1: The region r_1^* has one neighbour only and is not in contact with any other neighbour of r_2^*. Figure 5.3(a) shows an example.

Assume WLOG that a_2 is the neighbour of r_1. By axiom 10, there exist $d_1, d_2 \in A$ such that $a_2 = d_1 \oplus d_2$, $d_1 + r_1$, $d_1 + r_2$, $d_1 + -(a_1 + a_2)$ and $d_2 + -(a_1 + a_2)$ are connected. Then $a_1, d_1, d_2, a_3, \ldots, a_n$ is a j-partition radial about a_1 such that a_1 has at least three neighbours. Then by case 2 of this proof, there exist $b_1, b_2 \in A$ such that $a_1, d_1, d_2, a_3, \ldots, a_n, r_1, r_2 \sim a_1, d_1, d_2, a_3, \ldots, a_n, b_1, b_2$ and hence $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2$.

Case 3.2: The region r_1^* is in contact with exactly two neighbours of r_2^*.

Figures 5.3(b) and 5.3(c) show examples. Assume that a_2 is the neighbour common to r_1 and r_2. By axiom 11, there are $d_1, d_2 \in A$ such that $a_2 = d_1 \oplus d_2$, $d_1 + r_1$ and $d_2 + -(r_1 + a_2)$ are connected and $d_2 + r_1$ and $d_1 + -(r_1 + a_2)$ are disconnected. Then $d_2 + r_2$ is connected. By axiom 10, there exist $e_1, e_2 \in A$ such that $d_2 = e_1 \oplus e_2$ and $e_1 + d_1, e_2 + d_1, e_1 + -(a_1 + a_2)$ and $e_2 + -(a_1 + a_2)$ are connected. Then WLOG, $e_2 + a_1$ is connected and $e_1 + a_1$ is disconnected. Then $a_1, d_1 + e_1, e_2, a_3, \ldots, a_n$ is a j-partition radial about a_1 such that a_1 has at least three neighbours and r_1 and r_2 do not have any neighbours in $d_1 + e_1, e_2, a_3, \ldots, a_n$ in common. Then by case 1 of this proof there exist $b_1, b_2 \in A$ such that $a_1, d_1 + e_1, e_2, a_3, \ldots, a_n, r_1, r_2 \sim a_1, d_1 + e_1, e_2, a_3, \ldots, a_n, b_1, b_2$ and, therefore, $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2$. □

Claim 8. *Let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a j-region. Let $r_1, r_2 \in \mathbf{S}$ be j-regions such that $a_1 = r_1 \oplus r_2$. Then there are $b_1, b_2 \in A$ such that $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2$.*

Proof. By lemma 5.3.6 there exists a j-partition $a_1, c_1, \ldots, c_k$ which refines $a_1, \ldots, a_n$. By lemma 5.3.8, there exist $d_1, \ldots, d_m \in A$ such that $a_1, d_1, \ldots, d_m$ is a radial partition refining $a_1, c_1, \ldots, c_k$ such that a_1 has at least three neighbours in $d_1, \ldots, d_m$. By claim 7, there are $b_1, b_2 \in A$ such that $a_1, d_1, \ldots, d_m, r_1, r_2 \sim a_1, d_1, \ldots, d_m, b_1, b_2$. Hence, $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2$. □

Claim 9. *Let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a j-region. Let $r \in \mathbf{S}$ be a subset of a_1. Then there exists $b \in A$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$.*

Proof. If $r = \emptyset$ then let $b = \emptyset$. Assume $r \neq \emptyset$. By lemma 5.3.6, there exists a j-partition $r_1, \ldots, r_m \in \mathbf{S}$ or a_1 which refines r. It suffices to show that there are $b_1, \ldots, b_m \in A$ such that $a_1, \ldots, a_n, r_1, \ldots, r_m \sim a_1, \ldots, a_n, b_1, \ldots, b_m$. I proceed by induction on m. If $m = 1$ then $b_1 = a_1$ and the claim holds trivially. If $m > 1$, by lemma 3.1.15 on page 59, the regions $r_1, \ldots, r_n$ can be renumbered such that $r_2' = r_2 + \ldots + r_m$ is a j-region. Then $a_1 = r_1 \oplus r_2'$. By claim 8, there exist $b_1, b_2' \in A$ such that $a_1, \ldots, a_n, r_1, r_2 \sim a_1, \ldots, a_n, b_1, b_2'$. Since $\mathbf{S}$ is topologically homogeneous, there exist $s_2, \ldots, s_m \in \mathbf{S}$ such that $a_1, \ldots, a_n, r_1, r_2, \ldots, r_m \sim a_1, \ldots, a_n, b_1, s_2, \ldots, s_m$. Since $s_2, \ldots, s_m$ is a j-partition of b_2', it follows from the induction hypothesis that here exist $b_2, \ldots, b_m \in A$ such that $a_1, \ldots, a_n, r_1, \ldots, r_m \sim a_1, \ldots, a_n, b_1, \ldots, b_m$. □

Claim 10. *Let $a_1, \ldots, a_n \in A$ be a j-partition and $r \in \mathbf{S}$. Then there exists $b \in A$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$.*

Proof. Consider r as the sum of the regions $a_1 \cdot r, \ldots, a_n \cdot r$. A repeated application of claim 9 guarantees the existence of $b_1, \ldots, b_n \in A$ such that $a_1, \ldots, a_n, a_1 \cdot r, \ldots, a_n \cdot r \sim a_1, \ldots, a_n, b_1, \ldots, b_n$ and hence $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$ with $b = b_1 + \ldots + b_n$. □

Claim 11. *Let $a_1, \ldots, a_n \in A$ and $r \in \mathbf{S}$. Then there exists $b \in A$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$.*

Proof. Assume WLOG that $a_1 \neq 1$. By lemma 5.3.7, there exists a j-partition $c_1, \ldots, c_k \in A$ which refines $a_1, \ldots, a_n$. Then by claim 10 there exists $b \in A$ such that $c_1, \ldots, c_k, r \sim c_1, \ldots, c_k, b$. Hence, $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$. □

Claim 12. $\mathfrak{A} \preceq \mathfrak{S}$.

Proof. By claim 6, $\mathfrak{A}$ is a submodel of $\mathfrak{S}$. In order to show that $\mathfrak{A}$ is an elementary submodel of $\mathfrak{S}$, by the Tarski-Vaught-Lemma (Mendelson, 1997, Proposition 2.37) it is sufficient to show that for every formula $\phi(x_0, \ldots, x_n)$ in $\mathcal{L}$, $r \in \mathbf{S}$ and $a_1, \ldots, a_n \in A$, if $\mathfrak{S} \models \phi[r, a_1, \ldots, a_n]$ then there exists $b \in A$ such that $\mathfrak{S} \models \phi[b, a_1, \ldots, a_n]$. By claim 11, for any $r \in \mathbf{S}$, $a_1, \ldots, a_n \in A$ there exists $b \in A$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, b$. Since $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$ is a mereotopological language, if $\mathfrak{S} \models \phi[r, a_1, \ldots, a_n]$ then $\mathfrak{S} \models \phi[b, a_1, \ldots, a_n]$. □

The last claim completes the completeness proof. □

Corollary 5.3.14. *The deductive closure of $\mathcal{P}$ in the Δ-calculus is the theory of $\mathfrak{S}$ in the language $\mathcal{L}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$.*

Since the connectedness- and boundedness-relations and the Boolean operations and constants are $\mathcal{L}(\mathrm{C})$-definable in $\mathfrak{S}$, $\mathcal{P}$ can also be considered to be a complete axiomatisation of $\mathfrak{S}(\mathrm{C})$ in the Δ-calculus.

5.4 Conclusion

In this chapter, I introduced the Δ-calculus, which extends the predicate calculus by an additional infinitary rule of inference. This rule ensures—in the special case considered here—that any set of mereotopological formulae which is consistent in the Δ-calculus has a mereotopology as model all of whose elements have finitely many components. As a consequence, I could show the axiom system $\mathcal{P}$ to be complete in the Δ-calculus. Furthermore, the deductive closure in the Δ-calculus of $\mathcal{P}$ was shown to be the theory of $\mathfrak{S}(\mathrm{C})$. One of the properties that make $\mathfrak{S}(\mathrm{C})$ interesting for a common-sense representation space is that all of the regions in $\mathbf{S}$ have only finitely many components. However, proposition 4.3.6 on page 97 showed that there exists a model of $Th(\mathfrak{S}(\mathrm{C}))$ which does not omit Δ, i.e. some "regions" of this model have infinitely many "components". Thus, the first-order finitary mereotopological language $\mathcal{L}(\mathrm{C})$ is not sufficient to capture all of the nice properties of $\mathfrak{S}(\mathrm{C})$. So far, it has not been investigated whether a model of $\mathcal{P}$ that does not omit Δ can be a mereotopology. Such models are perhaps very exotic and cannot be constructed over the topological space $\mathbb{R}^2$. However, I conjecture:

Conjecture 5.4.1. *The mereotopology* $\mathfrak{J}(\mathrm{C})$ *is a model of* $\mathcal{P}$.

My belief in this conjecture is based on conjecture 3.2.10. If conjecture 3.2.10 is true, then axioms 1-9 have essentially already been shown to hold for $\mathfrak{J}(\mathrm{C})$. Since axioms 10 and 11 rely on connected regions and even Jordan regions, I expect them to hold in $\mathfrak{J}(\mathrm{C})$.

By theorem 4.3.3, $\mathfrak{J}(\mathrm{C})$ and $\mathfrak{S}(\mathrm{C})$ are not elementarily equivalent. The proof of this theorem shows that there exists a simple formula that distinguishes between $\mathfrak{J}(\mathrm{C})$ and $\mathfrak{S}(\mathrm{C})$. Immediately two questions arise:

1. Can we add some more formulae to $\mathcal{P}$ to construct an axiomatisation of $Th(\mathfrak{S}(\mathrm{C}))$?

2. Can $\mathcal{P}$ at least be extended to a theory $\mathcal{P}'$ that axiomatises the class of well-behaved mereotopologies that are models of $Th(\mathfrak{S}(\mathrm{C}))$?

Question 1 is answered in the next chapter in the negative. The axiom system $\mathcal{P}$ is shown to be incomplete. Furthermore, it is shown that no axiomatisation of $Th(\mathfrak{S}(\mathrm{C}))$ can possibly be found since $Th(\mathfrak{S}(\mathrm{C}))$ is undecidable. The undecidability proof provides the means for an interesting partial answer to question 2 which is given in section 6.2 below.

One interesting open question is whether other o-minimal structures, for example the o-minimal structure definable with the exponential function (cf. section 3.4), provide models of $Th(\mathfrak{S}(\mathrm{C}))$.

Chapter 6

The undecidability of mereotopological theories

This chapter shows almost all mereotopological theories which are considered in this thesis to be undecidable.

The decidability of mereotopological theories has been investigated before. Dornheim (1998) investigates the properties of a mereotopology which is similar to the mereotopology $\mathfrak{P}$. He encodes the Post-correspondence problem in the theory of his mereotopology, and thus shows its undecidability. Dornheim's undecidability proof relies on the fact that his spatial domain has regions with only finitely many components. Grzegorczyk (1951) shows a series of theories to be undecidable. His "theory of bodies" is essentially the theory of the mereotopology in the language $\mathcal{L}(\mathrm{C})$ of regular open sets in a second countable metric space. The paper does not give the full proof but only a sketch of a crucial induction. A closer inspection reveals that the proof relies on an important property of the spatial domain: it is closed under infinite sums.

It is possible to transfer these undecidability results to the theories of $\widetilde{\mathfrak{F}}(\mathrm{C})$ and $\widetilde{\mathfrak{S}}(\mathrm{C})$, and, given the expressivity results in section 4.1, also to the theory of $\mathfrak{S}^*(\leq, \mathrm{c})$. However, the undecidability of the theories of $\mathfrak{S}(\leq, \mathrm{c})$, $\widetilde{\mathfrak{I}}(\mathrm{C})$, $\widetilde{\mathfrak{I}}(\leq, \mathrm{c})$ and $\mathfrak{F}(\leq, \mathrm{c})$ cannot be inferred from either Dornheim's or Grzegorczyk's proof. Therefore, I will present a generalisation of Grzegorczyk's proof which shows the undecidability of $Th(\widetilde{\mathfrak{S}})$ and $Th(\widetilde{\mathfrak{F}})$. Moreover, I will show the stronger result that the mereotopological theories are hereditarily undecidable as defined below.

Definition 6.0.2. A theory T is *decidable* if the set of its consequences is recursive. A consistent theory T in a recursive first-order language $\mathcal{L}$ is *essentially* *undecidable*, if every consistent theory T' in $\mathcal{L}$ with $T \subseteq T'$ is undecidable. A theory T in a first-order language $\mathcal{L}$ is *hereditarily undecidable* if every subset of the deductive closure of T is an undecidable theory in $\mathcal{L}$.

The main result of this chapter is:

Theorem 6.0.3. *The mereotopological theories of $\widetilde{\mathfrak{F}}$, $\widetilde{\mathfrak{S}}$, $\widetilde{\mathfrak{P}}$ and $\widetilde{\mathfrak{Q}}$ are hereditarily undecidable.*

Since the proof of this theorem relies on the fact that the spatial domain is closed under sums of components, the result is only conjectured for $\widetilde{\mathfrak{J}}$.

Conjecture 6.0.4. *The mereotopological theory of $\widetilde{\mathfrak{J}}$ is hereditarily undecidable.*

It is a consequence of the expressivity results in section 4.1 that it is sufficient to show the following proposition in order to prove the above theorem.

Proposition 6.0.5. *The theories of $\mathfrak{F}^*(\leq, \mathrm{c})$ and $\mathfrak{S}^*(\leq, \mathrm{c})$ are hereditarily undecidable.*

However, I will show a stronger theorem that does not restrict the result to planar mereotopology:

Theorem 6.0.6. *Let $\mathfrak{M}$ be a mereotopology in the mereotopological language $\mathcal{L}(\leq, \mathrm{c})$ over the compact space $(\mathbb{R}^k)^*$ $(k \geq 2)$ such that*

(i) $(M, \subseteq)$ *is a Boolean subalgebra of* $RO((\mathbb{R}^k)^*)$,

(ii) if $r \in M$ *and* R *is a subset of the components of* r *then* $\sum R \in M$,

(iii) $(\mathbf{Q}_{\mathbb{R}^k})^* \subseteq M$, *where* $\mathbf{Q}_{\mathbb{R}^k}$ *stands for the set of regular open semi-linear sets in* $\mathbb{R}^k$ *with rational "corner points". In particular, for* $k = 2$, $\mathbf{Q}_{\mathbb{R}^2} = \mathbf{Q}$.

