COMPUTABILITY OF EUCLIDEAN SPATIAL LOGICS

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Abstract

In the last two decades, qualitative spatial representation and reasoning, and in particular *spatial logics*, have been the subject of an increased interest from the Artificial Intelligence community. By a *spatial logic*, we understand a formal language whose variables range over subsets of a fixed topological space, called *regions*, and whose non-logical primitives have fixed geometric meanings. A spatial logic for reasoning about regions in a Euclidean space is called a *Euclidean spatial logic*. We consider first-order and quantifier-free Euclidean spatial logics with primitives for topological relations and operations, the property of *convexity* and the ternary relation of being *closer-than*. We mainly focus on the computational properties of such logics, but we also obtain interesting model-theoretic results.

We provide a systematic overview of the computational properties of first-order Euclidean spatial logics and fill in some of the gaps left by the literature. We establish upper complexity bounds for the (undecidable) theories of logics based on Euclidean spaces of dimension greater than one, which yields tight complexity bounds for all but two of these theories. In contrast with these undecidability results, we show that the topological theories based on one-dimensional Euclidean space are decidable, but non-elementary.

We also study the computational properties of quantifier-free Euclidean spatial logics, and in particular those able to express the property of *connectedness*. It is known that when variables range over regions in the Euclidean plane, one can find formulas in these languages satisfiable only by regions with infinitely many connected components. Using this result, we show that the corresponding logics are undecidable. Further, we show that there exist formulas that are satisfiable in higher-dimensional Euclidean space, but only by regions with infinitely many connected components. We finish by outlining how the insights gained from this result were used (by another author) to show the undecidability of certain quantifier-free Euclidean spatial logics in higher dimensions.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Chapter 1

Introduction

Spatial knowledge is integral to our understanding of the world. We require spatial information for most of our activities, and our ability to acquire, manipulate and apply such information is one of the main aspects of what is considered intelligent behaviour. Naturally then, there have been significant efforts on the part of the Artificial Intelligence community to endow intelligent agents with the ability to represent and reason with spatial knowledge. The formalisms used to facilitate this intelligent behaviour are called *spatial logics*, and consist of logical languages for describing the properties of spatial entities and the relations that hold between them. But what entities, properties and relations should one consider?

The traditional approach in mathematics is to take points to be the primitive spatial entities and to define all other entities in terms of the points contained in them. Further, one considers *quantitative* spatial properties and relations such as the length of a line segment, the area of a triangle and the volume of a pyramid. Although this approach indisputably remains of great importance to us, this is not how we perceive the world around us. Much of the spatial information that we encounter in our everyday life concerns *regions* in space and is *qualitative* in nature. Indeed, we do not require *quantitative* description of the location, shape and orientation of the regions on a political map to see that Bulgaria and Greece share a common border, that Germany is part of Europe, that Russia resides on two continents and that Manchester is closer to Liverpool than it is to London. Hence, in replicating the spatial aspect of human intelligence, it has become common to consider formalisms for representing and deducing *qualitative* spatial knowledge about *regions*. We call these formalisms *spatial* *logics*.

More formally, by a *spatial logic*, we understand a logical language whose variables range over subsets of a fixed topological space, called regions, and whose non-logical primitives have fixed qualitative geometric meanings. From a practical point of view, the most interesting spatial logics are those for reasoning about regions in a Euclidean space; we call these *Euclidean spatial logics*. There are several aspects to consider when defining a spatial logic, including the logical syntax of the language, the non-logical symbols of the language (with their fixed interpretation), the topological spaces hosting the regions and the types of subsets of these topological spaces that are regarded as regions.

The spatial logics that we consider feature first-order or quantifier-free languages with primitives for Boolean operations and relations (such as the *part-of* relation and the operations *union*, *intersection* and *complementation*), topological properties and relations (such as the property of *connectedness* and the binary relation of being in *contact*), the property of *convexity* and the ternary relation of being *closer-than*.

Once a language $\mathcal{L}$ for representing spatial knowledge has been selected, we need to specify the collection of regions whose properties and relations will be described in that language. That is, we need to provide an *interpretation* for $\mathcal{L}$. So, we need to select a topological space $\mathcal{X}$ (arbitrary or a particular one such as $\mathbb{R}^3$), and then select a collection $\mathcal{M}$ of subsets of $\mathcal{X}$ whose members regarded as regions. It is well know that the collection $\wp(\mathcal{X})$ of all subsets of $\mathcal{X}$ is a complete Boolean algebra under the subset relation (a Boolean algebra is complete if any subset of its elements has a least-upper bound and a greatest lower bound), and it is common to choose $\mathcal{M}$ to be a Boolean subalgebra of $\wp(\mathcal{X})$, in which case we call $\mathcal{M}$ a *set algebra*. Evidently, for every topological space $\mathcal{X}$, $\wp(\mathcal{X})$ is a set algebra.

However, not every set algebra is suitable for modelling human-like spatial knowledge. Indeed, it is very unlikely that one would ever refer to a region in the set algebra $\wp(\mathbb{R}^3)$ consisting only of points with rational coordinates, for no physical object would occupy such a region. A common restriction is to consider as regions only subsets of $\mathcal{X}$ which are *regular closed* in $\mathcal{X}$. A region is regular closed if it is equal to the closure of its interior, or in other words, it is a closed set all of whose points can be approximated by points in its interior. The collection $\mathsf{RC}(\mathcal{X})$ of all regular closed sets in a topological space $\mathcal{X}$ is a complete Boolean algebra as well. We call a Boolean subalgebra of $\mathsf{RC}(\mathcal{X})$ a *region algebra*.

Once we have selected a collection of regions $\mathcal{M}$ and a language $\mathcal{L}$ for describing their properties and relations, we can determine whether descriptions written in $\mathcal{L}$, called *$\mathcal{L}$-formulas*, are realisable by regions in $\mathcal{M}$, or *satisfiable* in $\mathcal{M}$. In the special case when an $\mathcal{L}$-formula φ is a statement about the regions in $\mathcal{M}$ than can be either true or false, then φ is called an *$\mathcal{L}$-sentence*, and its truth in $\mathcal{M}$ is determined by its satisfiability in $\mathcal{M}$. The satisfiability of logical formulas plays an important role in mathematical logic and its applications, because checking for other important notions such as logical entailment (whether one $\mathcal{L}$-sentence entails another) and logical validity (whether an $\mathcal{L}$-sentece is true under all interpretations of $\mathcal{L}$) can be reduced to checking satisfiability of formulas.

The $\mathcal{L}$-sentences that are true in $\mathcal{M}$ are called $\mathcal{L}$-theorems of $\mathcal{M}$, and the set of all $\mathcal{L}$-theorems of $\mathcal{M}$ is referred to as the *$\mathcal{L}$-theory* of $\mathcal{M}$. Characterising logical theories is of great theoretical importance to mathematical logic. A logical theory is usually characterised by the means of an *axiomatisation*, which consists of a (finite or infinite) set of sentences in the theory, called *axioms*, and a finite set of *deduction rules*. Applying the deduction rules using the axioms and the sentences that have already been deduced, one will eventually deduce exactly the sentences in the theory. An axiomatisation of a logical theory gives us a procedure of systematically generating all theorems in the theory. Under certain conditions on the axiomatisation, this procedure can be turned into an algorithm, which can be implemented on a computer. If, in addition, the theory is *complete* (i.e. if it contains every sentence or its negation), one can check for every sentence φ in the language whether it is a theorem or not. This can be done by running the procedure that generates all theorems until either φ or its negation has been generated.

Determining the computational properties of a logical theory is of great importance for practical applications. The most general question is whether the satisfiability problem is *decidable*, i.e. whether there exists a (*reasoning*) algorithm which takes as input an $\mathcal{L}$-formula and tells us if it is satisfiable in $\mathcal{M}$. Restricting the input to $\mathcal{L}$-sentences, the algorithm decides whether an $\mathcal{L}$-sentence is in the $\mathcal{L}$-theory of $\mathcal{M}$. One way of showing that a complete theory is decidable is, of course, by providing a suitable axiomatisation. The mere existence of an algorithm which decides whether an $\mathcal{L}$-sentence is an $\mathcal{L}$-theorem of $\mathcal{M}$ tells us little about its performance in practice. In particular, some theories are inherently more difficult to compute than others, and this is reflected by the performance of the respective reasoning algorithms. In general, the more expressive the logical language is, the more difficult it is to compute. *Complexity theory* provides a means of measuring and comparing the computational complexity of different problems, such as the satisfiability problems for different spatial logics. Given a spatial logic with undesirable computational complexity, it is a major challenge, to identify fragments of that logic that exhibits low computational complexities. When choosing a language for reasoning about spatial knowledge, it is of great importance to have a complete classification of spatial logics according to their expressiveness and computational complexities.

Aims and Objectives

This thesis aims to study the model-theoretic and computational properties of spatial logics. It has the following objectives.

- To study the axiomatisability of spatial logics.
  - To survey the axiomatisations of general spatial logics—spatial logics of set algebras and region algebras over large collections of topological spaces (Section 3.2.1 and Section 4.2).
  - To survey the latest axiomatisations of Euclidean spatial logics (Section 3.2.2).
  - To show that the topological theory of all complete region algebras (set algebras) is different from that of all region algebras (set algebras), and that the collections of all complete region algebras and all complete set algebras are not first-order definable (Section 4.2.1).
- To study the computability and expressiveness of first-order spatial logics.
  - To study the relative expressiveness of first-order languages with topological, affine and metric primitives (Section 4.1).
  - To survey the latest results regarding the computability of first-order Euclidean spatial logics (Section 3.3.1).

  - To establish tight complexity bounds for these undecidable logics (Section 4.3.3 and Section 4.3.4).
  - To show that the first-order topological theories of region algebras over the real line are decidable but non-elementary (Section 4.3.1 and Section 4.3.2).

- To study the computability and expressiveness of quantifier-free topological spatial logics.
  - To survey the current state of the art (Section 3.3.2 and Section 5.1).
  - To show the undecidability of quantifier-free languages with connectedness when interpreted over the Euclidean plane (Section 5.4).
  - To show the sensitivity of the languages with connectedness to Euclidean regions with infinitely many components (Section 5.2 and Section 5.3).
  - To outline how the insights gained from above result were used (by another author) to show the undecidability of certain Euclidean spatial logics in higher dimensions (Section 5.5).

Thesis Structure

The thesis is organised as follows. Chapter 2 contains the mathematical preliminaries required for the rest of the thesis. We recall the basic definitions and results on many-sorted structures, Boolean algebras, computability theory, complexity theory and general and Euclidean topology. We then examine the computational complexity of the first- and second-order arithmetics of the natural, rational, algebraic and real numbers, and finish by introducing and establishing some basic properties of region algebras and set algebras.

In Chapter 3 we survey the literature on spatial logics and explain the relation between the findings of the thesis and the currently available results. First we discuss the currently available axiomatisations for general spatial logics and Euclidean spatial logics. Then we review the literature on the computability of first-order and quantifier-free Euclidean spatial logics.

In Chapter 4 we investigate first-order spatial logics. Firstly, we introduce and study the expressiveness of first-order languages with topological, affine and metric primitives. Then, we consider the axiomatisations of various topological theories and show that the topological theories of all complete region algebras (set algebras) are different from those of all region algebras (set algebras). Finally, we examine the computability of first-order Euclidean spatial logics. In particular, we establish tight complexity bounds for most of the logics that were known to be undecidable, and showed that the topological first-order spatial logics for reasoning about linear regions (regions in $\mathbb{R}$) are decidable but non-elementary.

In Chapter 5 we study the expressiveness, computability and complexity of quantifier-free Euclidean spatial logics with topological primitives. We consider languages with primitives for the Boolean relations and operations, and the property of being (interior-)connected. We recall some of the known expressiveness results about these languages. We show that there are formulas that are satisfiable in Euclidean region algebras, but only by tuples featuring regions with infinitely many connected components. We then establish the undecidability of the satisfiability problems for these languages interpreted over region algebras in the Euclidean plane. Finally, we outline how, by using some of the ideas presented in this section, it was shown by another author that, even in the higher-dimensional Euclidean spaces, some of the satisfiability problems for these languages are undecidable.

Chapter 2

Mathematical Preliminaries

In this chapter we recall some mathematical notions and results in the fields of model theory, topology, Boolean algebra, computability theory and complexity theory.

2.1 Model Theory

In this section we discuss logical theories of many-sorted structures. We assume prior knowledge of the corresponding definitions and results in the special case of one-sorted structures (see e.g. [Mar02]).

Signatures and Structures

Definition 1. A *signature* σ consists of:

1. a set of *sorts* $\mathcal{S} = \{0, \ldots, S\}$, for some $S \in \mathbb{N}$;
2. a set of *relational symbols* $\mathcal{R}$ with positive natural numbers n_R and tuples $s_R \in \mathcal{S}^{n_R}$ for each $R \in \mathcal{R}$;
3. a set of *functional symbols* $\mathcal{F}$ with positive natural numbers n_f and tuples $s_f \in \mathcal{S}^{n_f+1}$ for each $f \in \mathcal{F}$;
4. a set of constant symbols $\mathcal{C}$ with sorts $s_c \in \mathcal{S}$ for each $c \in \mathcal{C}$.

An example of a many-sorted signature is $\sigma_{MON} = (2, \in, \subseteq)$ having two sorts, a binary relational symbol '$\in$' with $s_\in = (0, 1)$, and a binary relational symbol '$\subseteq$' with $s_\subseteq = (1, 1)$. When we describe one-sorted signatures, we skip the information concerning the sort of the signatures. So, for example, the signature of Boolean algebras σ_{BA} is the tuple $(+, \cdot, -, 0, 1)$, where '+' and '·' are binary functional symbols, '−' is a unary functional symbol, and '0' and '1' are constant symbols.

Definition 2. Let σ be a signature. A *σ-structure* $\mathcal{M}$ consists of:

- disjoint nonempty sets M_s for each sort $s \in \mathcal{S}$, called the *domains* of $\mathcal{M}$;
- relations $R^{\mathcal{M}} \subseteq M_{s_R(0)} \times \cdots \times M_{s_R(n_R-1)}$ for each $R \in \mathcal{R}$;
- functions $f^{\mathcal{M}} : M_{s_f(1)} \times \cdots \times M_{s_f(n_f)} \rightarrow M_{s_f(0)}$ for each $f \in \mathcal{F}$;
- elements $c^{\mathcal{M}} \in M_{s_c}$ for each $c \in \mathcal{C}$.

A typical σ_{BA}-structure is the tuple $(\wp(X), \cup, \cap, \cdot^C, \emptyset, X)$, where X is any set; $\wp(X)$ is the powerset of X; '$\cup$', '$\cap$' and '$\cdot^C$' are the set-theoretical *union, intersection* and *complementation*; and $\emptyset$ is the *empty set*. A typical σ_{MON}-structure is the tuple $(X, \wp(X), \in, \subseteq)$, where X is any set, $\wp(X)$ is the powerset of X, '$\in$' is the *membership* relation, and '$\subseteq$' is the *subset* relation. Note that every structure uniquely determines its signature.

Logical Languages — Syntax and Semantics

Let σ be a signature, and fix disjoint sets of *variables* $\text{VAR}_s = \{x_i^s \mid i \in \mathbb{N}\}$ for each sort s in $\mathcal{S}$. For every $x \in \text{VAR}_s$, we say that x has a sort s.

Definition 3. The *σ-terms* $\mathcal{T}_s$, for s in $\mathcal{S}$, are the smallest sets such that:

- for every $c \in \mathcal{C}$, $c \in \mathcal{T}_{s_c}$;
- for every $s \in \mathcal{S}$ and every $v \in \text{VAR}_s$, $v \in \mathcal{T}_s$;
- for every $f \in \mathcal{F}$ with $s_f = (s_0, \ldots, s_{n_f})$, if $t_i \in \mathcal{T}_{s_i}$, $1 \leq i \leq n_f$, then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}_{s_0}$;

Every σ-structure $\mathcal{M}$ interprets σ-terms as functions on its domains in the following sense.

Definition 4. Let $\mathcal{M}$ be a σ-structure and t be a σ-term whose variables are among $(x_{i_1}^{s_1}, \ldots, x_{i_n}^{s_n})$, for some $n \in \mathbb{N}$, where $s_j \in \mathcal{S}$ for $1 \leq j \leq n$. For every tuple $\bar{a} = (a_{i_1}^{s_1}, \ldots, a_{i_n}^{s_n})$ with $a_{i_j}^{s_j} \in M_{s_j}$, $1 \leq j \leq n$, and every sub-term t' of t we inductively define $t'^{\mathcal{M}}(\bar{a})$:

- if t' is a constant symbol $c \in \mathcal{C}$, then $t'^{\mathcal{M}}(\bar{a}) := c^{\mathcal{M}}$;
- if t' is the variable $x_{i_j}^{s_j}$, for $1 \leq j \leq n$, then $t'^{\mathcal{M}}(\bar{a}) := a_{i_j}^{s_j}$;
- if t' is of the form $f(t_1^{s_1}, \ldots, t_{n_f}^{s_{n_f}})$ with $s_f = (s_0, \ldots, s_{n_f})$ and $t_k^{s_k} \in \mathcal{T}_{s_k}$, $1 \leq k \leq n_f$, then $t'^{\mathcal{M}}(\bar{a}) := f^{\mathcal{M}}\big(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a})\big)$.

Definition 5. The *atomic σ-formulas* are given by:

- $t_1 = t_2$, where $t_1, t_2 \in \mathcal{T}_s$ for some $s \in \mathcal{S}$;
- $R(t_1, \ldots, t_{n_R})$, where $R \in \mathcal{R}$, $s_R = (s_1, \ldots, s_{n_R})$ and $t_i \in \mathcal{T}_{s_i}$, $1 \leq i \leq n_R$.