Then the $\mathcal{L}(\leq, \mathrm{c})$-theory of $\mathfrak{M}$ is hereditarily undecidable.

By construction, the spatial domains $\mathbf{F}^*, \mathbf{J}^*$ and $\mathbf{S}^*$ satisfy conditions (i) and (iii). Condition (ii) is satisfied for $\mathbf{F}^*$, since $\mathbf{F}^*$ is closed under infinite sums, and for $\mathbf{S}^*$, since every region in $\mathbf{S}^*$ has only finitely many components and $\mathbf{S}^*$ is closed under finite sums. Condition (ii) is also believed to hold for $\mathbf{J}^*$ (cf. conjecture 3.2.13 on page 65).

From now on let $\mathfrak{M}$ be some mereotopology with domain M satisfying the conditions of theorem 6.0.6.

6.1 The proof

The proof of theorem 6.0.6 makes use of the fact that hereditary undecidability of a theory T can be proven by interpreting a finite essentially undecidable theory in T (see theorem 6.1.2 below). The formal definition of interpretation is given as follows.

Definition 6.1.1 (Hodges, 1993). Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two first-order languages, $\mathfrak{A}$ an $\mathcal{L}_1$-structure with domain A, and $\mathfrak{B}$ an $\mathcal{L}_2$-structure with domain B. An (n-dimensional) *interpretation* $\mathcal{I}$ of $\mathfrak{B}$ in $\mathfrak{A}$ is a triple $(\theta_\mathcal{I}, \cdot_\mathcal{I}, f_\mathcal{I})$ where

1. $\theta_\mathcal{I}(x_1, \dots, x_n) \in \mathcal{L}_1$ is a distinguished formula, the *domain formula* of $\mathcal{I}$,
2. $\cdot_\mathcal{I}$ is a function which maps each unnested atomic formula $\phi(x_1, \dots, x_m) \in \mathcal{L}_2$ to a formula $\phi_\mathcal{I}(\bar{x}_1, \dots, \bar{x}_m) \in \mathcal{L}_1$ where the $\bar{x}_i$ are disjoint n-tuples of distinct variables, and
3. $f_\mathcal{I} : \theta_\mathcal{I}(A^n) \to B$ is a surjective function such that for all unnested atomic formulae $\phi(x_1, \dots, x_m) \in \mathcal{L}_2$ and all $\bar{a}_i \in \theta_\mathcal{I}(A^n)$

$$\mathfrak{A} \models \phi_\mathcal{I}(\bar{a}_1, \dots, \bar{a}_m) \text{ iff } \mathfrak{B} \models \phi(f_\mathcal{I}(\bar{a}_1), \dots, f_\mathcal{I}(\bar{a}_m)) .$$

Theorem 6.1.2 (Hodges, 1993, Theorem 5.5.7). *Let T_1 be a finite and essentially undecidable theory in a first-order language $\mathcal{L}_1$ of finite signature. Let $\mathcal{L}_2$ be a recursive first-order language and T_2 a theory in $\mathcal{L}_2$. If $\mathcal{I}$ is an interpretation which interprets some model of T_1 in some model of T_2 then T_2 is hereditarily undecidable.*

A well-known finite and essentially undecidable theory is the theory T_A of finite arithmetic (Tarski et al., 1953). The theory T_A is a theory in a first-order language with equality and finite signature $\{0, S, +, \cdot\}$ and consists of the following sentences:

$$\begin{array}{ll} \forall x \forall y (S(x) = S(y) \to x = y) & \forall x (x \neq 0 \to \exists y (x = S(y))) \\ \forall x (x + 0 = x) & \forall x \forall y (x + S(y) = S(x + y)) \\ \forall x (x \cdot 0 = 0) & \forall x \forall y (x \cdot S(y) = (x \cdot y) + x) \end{array}$$

One model of T_A is the standard model of number theory $\mathfrak{N} = \langle \mathbb{N}, +, \cdot, S, 0 \rangle$ where S is the successor function and $+$, $\cdot$ and 0 have their usual interpretation. I will give a 1-dimensional interpretation of $\mathfrak{N}$ in the mereotopology $\mathfrak{M}$. Every

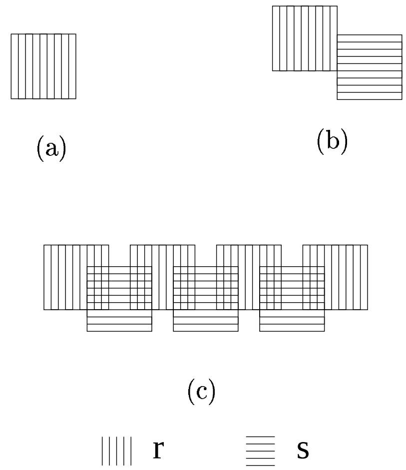


Figure 6.1: Examples of chains

natural number n will be associated by the interpretation with the set of regions with n components. Therefore, the domain formula $\theta_{\mathcal{I}}(x)$ of the interpretation $\mathcal{I}$ must be satisfied only by regions with finitely many components, and furthermore for each $n \in \mathbb{N}$ by at least one region with n components.
The next definition introduces the condition for a pair (r, s) of regions to be a *chain*. Examples of chains are given in figure 6.1. It will be shown below that the region r of a chain (r, s) has only finitely many components. The definition of chain, therefore, gives us a first idea of what the domain formula will look like.

Definition 6.1.3. Let $r, s \in M$. If $r \cdot s \neq \emptyset$ then r is said to *overlap* s. The region r is said to be a *component respecting part* of s if every component of r is also a component of s. A component s' of s is said to be a *neighbour of* r *in* s if r and s' overlap.

The pair (r, s) is called *chain* if

1. $r + s$ is connected and
2. either r is connected or the following conditions hold:
   (a) every component of s overlaps two components of r,
   (b) every component of r overlaps two components of s except for two components r_H and r_T of r, called *head* and *tail* of the chain, each of which overlaps exactly one component of s,

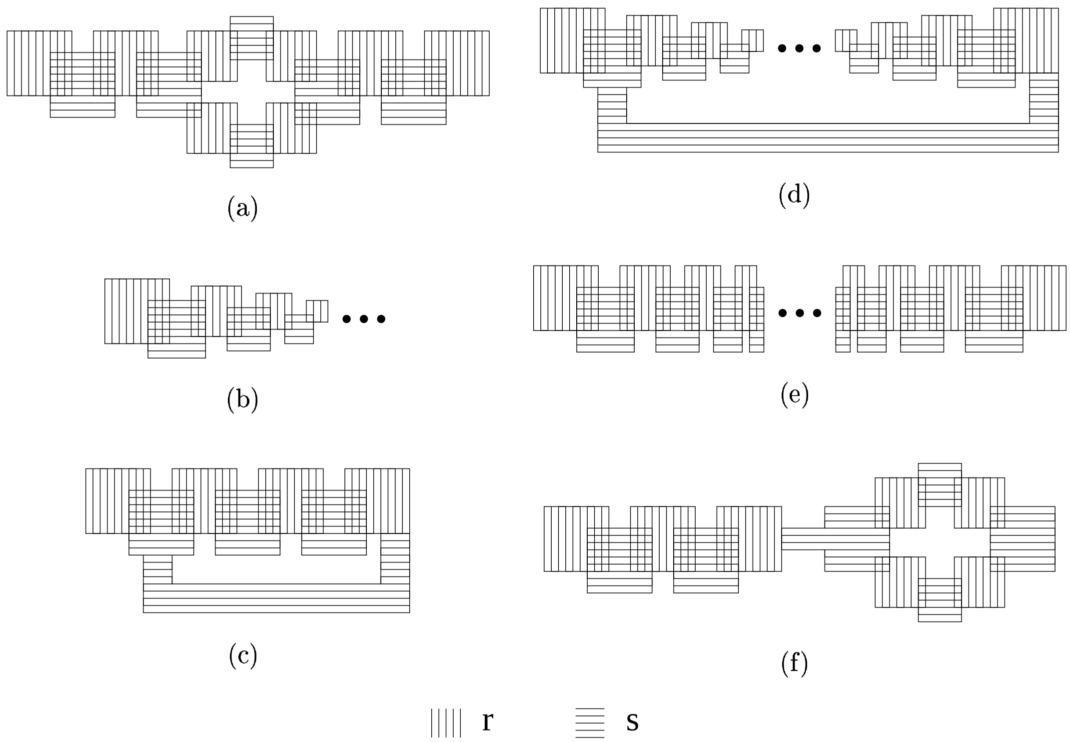


Figure 6.2: Counterexamples of chains

(c) for every component r' of r distinct from head or tail, $(r \cdot -r') + s$ has two components, one of which contains the head while the other contains the tail,

(d) for every component respecting proper part r' of r there exists a component s' of s such that s' overlaps r' and $r \cdot -r'$.

Certainly, the pair $(\emptyset, s)$ is a chain for any connected region s; so is $(r, \emptyset)$ for any connected region r. However, if r is disconnected, a chain (r, s) has to obey conditions 2a-2d in the above definition. Figure 6.2 depicts pairs (r, s) of regions that violate one or more of these conditions. The pair (r, s) depicted in subfigure (a) violates condition 2a and the pair depicted in subfigure (b) violates condition 2b, but both pairs obey all other conditions. Subfigure (c) satisfies all conditions except for condition 2c since there is a component r' of r such that $(r \cdot -r') + s$ is connected. Subfigure (d) violates conditions 2c and 2d while subfigures (e) and (f) only violate condition 2d. I will show that all chains are as simple as those depicted in figure 6.1.

Lemma 6.1.4. *Let $(r, s) \in M^2$ be a chain such that r is disconnected. Let R be the set of components of r, and let r_H and r_T denote head and tail of the chain respectively. Then*

(i) $(r \cdot -r_T) + s$ *is connected,*

(ii) the binary relation $\prec$ defined on R by

$$r_1 \prec r_2 \text{ iff } r_1 \text{ and } r_H \text{ lie in the same component of } (r \cdot -r_2) + s$$

is a strict linear ordering,

(iii) if $r_1, r_2 \in R$ and s' is a component of s such that $r_1 + s' + r_2$ is connected and $r_1 \prec r_2$ then for all $r_3 \in R \setminus \{r_1, r_2\}$ either $r_3 \prec r_1$ or $r_2 \prec r_3$.

Proof. (i) By condition 2d of the definition of chain, there exists a component s_T of s that overlaps r_T and $r \cdot -r_T$. Hence, for some component t of $(r \cdot -r_T) + s$, $r_T + t$ is connected. Assume $t \neq (r \cdot -r_T) + s$. By condition 2d, there exists a component s' of s such that s' overlaps $r_T + (t \cdot r)$ and $r \cdot -(r_T + (t \cdot r))$. It follows from condition 2a that $s' \neq s_T$. By condition 2b, s' overlaps $t \cdot r$. Thus, t is not a maximally connected proper subset of $(r \cdot -r_T) + s$ contradicting the assumption. Hence, $r_T + (t \cdot r)$ is connected.

(ii) Let $r' \in R \setminus \{r_H, r_T\}$. By condition 2c, $(r \cdot -r') + s$ has two components t_H and t_T containing head and tail respectively. By condition 2d, some component of s overlaps r' and $(t_H + t_T) \cdot r$. Thus, $r' + t_H$ or $r' + t_T$ is connected. Assume, WLOG, that $r' + t_H$ is connected. Again by condition 2d, some component s' of s overlaps $(r' + t_H) \cdot r$ and $r \cdot -((r' + t_H) \cdot r) = t_T \cdot r$. Since $(r \cdot -r') + s = t_H + t_T$ is disconnected, s' overlaps r'. Hence, $r' + t_T$ is connected as well. Note that by the definition of $\prec$, $r_H \prec r'$ and $r' \not\prec r_H$ and by condition 2c, $r' \prec r_T$ and $r_T \not\prec r'$.

For the following let $r_1, r_2, r_3 \in R$ be distinct and t_{iH} and t_{iT} be the components of $(r \cdot -r_i) + s$ $(i = 1, 2, 3)$ that contain the head and the tail respectively.

It follows directly from the definition of the relation $\prec$ that $r_1 \not\prec r_1$. Thus, $\prec$ is strict.

Assume $r_1 \not\prec r_2$. If $r_1 = r_T$ or $r_2 = r_H$ then $r_2 \prec r_1$. If $r_1, r_2 \in R \setminus \{r_H, r_T\}$ then $r_1 \subseteq t_{2T}$. Since $r_2 + t_{2H}$ is connected and $r_1 \cdot (r_2 + t_{2H}) = \emptyset$, r_2 lies in r_{1H}. Hence, $r_2 \prec r_1$. Thus, $\prec$ is total.

Assume $r_1 \prec r_2$. If $r_1 = r_H$ or $r_2 = r_T$ then $r_2 \not\prec r_1$. If $r_1, r_2 \in R \setminus \{r_H, r_T\}$ then $r_1 \not\subseteq t_{2T}$. Since $r_2 + t_{2T}$ is connected and $(r_2 + t_{2T}) \cdot r_1 = \emptyset$, $r_2 + t_{2T} \subseteq t_{1T}$ whence $r_2 \not\prec r_1$. Thus, $\prec$ is asymmetric.

If $r_1 \prec r_2$ and $r_2 \prec r_3$ then $r_3 \not\prec r_2$ and, thus, $t_{2H} \cdot r_3 = \emptyset$. Then $t_{2H} \subseteq t_{3H}$. Since $r_1 \subseteq t_{2H}$, $r_2 \prec r_3$. Thus, $\prec$ is transitive.

Hence, $\prec$ is a strict linear ordering on R.