Definition 6. The *unnested atomic σ-formulas* are given by:

- $c = x_i^{s_c}$, for $c \in \mathcal{C}$ and $i \in \mathbb{N}$;
- $x_j^s = x_i^s$, for $i, j \in \mathbb{N}$;
- $f(x_1, \ldots, x_{n_f}) = x_0$, for $f \in \mathcal{F}$, and $x_i \in \text{VAR}_{s_f(i)}$, $0 \leq i \leq n_f$;
- $R(x_0, \ldots, x_{n_R-1})$, for $R \in \mathcal{R}$, and $x_i \in \text{VAR}_{s_R(i)}$, $0 \leq i < n_R$.

Note that every unnested atomic formula is an atomic formula.

Definition 7. The *first-order σ-language $\mathcal{L}_\sigma$* is the smallest set containing the atomic σ-formulas such that:

- if $\varphi, \psi \in \mathcal{L}_\sigma$, then $\neg\varphi$ and $(\varphi \wedge \psi)$ are in $\mathcal{L}_\sigma$;
- if $\varphi \in \mathcal{L}_\sigma$, $s \in \mathcal{S}$ and $v \in \text{VAR}_s$, then $\exists v \varphi$ is in $\mathcal{L}_\sigma$.

A *unnested formula* is a formula whose atomic subformulas are unnested. Standardly, we abbreviate: $(\varphi \vee \psi)$ for $\neg(\neg\varphi \wedge \neg\psi)$; $(\varphi \to \psi)$ for $(\neg\varphi \vee \psi)$; $(\varphi \leftrightarrow \psi)$ for $((\varphi \to \psi) \wedge (\psi \to \varphi))$; and $\forall x \varphi$ for $\neg\exists x \neg\varphi$. Also standardly, we consider only formulas in which every variable is either quantified or free, but not both.

An $\mathcal{L}_\sigma$-formula with no free variables is called an *$\mathcal{L}$-sentence*. An $\mathcal{L}_\sigma$-formula with no quantifiers is called a *quantifier-free formula*. Note that first-order languages uniquely determine their signatures. If $\mathcal{L}$ is a first-order language and $\mathcal{M}$ is a structure, we say that $\mathcal{M}$ is an $\mathcal{L}$-structure if the signatures determined by $\mathcal{L}$ and $\mathcal{M}$ are the same. If $\mathcal{M}$ is a structure, we denote by $\mathcal{L}_{\mathcal{M}}$ the first-order language of $\mathcal{M}$.

Definition 8. Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ be an $\mathcal{L}$-structure. Also, let $\varphi(x_{i_1}^{s_1}, \ldots, x_{i_n}^{s_n})$ be an $\mathcal{L}$-formula with $s_j \in \mathcal{S}$ and $x_{i_j}^{s_j} \in \text{VAR}_{s_j}$, for $1 \leq j \leq n$, and let $\bar{a} = (a_{i_1}^{s_1}, \ldots, a_{i_n}^{s_n})$ with $a_{i_j}^{s_j} \in M_{s_j}$, for $1 \leq j \leq n$. Then $\bar{a}$ *satisfies* φ in $\mathcal{M}$, denoted by $\mathcal{M} \models \varphi[\bar{a}]$, if:

- if φ is $t_1 = t_2$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.
- if φ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$;
- if φ is $\psi_1 \wedge \psi_2$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{M} \models \psi_1[\bar{a}]$ and $\mathcal{M} \models \psi_2[\bar{a}]$;
- if φ is $\neg\psi$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{M} \not\models \psi[\bar{a}]$;
- if φ is $\exists x_j^s \psi(\bar{x}, x_j^s)$, for $j \in \mathbb{N}$ and $s \in \mathcal{S}$, then $\mathcal{M} \models \varphi[\bar{a}]$ if and only if for some $a_j^s \in M_s$, $\mathcal{M} \models \psi[\bar{a}, a_j^s]$.

Let $\mathcal{L}$ be a first-order languages and $\mathcal{M}$ be an $\mathcal{L}$-structure. An $\mathcal{L}$-formula $\varphi(\bar{x})$ is *satisfiable* in $\mathcal{M}$ if there is a tuple $\bar{a}$ in $\mathcal{M}$ satisfying $\varphi(\bar{x})$ in $\mathcal{M}$. We also say that $\mathcal{M}$ *is a model of* φ. If φ is an $\mathcal{L}$-sentence satisfiable in $\mathcal{M}$, then it is satisfied in $\mathcal{M}$ by the empty tuple, in which case we say that φ *is true in* $\mathcal{M}$, and we write $\mathcal{M} \models \varphi$. The *first-order theory* of $\mathcal{M}$, which is denoted by $T(\mathcal{M})$, is the set of all sentences in $\mathcal{L}$ which are true in $\mathcal{M}$. These notions naturally extend to classes of structures. Two formulas are *logically equivalent* if they are satisfied by the same models. Every (atomic) formula is logically equivalent to a unnested (atomic) formula.

The syntax of first-order languages can be restricted in different ways. For example one can consider only *universal sentences*—sentences obtained by universally quantifying the variables of quantifier-free formulas. The resulting logics are commonly referred to as *quantifier-free logics*, because the truth of every universal sentence is uniquely determined by the satisfiability of the negation of the corresponding quantifier-free formula. *Constraint satisfaction problems*, which are of great importance to Artificial Intelligence, are variants of the problem of determining the satisfiability of quantifier-free formulas. Hence, it has become much more common to consider the satisfiability of quantifier-free formulas, instead of the truth of universal sentences. For a signature σ and a σ-structure $\mathcal{M}$, $\text{Sat}(\sigma, \mathcal{M})$ denotes the set of quantifier-free σ-formulas satisfiable in $\mathcal{M}$. Similarly, for a class of σ-structures Σ, $\text{Sat}(\sigma, \Sigma)$ denotes the quantifier-free σ-formulas satisfiable in Σ.

Second-order languages are extensions of first-order languages that feature additional *second-order variables* interpreted as relations on the domains of the interpreting structures. There exist different second-order logics depending on the relations over which one is allowed to quantify. We will discuss the *"full" second-order logic*, the *monadic second-order logic* and the *weak-monadic second-order* logic of a given structure.

Let σ be a signature. In addition to the first-order variables, we also consider a set of second-order variables $\text{VAR}_s = \{x_i^s \mid i \in \mathbb{N}\}$ for each finite sequence of sorts $s = (s_0, \dots, s_k) \in \mathcal{S}^k$, $k \in \mathbb{N}^+$. As before, for $x \in \text{VAR}_s$, we say that x has a sort s. Note that we distinguish between the elements s of $\mathcal{S}$ and the elements (s) of $\mathcal{S}^1$. Hence, for $s \in \mathcal{S}$, VAR_s is the set of first-order variables of sort s, and $\text{VAR}_{(s)}$ is the set of second-order variables of sort (s)—i.e. there are no second-order variables of sort s. We abbreviate

$$\bar{\mathcal{S}} := \mathcal{S} \cup \bigcup_{k \in \mathbb{N}^+} \mathcal{S}^k.$$

The sets of terms is the same as in the first-order case. The set of atomic formulas is extended with

- $x_i^s = x_j^s$, for $i, j \in \mathbb{N}$, $k \in \mathbb{N}^+$ and $s \in \mathcal{S}^k$;
- $x_j^s(t_0, \dots, t_{k-1})$, for $j \in \mathbb{N}$, $k \in \mathbb{N}^+$, $s \in \mathcal{S}^k$, $x_j^s \in \text{VAR}_s$ and $t_i \in \mathcal{T}_{s(i)}$, $0 \leq i < k$.

The *second-order σ-language* $\mathcal{L}_\sigma^2$ is the smallest set containing the atomic σ-formulas such that:

- if $\varphi, \psi \in \mathcal{L}_\sigma^2$, then $\neg\varphi$ and $(\varphi \wedge \psi)$ are in $\mathcal{L}_\sigma^2$;
- if $\varphi \in \mathcal{L}_\sigma^2$, $s \in \bar{\mathcal{S}}$ and $v \in \text{VAR}_s$, then $\exists v \varphi$ is in $\mathcal{L}_\sigma^2$;

For a structure $\mathcal{M}$, we denote by $\mathcal{L}_\mathcal{M}^2$ the second-order language of $\mathcal{M}$.

Let $\mathcal{M}$ be a σ-structure. For every $s \in \mathcal{S}^k$, $k > 0$, we define

$$M_s := \wp(M_{s(0)} \times \cdots \times M_{s(k-1)}).$$

Let $\varphi(x_{i_1}^{s_1}, \dots, x_{i_k}^{s_k})$ be an $\mathcal{L}_\sigma^2$-formula with $s_j \in \bar{\mathcal{S}}$ and $x_{i_j}^{s_j} \in \text{VAR}_{s_j}$, $1 \leq j \leq k$. Further, let $\bar{a} = (a_{i_1}^{s_1}, \dots, a_{i_k}^{s_k})$ be such that $a_{i_j}^{s_j} \in M_{s_j}$, $1 \leq j \leq n$. Then, φ is satisfied by $\bar{a}$ in $\mathcal{M}$, denoted by $\mathcal{M} \models \varphi[\bar{a}]$, iff:

- if φ is $t_1 = t_2$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$;
- if φ is $R(t_1, \ldots, t_{n_R})$, for $R \in \mathcal{R}$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$;
- if φ is $x_{i_j}^{s_j}(t_1, \ldots, t_{k_\ell})$, for $1 \leq j \leq k$, $\ell \in \mathbb{N}^+$ and $s_j \in \mathcal{S}^\ell$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_\ell^{\mathcal{M}}(\bar{a})) \in a_{i_j}^{s_j}$;
- if φ is $\psi_1 \wedge \psi_2$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{M} \models \psi_1[\bar{a}]$ and $\mathcal{M} \models \psi_2[\bar{a}]$;
- if φ is $\neg\psi$, then $\mathcal{M} \models \varphi[\bar{a}]$ iff $\mathcal{M} \not\models \psi[\bar{a}]$;
- if φ is $\exists x_j^s \psi(\bar{x}, x_j^s)$, for $s \in \bar{\mathcal{S}}$ and $j \in \mathbb{N}$, then $\mathcal{M} \models \varphi[\bar{a}]$ if and only if for some $a_j^s \in M_s$, $\mathcal{M} \models \psi[\bar{a}, a_j^s]$.

We denote the second-order theory of a structure $\mathcal{M}$ by $T_2(\mathcal{M})$. The *monadic second-order logics* are restrictions of second-order logics in which the only second-order variables are those in $\text{VAR}_{(s)}$, for $s \in \mathcal{S}$. We refer to these variables as *set variables*. *Weak-monadic second-order logics* are variants of the monadic second-order logics interpreting set variables as *finite* subsets of the domains of the structure. This is done by setting $M_{(s)}$, $s \in \mathcal{S}$, to be the set of all finite subsets of M_s.

Definable sets. Interpretations of Theories

For the following definitions and results we refer to [HH93, Section 5].

Let σ be a signature with sorts $\mathcal{S}$, $\mathcal{M}$ be a σ-structure and $\mathcal{L}$ be a fragment of the second-order σ-language with variable sorts $\mathcal{S}_\mathcal{L} \subseteq \bar{\mathcal{S}}$. For an $\mathcal{L}$-formula $\varphi(x_0, \ldots, x_m)$, with $m \in \mathbb{N}$, $\bar{s} \in \mathcal{S}_\mathcal{L}^{m+1}$ and $x_i \in \text{VAR}_{\bar{s}(i)}$, $0 \leq i \leq m$, we define:

$$\varphi(\mathcal{M}) := \{\bar{a} \in M_{\bar{s}(0)} \times \cdots \times M_{\bar{s}(m)} \mid \mathcal{M} \models \varphi[\bar{a}]\}.$$

For $\bar{s} \in \mathcal{S}_\mathcal{L}^{m+1}$, $m \in \mathbb{N}$, a set $R \subseteq M_{\bar{s}(0)} \times \cdots \times M_{\bar{s}(m)}$ is *$\mathcal{L}$-definable* in $\mathcal{M}$ if there exists an $\mathcal{L}$-formula $\varphi(x_0, \ldots, x_m)$ such that $R = \varphi(\mathcal{M})$.

Definition 9. Let $\mathcal{M}$ and $\mathcal{N}$ be structures, and $\mathcal{L}_\mathcal{M}$ and $\mathcal{L}_\mathcal{N}$ be logical languages for $\mathcal{M}$ and $\mathcal{N}$. An $\mathcal{L}_\mathcal{M} \to \mathcal{L}_\mathcal{N}$*-interpretation* Γ of $\mathcal{M}$ in $\mathcal{N}$ consists of:

- $\mathcal{L}_\mathcal{N}$-formulas $\psi_s(x_1, \ldots, x_{k_s})$, for each variable sort s in $\mathcal{L}_\mathcal{M}$;

- $\mathcal{L}_{\mathcal{N}}$-formulas $\varphi_\Gamma(\bar{x}_1, \ldots, \bar{x}_k)$, for each unnested atomic $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(y_1, \ldots, y_k)$;
- surjective mappings $f^s_\Gamma : \psi_s(\mathcal{N}) \to M_s$, for each variable sort s in $\mathcal{L}_{\mathcal{M}}$,

such that for every unnested atomic $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(x^{s_1}_{i_1}, \ldots, x^{s_k}_{i_k})$, with $s_j \in \bar{\mathcal{S}}$, $1 \le j \le k$, and every $\bar{a}^{s_j}_{i_j} \in \psi_{s_j}(\mathcal{N})$,

$$\mathcal{N} \models \varphi_\Gamma[\bar{a}^{s_1}_{i_1}, \ldots, \bar{a}^{s_k}_{i_k}] \qquad \text{iff} \qquad \mathcal{M} \models \varphi[f^{s_1}_\Gamma(\bar{a}^{s_1}_{i_1}), \ldots, f^{s_k}_\Gamma(\bar{a}^{s_k}_{i_k})].$$

Γ is called *(polynomial-time) computable* if the mapping $\varphi \mapsto \varphi_\Gamma$ is (polynomial-time) computable.

Every $\mathcal{L}_{\mathcal{M}} \to \mathcal{L}_{\mathcal{N}}$-interpretation provides a satisfiability-preserving $\mathcal{L}_{\mathcal{N}}$ translation of the unnested atomic $\mathcal{L}_{\mathcal{M}}$-formulas, which can be extended to arbitrary unnested formulas.

Definition 10. Let $\mathcal{M}$ and $\mathcal{N}$ be structures, $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{N}}$ be logical languages for $\mathcal{M}$ and $\mathcal{N}$ and Γ be an $\mathcal{L}_{\mathcal{M}} \to \mathcal{L}_{\mathcal{N}}$-interpretation of $\mathcal{M}$ in $\mathcal{N}$. For every unnested $\mathcal{L}_{\mathcal{M}}$-formula φ, we define an $\mathcal{L}_{\mathcal{N}}$-formula φ_Γ recursively:

- φ_Γ is defined for unnested atomic φ;
- $\varphi_\Gamma := \psi_\Gamma \wedge \psi'_\Gamma$, if $\varphi = \psi \wedge \psi'$;
- $\varphi_\Gamma := \neg\psi_\Gamma$, if $\varphi = \neg\psi$;
- $\varphi_\Gamma := \exists x_1 \ldots \exists x_{n_s}(\varphi_s(x_1, \ldots, x_{n_s}) \wedge \psi_\Gamma)$, if $\varphi = \exists x^s_i \psi$, for $i \in \mathbb{N}$ and a variable sort s in $\mathcal{L}_{\mathcal{M}}$.

It is routine to show by structural induction the following lemma.

Lemma 11. *Let $\mathcal{M}$ and $\mathcal{N}$ be structures, and $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{N}}$ be logical languages for $\mathcal{M}$ and $\mathcal{N}$. Further, let Γ be an $\mathcal{L}_{\mathcal{M}} \to \mathcal{L}_{\mathcal{N}}$-interpretation of $\mathcal{M}$ in $\mathcal{N}$. Then, for every unnested $\mathcal{L}_{\mathcal{M}}$-sentence φ,*

$$\mathcal{M} \models \varphi \qquad \textit{iff} \qquad \mathcal{N} \models \varphi_\Gamma.$$

Definition 12. Let $\mathcal{M}$ and $\mathcal{N}$ be structures, and $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{N}}$ be logical languages for $\mathcal{M}$ and $\mathcal{N}$. If there exists a $\mathcal{L}_{\mathcal{M}} \to \mathcal{L}_{\mathcal{N}}$-interpretation of $\mathcal{M}$ in $\mathcal{N}$, we say that the $\mathcal{L}_{\mathcal{M}}$-theory of $\mathcal{M}$ is *definable* in the $\mathcal{L}_{\mathcal{N}}$-theory of $\mathcal{N}$. If, in addition, the $\mathcal{L}_{\mathcal{N}}$-theory of $\mathcal{N}$ is *definable* in the $\mathcal{L}_{\mathcal{M}}$-theory of $\mathcal{M}$, we say that the two theories are *inter-definable*.

In the sequel we make use of the following well-known result (see e.g. [HH93, Corollary 3.1.5, p.90]).

Theorem 13 (Downward Löwenheim-Skolem theorem)**.** *Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$ an $\mathcal{L}$-structure with domain M, A a subset of M, and λ a cardinal such that $|\mathcal{L}| + |A| \leq \lambda \leq |M|$. Then $\mathcal{M}$ has an elementary substructure $\mathcal{N}$ of cardinality λ whose domain contains A.*

2.2 Boolean Algebras

There are various definitions of the term Boolean algebra, which differ not only in the set of axioms that a structure has to satisfy, but also in the choice of signature for the structure. These definitions, despite their apparent differences, are equivalent in a sense that will be demonstrated below. In this treatment, we consider as Boolean algebras only structures whose signatures are either $\sigma_{BA} = (+, \cdot, -, 0, 1)$, or $\sigma_{\leq} = (\leq)$.

A σ_{BA}-structure $\mathcal{B} = (B, +, \cdot, -, 0, 1)$ is called a *σ_{BA}-Boolean algebra* if it satisfies the axioms:

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z);$$
$$\forall x \forall y (x + y = y + x \wedge x \cdot y = y \cdot x);$$
$$\forall x \forall y (x + (x \cdot y) = x \wedge x \cdot (x + y) = x);$$
$$\forall x \forall y \forall z (x + y \cdot z = (x + y) \cdot (x + z) \wedge x \cdot (y + z) = x \cdot y + x \cdot z);$$
$$\forall x (x + -x = 1 \wedge x \cdot -x = 0).$$

A $\sigma_{\leq}$-structure $(B, \leq)$ is called a *$\sigma_{\leq}$-Boolean algebra* if it satisfies the axioms for partial order:

$$\begin{aligned} \varphi_{refl} &:= \forall x (x \leq x); \\ \varphi_{asymm} &:= \forall x \forall y (x \leq y \wedge y \leq x \rightarrow x = y); \\ \varphi_{trans} &:= \forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z); \end{aligned}$$

together with the axioms in $\Phi_{BA\leq} := \{\varphi_{\exists+}, \varphi_{\exists\cdot}, \varphi_{\exists-}, \varphi_{\exists 1}, \varphi_{\exists 0}\}$, where:

$$\begin{aligned}
\varphi_{\exists+} &:= \forall x \forall y \exists z(\varphi_{+}(x,y,z));\\
\varphi_{\exists\cdot} &:= \forall x \forall y \exists z(\varphi_{\cdot}(x,y,z));\\
\varphi_{\exists-} &:= \forall x \exists y(\varphi_{-}(x,y));\\
\varphi_{\exists 1} &:= \exists x(\varphi_{1}(x) \land \forall y(\varphi_{1}(y) \to x = y));\\
\varphi_{\exists 0} &:= \exists x(\varphi_{0}(x) \land \forall y(\varphi_{0}(y) \to x = y));
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{\cdot}(x,y,z) &:= z \leq x \land z \leq y \land \forall z'(z' \leq x \land z' \leq y \to z' \leq z);\\
\varphi_{+}(x,y,z) &:= z \geq x \land z \geq y \land \forall z'(z' \geq x \land z' \geq y \to z' \geq z);\\
\varphi_{1}(x) &:= \forall y(y \leq x);\\
\varphi_{0}(x) &:= \forall y(x \leq y);\\
\varphi_{-}(x,y) &:= \forall z_0 \forall z_1(\varphi_0(z_0) \land \varphi_1(z_1) \to (\varphi_{+}(x,y,z_1) \land \varphi_{\cdot}(x,y,z_0))).
\end{aligned}$$

Denote by Σ_{BA} the class of σ_{BA}-Boolean algebras and by $\Sigma_{BA\leq}$ the class of $\sigma_{\leq}$-Boolean. For every $\mathcal{B} = (B, \leq)$ in $\Sigma_{BA\leq}$, we denote by $\pi(\mathcal{B})$ the structure $(B, +, \cdot, -, 0, 1)$, where $+$, $\cdot$ and $-$ are the functions with respective graphs $\varphi_{+}(\mathcal{B})$, $\varphi_{\cdot}(\mathcal{B})$ and $\varphi_{-}(\mathcal{B})$, and 0 and 1 are the only elements of the respective sets $\varphi_0(\mathcal{B})$ and $\varphi_1(\mathcal{B})$. Similarly, for every structure $\mathcal{B} = (B, +, \cdot, -, 0, 1)$ in Σ_{BA}, we denote by $\kappa(\mathcal{B})$ the structure $(B, \varphi_{\leq}(\mathcal{B}))$, where $\varphi_{\leq}(x,y) := x \cdot y = x$.

Fact 14. *The classes Σ_{BA} and $\Sigma_{BA\leq}$ are equivalent in the following sense:*

- *for every $\mathcal{B} \in \Sigma_{BA\leq}$, $\pi(\mathcal{B}) \in \Sigma_{BA}$;*
- *for every $\mathcal{B} \in \Sigma_{BA}$, $\kappa(\mathcal{B}) \in \Sigma_{BA\leq}$;*
- *for every $\mathcal{B} \in \Sigma_{BA\leq}$, $\kappa(\pi(\mathcal{B})) = \mathcal{B}$;*
- *for every $\mathcal{B} \in \Sigma_{BA}$, $\pi(\kappa(\mathcal{B})) = \mathcal{B}$.*

So, as long as first-order logics are concerned, it makes little difference which of the classes Σ_{BA} and $\Sigma_{BA\leq}$ we consider.

As we already saw, every Boolean algebra $\mathcal{B} = (B, +, \cdot, -, 0, 1)$ induces a partial order on its elements. For $a, b \in B$, $a + b$ is the least upper bound of a and b. It follows then, that the least upper bound of a finite subset $\{a_0, \ldots, a_{k+2}\}$

of B is the element $a_0+\cdots+a_{k+2}$ of B, which we also write as $\sum_{i=0}^{k+2} a_i$. However, the least upper bound of an infinite subset A of B need not exist. A *complete Boolean algebra* is a Boolean algebra for which every subset A of B has a least upper bound, denoted by $\sum A$.

We recall the following standard result, (see e.g. [Kop89]).

Theorem 15 (Stone's Representation Theorem). *Every Boolean algebra is isomorphic to a Boolean subalgebra of* $(\wp(X), \subseteq)$*, for some set* X.

2.3 Computability and Complexity

We now recall different notions of *computational reductions* (see e.g. [Koz06]). We also disuss how logical interpretations induce computational reducibility of logical theories (see e.g. [HH93, Section 5]). We assume familiarity with Turing machines, time and space complexity classes and related standard results.

Let A and B be two languages over the alphabets Σ and Γ. A is *many-one reducible* to B, denoted by $A \leq_m B$, if there exists a total recursive function $f : \Sigma^* \to \Gamma^*$ such that for all $x \in \Sigma^*$, $x \in A$ if and only if $f(x) \in B$. If f is polynomial time computable, then A is *polynomial-time many-one reducible* to B, denoted by $A \leq_m^p B$. If f is logspace-computable, then A is *logspace-computable many-one reducible* to B, denoted by $A \leq_m^{log} B$. Let $\leq$ be any of $\leq_m$, $\leq_m^p$ and $\leq_m^{log}$. A set A is said to be $\leq$-*hard* for a collection of sets $\mathcal{C}$ if $B \leq A$ for every $B \in \mathcal{C}$. A is $\leq$-*complete* for $\mathcal{C}$ if A is $\leq$-hard and $A \in \mathcal{C}$. Most complexity classes $\mathcal{C}$ which contain NP (e.g. NP, PSPACE, EXPTIME, EXPSPACE and NEXPTIME), are closed downward under $\leq_m^p$ and $\leq_m^{log}$ reducibilities, i.e. if $A \leq_m^p B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$. The class of decidable problems is closed downward under any of the three reducibilities. Finally, the complexity classes LOGSPACE and NLOGSPACE are closed downward under $\leq_m^{log}$ reducibility. We will extensively use the following result about computational reductions of logical theories.

Lemma 16. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be structures,* $\mathcal{L}_\mathcal{M}$ *and* $\mathcal{L}_\mathcal{N}$ *be logical languages for* $\mathcal{M}$ *and* $\mathcal{N}$ *with corresponding theories* $T_\mathcal{M}$ *and* $T_\mathcal{N}$*. If there is an* $\mathcal{L}_\mathcal{M} \to \mathcal{L}_\mathcal{N}$*-interpretation of* $\mathcal{M}$ *in* $\mathcal{N}$*, then* $T_\mathcal{M} \leq_m T_\mathcal{N}$.

Proof. Follows from Definition 10 and Lemma 11. □

If $\mathcal{M}$ and $\mathcal{N}$ are two structures, we denote by $\mathcal{M} \leq_m^p \mathcal{N}$ the fact that $T(\mathcal{M}) \leq_m^p T(\mathcal{N})$.

2.4 First- and Second-Order Arithmetics

We denote by Δ^0_ω and Δ^1_ω the first- and second-order theories of the ordered semiring of the natural numbers $(\mathbb{N}, <, +, \cdot, 0, 1)$. Δ^0_ω and Δ^1_ω are also known as the first- and second-order arithmetics. We also treat Δ^0_ω and Δ^1_ω as complexity classes in the following sense. A set A for which $\Delta^0_\omega \leq_m A$ is called Δ^0_ω-*hard*, and if in addition $A \leq_m \Delta^0_\omega$, then A is called Δ^0_ω-*complete*. Similarly, a set A for which $\Delta^1_\omega \leq_m A$ is called Δ^1_ω-*hard*, and if in addition $A \leq_m \Delta^1_\omega$, then A is called Δ^1_ω-*complete*.

To show that a theory T is Δ^0_ω-hard or Δ^1_ω-hard, it is sufficient to define in T (see Definition 12) another theory which is Δ^0_ω-hard or Δ^1_ω-hard, respectively. Although Δ^0_ω and Δ^1_ω are trivially such theories, it is more convenient to work with theories which are syntactically more economical. In Section 4.3.3 we will make use of the Δ^0_ω-hard first-order theory of the structure $(\mathbb{N}, +, \cdot)$ and the Δ^1_ω-hard first-order theory of the two-sorted structure $(\mathbb{N}, \wp(\mathbb{N}), +, \cdot, \in)$.

By contrast, to show for a theory T that $T \leq_m \Delta^0_\omega$ or $T \leq_m \Delta^1_\omega$, it is sufficient to show that T is definable in another theory with that property. Again, although the most natural such theories are Δ^0_ω and Δ^1_ω, they are not the most convenient to work with. Instead, in Section 4.3.4 we make use of the first-order theories of the structures $(\mathbb{Q}, \sigma^+_F)$ and $(\mathbb{A}, \sigma^+_F)$, both of which are definable in Δ^0_ω, and the first-order theory of $(\mathbb{R}, \sigma^+_F)$ and the monadic second-order theory of $(\mathbb{Q}, \sigma^+_F)$, both of which are definable in Δ^1_ω, where $\sigma^+_F = (<, +, \cdot, 0, 1, \pi, [\,], N)$ is defined latter on.

The following result is immediate.

Lemma 17. *The first-order theory of* $(\mathbb{N}, +, \cdot)$ *is* Δ^0_ω*-hard.*

We now show Δ^1_ω-hardness for the first-order theory of $(\mathbb{N}, \wp(\mathbb{N}), +, \cdot, \in)$.

Lemma 18. *The first-order theory of the two-sorted structure* $(\mathbb{N}, \wp(\mathbb{N}), +, \cdot, \in)$ *is* Δ^1_ω*-hard.*

Proof. We define Δ^1_ω in the first-order theory of $\mathcal{M} = (\mathbb{N}, \wp(\mathbb{N}); +, \cdot, \in)$ (recall Definition 12). The idea is to encode every n-ary relation R on $\mathbb{N}$ as a subset of $\mathbb{N}$ whose elements encode exactly the n-tuples in R. In the language $\mathcal{L}_\mathcal{M}$, we use the letters x, x_1, x_2, etc., for the variables of sort 1, and X, X_1, X_2, etc., for variables of sort 2. Fix in $\mathcal{M}$ a pairing mechanism and let the two families of formulas $\psi^n_\pi(x)$ and $\psi^n_{=\pi}(x_1, \ldots, x_n, x)$, $n \in \mathbb{N}$, be such that, $\mathcal{M} \models \psi^n_\pi[k]$

if and only if k encodes an n-tuple of natural numbers, $k, n \in \mathbb{N}$, and $\mathcal{M} \models \psi^n_{=\pi}[k_1, \ldots, k_n, k]$ iff k encodes the n-tuple $(k_1, \ldots, k_n)$, for $k_1, \ldots, k_n, k \in \mathbb{N}$. Let $s = \{1\}^n$, $n \in \mathbb{N}^+$, be the second-order variable sort in the language $\mathcal{L}^2_{\sigma_F}$ for n-ary relations. We now define the sets of natural numbers that encode n-ary relations using the formula:

$$\psi_s(X) := \forall x(X(x) \rightarrow \psi^n_\pi(x)).$$

For the $\mathcal{L}^2_{\sigma_F}$-atomic formula $\psi = x^s_i(x_1, \ldots, x_n)$, we define the $\mathcal{L}_\mathcal{M}$-formula

$$\psi^s_\in(x_1, \ldots, x_n, X) := \exists x(X(x) \wedge \psi^n_{=\pi}(x_1, \ldots, x_n, x)). \qquad \square$$

It is a standard result that in the first-order theory of $(\mathbb{N}, <, +, \cdot, 0, 1)$ one can interpret the first-order arithmetics of the integer and the rational numbers, by encoding each integer and rational number using a single natural number. In a standard way, using for example Gödel's β function ([Men97, p.186]), one can also define finite sequences of natural numbers. Before we state all this formally, consider the signature $\sigma^+_F = (<, +, \cdot, 0, 1, \pi, [\ ], N)$ and the structure $\mathcal{Q} = (\mathbb{Q}, \sigma^+_F)$. Here, the unary relation $\pi(x)$ and the binary function $x[y]$ are the means of encoding finite sequences of rational numbers. I.e. if $\pi(q)$, then $q[0]$ returns the length of the sequence encoded by q, and $q[n]$ returns the nth element of the sequence encoded by q, for every natural $1 \leq n \leq q[0]$. Hence, the following is a standard result.

Lemma 19. *$T(\mathcal{Q}) \leq_m \Delta^0_\omega$ and $T_2(\mathcal{Q}) \leq_m \Delta^1_\omega$.*

We now proceed to showing that the first-order theory of $(\mathbb{A}, \sigma^+_F)$ is definable in first-order theory of $(\mathbb{Q}, \sigma^+_F)$, where $\mathbb{A}$ is the set of algebraic numbers. Recall that a real number is algebraic, if it is a root of a polynomial with rational coefficients, i.e. a polynomial in $\mathbb{Q}[X]$. In order to do so, we need to encode polynomials with rational coefficients, and for that, we need to define some operations on finite sequences of rational numbers.

By the formula $\psi_{sum}(x, y)$, we define the set of pairs of rational numbers, the first encoding a finite sequence of rational numbers and the second being the sum those numbers.

$$\begin{aligned}\psi_{sum}(x,y) := \quad & \pi(x) \wedge \big((x[0] = 0 \wedge y = 0) \vee \\ & \exists z\big(\pi(z) \wedge x[0] = z[0] \wedge \\ & \forall t(N(t) \wedge 1 \leq t \wedge t < x[0] \rightarrow \\ & z[t+1] = x[t+1] + z[t] \wedge y = z[z[0]])\big)\big).\end{aligned}$$

By the formula $\psi_{bit\times}(x,y,z)$ we define the element by element multiplication of two sequences.

$$\begin{aligned}\psi_{bit\times}(x,y,z) := \quad & \pi(x) \wedge \pi(y) \wedge \pi(z) \wedge x[0] = y[0] \wedge y[0] = z[0] \wedge \\ & \forall t(N(t) \wedge 1 \leq t \wedge t \leq x[0] \rightarrow z[t] = x[t] \cdot y[t]).\end{aligned}$$

We identify each polynomial in $\mathbb{Q}[X]$ of degree n with the $n+1$ tuple of its coefficients. We define in $\mathcal{Q}$ the set of rational numbers that encode polynomials in $\mathbb{Q}[X]$ using the formula $\psi_{\mathbb{Q}[X]}$.

$$\psi_{\mathbb{Q}[X]}(x) \;:=\; \pi(x) \wedge x[0] > 0 \wedge x[x[0]] \neq 0$$

The formula $\psi_{PV}(x,y,z)$ defines the set of triples $(q_1,q_2,q_3) \in \mathbb{Q}^3$ for which q_1 encodes a polynomial $P \in \mathbb{Q}[X]$, and $P(q_2) = q_3$:

$$\psi_{PV}(x,y,z) := \psi_{\mathbb{Q}[X]}(x) \wedge \exists t \exists u(\psi_{x^n}(x[0],y,t) \wedge \psi_{bit\times}(x,t,u) \wedge \psi_{sum}(u,z)), \text{ where}$$

$$\begin{aligned}\psi_{x^n}(x,y,z) := \; & N(x) \wedge \pi(z) \wedge x = z[0] \wedge z[1] = 1 \wedge \\ & \forall t(N(t) \wedge 0 < t \wedge t < x \rightarrow z[t+1] = y \cdot z[t]).\end{aligned}$$

We represent algebraic numbers using *root isolation* (see Figure 2.1). A root isolation of an algebraic number r is a pair of a rational polynomial P and a bounded open rational interval i such that r is the unique root of P in i. First, we define in $\mathcal{Q}$ the bounded open rational intervals together with some operations on them. The formulas $\psi_{()}(x)$, $\psi_{(.)}(x,y)$ and $\psi_{(())}(x,y)$ define the set of intervals, the membership relation and the subset relation, respectively.

$$\begin{aligned}\psi_{()}(x) &:= \pi(x) \wedge x[0] = 2 \wedge x[1] < x[2]\\ \psi_{(.)}(x,y) &:= \psi_{()}(x) \wedge x[1] < y \wedge y < x[2]\\ \psi_{(())}(x,y) &:= \psi_{()}(x) \wedge \psi_{()}(y) \wedge \forall z(\psi(z,x)_{(.)} \rightarrow \psi(z,y)_{(.)})\end{aligned}$$