(iii) Assume, there exists a component s' of s such that $r_1+s'+r_2$ is connected and $r_1 \prec r_2$. Suppose there exists $r_3 \in R$ such that $r_1 \prec r_3 \prec r_2$. Then by condition 2c, $(r \cdot -r_3)+s$ is disconnected. Let t_H be the component of $(r \cdot -r_3)+s$ containing the head and let t_T the other component. Since $r_1 \subseteq t_H$ and $r_2 \subseteq t_T$, $s' \subseteq t_H$ and $s' \subseteq t_T$. Hence, $t_H + t_T$ is connected contradicting the assumption that (r, s) is a chain. □

Proposition 6.1.5. *Let $(r, s) \in M^2$ be a chain. Then r has finitely many components.*

Proof. Suppose for a proof by contradiction that the set R of components of r is infinite. Let r_H and r_T denote the two components of r which overlap only one component of s. Let the ordering $\prec$ on R be defined as in lemma 6.1.4(ii). For every component $r_i \in R$, choose a point p_i in r_i and let P be the set of these points. Then, P is infinite. Since $(\mathbb{R}^k)^*$ is compact, $[r]$ is compact and P has an accumulation point $q \notin P$ in $[r]$. Let an ordering $<$ on $P \cup \{q\}$ be defined by

$$\begin{array}{lll} p_i < p_j & \text{if} & p_i, p_j \in P \text{ and } r_i \prec r_j, \\ p_i < q & \text{if} & p_i \in P \text{ and the set } \{p \in P | p_i < p\} \text{ has accumulation point } q, \\ q < p_i & \text{if} & p_i \in P \text{ and the set } \{p \in P | p < p_i\} \text{ has accumulation point } q. \end{array}$$

Note that the ordering $(P \cup \{q\}, <)$ may be neither asymmetric nor anti-symmetric for q. However, for all $p_i \in P$, $p_i < q$ or $q < p_i$. In particular, $p_H < q < p_T$ and $p_T \not< q \not< p_H$. Let $R_{<q} = \{r_i \in R | p_i < q\}$ and $t_{<q} = \sum R_{<q} + \sum\{s' | s'$ is a component of s which overlaps $\sum R_{<q}\}$. Since $p_H < q$, $r_H \subseteq t_{<q}$. Let t_H be the component of $t_{<q}$ which contains r_H and let R_H the set of components of $r \cdot t_H$. By condition 2d, for some component s' of s and some $r_j, r_k \in R$, $r_j \subseteq t_H$, $r_k \subseteq r \cdot -\sum R_{<q}$ and s' overlaps r_j and r_k. By lemma 6.1.4(iii), there is no $r_l \in R$ with $r_j \prec r_l \prec r_k$ or $r_k \prec r_l \prec r_j$. Since $r_j \subseteq t_H \subseteq t_{<q}$, $p_j < q$, i.e. the set $\{p \in P | p_j < p\}$ has accumulation point q. If $p_j < p_k$, then $\{p \in P | p_k < p\} = \{p \in P | p_j < p\} \setminus \{p_j\}$. If $p_k < p_j$, then $\{p \in P | p_k < p\} = \{p \in P | p_j < p\} \cup \{p_j\}$. Either way, $p_k < q$. Hence, $r_k \in R_{<q}$. Since $r_j \subseteq t_H$ and $r_j + s' + r_k$ is connected, $r_k \subseteq t_H$ contradicting the maximality of t_H. Therefore, P cannot have an accumulation point q in $[r]$. Since $[r]$ is compact, P and therefore also R must be finite. Thus, r has finitely many components. □

The following two lemmas, whose straightforward proofs are omitted, introduce formulae that are used below to introduce a formula defining chains.

Lemma 6.1.6. *Let* comp(x, y) *stand for the formula*

$$x \leq y \wedge \mathrm{c}(x) \wedge \forall z(x \leq z \wedge z \leq y \wedge \mathrm{c}(z) \rightarrow x = z))$$

and $x \trianglelefteq y$ *stand for the formula*

$$\forall x'(\mathrm{comp}(x', x) \rightarrow \mathrm{comp}(x', y)) .$$

Then for $r, s \in M$*: (i)* $\mathfrak{M} \models \mathrm{comp}[r, s]$ *if and only if* r *is a component of* s *and (ii)* $\mathfrak{M} \models r \trianglelefteq s$ *if and only if* r *is a component respecting part of* s*.*

Lemma 6.1.7. *Let* $N(x, y, z)$ *stand for the formula*

$$(x \cdot y = 0 \wedge z = 0) \vee \forall y'(\mathrm{comp}(y', y) \wedge x \cdot y' \neq 0 \leftrightarrow \mathrm{comp}(y', z))$$

and $N_n(x, y)$ *for the formula*

$$x \cdot y \neq 0 \wedge \exists z \exists z_1 \ldots \exists z_n \Big(z = \sum_{i=1}^{n} z_i \wedge \bigwedge_{1 \leq i < j \leq n} z_i \neq z_j \wedge \bigwedge_{i=1}^{n} \mathrm{comp}(z_i, z) \wedge N(x, y, z) \Big)$$

where n *is some positive integer. Then for any* $r, s, t \in M$*:*
(i) $\mathfrak{M} \models N[r, s, t]$ *if and only if* $t = \sum \mathcal{N}_s(r)$*, where* $\mathcal{N}_s(r)$ *stands for the set of all neighbours of* r *in* s*,*
(ii) $\mathfrak{M} \models N_n[r, s]$ *if and only if* r *and* s *are not disjoint and* $||\mathcal{N}_s(r)|| = n$*.*

Lemma 6.1.8. *Let each expression on the left hand side of the following list stand as abbreviation for the indented formula directly below the respective expression.*

chain2a(x, y)

$$\forall y'(\mathrm{comp}(y', y) \rightarrow N_2(y', x))$$

chain2b(x, y)

$$\exists x_H \exists x_T \Big(\mathrm{comp}(x_H, x) \wedge \mathrm{comp}(x_T, x) \wedge x_H \neq x_T \wedge N_1(x_H, y) \wedge N_1(x_T, y) \\ \wedge \forall x'(\mathrm{comp}(x', x) \wedge x' \neq x_H \wedge x' \neq x_T \rightarrow N_2(x', y)) \Big)$$

chain2c(x, y)

$$\forall x' \Big(\mathrm{comp}(x', x) \rightarrow \\ \exists z_H \exists z_T \exists x_H \exists x_T (\mathrm{c}(z_H) \wedge \mathrm{c}(z_T) \wedge z_H \cdot z_T = 0 \wedge z_H + z_T = (x \cdot -x') + y \\ \wedge N_1(x_H, y) \wedge N_1(x_T, y) \wedge x_H \leq z_H \wedge x_T \leq z_T) \Big)$$

chain2d(x, y)

$$\forall x'\Big(x' \trianglelefteq x \wedge x' \neq x \rightarrow \exists y'(\text{comp}(y', y) \wedge y' \cdot x' \neq 0 \wedge y' \cdot (x \cdot -x') \neq 0)\Big)$$

chain(x, y)

$$\text{c}(x+y) \wedge (\text{c}(x) \vee (\text{chain2a}(x, y) \wedge \text{chain2b}(x, y) \wedge \text{chain2c}(x, y) \wedge \text{chain2d}(x, y)))$$

Then for $r, s \in M$, (r, s) is a chain if and only if $\mathfrak{M} \models \text{chain}[r, s]$.

Proof. Given lemmas 6.1.6 and 6.1.7, a comparison of the formulae chain2a(x, y), ..., chain2d(x, y) with the conditions 2a-2d in definition 6.1.3 shows the formula chain(x, y) to define the set of chains in $\mathfrak{M}$. □

The formula chain(x, y) will be used in lemma 6.1.14 to define the domain formula for the interpretation of $\mathfrak{N}$ in $\mathfrak{M}$. The next lemmas show that the domain M contains sufficiently many well-behaved regions to construct chains.

Lemma 6.1.9. *Let A be a closed set in $(\mathbb{R}^k)^*$ and γ be an arc in $(\mathbb{R}^k)^* \setminus A$. Then there exists a connected element $r \in M$ with connected complement such that $|\gamma| \subseteq r$, $A \cap [r] = \emptyset$ and the boundary of r is accessible from r and $-r$.*

Proof. Since γ is an arc, there exists a homeomorphism $h : (\mathbb{R}^k)^* \rightarrow (\mathbb{R}^k)^*$ such that $|\gamma|$ is mapped to a line segment. Let $\epsilon > 0$ be the minimal distance between $h(|\gamma|)$ and $h(A)$. Let $D = \bigcup\{B_{\frac{\epsilon}{2}}(p) | p \in h(|\gamma|)\}$. Then $[D]$ is connected and does not intersect $h(A)$. Moreover, $h(A)$ lies in one component of $(\mathbb{R}^k)^* \setminus [D]$. By condition (iii) of M, for every point $p \in |\gamma|$ there exists an open "cube" $r_p \subset h^{-1}([D])$. Thus, $V = \{r_p | p \in |\gamma|\}$ is an open cover of $|\gamma|$. Since $|\gamma|$ is closed and hence compact, there is a finite subcover V' of V. Then A lies in one component $-r$ of $-\sum V'$. Then r is the required region. □

Lemma 6.1.10. *Let $r_0, \ldots, r_n \in M$ be the components of the non-empty region $r \in M$ such that for $0 \leq i < j \leq n$, $[r_i] \cap [r_j] = \emptyset$ and the boundary of r_i is accessible from r_i and $-r_i$. Then there exist disjoint arcs $\gamma_1, \ldots, \gamma_n$ such that for $1 \leq i \leq n$, $\gamma_i(0) \in r_{i-1}$, $\gamma_i(1) \in r_i$ and $|\gamma_i| \cap [r_j] = \emptyset$ if $j \in \{0, \ldots, n\} \setminus \{i-1, i\}$.*

Proof. Since the boundary of $r_0 + r_1$ is accessible from $r_0 + r_1$ and $-(r_0 + r_1)$ and $(-r) + r_0 + r_1$ is arc-connected there exists an arc γ_1 in $(-r) + r_0 + r_1$ from some point in r_0 to some point in r_1 such that $(-r) \setminus |\gamma_1|$ is connected. Since the boundary of $r_1 + r_2$ is accessible from $r_1 + r_2$ and $-(r_1 + r_2)$ and $((-r) + r_1 + r_2) \setminus |\gamma_1|$ is arc-connected there exists an arc γ_2 in $((-r) + r_1 + r_2) \setminus |\gamma_1|$ from some point in r_1 to some point in r_2 such that $(-r) \setminus (|\gamma_1| \cup |\gamma_2|)$ is connected. A repeated application of the above argument guarantees the existence of all the required arcs. □

Lemma 6.1.11. *Let the non-empty region $r \in M$ have a connected complement, a boundary accessible from r and $-r$ and finitely many components $r_0, r_1, \ldots, r_n$ such that $[r_i] \cap [r_j] = \emptyset$ for $0 \leq i < j \leq n$. Then there exists $s \in M$ such that (r, s) is a chain.*

Proof. By lemma 6.1.10, there exist disjoint arcs $\gamma_1, \ldots, \gamma_n$ such that for $1 \leq i \leq n$, $\gamma_i(0) \in r_{i-1}$, $\gamma_i(1) \in r_i$ and $|\gamma_i| \cap [r_j] = \emptyset$ if $j \in \{0, \ldots, n\} \setminus \{i-1, i\}$. By lemma 6.1.9, there exist regions $s_1, \ldots, s_n \in M$ such that for $1 \leq i \leq n$, $|\gamma_i| \subseteq s_i$ and $[s_i] \cap [r_j] = \emptyset$ if $j \in \{0, \ldots, n\} \setminus \{i-1, i\}$. Then $(r, \sum_{i=1}^{n} s_i)$ is a chain. □

In the following, I write $|r|$ to denote the number of non-empty components of a region $r \in M$, i.e. $|\emptyset| = 0$ and $|r| \geq 1$ for $r \neq \emptyset$. Given $r, s \in M$, I write $s \ll r$ if $s \subseteq r$, $-s$ is connected, every component of r contains one component of s, and r is empty if s is empty. Obviously, $s \ll r$ implies $|s| = |r|$.

Lemma 6.1.12. *Let $r, s \in M$ be non-empty, have connected complements and the same finite number of components such that for any two distinct components r_1 and r_2 of r or s, $[r_1] \cap [r_2] = \emptyset$. Furthermore, let r and s such that the boundary of r is accessible from r and $-r$ and the boundary of s is accessible from s and $-s$. Then there exists $t \in M$ such that $r + s \subseteq t$, $r \ll t$ and $s \ll t$.*

Proof. Assume $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are the components of r and s respectively. By lemma 6.1.10, there are disjoint arcs $\gamma_1, \ldots, \gamma_n$ such that for $1 \leq i \leq n$, $\gamma_i(0) \in r_i$, $\gamma_i(1) \in s_i$ and for $1 \leq i < j \leq n$, $|\gamma_i| \cap [r_j + s_j] = \emptyset$. By lemma 6.1.9, there exist mutually disjoint connected regions $t_1, \ldots, t_n \in M$ such that $r_i + s_i \subseteq t_i$ for $1 \leq i \leq n$. Then $t = \sum_{i=1}^{n} t_i$ is the required region. □

Lemma 6.1.13. *Let $x \ll y$ stand for the formula*

$$(x = 0 \to y = 0) \wedge \mathrm{c}(-x) \wedge x \leq y \wedge \forall y'(\mathrm{comp}(y', y) \to \mathrm{comp}(x \cdot y', x))\,.$$

Then for $r, s \in M$, $\mathfrak{M} \models r \ll s$ if and only if $r \ll s$.

Proof. Straightforward. □

Lemma 6.1.14. *Let $\mathcal{F}(x)$ stand for the formula*

$$\exists y \exists z (y \ll x \wedge \mathrm{chain}(y, z))\,.$$

Then for $r \in M$, $\mathfrak{M} \models \mathcal{F}[r]$ if and only if r has finitely many components.

Proof. If-direction: If r is the empty set then with $\emptyset$ as witness for y and z, $\mathcal{F}(x)$ is satisfied by r in $\mathfrak{M}$. If r is non-empty, it is easy to see that by condition (iii) of M there exists $r' \in M$ whose boundary is accessible from r' and $-r'$ such that $r' \ll r$ and for every distinct pair r_1, r_2 of components of r', $[r_1] \cap [r_2] = \emptyset$. By lemma 6.1.13, $\mathfrak{M} \models r' \ll r$. By lemma 6.1.11, there exists $s \in M$ such that (r', s) is a chain. Then by lemma 6.1.8, $\mathfrak{M} \models \mathcal{F}[r]$.

Only-if-direction: Let r' and s be witnesses for y and z in $\mathcal{F}[r]$ respectively. By lemma 6.1.8, (r', s) is a chain. Therefore, by proposition 6.1.5, r' has finitely many components. Since $\mathfrak{M} \models r' \ll r$, $|r| = |r'|$. Hence, r has finitely many components. □

The formula $\mathcal{F}(x)$ will be the domain formula of the interpretation of $\mathfrak{N}$ in $\mathfrak{M}$. It remains to find formulae to mimic equality and the operators sum $+$, product $\cdot$ and successor S.