The formulas $\psi_{()+()}(x,y,z)$ and $\psi_{()\cdot'()}(x,y,z)$ define addition and multiplication of intervals, which will be used for defining addition and multiplication of algebraic numbers. The *sum* and the *product* of two intervals (p,q) and (r,s) are the intervals $(p+r, q+s)$ and $(p\cdot r, q\cdot s)$, respectively.

$$\psi_{()+()}(x,y,z) := \psi_{()}(x) \wedge \psi_{()}(y) \wedge \psi_{()}(z) \wedge x[1] + y[1] = z[1] \wedge x[2] + y[2] = z[2]$$

$$\psi_{()\cdot'()}(x,y,z) := \psi_{()}(x) \wedge \psi_{()}(y) \wedge \psi_{()}(z) \wedge x[1] \cdot y[1] = z[1] \wedge x[2] \cdot y[2] = z[2]$$

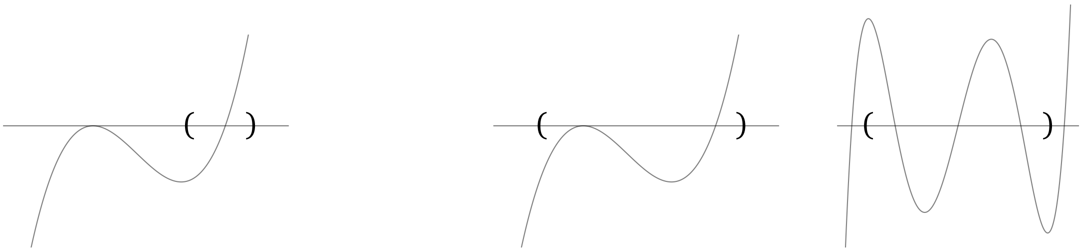
The formulas $\psi_{PV+}(x,y)$ and $\psi_{PV-}(x,y)$, are satisfied by pairs of rationals representing a polynomial and an interval such that the polynomial has, respectively, different and same signs at the endpoints of the interval. For $\# \in \{+,-\}$ we define:

$$\begin{aligned}\psi_{PV\#}(x,y) &:= \psi_{\mathbb{Q}[X]}(x) \wedge \psi_{()}(y) \wedge\\ &\qquad \exists y_1 \exists y_2 (\psi_{PV}(x,y[1],y_1) \wedge \psi_{PV}(x,y[2],y_2) \wedge (\#1) \cdot y_1 \cdot y_2 > 0).\end{aligned}$$

The set of algebraic numbers is defined by the formula $\psi_{\mathbb{A}}(x,y)$:

$$\begin{aligned}\psi_{\mathbb{A}}(x,y) := \psi_{\mathbb{Q}[X]}(x) \wedge \psi_{()}(y) \wedge \psi_{PV-}(x,y) \wedge \forall y'(\psi_{(())}(y',y) \wedge \psi_{PV+}(x,y') \rightarrow\\ \exists z(z > 0 \wedge \forall z' \forall z''(\psi_{(.)}(z',y') \wedge \psi_{PV}(x,z',z'') \rightarrow z'' > z \vee z'' > -z))).\end{aligned}$$

Let $\mathcal{Q} \models \psi_{\mathbb{A}}[p,q]$ for some $p,q \in \mathbb{Q}$. Then p represents a polynomial P in $\mathbb{Q}[X]$, and q represents a bounded open rational interval (r,s) such that $P(r)$ and $P(s)$ have different signs. So P has a root in (r,s). Further, for every subinterval (r',s') of (r,s) such that $P(r')$ and $P(s')$ have the same signs, the graph of P restricted to (r',s') is at a distance greater than 0 from the horizontal axis, hence avoiding situations as the one depicted in Figure 2.1b. So P has a unique root in (r,s). The root isolations represent the same algebraic number



(a) A positive example.

(b) Two negative examples.

Figure 2.1: Examples of root isolations.

if they satisfy the formula:

$$\psi_{\mathbb{A}\sim}(x,y,x',y') := \psi_{\mathbb{A}}(x,y) \wedge \psi_{\mathbb{A}}(x',y') \wedge \exists z\big(\psi_{(())}(z,y) \wedge \psi_{(())}(z,y') \wedge \psi_{\mathbb{A}}(x,z) \wedge \psi_{\mathbb{A}}(x',z) \wedge \forall z'\big(\psi_{(())}(z',z) \wedge \psi_{\mathbb{A}}(x,z') \rightarrow \psi_{\mathbb{A}}(x',z')\big)\big).$$

A root isolation (p,i) represents a rational number q if q is in the interval represented by i and q is a root of the polynomial represented by p. Clearly, if q is a natural number, then (p,i) represents a natural number.

$$\begin{aligned}
\psi_{\mathbb{A}\sim\mathbb{Q}}(p,i,q) &:= \psi_{\mathbb{A}}(p,i) \wedge i[1] < q \wedge q < i[2] \wedge \psi_{PV}(p,q,0);\\
\psi_{\mathbb{A}\sim\mathbb{N}}(p,i,q) &:= N(q) \wedge \psi_{\mathbb{A}\sim\mathbb{Q}}(p,i,q);\\
\psi_{\mathbb{A}_{\mathbb{Q}}}(p,i) &:= \exists q(\psi_{\mathbb{A}\sim\mathbb{Q}}(p,i,q));\\
\psi_{\mathbb{A}_{\mathbb{N}}}(p,i) &:= \exists q(\psi_{\mathbb{A}\sim\mathbb{N}}(p,i,q));
\end{aligned}$$

Trivially, a root isolation represents 0, if it satisfies the formula

$$\psi_{\mathbb{A}0}(x,y) := \psi_{\mathbb{A}\sim\mathbb{Q}}(x,y,0).$$

A root isolation (p,q) represents a positive or a negative algebraic number, if for some $r \in \mathbb{Q}$, (p,q,r) satisfies, respectively, the formulas $\psi_{\mathbb{A}^+}(x,y,z)$ or $\psi_{\mathbb{A}^-}(x,y,z)$, defined by:

$$\begin{aligned}
\psi_{\mathbb{A}^+}(x,y,z) &:= \psi_{\mathbb{A}\sim}(x,y,x,z) \wedge z[1] > 0,\\
\psi_{\mathbb{A}^-}(x,y,z) &:= \psi_{\mathbb{A}\sim}(x,y,x,z) \wedge z[2] < 0.
\end{aligned}$$

The addition operation on algebraic numbers and the multiplication operation on positive algebraic numbers are defined similarly using the formulas $\psi_{\mathbb{A}+\mathbb{A}}(x_1, y_1, x_2, y_2, x_3, y_3)$ and $\psi_{\mathbb{A}\cdot'\mathbb{A}}(x_1, y_1, x_2, y_2, x_3, y_3)$. For $\# \in \{+, \cdot'\}$, we define:

$$\begin{aligned}\psi_{\mathbb{A}\sharp\mathbb{A}}(x_1, y_1, x_2, y_2, x_3, y_3) \;:=&\\ \bigwedge_{n=1,2,3} \psi_{\mathbb{A}}(x_n, y_n) \wedge \exists z_1 \exists z_2 \exists z_3 \big(\psi_{()\sharp()}(z_1, z_2, z_3) \wedge&\\ \bigwedge_{n=1,2,3} \psi_{\mathbb{A}\sim}(x_n, y_n, x_n, z_n) \wedge \forall z_1' \forall z_2' \forall z_3' \big(\psi_{()\sharp()}(z_1', z_2', z_3') \wedge&\\ \bigwedge_{n=1,2} (\psi_{\mathbb{A}\sim}(x_n, y_n, x_n, z_n') \wedge \psi_{(())}(z_n', z_n)) \rightarrow \psi_{\mathbb{A}\sim}(x_3, y_3, x_3, z_3')\big)\big).&\end{aligned}$$

The negation operation on algebraic numbers defined by the formula

$$\psi_{-\mathbb{A}}(x, y, x', y') := \exists x'' \exists y'' (\psi_{\mathbb{A}0}(x'', y'') \wedge \psi_{\mathbb{A}+\mathbb{A}}(x, y, x', y', x'', y'')).$$

Once we have the formulas $\psi_{\mathbb{A}0}$, $\psi_{\mathbb{A}^+}$, $\psi_{\mathbb{A}^-}$, $\psi_{-\mathbb{A}}$ and $\psi_{\mathbb{A}\cdot'\mathbb{A}}$, it is straightforward to extend multiplication of positive algebraic numbers to multiplication of arbitrary algebraic numbers, using a formula, say, $\psi_{\mathbb{A}\cdot\mathbb{A}}(x_1, y_1, x_2, y_2, x_3, y_3)$.

Putting this all together, we get the following result.

Lemma 20. *The first-order theory of* $(\mathbb{A}, <, +, \cdot, 0, 1, \pi, [\,], \mathbb{N})$ *is definable in* Δ^0_ω.

We now proceed to showing that the first-order theory of the structure $(\mathbb{R}, <, +, \cdot, 0, 1, \pi, [\,], \mathbb{N})$ is definable in the monadic second-order theory of $(\mathbb{Q}, <, +, \cdot, 0, 1, \pi, [\,], \mathbb{N})$, and hence definable in Δ^1_ω. We encode real numbers in a standard way, by using *Dedekind cuts* (see e.g.[Pug10]).

A set of rational numbers represents a real number if and only if it satisfies the formula:

$$\begin{aligned}\psi_{\mathbb{R}}(X) \;:=\; & \exists x \exists y (x \in X \wedge y \notin X \wedge x < y) \wedge\\ & \forall x \forall y (x \in X \wedge y < x \rightarrow y \in X) \wedge\\ & \forall x (x \in X \rightarrow \exists y (x < y \wedge y \in X)).\end{aligned}$$

Two sets of rational numbers represent the same real number if they are the same sets, i.e.:

$$\psi_{\mathbb{R}\sim}(X, Y) \;:=\; X = Y.$$

For defining the arithmetic operations on real numbers in terms of Dedekind cuts, we refer to [Pug10].

A set of rational numbers A represents a rational number if the least upper bound of A is a rational number. We define:

$$\begin{aligned}
\psi_{ub}(x, X) &:= \forall y(y \in X \rightarrow y < x); \\
\psi_{lub}(x, X) &:= \psi_{ub}(x, X) \wedge \forall y(\psi_{ub}(y, X) \rightarrow x \leq y); \\
\psi_{\mathbb{R}\sim\mathbb{Q}}(x, X) &:= \psi_{lub}(x, X); \\
\psi_{\mathbb{R}_{\mathbb{Q}}}(X) &:= \exists x(\psi_{\mathbb{R}\sim\mathbb{Q}}(x, X)).
\end{aligned}$$

Trivially, a set of rational numbers represent a natural number if it represents a rational number that is a natural number.

$$\begin{aligned}
\psi_{\mathbb{R}\sim\mathbb{N}}(x, X) &:= N(x) \wedge \psi_{\mathbb{R}\sim\mathbb{Q}}(x, X); \\
\psi_{\mathbb{R}_{\mathbb{N}}}(X) &:= \exists x(\psi_{\mathbb{R}\sim\mathbb{N}}(x, X)).
\end{aligned}$$

We encode a finite sequence of real numbers $(r_1, \ldots, r_n)$ by a set $A \subseteq \mathbb{Q}$ whose elements encode n-tuples in such a way that, for $i = 1, \ldots, n$, r_i is encoded by the set $A_i := \{q[i] \mid q \in A\}$. To express this, we need a way of referring to the set A_i. We use the formula:

$$\psi_{\pi[i]}(X, x, Y) := \forall y(y \in X \rightarrow y[x] \in Y) \wedge \forall y(y \in Y \rightarrow \exists y'(y' \in X \wedge y'[x] = y))$$

If $A \subseteq \mathbb{Q}$ consists of rational numbers that encode n-tuples, $n \in \mathbb{N}$, if $0 < i \leq n$, and if $\mathbb{Q} \models \psi_{\pi[i]}[A, i, A_i]$, then $A_i = \{q[i] \mid q \in A\}$. A set of rational numbers encodes a sequence of real numbers if it satisfies the formula:

$$\begin{aligned}
\psi_{\mathbb{R}\pi}(X) := \; & \exists x(x \in X) \wedge \exists x\big(N(x) \wedge \forall y(y \in X \rightarrow \pi(y) \wedge y[0] = x) \wedge \\
& \forall Y \forall y(0 < y \wedge y < x + 1 \wedge \psi_{\pi[i]}(X, y, Y) \rightarrow \psi_{\mathbb{R}}(Y))\big).
\end{aligned}$$

Two sets of rational numbers encode the same sequence of real numbers if they satisfy the formula

$$\psi_{\mathbb{R}\pi\sim}(X, Y) := \forall x \forall X' \forall Y'\big(N(x) \wedge \psi_{\pi[i]}(X, x, X') \wedge \psi_{\pi[i]}(Y, x, Y') \rightarrow X' = Y'\big).$$

The binary index relation is defined by the formula:

$$\psi_{X[Y]}(X,Y,Z) \;:=\; \big(\exists x(x>0 \wedge \psi_{\mathbb{R}\sim\mathbb{N}}(x,Y) \wedge \psi_{\pi[i]}(X,x,Z))\big) \vee \big(\psi_{\mathbb{R}\sim\mathbb{N}}(0,Y) \wedge \forall x(X(x) \to \psi_{\mathbb{R}\sim\mathbb{N}}(x[0],Z))\big).$$

As a result we get:

Lemma 21. *The first-order theory of* $(\mathbb{R},<,+,\cdot,0,1,\pi,[\,],\mathbb{N})$ *is definable in* Δ^1_ω.

2.5 Topology

A *topological space* is a tuple $\mathcal{X}=(X,\tau)$, where X is a nonempty set, and τ is a collection of subsets of X that contains X and $\emptyset$, and that is closed under finite intersections and arbitrary unions. A subset of X is *open in* $\mathcal{X}$, if it is a member of τ. A *neighborhood* of a subset A of X is an open set containing A. A subset of X is *closed in* $\mathcal{X}$, if its complement is a member of τ. The collection of closed sets in $\mathcal{X}$ contains X and $\emptyset$, and is closed under finite unions and arbitrary intersections. The *interior* operation $\cdot^\circ$ is defined by $A^\circ = \bigcup\{B \in \tau \mid B \subseteq A\}$, for $A \subseteq X$. The *closure* operation $\cdot^-$ is defined by $A^- = \bigcap\{B \in \tau \mid A \subseteq B\}$, for $A \subseteq X$. Note that $A^- = X \setminus (X \setminus A)^\circ$ and $A^\circ = X \setminus (X \setminus A)^-$. The *boundary* of a set A, denoted by $\delta(A)$, is given by $A^- \setminus A^\circ$. A set A is *regular closed* if $A = A^{\circ -}$. A set A is regular open if $A = A^{-\circ}$. The collections $\mathsf{RC}(\mathcal{X})$ of regular closed sets and $\mathsf{RO}(\mathcal{X})$ of regular open sets, form complete Boolean algebras under set-theoretical inclusion (see e.g. [Kop89, pp.26,28]). The operations, relations and constants are given in Table 2.1.

A *base* for a topological space $\mathcal{X}$ is a collection of open sets ρ such that every open set in $\mathcal{X}$ is a union of members of ρ. For ρ we have that $\bigcup\rho = X$ and that for $A,B \in \rho$ and $p \in A \cap B$, there exists $C \in \rho$ such that $p \in C \subseteq A \cap B$. Moreover, a collection of subsets ρ which satisfies these conditions, determines a unique topology on X for which ρ is a base. It is sometimes easier to define a topological space by specifying a base for it. For example, the *metric topology* of a metric space (X,g) is the topology with base ρ which consists of all sets $B^r_x = \{y \in X \mid g(x,y) < r\}$, for $x \in X$ and $r \in \mathbb{R}^+$. In a Euclidean space $\mathbb{R}^n$, $n>0$, ρ consists of all n-dimensional open balls. We refer to the resulting topology as the *usual topology* for $\mathbb{R}^n$.

		$\mathsf{RC}(\mathcal{X})$	$\mathsf{RO}(\mathcal{X})$
$a+b$	$=$	$a \cup b$	$(a \cup b)^{-\circ}$
$a \cdot b$	$=$	$(a \cap b)^{\circ -}$	$a \cap b$
$-a$	$=$	$(X \setminus a)^{-}$	$(X \setminus a)^{\circ}$
1	$=$	X	X
0	$=$	$\emptyset$	$\emptyset$
$a \leq b$	iff	$a \subseteq b$	$a \subseteq b$
$\sum_{i \in I} a_i$	$=$	$\left(\bigcup_{i \in I} a_i^{\circ}\right)^{-}$	$\left(\bigcup_{i \in I} a_i^{-}\right)^{\circ}$
$\prod_{i \in I} a_i$	$=$	$-\left(\sum_{i \in I} -a_i\right)$	$-\left(\sum_{i \in I} -a_i\right)$

Table 2.1: The complete Boolean algebras $\mathsf{RC}(\mathcal{X})$ and $\mathsf{RO}(\mathcal{X})$ of a topological space $\mathcal{X}$.

Two sets A and B are separated if $A^{-} \cap B = \emptyset$ and $A \cap B^{-} = \emptyset$. Two sets A and B *separate* a set C (in $\mathcal{X}$), if A and B are separated and $C = A \cup B$. A subset of X is *connected* (in $\mathcal{X}$), if it cannot be separated by a pair of nonempty sets. The space $\mathcal{X}$ is connected, if X is connected in $\mathcal{X}$. A space is *locally-connected* if it has a base of connected sets. A set A is a *connected component* of B if A is a maximal connected subset of B. A set is *interior-connected* if its interior is connected. A set A is an *interior-component* of a set B if A is a maximal interior-connected subset of B.

We now show that a (regular) closed set can be separated only by (regular) closed sets.