Lemma 6.1.15. *Let* $\dot{\approx}(x, y)$ *stand for the formula*

$$\begin{aligned}&(x = 0 \land y = 0)\\ &\lor\Big(x \neq 0 \land y \neq 0 \land \mathcal{F}(x) \land \mathcal{F}(y) \land x \cdot y = 0\\ &\quad \land \exists x' \exists y' \exists z (x' \ll x \land y' \ll y \land x' \ll z \land y' \ll z)\Big).\end{aligned}$$

Then for $r, s \in M$, $\mathfrak{M} \models \dot{\approx}[r, s]$ *if and only if* r *and* s *are disjoint, have finitely many components and* $|r| = |s|$.

Proof. If-direction: If $|r| = |s| = 0$ then $r = \emptyset = s$ and $\mathfrak{M} \models \dot{\approx}[r, s]$. Assume, therefore, $|r| = |s| \geq 1$. By lemma 6.1.14 and the fact that r and s are disjoint, $\mathfrak{M} \models \mathcal{F}[r] \land \mathcal{F}[s] \land r \cdot s = 0$. It follows from condition (iii) of M that there exist $r', s' \in M$ such that $\mathfrak{M} \models r' \ll r \land s' \ll s$, the boundary of r' is accessible from r' and $-r'$, the boundary of s' is accessible from s' and $-s'$ and for all distinct components r_1, r_2 of r or s, $[r_1] \cap [r_2] = \emptyset$. By lemma 6.1.12, there exists $t \in M$ such that $r \ll t$ and $s \ll t$. By lemma 6.1.13, $\mathfrak{M} \models \dot{\approx}[r, s]$.

Only-if-direction: If $(r, s) \in M^2$ satisfies the first conjunct of $\dot{\approx}(x, y)$ then $r = \emptyset = s$. Hence r and s are disjoint and $|r| = |s| = 0$. If (r, s) satisfies the second conjunct of $\dot{\approx}(x, y)$ then r and s are non-empty and by lemma 6.1.14 have finitely many components. Let r', s' and t be witnesses for x', y' and z respectively. By lemma 6.1.13, $|r'| = |r|$ and $|s'| = |s|$. Since $r' \ll t$ and $s' \ll t$, $|r'| = |t| = |s'|$. Thus, $|r| = |s|$. □

Lemma 6.1.16. *Let* $\approx(x,y)$ *stand for the formula*

$$\mathcal{F}(x)\wedge\mathcal{F}(y)\wedge\exists x_1\exists x_2\exists y_1\exists y_2(x_1\ll x\wedge y_1\ll y\wedge\dot{\approx}(x_1,x_2)\wedge\dot{\approx}(y_1,y_2)\wedge\dot{\approx}(x_2,y_2))\,.$$

Then for $r,s\in M$, $\mathfrak{M}\models\approx[r,s]$ *if and only if* r *and* s *have finitely many components and* $|r|=|s|$.

Proof. If-direction: By lemma 6.1.14, $\mathfrak{M}\models\mathcal{F}[r]\wedge\mathcal{F}[s]$. If $|r|=|s|=0$ then r is a witness for x_1 and x_2 and s is a witness for y_1 and y_2 in $\approx(x,y)$ satisfied by (r,s) in $\mathfrak{M}$. Assume, therefore, $|r|=|s|=n\geq 1$. By condition (iii) of M, there exist $r_1,s_1\in M$ such that $r_1\ll r$ and $s_1\ll s$. Certainly, r_1 and s_1 can be chosen to be different from $(\mathbb{R}^k)^*$. If $r_1+s_1\neq(\mathbb{R}^k)^*$ then, by condition (iii) of M, there exist two disjoint regions $r_2,s_2\in M$ lying in $-(r+s)$ such that $|r'|=|s'|=n$. Then, by lemma 6.1.15, $\mathfrak{M}\models\approx[r,s]$. Otherwise $r_1\cdot -s_1$ and $s_1\cdot -r_1$ are non-empty and disjoint. By condition (iii) of M, there exist regions $r_2,s_2\in M$ such that $|r_2|=|s_2|=n$, $r_2\subseteq s_1\cdot -r_1$ and $s_2\subseteq r_1\cdot -s_1$. Then by lemma 6.1.15, $\mathfrak{M}\models\approx[r,s]$.

Only-if-direction: By lemma 6.1.14, r and s have finitely many components. Let r_1,r_2,s_1 and s_2 be witnesses for x_1,x_2,y_1 and y_2 respectively. Then by lemmas 6.1.13 and 6.1.15, $|r|=|r_1|=|r_2|=|s_2|=|s_1|=|s|$. □

At last I am in the position to define formulae defining successor, addition and multiplication.

Lemma 6.1.17. *Let* $S(x,y)$ *stand for the formula*

$$\mathcal{F}(x)\wedge\mathcal{F}(y)\wedge y\neq 0\wedge\exists y'(\mathrm{comp}(y',y)\wedge\approx(x,y\cdot -y'))\,,$$

$+(x,y,z)$ *for the formula*

$$\mathcal{F}(x)\wedge\mathcal{F}(y)\wedge\mathcal{F}(z)\wedge\exists x'\exists y'(x'\cdot y'=0\wedge\approx(x,x')\wedge\approx(y,y')\wedge\approx(x'+y',z))$$

and $\bullet(x,y,z)$ *for the formula*

$$\begin{aligned}&\mathcal{F}(x)\wedge\mathcal{F}(y)\wedge\mathcal{F}(z)\wedge\exists z'(z'\leq x\wedge\approx(z',z)\\&\wedge\forall x'(\mathrm{comp}(x',x)\wedge x'\neq 0\rightarrow\approx(x'\cdot z',y)))\,.\end{aligned}$$

Then for $r,s\in M$,

1. $\mathfrak{M}\models S[r,s]$ *iff* r *and* s *have finitely many components and* $|r|+1=|s|$,
2. $\mathfrak{M}\models +[r,s,t]$ *iff* r,s *and* t *have finitely many components and* $|r|+|s|=|t|$ *and*
3. $\mathfrak{M}\models\bullet[r,s,t]$ *iff* r,s *and* t *have finitely many components and* $|r|\cdot|s|=|t|$.

Proof. Since it is straightforward to show 1 and 2, I concentrate on 3.

For the if-direction assume first $|r| = |t| = 0$. Then t is a witness for z' in $\bullet(x, y, z)$ satisfied by (r, s, t) in $\mathfrak{M}$. Now assume $r_1, \ldots, r_n$ are the (non-empty) components of r. By condition (iii) of M, for each r_i $(1 \leq i \leq n)$ there exists $t_i \subset r_i$ with connected complement and $|t_i| = |s|$. Let $t' = \sum_{i=1}^{n} t_i$. Then $t' \subseteq r$ and every component of r contains $|s|$ components of t'. Hence, $|t'| = |r| \cdot |s| = |t|$ and $\mathfrak{M} \models \bullet[r, s, t]$ with t' as witness for z'. The only-if-direction is straightforward. □

Now an interpretation $\mathcal{I}$ of $\mathfrak{N}$ in $\mathfrak{M}$ can be given:

Proof of theorem 6.0.6. By lemmas 6.1.14, 6.1.16 and 6.1.17 a one-dimensional interpretation $\mathcal{I}$ of $\mathfrak{N}$ in $\mathfrak{M}$ is given by

1. $\mathcal{F}(x)$ is the domain formula,
2. $\cdot_{\mathcal{I}}$ maps $x = y$ to $\approx(x, y)$, $y = S(x)$ to $S(x, y)$, $x + y = z$ to $+(x, y, z)$, $x \cdot y = z$ to $\bullet(x, y, z)$ and 0 to 0,
3. $f_{\mathcal{I}} : \mathcal{F}(M) \to \mathbb{N}$ maps 0 to 0 and every non-empty region of $\mathcal{F}(M)$ to the number of its components.

Then by theorem 6.1.2, $Th(\mathfrak{M})$ is hereditarily undecidable. □

It is worth pointing out the difference between the above undecidability proof and Grzegorczyk's proof. To show the undecidability of the "theory of bodies", Grzegorczyk made use of two properties of the topological space $\mathbb{R}^n$: (i) the (regular) open sets are closed under infinite union (sum) and (ii) $\mathbb{R}^n$ is second countable, that is, $\mathbb{R}^n$ has a countable basis. Consequently, any open set in $\mathbb{R}^n$ has at most countably many components. Property (i) enables Grzegorczyk to define a set of structures which have finitely or infinitely many components. To be more precise, using the language $\mathcal{L}(\mathrm{C})$, Grzegorczyk is able to define the set of isolated sets, i.e. sets that do not contain their limit points. Moreover, property (i) allows him to introduce the operations for sum, product and successor for finite *and* infinite isolated sets. These operators together with property (ii) enable him to define the *finite* isolated sets: an isolated set S is finite if the cardinality of S is the same as the cardinality of the set S with one point removed. Then, given that the operators for sum, product and successor are defined for all isolated sets, there is an immediate interpretation of number theory in the theory of bodies. Thus, Grzegorczyk uses the properties of the spatial domain to define the operators sum, product and successor on representations for numbers, and then, using the defined operators, Grzegorczyk defines “finiteness”.

Since some of the spatial domains in this thesis are not closed under infinite sums, the undecidability proof that was given in this section could not follow Grzegorczyk’s argument. I could only define the operators for sum, product and successor *after* “finiteness” was defined. The existence of an $\mathcal{L}(\leq, \mathrm{c})$-formula $\mathcal{F}(x)$ that defines the set of regions which have only finitely many components is a fact which is interesting in its own right. The following section uses the formula $\mathcal{F}(x)$ to acquire further knowledge about mereotopologies over $\mathbb{R}^2$.

6.2 A characterisation of topological Δ-models

The spatial domain **S** is well-behaved in the sense that every region in **S** has finitely many components. This property is reflected in the fact that the mereotopology $\mathfrak{S}(\mathrm{C})$ omits $\Delta(x)$, i.e. the set of formulae

$$\left\{ \neg\exists x_1 \ldots \exists x_n (x = x_1 + \ldots + x_n \wedge \bigwedge_{i=1}^{n} \mathrm{c}(x_i)) \middle| n \geq 1 \right\}$$

where $\mathrm{c}(x)$ is an abbreviation for the formula defining connectedness as given in lemma 4.1.3. In section 5.2, a complete axiomatisation $\mathcal{P}$ of $Th(\mathfrak{S}(\mathrm{C}))$ in the Δ-calculus was presented. Proposition 4.3.6 on page 97 showed that there exist models of $Th(\mathfrak{S}(\mathrm{C}))$ that do not omit $\Delta(x)$. However, can such model be a mereotopology, i.e. a model defined over a topological space? In this section, I give a partial answer to this question.

Let any mereotopology $\mathfrak{M}(\mathrm{C})$ over $\mathbb{R}^k$ or $(\mathbb{R}^k)^*$ whose domain M satisfies the conditions of theorem 6.0.6 be called a *standard mereotopology*. It can be shown, following lemmas 4.1.2 and 4.1.3, that the subset-relation and connection are $\mathcal{L}(\mathrm{C})$-definable in $\mathfrak{M}$. Therefore, the $\mathcal{L}(\leq, \mathrm{c})$-formula $\mathcal{F}(x)$, as given in lemma 6.1.14, can be considered as an $\mathcal{L}(\mathrm{C})$-formula. It was shown that in every standard mereotopology the formula $\mathcal{F}(x)$ defines the set of regions with finitely many components. Therefore, for any standard mereotopology $\mathfrak{M}$, $\mathfrak{M} \models \forall x(\mathcal{F}(x))$ if and only if $\mathfrak{M}$ omits $\Delta(x)$. Thus, any mereotopology over $\mathbb{R}^k$ or $(\mathbb{R}^k)^*$ that is a model of $Th(\mathfrak{S}(\mathrm{C}))$ and that realizes Δ is an exotic mereotopology indeed. However the question for the existence of such exotic mereotopology is answered, the following theorem shows that elementary equivalence of standard mereotopologies that are models of $Th(\mathfrak{S}(\mathrm{C}))$ can be characterised by two simple axiom systems.

Theorem 6.2.1. *Let $\mathcal{P}$ be the axiom system as given in section 5.2. Consider $\mathcal{F}(x)$ as an $\mathcal{L}(\mathrm{C})$-formula where* c *and* $\leq$ *are abbreviations for the formulas given in lemmas 4.1.2 and 4.1.3. Let $\mathcal{P}_{\mathcal{F}} = \mathcal{P} \cup \{\forall x \mathcal{F}(x)\}$ and let $\mathcal{P}_s$ be the axiom system $\mathcal{P}$ together with the infinitary axiom schema*[1]

$$(\phi(0) \wedge \forall x \forall y (\phi(x) \wedge \mathrm{c}(y) \rightarrow \phi(x+y)) \rightarrow \forall x \phi(x) .$$

Then the theories $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{P}_s$ are ω-categorical with respect to the class of standard mereotopologies.

Proof. By lemma 6.1.14, a standard mereotopology which is a model of $\mathcal{P}_{\mathcal{F}}$ is Δ-model of $\mathcal{P}$. Let $\mathfrak{M}$ be a standard mereotopology that is a model of $\mathcal{P}_s$. It is easy to see that for $\phi(x)$ being the formula $\mathcal{F}(x)$ the antecedent of the implication in the axiom schema holds in $\mathfrak{M}$. Hence, $\forall x(\mathcal{F}(x))$ holds in $\mathfrak{M}$. Therefore, $\mathfrak{M}$ is a Δ-model of $\mathcal{P}$.

Since by proposition 4.3.9 on page 98, $\mathcal{P}$ is ω-categorical with respect to Δ-models, and all standard mereotopologies of $\mathcal{P}_{\mathcal{F}}$ or $\mathcal{P}_s$ are Δ-models, $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{P}_s$ are ω-categorical with respect to the class of standard mereotopologies. □

6.3 Conclusion

The theories of the mereotopologies $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{S}}$ were shown to be hereditarily undecidable. Thus, an application of the unrestricted mereotopological theories to topological reasoning is out of the question. However, promising complexity results for topological reasoning with restricted mereotopological languages have been presented in the literature (cf. section 2.6).

The undecidability of $Th(\mathfrak{S})$ shows that there exists no axiomatisation of $Th(\mathfrak{S})$ in first-order logic. Thus, the decision in chapter 5 to axiomatise the theory of $\mathfrak{S}$ in some other calculus than the predicate calculus, here the Δ-calculus, was a necessary one.