Lemma 22. *Let A be a set in a topological space $\mathcal{X}$, and let A_1 and A_2 be subsets of X that separate A. Then:*

$i)$ *A is closed if and only if A_1 and A_2 are closed;*

$ii)$ *A is regular closed if and only if A_1 and A_2 are regular closed.*

Proof. The right to left implications are immediate since the union of two (regular) closed sets is a (regular) closed set. Suppose A is closed. From $A_1^{-} \subseteq A^{-} = A = A_1 \cup A_2$ we have that $A_1^{-} \subseteq A_1^{-} \cap (A_1 \cup A_2) = (A_1^{-} \cap A_1) \cup (A_1^{-} \cap A_2) = A_1^{-} \cap A_1 = A_1$. Similarly, $A_2^{-} = A_2$. Suppose A is regular closed. From $i)$ it follows that A_1 and A_2 are closed. We will show that $A^{\circ} = A_1^{\circ} \cup A_2^{\circ}$, because it then follows that $A_1 = A \cap (X \setminus A_2) = A^{\circ -} \cap (X \setminus A_2) = (A_1^{\circ -} \cup A_2^{\circ -}) \cap (X \setminus A_2) = A_1^{\circ -} \cap (X \setminus A_2) = A_1^{\circ -}$. The inclusion $A^{\circ} \supseteq A_1^{\circ} \cup A_2^{\circ}$ is trivial. Let $p \in A^{\circ}$ and without loss of generality let $p \in A_1$. Because $A_1 \cap A_2 = \emptyset$, p must be in $X \setminus A_2$. Hence, p is in the set $A^\circ \cap (X \setminus A_2)$, which is open and contained in A_1. Thus, $p \in A_1^\circ$. $\square$

Consider the following fact about connected subsets of connected sets.

Lemma 23. (Theorem 4 [Kur68, p.133]) *Let A be a connected set in a connected topological space $\mathcal{X}$. Further, let B and C be separated sets such that $A \cup B \cup C = X$. Then, $A \cup B$ and $A \cup C$ are connected.*

Using Lemma 23, one can show the following.

Lemma 24. (Theorem 5 [Kur68, p.140]) *Let $\mathcal{X}$ be a connected topological space, $A \subseteq X$ be a connected set and B be a component of $X \setminus A$. Then $X \setminus B$ is also connected.*

We now discuss different properties of topological spaces. An *open cover* of $\mathcal{X}$ is a collection of open sets covering X. A space $\mathcal{X}$ is *compact* if every open cover has a finite subcover. $\mathcal{X}$ is T_1 if each of every two points in X has a neighbourhood not containing the other. $\mathcal{X}$ is T_2 (also *Hausdorff*), if every two points in X are contained in disjoint open sets. $\mathcal{X}$ is T_3 if every point and every closed set not containing that point are contained in disjoint open sets. A space is T_4 if every two disjoint closed sets are contained in disjoint open sets. $\mathcal{X}$ is *regular* if it is T_3 and T_1. $\mathcal{X}$ is *normal* if it is T_4 and T_1. $\mathcal{X}$ is *semi-regular* if it has a base of regular open sets. $\mathcal{X}$ is *weakly regular* [DW05] if it is semi-regular and if every non-empty open set contains a non-empty regular closed set.

A space $\mathcal{X}$ is *unicoherent* [Kur68, p.162] if it is connected and if every two connected closed sets that cover X have a connected intersection. The following fact follows easily from this definition.

Lemma 25. *Let $\mathcal{X}$ be a unicoherent topological space, $A \subseteq X$ be a connected set and B be a component of $X \setminus A$. Then $\delta(B)$ is also connected.*

Proof. By Lemma 24 it follows that $C := X \setminus B$ is connected. Hence, B^- and C^- are connected closed sets that cover X. Since $\mathcal{X}$ is unicoherent, we get that $\delta(B) = B^- \cap C^-$ must also be connected. $\square$

A much more elaborate argument is required to show that Euclidean spaces are unicoherent.

Lemma 26. *For $n \geq 0$, $\mathbb{R}^n$ with the usual topology is a unicoherent space.*

Proof. A direct consequence of Theorem 9 [Kur68, p.435] and Theorem 2 [Kur68, p.437]. □

A *Jordan arc* in a topological space $\mathcal{X}$ is an injective continuous function from the unit interval $[0, 1]$ to $\mathcal{X}$. A *Jordan curve* in $\mathcal{X}$ is an injective continuous function from the unit circle (the points in the Euclidean plane satisfying the equation $x^2 + y^2 = 1$) to $\mathcal{X}$. We usually identify Jordan arcs and Jordan curves with their images. An *end-cut* in a subset A of a topological space is a Jordan arc contained in A° except for one of its endpoints which is contained in $\delta(A)$. A *cross-cut* in a subset A of a topological space is a Jordan arc contained in A° except for its endpoints, which are contained in $\delta(A)$. We will implicitly use the following facts about subsets of $\mathbb{R}^2$. A subset A of a topological space $\mathcal{X}$ is said to have the *curve-selection* property if for every point p on its boundary, there exists an end-cut α in A such that $\alpha \cap \delta(A) = \{p\}$.

Lemma 27. *[New64, p.112, Theorem 9.2] Let F, G be disjoint, closed subsets of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus F$ and $\mathbb{R}^2 \setminus G$ are connected. Then $\mathbb{R}^2 \setminus (F \cup G)$ is connected.*

Theorem 28 (Jordan Curve Theorem)**.** *[New64, p.115, Theorem 10.2] The complement of a Jordan curve Γ in $\mathbb{R}^2$ has two connected components having Γ as their boundary.*

Lemma 29. *[New64, p.115, Theorem 11.6] If α is an end-cut in an open connected set $A \subseteq \mathbb{R}^2$, then $A \setminus \alpha$ is connected.*

Lemma 30. *[New64, p.115, Theorem 11.7] If α is a cross-cut in an open connected set $A \subseteq \mathbb{R}^2$ such that the endpoints of α lie on the same component of $\delta(A)$, then $A \setminus \alpha$ has two connected components.*

Lemma 31. *[New64, p.120] Let $A \subseteq \mathbb{R}^2$ be an open connected set, and let α be a cross-cut in A such that the endpoints of α lie on different components of $\delta(A)$. Then $A \setminus \alpha$ is connected.*

2.6 Boolean Algebras of Regions

Fix a topological space $\mathcal{X} = (X, \tau)$. A subset $\mathcal{M}$ of $\wp(X)$ is called a (*dense*) *region algebra* over $\mathcal{X}$, if $\mathcal{M}$ is a (dense) Boolean subalgebra of $\mathsf{RC}(\mathcal{X})$. A *complete* *region algebra* is a region algebra which is a complete Boolean algebra. A subset $\mathcal{M}$ of $\wp(X)$ is called a (*dense*) *open region algebra* over $\mathcal{X}$, if $\mathcal{M}$ is a (dense) Boolean subalgebra of $\mathsf{RO}(\mathcal{X})$. A *complete open region algebra* is an open region algebra which is a complete Boolean algebra. A subset $\mathcal{M}$ of $\wp(X)$ is called a (*dense*) *set algebra* over $\mathcal{X}$, if $\mathcal{M}$ is a (dense) Boolean subalgebra of $\wp(X)$. A *complete set algebra* is a set algebra which is a complete Boolean algebra. Let $\mathcal{M}$ be a region algebra, an open region algebra or a set algebra. We refer to the elements of $\mathcal{M}$ as *regions*. If, for every region A in $\mathcal{M}$ and every connected component B of A, B is in $\mathcal{M}$, then, we say that $\mathcal{M}$ *respects components*. If every region A in $\mathcal{M}$ has only finitely many connected components, then $\mathcal{M}$ is *finitely decomposable*.

We now show an analogue of Lemma 22 for region algebras and set algebras.

Lemma 32. *Let $\mathcal{X}$ be a topological space, and let $\mathcal{M}$ be either a region algebra or a set algebra which respects components and which is finitely decomposable. Also, let A, B and C be subsets of X such that B and C separate A. Then A is in $\mathcal{M}$ if and only if B and C are in $\mathcal{M}$.*

Proof. The right to left implication follows from $\mathcal{M}$ being a Boolean algebra. Now, suppose that A is in $\mathcal{M}$, and since $\mathcal{M}$ is finitely decomposable and respects components, let $A_1, \ldots, A_n \in \mathcal{M}$, $n \in \mathbb{N}$, be the connected components of A. By Lemma 22, B and C are disjoint sets. Since, A_i are connected sets that are contained in $B \cup C$, each of them must be contained in B or C. Because each of B and C is covered by the A_i, it has to be equal to the union of some of the A_i, and hence in $\mathcal{M}$. $\square$

We now show some properties for region algebras over unicoherent topological spaces. First consider the following variant of Lemma 24.

Lemma 33. *Let $\mathcal{X}$ be a connected topological space, let a be a connected region in $\mathsf{RC}(\mathcal{X})$, and let b be a component of $-a$. Then $-b$ is connected.*

Proof. (The proof is almost identical to the one of Lemma 24.) By Lemma 22, $-b$ can only be separated by regions in $\mathsf{RC}(\mathcal{X})$. Let b_1 and b_2 be two such regions ($-b = b_1 + b_2$ and $b_1 \cap b_2 = \emptyset$). By Lemma 23, both $-b + b_1$ and $-b + b_2$ are connected. Since a is connected, $a \leq b_1 + b_2$ and b_1 and b_2 are separated, it can be assumed that a and b_1 are disjoint, and thus $b \leq b + b_1 \leq -a$. Since b is a component of $-a$ we get that $b_1 = 0$. $\square$

We can now show the following variant of Lemma 25.

Lemma 34. *Let a be a connected region in a unicoherent topological space $\mathcal{X}$, and let b be a component of $-a$. Then the boundary of b is connected.*

Proof. By Lemma 33, $-b$ is connected, and since $\mathcal{X}$ is unicoherent, $\delta(b) = b \cap (-b)$ must also be connected. □

Fix $n > 0$. A region algebra over $\mathbb{R}^n$ is called a *Euclidean region algebra*. We define different Euclidean region algebras that are finitely decomposable and respect components. The rest of the section is similar to [PH07, Section 2.3], where different properties for Euclidean open region algebras were shown. We start with the region algebra of regular closed *semi-algebraic sets*.

A subset of $\mathbb{R}^n$ is *semi-algebraic* if it can be presented as:

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n \mid f_{i,j}(x) *_{i,j} 0\},$$

where $f_{i,j} \in \mathbb{R}[X_1, \ldots, X_n]$ and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$. We denote by $\mathsf{RCS}(\mathbb{R}^n)$ the collection of regular closed semi-algebraic subsets of $\mathbb{R}^n$.

Lemma 35. $\mathsf{RCS}(\mathbb{R}^n)$ *is a dense region algebra over* $\mathbb{R}^n$.

Proof. We have to show that $\mathsf{RCS}(\mathbb{R}^n)$ is a dense Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$. It is well-know that semi-algebraic sets are closed under the operations union, intersection, complement, closure and interior (for the operations closure and interior see Proposition 2.2.2 [BCR98, p.27]). So if $A, B \in \mathsf{RC}(\mathbb{R}^n)$, so are $A + B = A \cup B$, $A \cdot B = (A \cup B)^{\circ -}$, and $-A = (\mathbb{R}^n \setminus A)^-$. Clearly, $\emptyset$ and $\mathbb{R}^n$ are semi-algebraic. The fact that $\mathsf{RCS}(\mathbb{R}^n)$ is dense in $\mathsf{RC}(\mathbb{R}^n)$, follows from the fact that every closed n-ball is semi-algebraic. □

We recall a fundamental property of semi-algebraic sets.

Lemma 36. *[BCR98, Theorem 2.4.5] Every semi-algebraic set has a finite number of connected components which are semi-algebraic.*

We can now show the following.

Lemma 37. $\mathsf{RCS}(\mathbb{R}^n)$ *is finitely decomposable and respects components.*

Proof. From Lemma 36 it follows that $\mathsf{RCS}(\mathbb{R}^n)$ is finitely decomposable. To show that $\mathsf{RCS}(\mathbb{R}^n)$ respects components, let $A \in \mathsf{RCS}(\mathbb{R}^n)$ be a disconnected set, and let B be a connected component of A. A simple induction on the number of components of A (which are finitely many) shows that B and $(A \setminus B)^-$ are disjoint. Hence, B and $A \setminus B$ separate A. Now, by Lemma 22, B and $A \setminus B$ are regular closed, but, by Lemma 36, B is also semi-algebraic. Hence $B \in \mathsf{RCS}(\mathbb{R}^n)$. □

We now turn to region algebras of regular-closed *semi-linear sets*. A *half-space* in $\mathbb{R}^n$ is a set defined by an inequality:

$$a_0 + a_1 \cdot x_1 + \cdots + a_n \cdot x_n \leq 0,$$

with coefficients $a_i \in \mathbb{R}$, $0 \leq i \leq n$. An *algebraic half-space* is a half-space that is defined by an inequality with coefficients in $\mathbb{A}$. An *rational half-space* is a half-space that is defined by an inequality with coefficients in $\mathbb{Q}$. A *basic polytope* in $\mathbb{R}^n$ is the product, in $\mathsf{RC}(\mathbb{R}^n)$, of finitely many half-spaces. A *polytope* in $\mathbb{R}^n$ is the sum, in $\mathsf{RC}(\mathbb{R}^n)$, of finitely many basic polytopes. Similarly, we define *basic algebraic polytopes* and *algebraic polytopes*, and *basic rational polytopes* and *rational polytopes*. Denote by $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ the collections of polytopes, algebraic polytopes and rational polytopes, respectively.

Lemma 38. $\mathsf{RCP}(\mathbb{R}^n)$*,* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ *are dense region algebras.*

Proof. We have to show that $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ are dense Boolean subalgebras of $\mathsf{RC}(\mathbb{R}^n)$. That $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ are Boolean subalgebras of $\mathsf{RC}(\mathbb{R}^n)$ is immediate from the definitions. That these Boolean algebras are dense in $\mathsf{RC}(\mathbb{R}^n)$ follows from the fact that all rational n-dimensional hypercubes are basic rational polytopes. □

We also have the following.

Lemma 39. $\mathsf{RCP}(\mathbb{R}^n)$*,* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ *are finitely decomposable and respect components.*

Proof. Note that the nonempty basic polytopes are convex and hence connected. So the elements of $\mathsf{RCP}(\mathbb{R}^n)$, and in particular those of $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, have finitely many components as unions of finitely many connected sets. Further, a simple induction on k shows that, for every k-tuple $a_1, \ldots, a_k$ of basic polytopes in $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ or $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, the connected components of $a_1 + \cdots + a_k$ are also in $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, respectively. □

The following lemma captures the fact that the properties and relations on Euclidean regions are determined by the rational points that are contained in those regions. This fact will be extensively used in Section 4.3 for proving upper complexity bounds on various theories of Euclidean region algebras.

Lemma 40. *For* $a, b, c \in \mathsf{RC}(\mathbb{R}^n)$*,*

$$\begin{array}{lcl} a \leq b & \Longleftrightarrow & \text{for all } p \in \mathbb{Q}^n\text{, if } p \in a\text{, then } p \in b; \\ \mathsf{closer}(a, b, c) & \Longleftrightarrow & \text{for all } p, q \in \mathbb{Q}^n\text{, if } p \in a^\circ \text{ and } q \in c^\circ\text{, then there exist} \\ & & p' \in a^\circ \cap \mathbb{Q}^n \text{ and } q' \in b^\circ \cap \mathbb{Q}^n \text{ such that } d(p', q') \leq d(p, q), \end{array}$$

where for $p, q \in \mathbb{R}^n$*,* $d(p, q)$ *is the distance between the points* p *and* q*.*

Chapter 3

Related Work

This chapter contains a brief summary of the developments of *spatial logics* with emphasis on the results which are closely related to the findings of this thesis. We begin with a definition and some examples of spatial logics; we then discuss axiomatisations of some spatial theories; and we end with a discussion on the computability of first-order and quantifier-free spatial logics.

3.1 Spatial Logics

In this thesis we study formal languages for reasoning about regions in space. In particular, we consider logical languages interpreted over collections of region algebras or set algebras (possibly containing a single region algebra or set algebra), and we call the resulting logics *spatial logics*. If the underlying space of the set or region algebras in question is a Euclidean space, then we call these logics *Euclidean spatial logics*. The restriction that we interpret logical languages over Boolean algebra of subsets of some topological space is common, and indeed, only a few of the results that we consider in the thesis concern structures which are not set algebras or region algebras. A simple example of a spatial logic is the first-order language of Boolean algebras interpreted over the class of all set algebras. Of course, since the topological information in set algebras is inaccessible in this language, the resulting logic is just the first-order logic of Boolean algebras, or in *mereological* parlance, the first-order logic of the *part-of* relation.

The roots of spatial logics that acknowledge the topological nature of the interpreting structures can be traced back to the philosophers Whitehead and de Laguna. Whitehead introduced in [Whi29] the binary relation 'x is connected to y', now widely referred to as the *contact relation*, while de Laguna considered the ternary relation 'x connects y to z'. Both authors provided *geometric postulates* (or axioms) for their languages justified on an intuitive level. Strictly speaking, neither of the two systems fit the above definition of a spatial logic, because no formal semantics was provided for either of the two languages. However, the two systems, especially the one proposed by Whitehead, have had a significant influence on the development of spatial logics. The *contact* relation, after its introduction in [Whi29], became an integral part of many formalisms for spatial reasoning, and its model-theoretic and computational properties in different contexts are now well studied (see e.g. [Roe97, DW05, DV06, Nen09, Sch99, Grz51, WZ00, KPHWZ10, NPH10]).