A formula $\mathcal{F}(x)$ of $\mathcal{L}(\leq, \mathrm{c})$ was presented which defines the set of regions with finitely many components in standard mereotopologies, i.e. well-behaved mereotopologies over $\widetilde{\mathbb{R}^k}$. As a consequence, two extensions of the axiom system

[1] The axiom schema was suggested by Professor J. Paris.

$\mathcal{P}$, the axiom systems $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{P}_s$, could be shown to be ω-categorical with respect to the class of standard mereotopologies.

The undecidability proof of the mereotopological theories relied on the fact that natural numbers could be represented as classes of regions with finitely many components. Thus, on the one hand the undecidability proof relied on the definability of components, and on the other hand on the existing variety of regions in the spatial domains. Therefore, a decidable mereotopological theory either only admits models with regions which have a maximum of n components where n is some fixed natural number, or the mereotopological language is restricted in syntax or semantics. All of these restrictions, however, are severe.

Chapter 7

Points in point-free mereotopologies

It is possible to construct mereotopologies which contain points as well as regions in their spatial domain. I will introduce such mereotopology below. However, one of the main reasons why the AI-community is interested in mereotopology is that mereotopology, being a region-based approach to topology, can avoid the ontological questions regarding points and boundaries (cf. section 2.2). I will show in this chapter that the ontological simplicity of mereotopology is only superficial; even in certain point-free mereotopologies it is possible to refer to points and boundaries, thereby allowing the ontological questions to enter through the back door.

It has been shown that points can be reconstructed from regions by identifying points as sets of (sets of) regions (cf. Whitehead's abstractive elements in section 2.7 or the use of ultrafilters in section 4.2). Thus, points can be represented as higher-order constructs. It appears difficult if not impossible to refer to such "points" in a first-order language. Nevertheless, I will show that there are mereotopological languages which are expressive enough to refer to points represented by definable sets of regions. Moreover, I will show that it is possible to refer not only to points but also to boundaries, open regions and closed regions. To make this idea concrete I introduce an ontologically rich mereotopology as follows.

Let $\mathbf{S}_f$ be the spatial domain of all semi-algebraic subsets of $\mathbb{R}^2$, and let $\mathfrak{S}_f$ be the model-theoretic $\mathcal{L}(\leq, \mathrm{C})$-structure with domain $\mathbf{S}_f$ where the mereotopological symbols $\leq$ and C have their standard interpretation, i.e. $[\leq]^{\mathfrak{S}_f} = \{(r_1, r_2) \in \mathbf{S}_f | r_1 \subseteq r_2\}$ and $[\mathrm{C}]^{\mathfrak{S}_f} = \{(r_1, r_2) \in \mathbf{S}_f | [r_1] \cap [r_2] \neq \emptyset\}$. Hence, $\mathfrak{S}_f$ is a mereotopology containing points, while the mereotopology $\mathfrak{S}$ is point-free. Note that the language $\mathcal{L}(\mathrm{C})$ is not expressive enough to define the subset relation over $\mathbf{S}_f$ as the next lemma shows.

Lemma 7.0.1. *The set* $\{(r,s) \in \mathbf{S}_f^2 | r \subseteq s\}$ *is not definable in* $\mathfrak{S}_f(\mathrm{C})$.

Proof. Let the function $f : \mathbf{S}_f \to \mathbf{S}_f$ be defined by

$$f(x) = \begin{cases} x^\circ \cup (\partial(x) \setminus x) & \text{if } x^\circ = [x]^\circ \text{and } [x] = [x^\circ] \\ x & \text{otherwise} \end{cases} .$$

Thus, f "toggles" the boundaries of semi-regular sets and maps all other sets to themselves. Hence, f is a bijection and $[r] = [f(r)]$ for all $r \in \mathbf{S}_f$. Therefore, f is a model automorphism. However, for a semi-regular set $r \in \mathbf{S}_f$, $r^\circ \subset [r]$ but $f([r]) \subset f(r^\circ)$. □

It is easy to see that the set of singletons, i.e. points, in $\mathbf{S}_f$ is defined in $\mathfrak{S}_f$ by the $\mathcal{L}(\mathrm{C})$-formula

$$\exists y(\mathrm{C}(x,y)) \wedge \forall y \forall z(\mathrm{C}(x,y) \wedge \mathrm{C}(x,z) \to \mathrm{C}(y,z))$$

which will be abbreviated by $\texttt{point}(x)$. Obviously, $\texttt{point}(x)$ is not satisfied in $\mathfrak{S}$, since $\mathbf{S}$ is an atomless Boolean algebra. However, the language $\mathcal{L}(\mathrm{C})$ is expressive enough to refer to "imaginary points" in $\mathfrak{S}$. Moreover, it will be shown that every element in $\mathbf{S}_f$ corresponds to an "imaginary element" in $\mathbf{S}$. Thus, the "imaginary domain" of $\mathfrak{S}$ is as rich as the domain of $\mathfrak{S}_f$, although $\mathbf{S}$ is not as rich as $\mathbf{S}_f$. Furthermore, the correspondence between $\mathbf{S}_f$ and the imaginary domain of $\mathfrak{S}$ will entail that every $\mathcal{L}(\leq, \mathrm{C})$-formula that is satisfied in $\mathfrak{S}_f$ corresponds to an $\mathcal{L}(\mathrm{C})$-formula that is satisfied in $\mathfrak{S}$.

It is an old idea to define a point p either by two lines which intersect in p or as two regions r_1 and r_2 which touch only at the point p (cf. Galton, 1996). However, the pair (r_1, r_2) cannot be taken as the unique representation of p, since there exist many pairs of regions that touch only in p. Thus, there would be too many points. However, the set P of all pairs that touch in p will do nicely as the representation of p. If the set P is definable in $\mathfrak{S}(\mathrm{C})$ with parameters r_1 and r_2, then, in model theory, P is called an "imaginary element" of $\mathfrak{S}(\mathrm{C})$ (see Hodges, 1993). It will be shown below that P is definable in $\mathfrak{S}(\mathrm{C})$ with parameters.

It is important to realise the difference between the representation of a point as a set of (sets of) regions and the representation of a point as an imaginary element. While quantification over general sets in a first-order language is impossible, quantification over tuples of the domain is possible and facilitates quantification over imaginary elements. As a consequence, it will be possible to define not only points but all elements of $\mathbf{S}_f$ as imaginary elements in $\mathfrak{S}(\mathrm{C})$. More precisely, I will show that the mereotopology $\mathfrak{S}_f(\leq, \mathrm{C})$ can be interpreted in the mereotopology $\mathfrak{S}(\mathrm{C})$ and vice versa in the sense of definition 6.1.1 on page 131.

In order to interpret $\mathfrak{S}(\mathrm{C})$ in $\mathfrak{S}_f(\leq, \mathrm{C})$, the set of regular open sets must be definable in $\mathfrak{S}_f$. Since $(\mathbf{S}_f, \subseteq)$ is a Boolean algebra, the Boolean operators $\cap, \cup$ and $^-$ are definable in $\mathfrak{S}_f(\leq, \mathrm{C})$. Since the set of all singletons is defined in $\mathfrak{S}_f(\leq, \mathrm{C})$ by the formula $\mathtt{point}(x)$, it is easy to see that the formula $\forall y(\mathtt{point}(y) \wedge \mathrm{C}(x, y) \rightarrow y \leq x)$ defines the closed sets in $\mathbf{S}_f$. Hence, the closure- and interior-operators, $\mathtt{cl}(x)$ and $\mathtt{int}(x)$, are definable in $\mathfrak{S}_f(\leq, \mathrm{C})$. Then the set of regular open sets is defined by the formula $x = \mathtt{int}(\mathtt{cl}(x))$ which is the domain formula of the trivial interpretation of $\mathfrak{S}(\mathrm{C})$ in $\mathfrak{S}_f(\leq, \mathrm{C})$. The interpretation function $\cdot_{\mathcal{I}}$ maps $\mathrm{C}(x, y)$ to $\mathrm{C}(x, y)$ and $x \leq y$ to $\forall z(\mathrm{C}(z, x) \rightarrow \mathrm{C}(z, y))$ (cf. lemma 4.1.2 on page 89).

The interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ requires more effort, as shown in the following section.

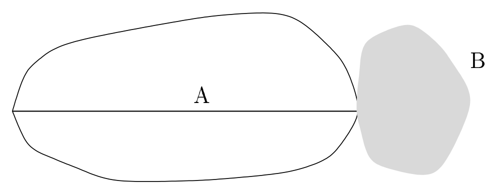
7.1 An interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$

The basis for the interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ is that every point in $\mathbb{R}^2$ is representable as a definable set of pairs of regions of $\mathbf{S}$. The next two lemmas ensure that, for every 0-cell and every 1-cell in $\mathbb{R}^2$, a representation by pairs of regular open semi-algebraic sets can be found. Then it will be shown that every semi-algebraic set can be represented by a set of 5-tuples of regular open semi-algebraic sets.

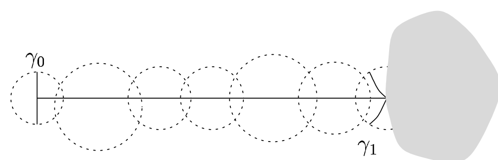
Lemma 7.1.1 (Existence of 0-cell representations). *Let $A \subset \mathbb{R}^2$ be a closed semi-algebraic set and let $p \in \mathbb{R}^2 \setminus A$. Then there exist regular open semi-algebraic sets $U_1, U_2 \subset \mathbb{R}^2$ with $[U_1] \cap [U_2] = \{p\}$ and $[U_1 \cup U_2] \cap A = \emptyset$.*

Proof. Since p is a point of the open set $\mathbb{R}^2 \setminus A$, there exists an open disc B around p with $[B] \cap A = \emptyset$. Certainly, any two orthogonal lines through p define four semi-algebraic regular open quadrants, which sum to B. Let U_1 and U_2 be two non-adjacent quadrants. Then U_1 and U_2 are as required. □

Lemma 7.1.2 (Existence of 1-cell representations). *Let the semi-algebraic set $A \subset \mathbb{R}^2$ be a 1-cell and let $B \subset \mathbb{R}^2$ be a closed semi-algebraic set disjoint from*



(a)



(b)

Figure 7.1: The construction of a 1-cell in $(\mathbb{R}^2)^*$

A. Then there exist disjoint regular open semi-algebraic sets $U_1, U_2 \subset \mathbb{R}^2$ *such that* $[U_1] \cap [U_2] = [A]$ *and* $[U_1 \cup U_2] \cap B$ *is a finite set.*

Proof. Two cases have to be considered:

Case 1: A separates the plane.

Since A and B are disjoint closed sets, there exists by the normality of $\mathbf{S}$ (proposition 3.3.11 on page 70) a regular open semi-algebraic set D such that $A \subseteq D$ and $[D] \cap B = \emptyset$. Since A separates the plane and $A \subseteq D$, A separates D in two regular open semi-algebraic sets U_1 and U_2 such that $[U_1] \cap [U_2] = [A] = A$ and $[U_1 \cup U_2] \cap B = [D] \cap B = \emptyset$.

Case 2: A does not separate the plane.

It will simplify the proof to consider A as a 1-cell in the one-point compactification $(\mathbb{R}^2)^*$ of $\mathbb{R}^2$. Let p_0 and p_1 be the endpoints of A. Note that one of p_0 and p_1 might be the point at infinity, and hence $[A]$ and B considered as subsets of $(\mathbb{R}^2)^*$ may share this point. Furthermore, the points p_0 and p_1 might be components of B. By the accessibility of the boundaries of A and B, it seems obvious that there are semi-algebraic arcs from p_0 to p_1 which define together with A the required regions (cf. figure 7.1(a)). However, I present a formal proof.

For each point $p \in A$ let U_p be an open disc around p not intersecting B or the endpoints of A. Let U_{p_0} and U_{p_1} be open discs around p_0 and p_1 respectively such that U_{p_0} and U_{p_1} do not include any component of B except for possibly p_0 and p_1. Then $\mathcal{U} = \{U_p | p \in A\} \cup \{U_{p_0}, U_{p_1}\}$ is an open cover of $[A]$. Since $[A]$ is compact there is a finite subcover $\mathcal{U}'$ of $\mathcal{U}$ containing U_{p_0} and U_{p_1}. The regular open semi-algebraic set $V = \sum \mathcal{U}'$ includes $[A]$, but does not include any component of B except for possibly p_0 and p_1. Since the boundary of a semi-algebraic set is accessible, there exist two disjoint semi-algebraic cross-cuts γ_0 and γ_1 of V going through p_0 and p_1 respectively but not through any point of B as shown in figure 7.1(b). Let the two components of $V \setminus (A \cup |\gamma_0| \cup |\gamma_1|)$ adjacent to both crosscuts be denoted by U_1 and U_2. Then $[U_1] \cap [U_2] = [A]$ and $[U_1 \cup U_2] \cap B \subseteq \{p_0, p_1\}$. Then $U_1 \setminus \{p_\infty\}$ and $U_2 \setminus \{p_\infty\}$ are the required regions in $\mathbf{S}$. □

Now I will use the existence of 0-cell- and 1-cell-representations to define for each semi-algebraic set a representation by regular open semi-algebraic sets.

Let the function $f : \mathbf{S}^5 \to \mathbf{S}_f$ be defined by

$$f(x_0, \ldots, x_4) = (x_0 \cup l(x_1, x_2) \cup p(x_3, x_4)) \setminus (x_0 \cap (l(x_1, x_2) \cup p(x_3, x_4)))$$

where $l(x_1, x_2) = ([x_1] \cap [x_2]) \setminus [-(x_1 + x_2)]$ and $p(x_3, x_4) = [x_3] \cap [x_4]$. Note that if x_1 and x_2 are disjoint then $l(x_1, x_2)$ is the finite union of 0- and 1-cells, and if x_3 and x_4 are disjoint and no two components of x_3 and x_4 have a connected sum then $p(x_3, x_4)$ is a finite set, i.e. a finite union of 0-cells.

Proposition 7.1.3 (Existence of representations for all $s \in \mathbf{S}_f$).
For any $s \in \mathbf{S}_f$ there exist $s_0, \ldots, s_4 \in \mathbf{S}$ such that $s = f(s_0, \ldots, s_4)$.

Proof. Let s_0 be the regular open semi-algebraic set $[s]^\circ$. Let $s^+ = s \setminus s_0$ and $s^- = s_0 \setminus s$. Then s^+ and s^- are disjoint semi-algebraic sets. Since s^+ as well as s^- have empty interior, by the cell stratification theorem on page 67, $s^+ \cup s^-$ is the union of finitely many mutually disjoint 1-cells $l_0, \ldots, l_m$ and 0-cells $p_0, \ldots, p_n$ such that $[l_i] \cap l_j = \emptyset$ for $1 \le i < j \le m$. A repeated application of lemma 7.1.2 guarantees the existence of $s_1^{(0)}, s_2^{(0)}, s_1^{(1)}, s_2^{(1)}, \ldots, s_1^{(m)}, s_2^{(m)} \in \mathbf{S}$ such that for $0 \le j \le m$, $[s_1^{(j)}] \cap [s_2^{(j)}] = [l_j]$ and $[s_1^{(j)} \cup s_2^{(j)}] \cap \bigcup_{i=0}^{j-1} [s_1^{(i)} \cup s_2^{(i)}]$ is a finite set. Let $s_1 = \bigcup_{i=0}^m s_1^{(i)} = \sum_{i=0}^m s_1^{(i)}$ and $s_2 = \bigcup_{i=0}^m s_2^{(i)} = \sum_{i=0}^m s_2^{(i)}$. Then $([s_1] \cap [s_2]) \setminus [-(s_1 + s_2)] = \bigcup_{i=0}^m l_i$. Analogously, a repeated application of lemma 7.1.1 guarantees the existence of $s_3, s_4 \in \mathbf{S}$ with $[s_3] \cap [s_4] = \bigcup_{i=0}^n p_i$. Then $s^+ \cup s^- = (([s_1] \cap [s_2]) \setminus [-(s_1 + s_2)]) \cup ([s_3] \cap [s_4])$ and $s = (s_0 \cup s^+) \setminus s^- = f(s_0, \ldots, s_4)$. □

Thus, the function f maps every 5-tuple of regular open sets to a semi-algebraic set and every semi-algebraic set s can be identified with the set $\{(s_0, \ldots, s_4) \in \mathbf{S}^5 | f(s_0, \ldots, s_4) = s\}$. In the interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$, I will employ exactly this identification. The interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ requires a number of definability results. In the following list, let each expression on the left stand as an abbreviation for the indented formula directly below the respective expression.

1. $\mathtt{pt}(x_1, x_2)$

   $\mathrm{C}(x_1, x_2) \wedge \forall x_1' \forall x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge \mathrm{C}(x_1', x_2) \wedge \mathrm{C}(x_1, x_2') \rightarrow \mathrm{C}(x_1', x_2'))$

2. $\langle x_1, x_2 \rangle \approx \langle y_1, y_2 \rangle$

   $\mathtt{pt}(x_1, x_2) \wedge \mathtt{pt}(y_1, y_2)$
   $\wedge \forall x_1' \forall x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge \mathrm{C}(x_1', x_2') \rightarrow \mathrm{C}(x_1', y_1) \wedge \mathrm{C}(x_2', y_2))$

3. $\langle x_1, x_2 \rangle \in y$

   $\exists z_1 \exists z_2 (\langle x_1, x_2 \rangle \approx \langle z_1, z_2 \rangle \wedge \neg \mathrm{C}(-y, z_1 + z_2))$

4. $\langle x_1, x_2 \rangle \in [y_1] \cap [y_2]$

   $\exists z_1 \exists z_2 (\langle x_1, x_2 \rangle \approx \langle z_1, z_2 \rangle \wedge z_1 \leq y_1 \wedge z_2 \leq y_2)$

5. $\mathtt{inls}(x_1, x_2, y_0, \ldots, y_4)$

   $\langle x_1, x_2 \rangle \in [y_1] \cap [y_2] \wedge \neg(\langle x_1, x_2 \rangle \in [-(y_1 + y_2)] \cap [1])$

6. $\langle x_1, x_2 \rangle \in \langle y_0, \ldots, y_4 \rangle$

   $(\langle x_1, x_2 \rangle \in y_0 \vee \mathtt{inls}(x_1, x_2, y_1, y_2) \vee \langle x_1, x_2 \rangle \in [y_3] \cap [y_4])$
   $\wedge \neg(\langle x_1, x_2 \rangle \in y_0 \wedge (\mathtt{inls}(x_1, x_2, y_1, y_2) \vee \langle x_1, x_2 \rangle \in [y_3] \cap [y_4]))$

7. $\langle x_0, \ldots, x_4 \rangle \leq \langle y_0, \ldots, y_4 \rangle$

   $\forall z_1 \forall z_2 (\langle z_1, z_2 \rangle \in \langle x_0, \ldots, x_4 \rangle \rightarrow \langle z_1, z_2 \rangle \in \langle y_0, \ldots, y_4 \rangle)$

8. $\mathtt{adhpt}(x_1, x_2, y_0, \ldots, y_4)$

   $\forall y \Big(\langle x_1, x_2 \rangle \in y \rightarrow \exists z_1 \exists z_2 (\langle z_1, z_2 \rangle \in y \wedge \langle z_1, z_2 \rangle \in \langle y_0, \ldots, y_4 \rangle) \Big)$

9. $\mathrm{C}(x_0, \ldots, x_4, y_0, \ldots, y_4)$

   $\exists z_1 \exists z_2 (\mathtt{adhpt}(z_1, z_2, x_0, \ldots, x_4) \wedge \mathtt{adhpt}(z_1, z_2, y_0, \ldots, y_4))$

Let $(r_1, r_2) \in \mathbf{S}^2$. If $[r_1] \cap [r_2]$ is a singleton then the pair (r_1, r_2) is said to *represent the point* $p \in [r_1] \cap [r_2]$ and I write $\langle r_1, r_2 \rangle \in U$ if $p \in U$ for $U \subseteq \mathbb{R}^2$.

Lemma 7.1.4. *The formulae given in the above list define the following sets in* $\mathfrak{S}$*:*

1. $\mathtt{pt}(x_1, x_2)$	$\{(r_1, r_2) \in \mathbf{S}^2 \mid \langle r_1, r_2\rangle \in \mathbb{R}^2\}$
2. $\langle x_1, x_2\rangle \approx \langle y_1, y_2\rangle$	$\left\{(r_1, r_2, s_1, s_2) \in \mathbf{S}^4 \,\middle|\, \begin{array}{l}\langle r_1, r_2\rangle \in \mathbb{R}^2 \text{ and} \\ [r_1] \cap [r_2] = [s_1] \cap [s_2]\end{array}\right\}$
3. $\langle x_1, x_2\rangle \in y$	$\{(r_1, r_2, s) \in \mathbf{S}^3 \mid \langle r_1, r_2\rangle \in s\}$
4. $\langle x_1, x_2\rangle \in [y_1] \cap [y_2]$	$\{(r_1, r_2, s_1, s_2) \in \mathbf{S}^4 \mid \langle r_1, r_2\rangle \in [s_1] \cap [s_2]\}$
5. $\mathtt{inls}(x_1, x_2, y_0, \ldots, y_4)$	$\{(r_1, r_2, s_0, \ldots, s_4) \in \mathbf{S}^7 \mid \langle r_1, r_2\rangle \in l(s_1, s_2)\}$
6. $\langle x_1, x_2\rangle \in \langle y_0, \ldots, y_4\rangle$	$\{(r_1, r_2, s_0, \ldots, s_4) \in \mathbf{S}^7 \mid \langle r_1, r_2\rangle \in f(s_0, \ldots, s_4)\}$
7. $\langle x_0, \ldots, x_4\rangle \leq \langle y_0, \ldots, y_4\rangle$	$\{(r_0, \ldots, r_4, s_0, \ldots, s_4) \in \mathbf{S}^{10} \mid f(\bar{r}) \subseteq f(\bar{s})\}$
8. $\mathtt{adhpt}(x_1, x_2, y_0, \ldots, y_4)$	$\{(r_1, r_2, s_0, \ldots, s_4) \in \mathbf{S}^7 \mid \langle r_1, r_2\rangle \in [f(\bar{s})]\}$
9. $\mathrm{C}(x_0, \ldots, x_4, y_0, \ldots, y_4)$	$\{(r_0, \ldots, r_4, s_0, \ldots, s_4) \in \mathbf{S}^{10} \mid [f(\bar{r})] \cap [f(\bar{s})] \neq \emptyset\}$

Proof. 1. Assume $(r_1, r_2) \in \mathbf{S}^2$ satisfies $\mathtt{pt}(x_1, x_2)$ in $\mathfrak{S}$. Since $\mathfrak{S} \models \mathrm{C}[r_1, r_2]$, $[r_1] \cap [r_2]$ is non-empty. Suppose $[r_1] \cap [r_2]$ contains two distinct points p and q. Since $\mathfrak{S}$ is regular, there exist two regular open semi-algebraic sets r_1' and r_2' with disjoint closures such that $p \in r_1'$ and $q \in r_2'$. Although $(r_1' \cdot r_1, r_2' \cdot r_2)$ satisfies the antecedent of the implication in $\mathtt{pt}[r_1, r_2]$, it does not satisfy the consequent. Hence, (r_1, r_2) must be a point representation. Conversely, if (r_1, r_2) represents the point p then r_1 and r_2 are in contact but disjoint. Then any two regular open semi-algebraic subsets r_1' and r_2' of r_1 and r_2 respectively must share the boundary point p if r_1' is in contact with r_2 and r_2' is in contact with r_1.

2. Assume $(r_1, r_2, s_1, s_2) \in \mathbf{S}^4$ satisfies $\langle x_1, x_2\rangle \approx \langle y_1, y_2\rangle$ in $\mathfrak{S}$. Suppose (r_1, r_2) and (s_1, s_2) represent distinct points p and q respectively. Then it follows from lemma 3.3.20 on page 72 that there exist regular open semi-algebraic subsets r_1' and r_2' of r_1 and r_2 respectively such that (r_1', r_2') represents p and $r_1' + r_2'$ is in contact with at most one of s_1 and s_2. Hence, (r_1, r_2) and (s_1, s_2) represent the same point. The converse direction is trivial.

3. Assume $(r_1, r_2) \in \mathbf{S}^2$ represents a point p in $s \in S$. Then since $\mathfrak{S}$ is regular, there exists $s' \in \mathbf{S}$ with $p \in s'$ and $[s'] \subset s$. Then $(s \cdot r_1, s \cdot r_2)$ represents the point p and $s \cdot (r_1 + r_2)$ is not in contact with $-s$. The converse direction is trivial.

4. Let $r_1, r_2, s_1, s_2 \in \mathbf{S}$ such that (r_1, r_2) represents $p \in [s_1] \cap [s_2]$. By lemma 3.3.20 on page 72, there exist $t_1, t_2 \in \mathbf{S}$ such that $[t_1] \cap [t_2] = \{p\}$, $t_1 \subseteq s_1$ and $t_2 \subseteq s_2$. The converse direction is trivial.

5. Remember that $l(r_1, r_2)$ is an abbreviation for $[r_1] \cap [r_2] \setminus [-(r_1 + r_2)]$. Then this definability result follows directly from the previous ones.

6. and 7. are trivial given the previous results.

8. The formula standing for $\mathtt{adhpt}(x_1, x_2, y_0, \ldots, y_4)$ is a formal rendering of the definition of adherent point: every (regular) open neighbourhood of the point represented by (x_1, x_2) has a point, represented by (z_1, z_2), with $f(y_0, \ldots, y_4)$ in common. Hence, $(r_1, r_2, s_0, \ldots, s_4) \in \mathbf{S}^7$ satisfies $\mathtt{adhpt}(x_1, x_2, y_0, \ldots, y_4)$ if and only if (r_1, r_2) represents an adherent point of $f(s_0, \ldots, s_4)$, i.e. a point in $[f(s_0, \ldots, s_4)]$.

9. Trivial given the previous result. □

Now the 5-dimensional interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ is given by

1. the domain formula $x_0 = x_0 \wedge x_1 = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge x_4 = x_4$,

2. the map of
   $x \leq y$ to $\langle x_0, \ldots, x_4 \rangle \leq \langle y_0, \ldots, y_4 \rangle$ and $\mathrm{C}(x, y)$ to $\mathrm{C}(x_0, \ldots, x_4, y_0, \ldots, y_4)$, and

3. the surjective function $f : \mathbf{S}^5 \to \mathbf{S}_f$ as defined on page 149. Lemma 7.1.4 ensures that for all $(r_0, \ldots, r_4), (s_0, \ldots, s_4) \in \mathbf{S}^5$

$$\begin{array}{lll} \mathfrak{S} \models \langle r_0, \ldots, r_4 \rangle \leq \langle s_0, \ldots, s_4 \rangle & \text{iff} & \mathfrak{S}_f \models f(r_0, \ldots, r_4) \leq f(s_0, \ldots, s_4) \\ \mathfrak{S} \models \mathrm{C}[r_0, \ldots, r_4, s_0, \ldots, s_4] & \text{iff} & \mathfrak{S}_f \models \mathrm{C}[f(r_0, \ldots, r_4), f(s_0, \ldots, s_4)] \,. \end{array}$$

7.2 A complete axiomatisation of $Th(\mathfrak{S}_f(\leq, \mathrm{C}))$

The interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ provides an easy means for a complete axiomatisation of $Th(\mathfrak{S}_f(\leq, \mathrm{C}))$. In chapter 5, I presented an axiom system $\mathcal{P}$ which axiomatises the theory of $\mathfrak{S}(\mathrm{c}, \mathrm{b}, +, \cdot, -, 0, 1)$ in the Δ-calculus. Given the expressivity results in section 4.1, $\mathcal{P}$ can be considered as a complete axiomatisation in the Δ-calculus of the theory of $\mathfrak{S}(\mathrm{C})$. I will show that an axiom system $\mathcal{P}_f$ consisting of a translation of the axioms of $\mathcal{P}$ together with four new axioms provides a complete axiomatisation of $Th(\mathfrak{S}_f(\leq, \mathrm{C}))$ in the Δ'-calculus where Δ' differs slightly from Δ.