One of the first formal systems that falls into the above definition of a spatial logic is given by McKinsey and Tarski in [MT44]. The authors considered the first-order language of an extension of the signature of Boolean algebras with a unary functional symbol interpreted over set algebras as the 'closure' operator. Again Tarski considered in [Tar56] a second-order language with two relational symbol interpreted over the region algebra $\mathsf{RC}(\mathbb{R}^3)$ as the mereological relation *part-of* and the property of being spherical. (More famously, of course, Tarski provided in [Tar59] a complete first-order axiomatisation of Euclidean geometry, establishing in addition its decidability.)

In the last two decades interest in spatial logics, particularly from the AI community, has intensified. There is a vast diversity of spatial logics that has been considered in the literature, varying in their logical syntax (e.g. first-order, second-order, quantifier-free and modal logics), non-logical primitives (e.g. topological, affine and metric) and interpretations. In the following two sections we consider the model-theoretic and computational properties of those spatial logics most closely related to the main results in this thesis.

3.2 Axiomatisations

In this section we discuss the axiomatisations of the theories of different spatial logics. The discussion is divided into two parts covering, respectively, the theories of large classes of region algebras and set algebras, and the theories of particular Euclidean region algebras.

3.2.1 General Spatial Logics

One way of establishing complete axiomatisations of spatial logics is by means of representation theorems. The first such result is Stone's representation theorem for Boolean algebras [Sto36], which states that every Boolean algebra $\mathcal{B}$ is isomorphic to a *field of sets* over the set of *ultrafilters* of $\mathcal{B}$. From the point of view of spatial logics, this theorem provides a complete axiomatisation for the $\mathcal{L}_{BA}$-theory of the class of all set algebras, where $\mathcal{L}_{BA}$ is the first-order language of Boolean algebras. This axiomatisation plays a fundamental role for the axiomatisations of other notable examples of spatial logics.

One such example is the first-order logic of the class of *closure algebras* established by McKinsey and Tarski in [MT44]. The language of the logic $\mathcal{L}_{cl}$ extends the first-order language of Boolean algebras with a unary functional symbol for the *closure* operation, and is interpreted over the class of all set algebras. For the axiomatisation of this logic, the authors employed a slight modification of Stone's representation theorem. The axiomatisations of other spatial logics, however, require significantly more involved extensions of Stone's representation theorem. In [Roe97, DW05, DV06] the authors axiomatised the $\mathcal{L}_{\mathsf{C}}$-theories of region algebras over different collections of topological spaces, where $\mathcal{L}_{\mathsf{C}}$ is the first-order language of Boolean algebra extended with the Whitehead's relational symbol interpreted as the *contact* relation. To show completeness of their axiomatisations, the authors used variants of Stone's representation theorem, which, in addition to the ultrafilters used in the original proof, required additional types of abstract points. We give a more detailed account of these results in Section 4.2.1.

Despite the intense interest in various first-order theories of region algebras and set algebras, there are no results in the literature regarding first-order theories of *complete region algebras* and *complete set algebras*. We address this deficit in Section 4.2.1, where we show that the $\mathcal{L}_{\mathsf{C}}$-theory of *complete* region algebras is different from the $\mathcal{L}_{\mathsf{C}}$-theory of all region algebras, and similarly for the $\mathcal{L}_{cl}$-theory of *complete* set algebras. We also show that the class of complete region algebras and the class of complete set algebras are not first-order definable, i.e. one cannot prove a representation theorem for either of the two classes. Some of these results appeared in [Nen09].

So far we have discussed only first-order spatial logics. There are axiomatisations for some propositional spatial logics as well. In [BTV07], Balbiani et al.

provided complete axiomatisations for the propositional fragment $\mathcal{C}$ of $\mathcal{L}_{\mathsf{C}}$ with respect to the classes of region algebras considered in [Roe97, DW05, DV06]. (Note that the language $\mathcal{C}$ was first introduced in [KPHWZ08a] as an alternative to the equally expressive language $\mathcal{BRCC}8$ introduced in [WZ00]; see Section 5.1.) The axiomatisations of these propositional spatial logics comprise all universal axioms in the respective first-order logics and finitary rules replacing all other axioms. It turns out, however, that the finitary rules are all *admissible*, i.e. every formula that can be proved with the use of these rules, can also be proved without them. This implies that the considered classes of region algebras, despite having different first-order theories, have the same propositional theories. In [TV10], Tinchev and Vakarelov axiomatised the propositional logics for the extension $\mathcal{C}$cc of $\mathcal{C}$ with predicates for counting connected components, a language first introduced in [PH02]. Although the predicates for component counting are $\mathcal{L}_{\mathsf{C}}$-definable, they are not $\mathcal{C}$-definable, because some of the "defining" $\mathcal{L}_{\mathsf{C}}$-formulas involve existential quantifiers. To acquire complete axiomatisations of the considered $\mathcal{C}$cc-logics, Tinchev and Vakarelov added to the axiomatisations of $\mathcal{C}$ from [BTV07] the "defining" $\mathcal{L}_{\mathsf{C}}$-formulas which are universal, and finitary rules of inference for the "defining" $\mathcal{L}_{\mathsf{C}}$-formulas involving existential quantifiers. The authors showed that the finitary rules from the $\mathcal{C}$-axiomatisations are admissible even in the more expressive language $\mathcal{C}$cc, and hence showing that the considered classes of region algebras have the same $\mathcal{C}$cc-theories.

3.2.2 Euclidean Spatial Logics

We now discuss the axiomatisations of first-order theories of particular Euclidean region algebras. An early axiomatisation of such a theory was established by Tarski in [Tar56]. He considered the $\mathcal{L}_{sph}$-theory of $\mathsf{RC}(\mathbb{R}^3)$, where $\mathcal{L}_{sph}$ extends the second-order language of Boolean algebras with a relational symbol for the property of being a closed ball. Tarski showed that the proposed axiom system, which he calls the Geometry of Solids, is categorical. The result was latter refined in [GP08], and another axiomatisation for the same theory was provided in [Ben01]. We now turn our attention to axiomatisations of spatial logics of region algebras over the Euclidean plane.

2D Regions with Connectedness

Axiom systems for first-order topological languages interpreted over planar region algebras were presented in [PS98, Sch99, PH07]. In [PS98], Pratt and Schoop established a complete axiomatisation, which we denote by Φ, of the $\mathcal{L}_{\mathsf{c}^\circ}$-theory of the region algebra $\mathsf{RCP}(\mathbb{R}^2)$, where $\mathcal{L}_{\mathsf{c}^\circ}$ is the first-order relational language with non-logical symbols $\leq$ and c° interpreted, respectively, as the part-of relation and the property of having connected interior. In [Sch99], this axiomatisation was adapted for the first-order language $\mathcal{L}_{\mathsf{C}}$, which extends the language of Boolean algebras with a single non-logical symbol C interpreted as the contact relation. In [PH07], it was shown that the complete axiomatisation Φ is sound with every planar region algebra satisfying certain properties, which are discussed below. In particular it follows that all such planar region algebras, including $\mathsf{RCS}(\mathbb{R}^2)$, $\mathsf{RCP}(\mathbb{R}^2)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ (see Section 2.6), have the same $\mathcal{L}_{\mathsf{c}^\circ}$-theories.

The main result in [PH07] is that Φ is the complete axiomatisation of the $\mathcal{L}_{\mathsf{c}^\circ}$-theory of every *finitely-decomposable*, *splittable* region algebra over $\mathbb{R}^2$ having *curve-selection*. A region algebra $\mathcal{M}$ has curve-selection, if every region in $\mathcal{M}$ has the curve-selection property (see Section 2.5). A region algebra $\mathcal{M}$ is *splittable* if $\mathcal{M}$ satisfies an $\mathcal{L}_{\mathsf{c}^\circ}$-formula ψ_{split}, which essentially ensures that $\mathcal{M}$ contains regions of various "shapes". In addition to the axioms for Boolean algebras and some basic axioms for c°, Φ also includes the axiom ψ_{split} and two axioms ensuring that the non-planar graph $K_{3,3}$ and K_5 are not embeddable in the topological space underlining the region algebra. Note that unlike the properties of being splittable and planar, the properties of having curve-selection and being finitely-decomposable are not $\mathcal{L}_{\mathsf{c}^\circ}$-definable. Of course, since Φ is a complete axiom system, it does reflect these two properties. Firstly, Φ contains two axioms which are entailed by the curve-selection property, but which turn out to be sufficient to capture that property. The first axiom ensures that if the sum of n regions has a connected interior, then the first of these regions forms an interior-connected sum with at least one of the other regions. The second axiom ensures that there are sufficiently many regions, by insisting that if a region r forms an interior-connected sums with regions s and t, then r can be partitioned into two regions each forming interior-connected sums with s and t. Finally, to capture the property of being finitely-decomposable, Φ contains an infinite rule of inference insisting that if a formula is satisfiable by all regions with finitely many components, then it is satisfiable by all regions. The axiom system Φ is clearly sound with respect to each finitely-decomposable, splittable region algebras over $\mathbb{R}^2$ having curve-selection. The difficult part, of course, is to show that it is complete with at least one such region algebra, and in this case $\mathsf{RCP}(\mathbb{R}^2)$. Using the infinite rule, the authors apply the omitting types theorem (see e.g. [Mar02, Theorem 2.4.3]) to get a countable and finitely-decomposable model $\mathcal{M}$. Using the planarity and curve-selection axioms, it is then showed that $\mathcal{M}$ is isomorphically embeddable in $\mathsf{RCP}(\mathbb{R}^2)$. Finally, using that $\mathcal{M}$ is splittable, i.e. contains regions of various shapes, it is shown by application of Tarski-Vaught test (see e.g. [Mar02, Proposition 2.3.5]) that $\mathcal{M}$ is in fact an elementary substructure of $\mathsf{RCP}(\mathbb{R}^2)$.

As we have just discussed, [PH07] showed that the region algebras $\mathsf{RCS}(\mathbb{R}^2)$, $\mathsf{RCP}(\mathbb{R}^2)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ have the same $\mathcal{L}_{\mathsf{c}^\circ}$-theories. In fact, this follows from a more general result established also in [PH07]. If σ is a topological signature, then $\mathsf{RCS}(\mathbb{R}^2)$, $\mathsf{RCP}(\mathbb{R}^2)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ have the same first-order σ-theories. The result uses Tarski-Vaught test and is based on the notions of *homogeneous region algebras* and *homogeneous region subalgebras*. Two tuples of regions $\bar{r}$ and $\bar{s}$ in a region algebra over a topological space $\mathcal{X}$ are *similarly situated*, denoted by $\bar{r} \sim \bar{s}$, if there exists a homeomorphism from $\mathcal{X}$ onto itself sending $\bar{r}$ to $\bar{s}$. A region algebra $\mathcal{M}$ over a topological space $\mathcal{X}$ is called *homogeneous*, if for all similarly situated tuples of regions $\bar{r}$ and $\bar{s}$ and every region r (all in $\mathcal{M}$), there exists a region s (also in $\mathcal{M}$) such that $\bar{r}, r \sim \bar{s}, s$. A region subalgebra $\mathcal{N}$ of $\mathcal{M}$ is called a *homogeneous region subalgebra*, if for every tuple $\bar{r}$ of regions in $\mathcal{N}$, and every region $r \in \mathcal{M}$, there exists a region $s \in \mathcal{N}$ such that $\bar{r}, r \sim \bar{r}, s$. The notion of homogeneous region subalgebra almost directly corresponds to the conditions of Tarski-Vaught test, and by showing that each of $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2)$, $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RCS}(\mathbb{R}^2)$ is a homogeneous region subalgebra of the next, one gets that $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2) \prec \mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2) \prec \mathsf{RCP}(\mathbb{R}^2) \prec \mathsf{RCS}(\mathbb{R}^2)$ for any topological signature σ.

2D Regions with Convexity

A complete axiom system for the $\mathcal{L}_{conv}$-theory of the region algebra $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ was presented by Trybus in [Try10], where $\mathcal{L}_{conv}$ is the first-order language of the signature $(\mathsf{conv}, \leq)$. The result was given for the structure $\mathsf{ROP}_{\mathbb{Q}}(\mathbb{R}^2)$, but the $\mathcal{L}_{conv}$-structures $\mathsf{ROP}_{\mathbb{Q}}(\mathbb{R}^2)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ are isomorphic due to the fact that the closures and the interiors of convex sets are convex. The axiomatisation is based on properties of $\mathcal{L}_{conv}$ which were previously discovered in [DGC99, Pra99, Dav06]. The first observation is that one can fix in $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ a coordinate system by three (*rational*) *half-planes* (p, q, r) (regions satisfying $\psi_{/}(x) := \mathsf{conv}(x) \wedge \mathsf{conv}(-x)$) whose boundaries are straight lines intersecting each other in three distinct points. The next step is to introduce a (countably) infinite sequence of formulas, called *fixing formulas*, such that each fixing formula is satisfiable by exactly one half-plane, and each half-plane satisfies exactly one fixing formula. This is also reflected in the axiom system, which contains axioms insisting that every fixing formula is satisfiable by exactly one half-plane. The other direction is captured by an infinite rule of inference insisting that if a formula is satisfiable by every half-plane which satisfies a fixing formula, then the formula is satisfiable by every half-plane. The axiom system features another infinite rule of inference insisting that if a formula is satisfiable by all regions that are Boolean combinations of half-planes, then the formula is satisfiable by all regions. These two inference rules play a crucial role in proving that the system is complete. In particular, Trybus used a non-standard version of the omitting types theorem to construct a countable model $\mathcal{M}$ in which, by the two non-standard inference rules, every element in $\mathcal{M}$ satisfying $\psi_{/}(x)$ also satisfies a fixing formula, and every element in $\mathcal{M}$ can be expressed as a Boolean combination of elements satisfying $\psi_{/}(x)$. These two facts about $\mathcal{M}$ allow the author to show that $\mathcal{M}$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ are in fact isomorphic, which implies the completeness of the system.

A natural question that is raised by Trybus in [Try10] is whether one can apply similar techniques to establish axiomatisations of the $\mathcal{L}_{conv}$-theories of other Euclidean region algebras, and in particular the region algebras $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$. Before tackling this question however, one needs to show that these Euclidean region algebras have different $\mathcal{L}_{conv}$-theories. This leads us to two of the contributions of this thesis. In Section 4.2.1 we show that the first-order theories of $(\mathsf{RC}(\mathbb{R}^n), \mathsf{C})$ and $(\mathcal{M}, \mathsf{C})$ are different, where $\mathcal{M}$ is any of the region algebras $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, and C is the contact relation. Now, since C is $\mathcal{L}_{conv}$-definable in each of the above region algebras (see Section 4.1), it follows that the $\mathcal{L}_{conv}$-theories of $\mathsf{RC}(\mathbb{R}^n)$ and $\mathcal{M}$ are also different. In [Pra99] it was shown that the $\mathcal{L}_{conv}$-theory of $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ is different from the $\mathcal{L}_{conv}$-theories of RCS($\mathbb{R}^n$), RCP($\mathbb{R}^n$) and RCP$_{\mathbb{A}}$($\mathbb{R}^n$). As a corollary of the complexity bounds that we establish in Section 4.3.4, we get that the $\mathcal{L}_{conv}$-theory of RCP$_{\mathbb{A}}$($\mathbb{R}^n$) is different from the $\mathcal{L}_{conv}$-theories of RCP($\mathbb{R}^n$) and RCS($\mathbb{R}^n$). So, all of the above region algebras have different $\mathcal{L}_{conv}$-theories, except for RCS($\mathbb{R}^n$) and RCP($\mathbb{R}^n$), for which we do not know.

Recall now that the axiomatisation of the $\mathcal{L}_{conv}$-theory of RCP$_{\mathbb{Q}}$($\mathbb{R}^2$) given in [Try10] depends on the fact that each half-plane and each region in RCP$_{\mathbb{Q}}$($\mathbb{R}^2$) can be fixed by a unique *fixing formula*. Since $\mathcal{L}_{conv}$ is countable, this technique can only be applied for countable structures. That is, any axiomatisations of the $\mathcal{L}_{conv}$-theories of the region algebras RC($\mathbb{R}^n$), RCS($\mathbb{R}^n$) and RCP($\mathbb{R}^n$) will use "essentially" different techniques. So the only $\mathcal{L}_{conv}$-theories that can be addressed in a similar way are the theories of RCP$_{\mathbb{Q}}$($\mathbb{R}^n$) and RCP$_{\mathbb{A}}$($\mathbb{R}^n$), $n \geq 2$.

3.3 Computability of Spatial Logics

In this section, we discuss work related to computability of spatial logics. The presentation is naturally divided into two parts, discussing the computability of first-order and quantifier-free logics, respectively.

3.3.1 First-Order Logics

Probably the first results about the computational properties of first-order (topological) spatial logics were established by Grzegorczyk in [Grz51]. He used reductions from first-order arithmetic to show the undecidability of the first-order logics of region algebras and set algebras over a large class of topological spaces, including Euclidean spaces of dimension greater than one. The techniques used in [Grz51] underlie many of the undecidability results regarding spatial logics established since then. First, Grzegorczyk considered the first-order language $\mathcal{L}_{cl}$, which extends the language of Boolean algebras with a unary functional symbol for the operation *closure*. He showed the Δ^0_ω-hardness of the $\mathcal{L}_{cl}$-theory of every non-trivial, atomic set algebra $\mathcal{M}$ over a connected, normal and second-countable topological space such that every two disjoint discrete regions in $\mathcal{M}$ (a region in $\mathcal{M}$ is discrete if it is a discrete set) are contained in two disjoint connected open regions in $\mathcal{M}$. Note that the conditions imposed on the topological space are satisfied by all Euclidean spaces except $\mathbb{R}$, which clearly fails the last condition. Indeed, no two disjoint open intervals contain the even and odd natural numbers respectively. To show the Δ^0_ω-hardness of each set algebra satisfying the above conditions, Grzegorczyk encoded each natural number n by the class of all discrete regions in $\mathcal{M}$ consisting of exactly n points. Whether two discrete regions a and b have the same number of points is $\mathcal{L}_{cl}$-definable. Indeed, this is determined by the existence of an open region d whose components are "pairing" the points in $a \cdot (-d)$ and $b \cdot (-d)$ in the sense that every component of d contains exactly one point in each of $a \cdot (-d)$ and $b \cdot (-d)$, and every point in $a \cdot (-d)$ and $b \cdot (-d)$ is contained in some component of d. Once the congruence relation is defined, it is then relatively easy to define in $\mathcal{L}_{cl}$ the arithmetic operations and relations. Extending this idea, Grzegorczyk showed the Δ^0_ω-hardness of the $\mathcal{L}_{\mathsf{C}}$-theory of every region algebra that satisfies certain axioms, and whose underlying topological space satisfy the above mentioned properties. Since the regions in region algebras are generally not discrete, in order to apply the techniques used in the case of set algebras, Grzegorczyk first encodes each discrete set by the pairs of regions the intersection of whose boundaries is exactly that set. The additional axioms mentioned above ensure that the region algebra contains enough regions to encode every discrete set in the underlying topological space. Once this is done, the encodings of the natural numbers and the arithmetic operations and relations are almost as in the case of set algebras.

The lower complexity bounds established in [Grz51] apply to the $\mathcal{L}_{\mathsf{C}}$-theories of the Euclidean region algebras $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, $n > 1$. However, they do not apply to the $\mathcal{L}_{\mathsf{C}}$-theories of $\mathsf{RC}(\mathbb{R})$, $\mathsf{RCS}(\mathbb{R})$, $\mathsf{RCP}(\mathbb{R})$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$, and to the best of our knowledge no complexity bounds for these logics have been established in the literature. We show in Section 4.3.1 and Section 4.3.2 that reasoning in these logics is decidable but non-elementary. Note also that the lower bounds established in [Grz51] are not shown to be tight, and, as we show in Section 4.3.3 and Section 4.3.4, some of these logics turn out to be as expressive as second-order arithmetic. These results appeared in [NPH10].

Grzegorczyk's idea to encode natural numbers as pairs of spatial regions intersecting in finitely many points was employed by Schaefer and Štefankovič in [Sv04]. The authors showed that the $\mathcal{L}_{\mathrm{DIAG}}$-theory of the collection DIAG of disc-homeomorphs in $\mathbb{R}^2$ (all planar sets that are homeomorphic to the closed unit disc $\{(x, y) \mid x^2 + y^2 \leq 1\}$) is as expressive as second-order arithmetic, and hence Δ^1_ω-complete. The language $\mathcal{L}_{\text{DIAG}}$ is equivalent to the first-order language of the binary relational symbol C interpreted as the contact relation. To show that the $\mathcal{L}_{\text{DIAG}}$-theory of DIAG is Δ^1_ω-hard, Schaefer and Štefankovič first interpreted the $\mathcal{L}_{\text{STRINGS}}$-theory of the set STRINGS of the Jordan arcs in $\mathbb{R}^2$, called *strings*, where $\mathcal{L}_{\text{STRINGS}}$ is the first-order language whose only binary relational symbol intersect is interpreted in the obvious way. To show that the $\mathcal{L}_{\text{STRINGS}}$-theory of STRINGS is Δ^1_ω-hard, Schaefer and Štefankovič defined the pairs (u, v) of strings that intersect each other in $\omega+1$ points, called intersection points. Each natural number n is encoded by the strings w that intersect u in the first n intersection points on u not used by the encodings of the numbers $0, \ldots, n-1$. A set of natural numbers A is then encoded by the strings that intersect u in the intersection points of u with the encodings of the members of A. The membership relation and the arithmetic relations and operations are then encoded relatively easily.

The result of Schaefer and Štefankovič relies on the fact that there is a pair of regions in DIAG whose boundaries intersect infinitely many times. Davis in [Dav06] argues that such "pathological" regions are not suitable for reasoning about regions occupied by physical objects. In the same paper, he considers languages featuring not only topological but also affine and metric primitives. He shows that the theories of these languages when interpreted over a wide range of Euclidean region algebras are either Δ^0_ω- or Δ^1_ω-hard. The two first-order languages in consideration are $\mathcal{L}_{conv}$ and $\mathcal{L}_{closer}$ over the signatures $(\mathsf{C}, \mathsf{conv})$ and (closer), respectively, where C, conv and closer are interpreted as the relation of being in *contact*, the property of *convexity* and the ternary relation of being *closer-than*. Consider first the language $\mathcal{L}_{conv}$ interpreted over a planar region algebra $\mathcal{M}$ extending $\mathsf{RCP}(\mathbb{R}^2)$. Extending the techniques used in [DGC99, Pra99], Davis constructs in $\mathcal{M}$ a two-dimensional affine coordinate system, and consequently encodes the real numbers as the points on one of the axes. He then defines the arithmetic operations and relations, and, very importantly, a predicate identifying all natural numbers, hence establishing the Δ^0_ω-hardness. Having encoded the arithmetic of natural numbers, he then defines each set of natural numbers as a single real number, hence establishing the Δ^1_ω-hardness. The result can be generalised to every region algebra of closed sets in $\mathbb{R}^n$, $n > 1$, extending $\mathsf{RCP}(\mathbb{R}^n)$. Note that in the above reduction of second-order arithmetic, sets of natural numbers are encoded as real numbers, and infinite sets of natural numbers are encoded as transcendental numbers. Hence it is essential for the Δ^1_ω-hardness result that $\mathsf{RCP}(\mathbb{R}^n)$ is a subset of $\mathcal{M}$. Alternatively, if one only requires that $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^n)$ is a subset of $\mathcal{M}$, the above reduction will work only up to the point where one has to encode sets of natural numbers. That is because instead of having encoded all real numbers, one will have encoded all rational numbers, and, of course, the countably many rational numbers cannot encode the uncountably many sets of natural numbers. Nevertheless, one still gets that the $\mathcal{L}_{conv}$-theory of every region algebra that extends $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^n)$ is Δ^0_ω-hard. Regarding the language $\mathcal{L}_{closer}$, Davis shows that it is strictly more expressive than $\mathcal{L}_{conv}$ when interpreted over region algebras over Euclidean spaces of dimension higher than one. Hence the established lower complexity bounds for the $\mathcal{L}_{conv}$-theories of the above region algebras also hold for their $\mathcal{L}_{closer}$-theories. Using significantly different techniques, Davis also shows that the $\mathcal{L}_{closer}$-theory of every region algebra over $\mathbb{R}$ that extends $\mathsf{RCP}_\mathbb{Q}(\mathbb{R})$ is Δ^0_ω-hard, and that the $\mathcal{L}_{closer}$-theory of every region algebra over $\mathbb{R}$ extending $\mathsf{RCP}(\mathbb{R})$ is Δ^1_ω-hard.

The lower complexity bounds established by Davis capture the $\mathcal{L}_{conv}$- and $\mathcal{L}_{closer}$-theories of the region algebras $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_\mathbb{A}(\mathbb{R}^n)$ and $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^n)$. The only exception is the $\mathcal{L}_{conv}$-theories of one-dimensional region algebras. In Section 4.3.4 we show that the theories of all countable region algebras are definable in first-order arithmetic and that the theories of all uncountable region algebras are definable in second-order arithmetic, yielding tight complexity bounds for all these theories. For the $\mathcal{L}_{conv}$-theories of one-dimensional region algebras, note that the properties of being convex and connected coincide, except for the empty region. It follows then that the $\mathcal{L}_{conv}$-theory of a one-dimensional region algebra $\mathcal{M}$ is $\mathcal{L}_\mathsf{C}$-definable in $\mathcal{M}$, and hence by the results established in Section 4.3.1 it is decidable. These results appeared in [NPH10].

We now mention another undecidability result regarding the region algebra $\mathsf{RCP}(\mathbb{R}^2)$. In [Dor98] Dornheim considered the first-order language $\mathcal{L}_{disc}$, which is an extension of the language of Boolean algebras with a unary relational symbol interpreted as the property of *being bounded and convex*. He established a reduction from the Post Correspondence Problem to the $\mathcal{L}_{disc}$-theory of $\mathsf{RCP}(\mathbb{R}^2)$, establishing in this way r.e.-hardness of the latter. This, of course,

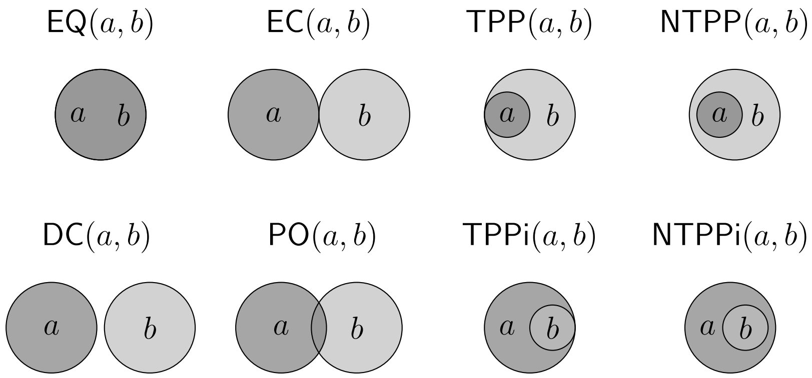


Figure 3.1: The eight $\mathcal{RCC}8$ relations satisfied by discs in $\mathbb{R}^2$.

is a much weaker result than the ones established by Grzegorczyk in [Grz51] and Davis in [Dav06], especially in the view of the recently emerged fact that even the quantifier-free-fragment of the logic is r.e.-hard (see [KNPHZ11a]).

3.3.2 Quantifier-Free Logics

The spatial logics that have attracted most of the attention of the AI community are quantifier-free, because the corresponding first-order logics are generally undecidable. The most intensively studied quantifier-free spatial language $\mathcal{RCC}8$ consists of eight binary relational symbols (DC, EQ, EC, PO, TPP, NTPP, TPPi, NTPPi) for describing how regions are topologically related (see Figure 3.1). The first to observe that the satisfiability of $\mathcal{RCC}8$ formulas over arbitrary topological spaces is decidable was Bennett in [Ben94]. He translated $\mathcal{RCC}8$ in the intuitionistic logic $\mathcal{I}$ with its topological interpretation. Each $\mathcal{RCC}8$-constraint is translated as a pair of a positive and a negative $\mathcal{I}$-constraints. Later in [Ben96], Bennett showed that $\mathcal{RCC}8$ can also be translated in the modal logic S4, using McKinsey and Tarski's topological interpretation of S4 [MT44, MT48]. Again positive and negative S4 constraints were used. In [RN99], Renz and Nebel corrected Bennett's translation from $\mathcal{RCC}8$ in S4 ensuring that regions are regular closed. The fact that the resulting translation preserves satisfiability was formally proved in [Nut99].

In Bennett's translation from $\mathcal{RCC}8$ to S4, the negative S4-constraints are required *not to be true*. In the topological semantics of S4, however, this is different from saying that the negation of the constraint is true. Hence, as noted by

Bennett, the procedures for consistency checking and determination of negative constraints involve some meta-level reasoning. To capture negation within the logical formalism, Bennett in [Ben96] extended **S4** with an **S5**-modality, which can be used to test the non-validity of the negative S4-constraints. As noted in [WZ00], the resulting bimodal logic $\mathbf{S4_u}$ was introduced in [GP92], where it was also shown to be decidable. That the new translation also preserves the satisfiability of $\mathcal{RCC}8$-formulas was confirmed in [Nut99].

The satisfiability problem for $\mathcal{RCC}8$ was shown to be NP-complete by Renz and Nebel in [RN97]. To show membership in NP, Renz and Nebel used Bennett's (corrected) translation from $\mathcal{RCC}8$ in $\mathbf{S4_u}$ and showed that the translation of every satisfiable $\mathcal{RCC}8$-formula is satisfiable by a Kripke frame having a particular structure and size that is polynomial in the length of the original formula. The Kripke frames in question are based on partial orders of depth two in which every element has at most one predecessor. In [Ren98] Renz simplified the structure of the Kripke frames to partial orders of depth one in which every element has at most one predecessor. As we see later, Kripke frames of similar structures have been used to show decidability of other quantifier-free spatial logics. Another interesting result shown in [Ren98] is that every satisfiable $\mathcal{RCC}8$-formula is satisfiable in all Euclidean spaces. The fact that $\mathcal{RCC}8$ is NP-hard, of course, is easy to establish. Indeed, already the fragment $\mathcal{RCC}5$ of $\mathcal{RCC}8$ based on the relations EQ, PO, DC∪EC, TPP∪NTPP and TPPi∪NTPPi is NP-hard. However, tractable fragments of $\mathcal{RCC}8$ and $\mathcal{RCC}5$ were identified in [RN99, JD97] as it was done for the Allen's interval calculus in [NB95, DJ96]. The fragments consist of positive formulas in conjunctive normal form in which only certain types of disjunctions occur. In [RN99, JD97], the authors established a complete classification of the maximal tractable fragments of $\mathcal{RCC}8$ and $\mathcal{RCC}5$, where a fragment is considered maximal if it cannot be extended with a new type of disjunctions.

An alternative proof of the fact that the satisfiability problem for $\mathcal{RCC}8$ is in NP was established by Griffiths in [Gri08]. Griffiths showed that the satisfiability problem for the conjunctive fragment of $\mathcal{RCC}8$ (all conjunctions in $\mathcal{RCC}8$) is reducible in logarithmic space to the graph reachability problem and, by [Jon75], is in NLogSpace. By counting the maximal number of disjunctions in the conjunctive normal form of an $\mathcal{RCC}8$-formula, he then shows that the satisfiability problem for $\mathcal{RCC}8$ is in NP.

The language $\mathcal{RCC}8$ has a rather limited expressiveness. Although one can express in it that Germany and France share a common boundary and that they are both in Europe, one cannot express, for example, that the United Kingdom consists of England, Northern Ireland, Scotland and Wales. To accommodate such expressiveness, Wolter and Zakharyaschev in [WZ00] considered the language $\mathcal{BRCC}8$ which extends $\mathcal{RCC}8$ with functional symbols for taking unions, intersections and complements of regions. Based on Bennett's translation from $\mathcal{RCC}8$ to $\mathbf{S4_u}$, Wolter and Zakharyaschev provided a satisfiability preserving translation from $\mathcal{BRCC}8$ to $\mathbf{S4_u}$. They also showed that the $\mathbf{S4_u}$-translation of every satisfiable $\mathcal{RCC}8$-formula is satisfiable in a Kripke frame based on a partial order of depth at most one and width at most two, and whose size is linear in the size of the original $\mathcal{RCC}8$ formula. Hence the satisfiability problem for $\mathcal{BRCC}8$ over arbitrary topological spaces is NP-complete. A Kripke frame based on a partial order of depth at most one and width at most two is called a *quasi-saw*.

Unlike $\mathcal{RCC}8$, not every satisfiable $\mathcal{BRCC}8$-formula is satisfiable in a Euclidean space. In fact, as shown in [WZ00], there exists a simple $\mathcal{BRCC}8$-formula that distinguishes connected and disconnected topological spaces. It was also shown that $\mathcal{BRCC}8$ has the same satisfiability problem over $\mathsf{RC}(\mathbb{R}^n)$, for $n \geq 1$, and over the class of all complete region algebras over connected topological spaces. Moreover, the problem turns out to be PSPACE-complete. To establish membership in PSPACE, Wolter and Zakharyaschev showed that if a $\mathcal{BRCC}8$-formula is satisfiable over a connected topological space, then it is satisfiable over a quasi-saw of size exponential in the size of the formula. Then they present an algorithm which, for every $\mathcal{BRCC}8$-formula φ, non-deterministically guesses a linear quasi-saw model of the $\mathbf{S4_u}$-translation of φ and then tries to make that model connected without violating φ. Although making the model connected generally requires exponentially many new points, the algorithm acts locally, and requires only polynomial part of the whole model to be kept in the memory at each point in time. Hence the algorithm runs in PSPACE. To show that the satisfiability problem is PSPACE-hard, the authors established a reduction from every language $\mathcal{L}$ in PSPACE to $\mathsf{Sat}(\mathcal{BRCC}8, \mathsf{RC}(\mathbb{R}))$ by encoding runs of a Turing machine recognising $\mathcal{L}$ in PSPACE.

The topological language $\mathcal{BRCC}8$, although significantly more expressive than $\mathcal{RCC}8$, still cannot express the fundamental topological property of *connectedness*. Early complexity results regarding a topological language with primitives for expressing connectedness constraints were given by Pratt-Hartmann in [PH02]. The language is essentially $\mathbf{S4_u cc}$, an extension of $\mathbf{S4_u}$ with unary relational symbols $\mathsf{c}^{\leq k}$ and $\mathsf{c}^{\geq k}$ $(k \geq 1)$ for bounding the number of connected components of regions from above and below. $\mathbf{S4_u cc}$ is interpreted over the collection of all complete set algebras and the corresponding satisfiability problem is shown to be NEXPTIME-complete. To establish membership in NEXPTIME, Pratt-Hartmann used a technique similar to the filtration method in modal logic, and showed that every satisfiable formula is satisfiable in the set algebras of a topological space whose size is exponential in the size of the formula. To show that the satisfiability problem is NEXPTIME-hard, Pratt-Hartmann established a reduction from the n-tiling problem, which is known to be NEXPTIME-hard (see e.g. [Pap93, p. 501]).