As noted above, the operators $\cap, \cup, ^{-}$ and the operators $\mathtt{cl}(x)$ and $\mathtt{int}(x)$ with their set-theoretical interpretation are $\mathcal{L}(\leq, \mathrm{C})$-definable in $\mathfrak{S}_f$. It follows immediately that the bottom element 0 of $(\mathbf{S}_f, \subseteq)$ as well as the set of connected elements are definable in $\mathfrak{S}_f$. Let $\mathtt{RO}(x)$ abbreviate the formula $x = \mathtt{int}(\mathtt{cl}(x))$ defining the regular open sets in $\mathbf{S}_f$ where $\mathtt{int}(x)$ and $\mathtt{cl}(x)$ are the interior and closure operators as introduced in the beginning of this chapter. In the sequel, let $x + y$ stand for $\texttt{int}(\texttt{cl}(x \cup y))$ and let $f(x, x_0, \ldots, x_4)$ stand for

$$\bigwedge_{i=0}^{4} \texttt{RO}(x_i) \wedge x = (x_0 \cup l(x_1, x_2) \cup p(x_3, x_4)) \cap \overline{(x_0 \cap (l(x_1, x_2) \cup p(x_3, x_4)))}$$

where $l(x_1, x_2)$ stands for $(\texttt{cl}(x_1) \cap \texttt{cl}(x_2)) \cap \overline{x_1 + x_2}$ and $p(x_3, x_4)$ stands for $\texttt{cl}(x_3) \cap \texttt{cl}(x_4)$.

Given $\mathcal{L}(\mathrm{C})$-formulae $\phi(\bar{x})$ and $\psi(\bar{x})$, let the function $' : \mathcal{L}(\mathrm{C}) \to \mathcal{L}(\leq, \mathrm{C})$ be defined as follows.

$$\begin{array}{ll}
(x = y)' \mapsto x = y & (\mathrm{C}(x, y))' \mapsto \mathrm{C}(x, y) \\
(\neg\phi(\bar{x}))' \mapsto \neg(\phi(\bar{x}))' & (\phi(\bar{x}) \wedge \psi(\bar{x}))' \mapsto (\phi(\bar{x}))' \wedge (\psi(\bar{x}))' \\
(\phi(\bar{x}) \vee \psi(\bar{x}))' \mapsto (\phi(\bar{x}))' \vee (\psi(\bar{x}))' & (\phi(\bar{x}) \to \psi(\bar{x}))' \mapsto (\phi(\bar{x}))' \to (\psi(\bar{x}))' \\
(\exists x \phi(\bar{x}))' \mapsto \exists x(\texttt{RO}(x) \wedge (\phi(\bar{x}))') & (\forall x \phi(\bar{x}))' \mapsto \forall x(\texttt{RO}(x) \to (\phi(\bar{x}))')
\end{array}$$

Then the axiom system $\mathcal{P}_f$ consists of the following axioms:

A($\mathcal{P}'$) The set of axioms $\mathcal{P}' = \{\phi' | \phi \in \mathcal{P}\}$.

A(i) $\forall x \exists x_0 \ldots \exists x_4 \Big(\bigwedge_{i=0}^{4} \texttt{RO}(x_i) \wedge f(x, x_0, \ldots, x_4) \Big)$

A(ii) $\forall x_0 \ldots \forall x_4 \exists x \Big(\bigwedge_{i=0}^{4} \texttt{RO}(x_i) \to f(x, x_0, \ldots, x_4) \Big)$

A(iii) $\forall x \forall y \forall x_0 \ldots \forall x_4 \forall y_0 \ldots \forall y_4 \Big(f(x, x_0, \ldots, x_4) \wedge f(y, y_0, \ldots, y_4) \to$
$\Big((\mathrm{C}(x, y) \leftrightarrow (\mathrm{C}(x_0, \ldots, x_4, y_0, \ldots, y_4))') \wedge$
$(x \leq y \leftrightarrow (\langle x_0, \ldots, x_4 \rangle \leq \langle y_0, \ldots, y_4 \rangle)') \Big) \Big)$
where $\mathrm{C}(x_0, \ldots, x_4, y_0, \ldots, y_4)$ and $\langle x_0, \ldots, x_4 \rangle \leq \langle y_0, \ldots, y_4 \rangle$ are the formulae defined in the previous section.

A(iv) $\forall x \forall y (x \leq y \wedge y \leq x \leftrightarrow x = y)$

Theorem 7.2.1. *Let for $n \geq 1$, $\delta'_n(x)$ stand for the formula*

$$\texttt{RO}(x) \to \exists x_1 \ldots \exists x_n \Big(x = x_1 + \ldots + x_n \wedge \bigwedge_{i=1}^{n} \mathrm{c}(x_i) \wedge \texttt{RO}(x_i) \Big)$$

and let $\Delta'(x) = \{\neg\delta'_n(x) | n \geq 1\}$. Then the axiom system $\mathcal{P}_f$ is consistent and complete in the Δ'-calculus. Furthermore, $\mathcal{P}_f$ axiomatises the theory of $\mathfrak{S}_f(\leq, \mathrm{C})$ in the Δ'-calculus.

Proof. Consistency: By lemma 3.3.12 on page 70 and theorem 5.3.1 on page 112, $\mathfrak{S}$ is a model of $\mathcal{P}$ omitting $\Delta(x)$. Therefore, $\mathfrak{S}_f$ is a model of $\mathcal{P}' = \{\phi' | \phi \in \mathcal{P}\}$ omitting $\Delta'(x)$. By proposition 7.1.3 and the fact that $(\mathbf{S}_f, \subseteq)$ is a Boolean algebra, $\mathfrak{S}_f$ satisfies axioms A(i), A(ii) and A(iv). By lemma 7.1.4, $\mathfrak{S}_f$ satisfies axiom A(iii). Hence, by the consistency theorem of the Δ'-calculus, $\mathcal{P}_f$ is consistent in the Δ'-calculus.

Completeness: Let $\mathfrak{A}'$ and $\mathfrak{B}'$ be models of $\mathcal{P}_f$ omitting $\Delta'(x)$. By the downwards Löwenheim-Skølem theorem, there exist countable models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{A}'$ and $\mathfrak{B} \prec \mathfrak{B}'$. I show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

Let $\mathfrak{A}_{\mathtt{RO}}$ be the structure $\mathfrak{A}$ restricted to the language $\mathcal{L}(\mathrm{C})$ and the domain $\mathtt{RO}(A)$ and let $\mathfrak{B}_{\mathtt{RO}}$ be the structure $\mathfrak{B}$ restricted to the language $\mathcal{L}(\mathrm{C})$ and the domain $\mathtt{RO}(B)$. By proposition 4.3.9 on page 98, there exists a bijection $\alpha : \mathtt{RO}(A) \to \mathtt{RO}(B)$ such that for all $a_1, \ldots, a_n \in \mathtt{RO}(A)$ and all formulae $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(\mathrm{C})$, $\mathfrak{A} \models \phi[a_1, \ldots, a_n]$ if and only if $\mathfrak{B} \models \phi[\alpha(a_1), \ldots, \alpha(a_n)]$.

Let the possibly partial function $\beta : A \to B$ be defined by $\beta(a) = \alpha(a)$ if $a \in \mathtt{RO}(A)$ and $\beta(a) = b$ if there are $a_0, \ldots, a_4 \in \mathtt{RO}(A)$ such that $\mathfrak{A} \models f[a, a_0, \ldots, a_4]$ and $\mathfrak{B} \models f[b, \alpha(a_0), \ldots, \alpha(a_4)]$. I show that β is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Let $a \in A$. By axiom A(i), there exist $a_0, \ldots, a_4 \in \mathtt{RO}(A)$ such that $\mathfrak{A} \models f[a, a_0, \ldots, a_4]$. It follows from axiom A(ii) that there is $b \in B$ such that $\mathfrak{B} \models f[b, \alpha(a_0), \ldots, \alpha(a_4)]$. Thus, β is a well-defined function.

Let $b \in B$. By axiom A(i), there exist $b_0, \ldots, b_4 \in \mathtt{RO}(B)$ such that $\mathfrak{B} \models f[b, b_0, \ldots, b_4]$. Since $\alpha^{-1}(b_0), \ldots, \alpha^{-1}(b_4) \in \mathtt{RO}(A)$, by axiom A(ii) there exists $a \in A$ such that $\mathfrak{A} \models f[a, \alpha^{-1}(b_0), \ldots, \alpha^{-1}(b_4)]$. Since the function β is well-defined, $\mathfrak{B} \models f[\beta(a), \alpha(\alpha^{-1}(b_0), \ldots, \alpha(\alpha^{-1}(b_4))]$. Since $\mathfrak{B} \models f[b, b_0, \ldots, b_4] \wedge f[\beta(a), b_0, \ldots, b_4]$, it follows from axioms A(iii) and A(iv) that $\beta(a) = b$. Thus, β is surjective.

Assume that for some $a, a' \in A \setminus \mathtt{RO}(A)$, $\beta(a) = \beta(a')$. By the definition of β, there are $a_0, \ldots, a_4, a'_0, \ldots, a'_4 \in \mathtt{RO}(A)$ such that $\mathfrak{A} \models f[a, a_0, \ldots, a_4] \wedge f[a', a'_0, \ldots, a'_4]$ and $\mathfrak{B} \models f[\beta(a), \alpha(a_0), \ldots, \alpha(a_4)] \wedge f[\beta(a'), \alpha(a'_0), \ldots, \alpha(a'_4)]$. By axioms A(iii) and A(iv), $\mathfrak{B} \models (\langle \alpha(a_0), \ldots, \alpha(a_4) \rangle \leq \langle \alpha(a'_0), \ldots, \alpha(a'_4) \rangle)'$ and $\mathfrak{B} \models (\langle \alpha(a'_0), \ldots, \alpha(a'_4) \rangle \leq \langle \alpha(a_0), \ldots, \alpha(a_4) \rangle)'$. Since $\mathfrak{B} \models \phi(\alpha(\bar{a}))'$ iff $\mathfrak{B}_{\mathtt{RO}} \models \phi(\alpha(\bar{a}))$ iff $\mathfrak{A}_{\mathtt{RO}} \models \phi(\bar{a})$ iff $\mathfrak{A} \models \phi(\bar{a})'$, it follows from axiom A(iii) that $\mathfrak{A} \models a \leq a'$ and $\mathfrak{A} \models a' \leq a$. By axiom A(iv), $a = a'$. Thus, β is injective.

Hence, $\beta : A \to B$ is a bijection and it follows from axiom A(iii) that β is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. □

7.3 Conclusion

I have shown that the mereotopologies $\mathfrak{S}(\mathrm{C})$ and $\mathfrak{S}_f(\leq, \mathrm{C})$ can be interpreted in each other. Therefore, although $\mathfrak{S}(\mathrm{C})$ and $\mathfrak{S}_f(\leq, \mathrm{C})$ are mereotopologies with different spatial domains—the former being point-free, the latter containing points—both mereotopologies allow us to refer to the same spatial entities. Thus, even in $\mathfrak{S}(\mathrm{C})$ it is possible to speak about points and boundaries and to distinguish between open and closed regions. Therefore, the ontological simplicity of some mereotopologies is superficial, and the unpleasant ontological questions regarding boundaries are persistent.

On a more positive note, the results of this chapter show that a mereotopology containing points can be equally well-behaved as a point-free mereotopology. It follows from the interpretation of $\mathfrak{S}_f(\leq, \mathrm{C})$ in $\mathfrak{S}(\mathrm{C})$ that the pleasing properties of $\mathfrak{S}(\mathrm{C})$ also apply to $\mathfrak{S}_f(\leq, \mathrm{C})$. Moreover, points *and* regions exist side by side as primitives in $\mathbf{S}_f$. Therefore, the problems occurring in point-based approaches to spatial representation are avoided (cf. section 2.1). For instance, $\mathbf{S}_f$ does not contain any fractals although it contains points.

Note that the results of this chapter rely on the expressivity of the mereotopological language $\mathcal{L}(\mathrm{C})$. It follows from lemma 4.1.4 that equivalent results cannot be obtained for the open plane mereotopology $\mathfrak{S}(\leq, \mathrm{c})$.

Chapter 8

Conclusion

In this thesis, I have taken a model-theoretic approach to mereotopology and addressed the following major issues:

- the representation of regions
- the expressivity of mereotopological languages
- the characterisation of the sentences in a mereotopological language which are true with respect to a specific mereotopology
- the decidability of the truth of these sentences

Regions were represented by regular open sets in the real open and closed plane. Thus, the spatial domains were kept ontologically simple, excluding points, boundaries and avoiding the distinction between open and closed sets. The spatial domain $\widetilde{\mathbf{F}}$ of all regular open sets was shown to exhibit wild phenomena. These phenomena show some elements of $\widetilde{\mathbf{F}}$ to be inappropriate for a common-sense representation of everyday objects. The spatial domain $\widetilde{\mathbf{S}}$ of all semi-algebraic regular open sets in $\widetilde{\mathbf{F}}$ was shown to be extremely well-behaved.

First-order mereotopological languages with predicate symbols expressing parthood, contact, connection and boundedness were interpreted over the spatial domains introduced in this thesis. It was shown that parthood and connection can be expressed in terms of the contact-predicate in the mereotopologies $\widetilde{\mathfrak{F}}$, $\widetilde{\mathfrak{I}}$ and $\widetilde{\mathfrak{S}}$. Conversely, contact was shown to be definable in terms of parthood and connection in the closed plane mereotopology $\mathfrak{S}^*$ and definable in terms of parthood, connection and boundedness in the open plane mereotopology $\mathfrak{S}$. Infinitary versions of the mereotopological languages were shown to be topologically adequate in the sense that every subset of the domain $\widetilde{\mathbf{S}}$ which is closed in $\widetilde{\mathbf{S}}$ under topological equivalence is definable in at least one of the mereotopological languages. These expressivity results were used to show that it is possible to refer to points, boundaries and closed regions in $\mathbf{S}$ although such entities do not exist in $\mathbf{S}$. Thus, the intended ontological simplicity of the spatial domain $\mathbf{S}$ is only superficial.

The theory of the mereotopology $\mathfrak{S}(\mathrm{C})$ was characterised by an axiom system $\mathcal{P}$ that is complete in an extension of the predicate calculus which contains an additional infinitary rule of inference. The theories of $\widetilde{\mathfrak{S}}$ and $\widetilde{\mathfrak{F}}$ were shown to be hereditarily undecidable.

Of course, a number of issues in mereotopology remain to be addressed. First of all, it is an open question whether the conjectures regarding the spatial domain $\widetilde{\mathbf{J}}$ hold. A confirmation of the conjectures would extend a number of results about $\widetilde{\mathfrak{S}}$ to $\widetilde{\mathfrak{J}}$. For instance, $\mathfrak{J}$ would provide a model of the axiom system $\mathcal{P}$ which is not a standard mereotopology over $\mathbb{R}^2$. Related to this observation is the question whether the spatial domain of all 2-dimensional regular open sets of any o-minimal structure over $\mathbb{R}$ provides a model of $\mathcal{P}$.

In this thesis, only 2-dimensional regions in the real open and real closed plane have been considered. The question is, whether the results that were attained can be transferred to spatial domains which are only locally planar, such as spatial domains over the Möbius-strip or the torus, or to the spatial domain of regular open sets in $\mathbb{R}^3$. Some of the results are already known to fail in the 3-dimensional case. For example, it was shown that there are only finitely many connected partitions in $\mathbf{S}$ up to topological equivalence. It is easy to see that this result fails for the semi-algebraic regular open sets in $\mathbb{R}^3$. Apart from the technical details, I do not expect the 3-dimensional case to provide new interesting results in mereotopology. However, it would be interesting to generalise the results of this thesis to non-Euclidean spaces in order to recognise which role the regularity of the Euclidean spaces plays for the results in this thesis.

An issue which has been only rudimentarily explored in this thesis is the reconstruction of points. It would be interesting to investigate under which conditions points can be reconstructed at all, and whether it is always possible to reconstruct a mereotopology from its reconstructed points.

The applicability of the results of this thesis is limited. Although logic-based approaches to spatial representation and reasoning might be very useful for Geographical Information Systems (Casati et al., 1998; Egenhofer, 1991, 1994; Egenhofer and Mark, 1995) and spatial databases (Bimbo et al., 1993; Papadimitriou et al., 1996; Papadimitriou, 1997; Paredaens, 1998), it remains to be shown that they offer a realistic alternative to standard reasoning methods such as provided by computational geometry (see e.g. Preparata, 1985). In particular, the undecidability results in mereotopology show that there is no effective approach to spatial reasoning with unrestricted first-order mereotopological languages. However, the complexity results in qualitative spatial reasoning in restricted first-order languages are promising (cf. section 2.6). Apart from actual reasoning, first-order mereotopology might be used to specify the semantics of graphical languages such as Pictorial Janus (cf. Cohn and Gooday, 1994). Furthermore, the close link between well-behaved mereotopological structures and o-minimal structures might turn out to be fruitful for mereotopology as well as for tame topology.

Glossary

$[u]$ closure of u

u° interior of u

$\partial(u)$ boundary of u

$|\gamma|$ LOCUS OF A PATH γ

Δ-model a model omitting the set of formulae Δ

p_∞ the point at infinity of a ONE-POINT COMPACTIFICATION

$r_1, \ldots, r_n \sim s_1, \ldots, s_n$ the tuples $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are TOPOLOGICALLY EQUIVALENT

$\mathfrak{A} \equiv \mathfrak{B}$ the structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent

$\mathfrak{A} \cong \mathfrak{B}$ the structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic

$\mathcal{L}(\Sigma)$ first-order language with signature Σ

$\mathcal{L}_{mt}$ refers indifferently to one of the MEREOTOPOLOGICAL LANGUAGES $\mathcal{L}(\mathrm{C})$, $\mathcal{L}(\leq, \mathrm{C})$, $\mathcal{L}(\leq, \mathrm{c})$ or $\mathcal{L}(\leq, \mathrm{c}, \mathrm{b})$.

$\mathbf{F}, \mathbf{F}^*$ SPATIAL DOMAIN of regular open sets in the OPEN PLANE and CLOSED PLANE respectively

$\widetilde{\mathbf{F}}$ refers indifferently to either $\mathbf{F}$ or $\mathbf{F}^*$

$\mathfrak{F}, \mathfrak{F}^*$ MEREOTOPOLOGY of the regular open semi-algebraic sets in the OPEN PLANE and CLOSED PLANE respectively

$\widetilde{\mathfrak{F}}$ refers indifferently to either $\mathfrak{F}$ or $\mathfrak{F}^*$

$\widetilde{\mathbb{R}^2}$ refers indifferently to either $\mathbb{R}^2$ or $(\mathbb{R}^2)^*$

$\mathbf{R}, \mathbf{R}^*$ refers indifferently to $\mathbf{F}$ or $\mathbf{S}$, and $\mathbf{F}^*$ or $\mathbf{S}^*$ respectively

$\widetilde{\mathbf{R}}$ refers indifferently to $\mathbf{R}$ or $\mathbf{R}^*$

$\mathfrak{R}, \mathfrak{R}^*$ refers indifferently to $\mathfrak{F}$ or $\mathfrak{S}$, and $\mathfrak{F}^*$ or $\mathfrak{S}^*$ respectively

$\mathbf{S}, \mathbf{S}^*$ SPATIAL DOMAIN of regular open semi-algebraic sets in the OPEN PLANE and CLOSED PLANE respectively

$\widetilde{\mathbf{S}}$ refers indifferently to either $\mathbf{S}$ or $\mathbf{S}^*$

$\mathfrak{S}, \mathfrak{S}^*$ MEREOTOPOLOGY of the regular open semi-algebraic sets in the OPEN PLANE and CLOSED PLANE respectively

$\widetilde{\mathfrak{S}}$ refers indifferently to either $\mathfrak{S}$ or $\mathfrak{S}^*$

$\mathfrak{S}_f$ MEREOTOPOLOGY of the semi-algebraic sets in $\mathbb{R}^2$

$Th(\mathfrak{A})$ theory of the $\mathcal{L}$-structure $\mathfrak{A}$, i.e. the set of sentences in $\mathcal{L}$ which hold in $\mathfrak{A}$

X^* ONE-POINT COMPACTIFICATION of the TOPOLOGICAL SPACE X.

accessible boundary The boundary B of a set U is accessible (from U) if for every point $p \in B$ there exists an END-CUT from some point in U to p.

accumulation point Given a subset U of a topological space X, a point $p \in X$ is an accumulation point of U if every open neighbourhood of p contains infinitely many points of U.

adherent point Given a subset U of a topological space X, a point $p \in X$ is an adherent point of U if every open neighbourhood of p contains a point of U. The closure of U is the set of all its adherent points. Cf. LIMIT POINT.

arc an injective PATH

arc, semi-algebraic an arc which is a semi-algebraic function

arc-connected A set S is arc connected if any two distinct points of S are joined by an arc lying in S.

atomic model A model $\mathfrak{A}$ is atomic if every n-tuple in A satisfies a COMPLETE FORMULA in $Th(\mathfrak{A})$.

(κ-)categorical A theory is κ-categorical if any models of power κ are isomorphic; the theory is categorical if it is κ-categorical in any power κ.

cell, n-cell a semi-algebraic subset of $\mathbb{R}^k$ $(k \geq n)$ homeomorphic to $(0,1)^n$ where $(0,1)^0$ is taken to be a point

closed plane the 2-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}$, homeomorphic to $(\mathbb{R}^2)^*$

compact A subset U of a topological space X is compact if every open COVER of U has a finite subcover.

connected A subset U of a topological space is connected if for any two non-empty sets V_1, V_2 such that $V_1 \cup V_2 = U$, either $[V_1] \cap V_2 \neq \emptyset$ or $V_1 \cap [V_2] \neq \emptyset$.

connected partition a PARTITION all of which REGIONS are CONNECTED

complete formula A formula $\phi(x_1, \ldots, x_n)$ is complete in a theory T if for every formula $\psi(x_1, \ldots, x_n)$ exactly one of $T \models \phi(x_1, \ldots, x_n) \rightarrow \psi(x_1, \ldots, x_n)$ and $T \models \phi(x_1, \ldots, x_n) \rightarrow \neg\psi(x_1, \ldots, x_n)$ holds.

complete theory a THEORY whose set of consequences is maximal consistent

cover, subcover A collection $\mathcal{U}$ of open sets of a topological space X is an open cover of a set $U \subseteq X$ if $U \subseteq \bigcup \mathcal{U}$. A subcover of $\mathcal{U}$ is a subset of $\mathcal{U}$.

cross-cut an arc γ in some set $U \cup \{\gamma(0), \gamma(1)\}$ of a topological space X such that $\gamma((0,1)) \subseteq U^\circ$ and $\gamma(0), \gamma(1) \in \partial(U)$

decidable A set A is decidable if there is a mechanical procedure that determines, for any given element a, whether or not $a \in A$. By Church's thesis, A is decidable if A is RECURSIVE.

end-cut an arc γ in some set $U \cup \{(1)\}$ of a topological space X such that $\gamma([0,1)) \subseteq U^\circ$ and $\gamma(1) \in \partial(U)$

Hausdorff A topological space is Hausdorff if any two points of the space lie in disjoint open sets (see also definition 3.3.9 on page 69).

homeomorphism continuous bijection between two TOPOLOGICAL SPACES which has a continuous inverse.

Jordan arc another name for ARC or SIMPLE ARC

Jordan curve a HOMEOMORPHISM whose domain is the unit circle in $\mathbb{R}^2$; a tOPOLOGICAL SPACE which is homeomorph to the unit circle; another name for SIMPLE CLOSED CURVE

Jordan region bounded open subset of the OPEN or CLOSED PLANE whose boundary is a JORDAN CURVE; homeomorph to an open disk

j-partition PARTITION all of which REGIONS are J-REGIONS

j-region REGION in $\widetilde{\mathbf{F}}$ which is bounded by a JORDAN CURVE

limit point Given a subset U of a topological space X, a point $p \in X$ is a limit point of U if every open neighbourhood of p contains a point of U distinct from p. Cf. ADHERENT POINT.

locally compact A topological space X is locally compact if every point of X has an open neighbourhood whose closure is compact.

locally connected A topological space X is locally connected if for every point $p \in X$ and every open NEIGHBOURHOOD u of p, there exists a connected open NEIGHBOURHOOD of p lying in u.
A subset u of a metric space X with metric ρ is locally connected at the point p, if, given a positive ϵ, there exists a positive δ such that any two points of $u \cap B_\delta(p)$ lie in a connected set lying in $u \cap B_\epsilon(p)$

locus of a path the range $\gamma([0,1])$ of a path γ; denoted $|\gamma|$

$\mathcal{L}$-structure a structure in the sense of model theory; consisting of a domain and an interpretation which interprets the predicate and function symbols of the formal language $\mathcal{L}$ over the domain

manifold topological space which is locally homeomorphic to a Euclidean space of given dimension

mereology the formal study of the relation between a whole and its parts; axiomatic mereology was established by Leśniewski (1929)

mereological property property of inclusion: regions x is part of region y

mereotopological language formal (first-order) language whose predicate symbols are intended to define mereological or topological properties.

mereotopological property mereological or topological property

mereotopology 1. the discipline that investigates the properties of space which remain invariant under continuous change where REGIONS are the primary spatial entities; 2. an $\mathcal{L}$-structure whose domain is a SPATIAL DOMAIN and where $\mathcal{L}$ is a MEREOTOPOLOGICAL LANGUAGE.

metaphysics the philosophical study of being

neighbourhood, open An open neighbourhood of the point p is an open set containing p.

normal A topological space is normal if it obeys the T_1- and T_4-separation axioms (see definition 3.3.9 on page 3.3.9).

one-point compactification The one-point compactification of a topological space (X, τ) is the topological space (X^*, τ^*) where $X^* = X \cup p_\infty$, p_∞ is a point not in X and $U \subseteq X^*$ is an element of τ^* if U is open in X or $X \setminus U$ is closed and compact in X.

ontology 1. the metaphysical study of nature and existence; 2. the (set of) entities whose existence is implicit in any given theory

open plane another name for the real plane $\mathbb{R}^2$, cf. CLOSED PLANE

partition (of $r \in \widetilde{\mathbf{F}}$) tuple $r_1, \ldots, r_n \in \widetilde{\mathbf{F}}$ such that $r_i \cdot r_j = \emptyset$ $(1 \leq i <\leq j)$ and $r = r_1 + \ldots + r_n$; "$r_1, \ldots, r_n$ is a partition" is shorthand for "$r_1, \ldots, r_n$ is a partition of $\widetilde{\mathbb{R}^2}$"

path continuous function γ from the unit interval $[0, 1]$ into a TOPOLOGICAL SPACE X

path-connected A subset U of a topological space is path-connected if every two point of U are joined by a path with locus in U
(Note that any connected open set in a Euclidean space is path-connected.)

radial partition a J-PARTITION $r_1, \ldots, r_n$ such that $-(r_i + r_j)$ is connected $(1 \leq i <\leq j \leq n)$

recursive A function f is said to be recursive if it can be obtained from initial functions (zero function, successor function, projection functions) by application of substitution, recursion and the restricted μ-operator (see Mendelson, 1997, Section 3.3).

A set A is recursive if the characteristic function f_C defined by $f_C(a) = 0$ if $a \notin A$ and $f_C(a) = 1$ if $a \in A$ is recursive.

region an element of a SPATIAL DOMAIN

regular A topological space is regular if it obeys the T_0- and T_3-separation axioms (see definition 3.3.9 on page 69).

regular open set open set which is identical to the interior of its closure

semi-regular set set u such that either $[u] = [u^\circ]$ or $u^\circ = [u]^\circ$

separation The pair of sets (v_1, v_2) forms a separation of the set u if v_1 and v_2 are disjoint and $u = v_1 \cup v_2$.

separation axioms see HAUSDORFF, REGULAR, NORMAL

simple arc another term for ARC or JORDAN ARC

simple closed curve see JORDAN CURVE

spatial domain a set of subsets of a TOPOLOGICAL SPACE

theory set of sentences in a formal language

topologically equivalent Two tuples $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ of a spatial domain over the open (closed) plane are topologically equivalent, written $r_1, \ldots, r_n \sim s_1, \ldots, s_n$, if there exists a HOMEOMORPHISM from the OPEN (CLOSED) PLANE onto itself taking r_i to s_i $(1 \leq i \leq n)$.

topological space a pair (X, τ) where X is a set, τ, the set of open sets, is a subset of $\wp(X)$ containing $\emptyset$ and X, and is closed under union and finite intersection

topologically homogeneous A spatial domain M is topologically homogeneous if for any $r_0, r_1, \ldots, r_n, s_1, \ldots, s_n \in M$ such that $r_1, \ldots, r_n \sim s_1, \ldots, s_n$ there exists s_0 such that $r_0, r_1, \ldots, r_n \sim s_0, s_1, \ldots, s_n$.

topological property property of a region that is invariant under continuous spatial change; examples: 'region x is connected', 'region x is hollow' but only depending on the type of hole 'region x has one hole'

topologist's sine curve the function $\sin(\frac{1}{x})$

topology formal study of the properties of space that are invariant under continuous change

type A *type* $\chi(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ is a maximal consistent set of formulae in the variables $x_1, \ldots, x_n$.

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