The study of $\mathbf{S4_u cc}$ was just the beginning of a whole series of similar investigations. In [KPHWZ08a, KPHWZ08b, KPHWZ10], various languages with connectedness predicates were considered for region algebras and set algebras over arbitrary and Euclidean topological spaces. For set algebras, the authors considered the fragment $\mathbf{S4_u c}$ of $\mathbf{S4_u cc}$, extending $\mathbf{S4_u}$ only with a symbol for expressing the property of being connected. In contrast to $\mathbf{S4_u cc}$, the complexity of $\mathbf{S4_u c}$ when interpreted over arbitrary or connected topological spaces, falls from NEXPTIME-complete to EXPTIME-complete. Membership in EXPTIME was shown by a reduction to a variant of $\mathcal{PDL}$ (namely $\mathcal{PDL}$ with converse and nominals) known to be in EXPTIME. For reasoning about region algebras, Kontchakov et al. considered the three basic languages $\mathcal{RCC}8$, $\mathcal{B}$ (the quantifier-free language of Boolean algebras) and $\mathcal{C}$ (an equally expressive variant of $\mathcal{BRCC}8$) and extended each of them either with the relational symbols $\mathsf{c}^{\leq k}$ and $\mathsf{c}^{\geq k}$ $(k \geq 1)$ or with the relational symbol c. The resulting extensions are denoted by $\mathcal{RCC}8\mathsf{c}$, $\mathcal{RCC}8\mathsf{cc}$, $\mathcal{B}\mathsf{c}$, $\mathcal{B}\mathsf{cc}$, $\mathcal{C}\mathsf{c}$ and $\mathcal{C}\mathsf{cc}$, and the satisfiability problem for each of them is polynomially reducible to the satisfiability problem for the corresponding extension of $\mathbf{S4_u}$, thus inheriting the latter's complexity upper bound. Kontchakov et al. showed that reasoning in $\mathcal{C}\mathsf{c}$ and $\mathcal{C}\mathsf{cc}$ about region algebras over arbitrary or connected topological spaces is as hard as reasoning, respectively, in $\mathbf{S4_u c}$ and $\mathbf{S4_u cc}$ about set algebras over arbitrary or connected topological spaces. Furthermore it was shown that every satisfiable formula in

$\mathcal{C}$c ($\mathcal{C}$cc) can be translated in equally satisfiable formula in $\mathcal{B}$c ($\mathcal{B}$cc). Hence, reasoning in $\mathcal{B}$c ($\mathcal{B}$cc) about region algebras over arbitrary or connected topological spaces is ExpTime-complete (NExpTime-complete). Regarding the languages $\mathcal{RCC}$8c and $\mathcal{RCC}$8cc, it is shown that reasoning about region algebras over arbitrary or connected topological spaces is NP-complete.

In addition to establishing the exact complexity of reasoning with connectedness predicates over arbitrary or connected topological spaces, Kontchakov et al. observed an interesting fact about the expressiveness of these languages. In particular, it was shown that the languages $\mathcal{RCC}$8c, $\mathcal{B}$c and $\mathcal{C}$c can distinguish between RC($\mathbb{R}$), RC($\mathbb{R}^2$) and RC($\mathbb{R}^n$) ($n > 2$). This is in stark contrast to the languages $\mathcal{RCC}$8, $\mathcal{B}$ and $\mathcal{C}$, for which every formula satisfiable in a region algebra over some (connected) topological space, is also satisfiable in each of the region algebras RC($\mathbb{R}^n$) ($n \geq 1$). Kontchakov et al. showed that the complexity of reasoning in $\mathcal{RCC}$8c and $\mathcal{B}$c about RC($\mathbb{R}$) is NP-complete, while in the case of $\mathcal{C}$c it is PSpace-complete. Reasoning in $\mathcal{RCC}$8c in higher dimensions was shown to remain NP-complete. For this latter result, the authors made use of the remarkable result established by Schaefer, Sedgwick and Štefankovič in [SSS03] regarding the satisfiability of $\mathcal{RCC}$8 by disc-homeomorphs in the Euclidean plane—a similar adaptation was previously made in [Gri08, p. 105, Corollary 6.4.19]. Reasoning in $\mathcal{B}$c and $\mathcal{C}$c in higher dimensions is shown to inherit the ExpTime lower complexity bound for reasoning about region algebras over connected topological spaces. In this case, however, no decidable upper bound was established. As we show in Section 5.4, for region algebras over $\mathbb{R}^2$, no such upper bound exists, for the two satisfiability problem turn out to be undecidable—a result that appeared in [KNPHZ11a, KNPHZ11b].

Another important observation made by Kontchakov et al. in [KPHWZ08a, KPHWZ08b, KPHWZ10, KPHZ10] was that the complete region algebras RC($\mathbb{R}^n$) and the region algebras of regular closed polytopes RCP($\mathbb{R}^n$), for $n = 1, 2$, do not satisfy the same $\mathcal{RCC}$8c-, $\mathcal{B}$c- or $\mathcal{C}$c-formulas. This was shown by finding formulas satisfiable in RC($\mathbb{R}^n$), $n = 1, 2$, but only by tuples containing regions with infinitely many components. The arguments that were used to show these expressiveness results fail in higher dimensions, and, it was an open problem whether the results were true for RC($\mathbb{R}^n$), $n > 2$. It turns out that in each of the languages $\mathcal{C}$c and $\mathcal{B}$c, there exists a formula which is satisfiable in region algebras over $\mathbb{R}^n$, $n > 1$, but only by tuples some of whose regions have infinitely many components. This result, which is based on the fact that Euclidean spaces are *unicoherent*, is presented in Section 5.2 and appeared in [KNPHZ11a, KNPHZ11b]. We sketch in Section 5.5 how the insights gained from it were used in an unpublished work of Pratt-Hartmann to show the undecidability of the satisfiability problem for $\mathcal{B}$c and $\mathcal{C}$c when interpreted over the region algebras RCP($\mathbb{R}^n$), $n \geq 2$. It is unclear, however, how these complexity results can be extended to the case of region algebras RC($\mathbb{R}^n$), $n \geq 3$.

In [KPHZ10], Kontchakov et al. considered the effect of adding to the basic languages $\mathcal{B}$, $\mathcal{RCC}8$ and $\mathcal{C}$ the property of *having a connected interior*. The resulting languages, denoted by $\mathcal{RCC}8$c$^\circ$, $\mathcal{B}$c$^\circ$ and $\mathcal{C}$c$^\circ$, turn out to have expressiveness similar to the one of the languages $\mathcal{RCC}8$c, $\mathcal{B}$c and $\mathcal{C}$c, in distinguishing between RC($\mathbb{R}$), RC($\mathbb{R}^2$) and RC($\mathbb{R}^n$) ($n > 2$), and in forcing regions with infinitely many components in RC($\mathbb{R}$) and RC($\mathbb{R}^2$). That the language $\mathcal{B}$c$^\circ$ can force regions with infinitely many components in the region algebra RC($\mathbb{R}^2$) is shown in Section 5.3. In Section 5.2, we show that $\mathcal{C}$c$^\circ$ can force regions with infinitely many components in each region algebra RC($\mathbb{R}^n$), $n \geq 2$. This result, however, cannot be extended for the language $\mathcal{B}$c$^\circ$. Still, there are formulas in $\mathcal{B}$c$^\circ$ which can distinguish between RC($\mathbb{R}^n$) and RCP($\mathbb{R}^n$), $n > 1$, and that is based on the existence in RC($\mathbb{R}^n$) of regions whose boundaries are pathological in some sense (e.g. there are regions in RC($\mathbb{R}^n$) whose boundaries contain the topologist's sine curve, and which is not locally connected). For region algebras over $\mathbb{R}$, the satisfiability problem for the languages with interior-connectedness are the same as for the languages with connectedness, merely because connectedness and interior-connectedness coincide. Reasoning in $\mathcal{C}$c$^\circ$ about region algebras over Euclidean spaces in dimensions higher than one is shown to be ExpTime-hard, and again no matching upper bound was established. It turns out that the languages $\mathcal{B}$c$^\circ$ and $\mathcal{C}$c$^\circ$ when interpreted over RC($\mathbb{R}^2$) and RCP($\mathbb{R}^2$) are again undecidable. This result is presented in Section 5.4 and appeared in [KNPHZ11a, KNPHZ11b].

Chapter 4

First-Order Spatial Logics

Consider the veracity of the following statements about (closed) regions of a topological space.

A1 *The union of two intersecting connected regions is connected.*

A2 *If two non-empty regions fill the whole space, then they have a non-empty intersection.*

A3 *In the interior of every non-empty region lies another non-empty region.*

A1 is a standard theorem in topology, and holds in every region algebra. By contrast, A2 and A3 hold only in some region algebras. In particular, it is not difficult to see that, A2 holds exactly in the region algebras over connected topological spaces, whereas, due to a much more elaborate argument [DW05], A3 holds exactly in the region algebras over weakly-regular topological spaces (see Section 2.5).

By choosing a suitable signature, it is routine to translate A1-A3 into first-order logic. Consider, for example, the signature σ that extends the signature of Boolean algebras with a predicate $\mathsf{c}(x)$ interpreted as "x is *connected*", and the binary relational symbol $\mathsf{C}(x, y)$ interpreted as "x and y have a non-empty intersection"—also known as the *contact* relation. Then an $\mathcal{L}_\sigma$-translation of A1-A3 would be:

$$\begin{aligned}
\psi_1 &:= \forall x \forall y (\mathsf{c}(x) \wedge \mathsf{c}(y) \wedge x \cdot y > 0 \rightarrow \mathsf{c}(x + y));\\
\psi_2 &:= \forall x \forall y (x > 0 \wedge y > 0 \wedge x + y = 1 \rightarrow \mathsf{C}(x, y));\\
\psi_3 &:= \forall x (x > 0 \rightarrow \exists y (\neg \mathsf{C}(y, -x))).
\end{aligned}$$

So, if Σ is the class of all region algebras considered as σ-structures, then ψ_1 is in the first-order theory $T(\Sigma)$ of Σ. Similarly, ψ_2 is in the first-order theory of the class of all region algebras over connected topological spaces, and ψ_3 is in the first-order theory of the class of all region algebras over weakly regular topological spaces.

A natural question, then, is, given a logical signature σ, how to describe the first-order σ-theory of a collection of region algebras (possibly containing a single structure). Another interesting question is whether two (collections of) region algebras have the same first-order σ-theories. From a computability theory point of view, an intriguing question is whether the first-order σ-theory T of a collection of region algebras is decidable, i.e. whether there exists an algorithm which can determine the membership of a first-order σ-sentence in T. Additionally, one can ask for exact classification of the complexity of every such computational problem.

In this chapter we address these and other questions concerning first-order spatial logics. The main results are separated in two parts. In the first part we discuss axiomatisations of different first-order theories, and in the second part, we discuss computability of first-order theories of set and region algebras over Euclidean spaces. We start, however, by proving various expressiveness results about first-order spatial logics.

4.1 Languages and Expressiveness

We now discuss the relative expressiveness of first-order Euclidean spatial logics. The first-order languages that we consider feature, in addition to Boolean primitives as defined in Table 2.1, topological, affine and metric properties and relations. The topological primitives that we consider are the property $\mathsf{c}(x)$ of being topologically connected, and the Whitehead's contact relation $\mathsf{C}(x, y)$ comprising the pairs of intersecting regions. The affine primitive that we consider is the property $\mathsf{conv}(x)$ of being convex, which was also examined in [Pra99, DGC99, Dav06]. Finally, we consider the metric relation $\mathsf{closer}(x, y, z)$, introduced in [Dav06], comprising of all triples (a, b, c) such that a is closer to b than it is to c.

We interpret first-order languages featuring the above topological, affine and metric primitives over Euclidean region algebras, and, in particular, the region algebra $\mathsf{RC}(\mathbb{R}^n)$ together with its *tame* Boolean subalgebras $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ (see Section 2.6). This results in a large problem space that is summarised in Table 4.1. For each of these Euclidean region algebras, we have that topological languages (those featuring primitives preserved under homeomorphic transformations) are less expressive than languages featuring the convexity predicate, which in turn are less expressive than languages featuring the predicate closer-than. The only exceptions are Euclidean region algebras over $\mathbb{R}$, in which the convex regions are simply the non-empty connected regions, and in this case topological languages and languages featuring the convexity predicate become equally expressive.

It is well known that first-order topological languages, and consequently all first-order languages considered here, are sufficiently expressive to distinguish between $\mathsf{RC}(\mathbb{R}^n)$ and its tame Boolean subalgebras $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$. In fact, this is true even for certain quantifier-free topological languages (see Chapter 5). However, first-order topological languages lack the expressive power to distinguish between the tame region algebras over the Euclidean spaces $\mathbb{R}$ (see Lemma 59) or $\mathbb{R}^2$ (see [PH07]). Whether this is true in higher-dimensional Euclidean spaces is an open problem. By contrast, the languages featuring the convexity predicate, and consequently those featuring the predicate closer-than, are sufficiently expressive to distinguish between rational and algebraic polygons (see [Pra99]), and between algebraic and arbitrary polygons (see Theorem 104). Both results also hold for dimensions higher than 2. Whether these languages can distinguish between polytopes and semi-algebraic regions (i.e. between the region algebras $\mathsf{RCS}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$), is an open problem.

We now proceed with a formal presentation of the above results. The proofs of Lemma 41—Lemma 45 are almost identical to the one given in [PH07, Section 2.3] for the case of open region algebras. Let $\mathcal{L}_{\mathsf{C}}$ be the first-order language of the signature (C) whose only relational symbol C represents the binary relation "contact". Two regions a and b are in contact if $a \cap b \neq \emptyset$. Note that $a \cdot b \neq 0$ implies $\mathsf{C}(a, b)$, but not vice versa (see Table 2.1). The property of being connected is $\mathcal{L}_{\mathsf{C}}$-definable. Consider the $\mathcal{L}_{\mathsf{C}}$-formula:

$$\psi_c(x) := \forall y \forall z (y > 0 \land z > 0 \land x = y + z \to \mathsf{C}(x, y)).$$

Lemma 41. *Let $\mathcal{M}$ be a region algebra over a topological space $\mathcal{X}$ such that, $\mathcal{M}$ is* *a complete region algebra or a finitely decomposable region algebra that respects components. Then the property of being connected is $\mathcal{L}_{\mathsf{C}}$-definable in $\mathcal{M}$ by the formula $\psi_c(x)$.*

Proof. Let $a \in \mathcal{M}$ be such that $\mathcal{M} \models \neg\psi_c[a]$. Then, there exist $b, c \in \mathcal{M}$ that separate a, and so a is disconnected. Let $a \in \mathcal{M}$ be disconnected. Then, there exist subsets b and c of $\mathcal{X}$ that separate a. By Lemma 22 and Lemma 32, b and c are in $\mathcal{M}$, so $\mathcal{M} \models \neg\psi_c[a]$. □

Corollary 42. *The property of being connected is $\mathcal{L}_{\mathsf{C}}$-definable in* $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$.

Proof. Lemma 41, Lemma 37 and Lemma 39. □

We now show that binary relation part-of is $\mathcal{L}_{\mathsf{C}}$-definable. Consider the $\mathcal{L}_{\mathsf{C}}$-formula:

$$\psi_{\leq}(x, y) := \forall z(\mathsf{C}(z, x) \rightarrow \mathsf{C}(z, y)).$$

Lemma 43. *Let $\mathcal{M}$ be a dense region algebra over a weakly-regular topological space $\mathcal{X}$. Then the part-of relation is $\mathcal{L}_{\mathsf{C}}$-definable in $\mathcal{M}$ by the formula $\psi_{\leq}(x, y)$.*

Proof. Clearly, if $a \leq b$, for $a, b \in \mathcal{M}$, then $\mathcal{M} \models \psi_{\leq}[a, b]$. Suppose that $c := a \cdot (-b)$ is nonempty. Because $\mathcal{X}$ is weakly-regular, there exists a nonempty open set A such that $A^- \subseteq c^\circ$. Since $\mathcal{M}$ is a dense region algebra and A^- is regular closed, there exists a nonempty $c' \in \mathcal{M}$ such that $c' \subseteq A^-$. So, c' is in contact with a, as a nonempty subset of a, and disjoint from b, as a subset of A^-. Hence, c' is a witness for $\mathcal{M} \not\models \psi_{\leq}[a, b]$. □

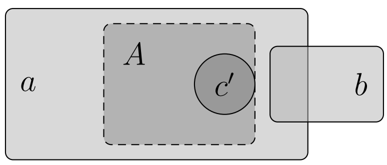


Figure 4.1: Defining the part-of relation using $\mathcal{L}_{\mathsf{C}}$.

Corollary 44. *The part-of relation is $\mathcal{L}_{\mathsf{C}}$-definable in* $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$.

Proof. Lemma 43, Lemma 38 and Lemma 35. □

Consider the first-order language $\mathcal{L}_c$ of the signature $(\mathsf{c}, \leq)$, where c is a unary predicate interpreted as the property of being "connected", and $\leq$ is a binary predicate interpreted as the relation "part-of" (see Section 2.5).

Lemma 45. *Let $\mathcal{M}$ be finitely decomposable region algebra that respects components. Then the contact relation is $\mathcal{L}_c$-definable in $\mathcal{M}$.*

Proof. To define the contact relation we use the $\mathcal{L}_c$-formula:

$$\psi_C(x, y) := \exists x' \exists y' (x' \neq 0 \land x' \leq x \land y' \neq 0 \land y' \leq y \land \mathsf{c}(x' + y')).$$

Clearly, if $\mathcal{M} \models \psi_C[a, b]$, for $a, b \in \mathcal{M}$, then a and b are in contact. Suppose that the regions a and b are in contact. Since $\mathcal{M}$ is finitely decomposable, there exist components a' and b' of a and b which are also in contact. Since $\mathcal{M}$ respects components, a' and b' are in $\mathcal{M}$, and, as such, they are witnesses for $\mathcal{M} \models \psi_C[a, b]$. □

Let $\mathcal{L}_{conv}$ and $\mathcal{L}_{closer}$ be the first-order languages of the signatures $(\mathsf{conv}, \leq)$ and (closer), respectively, where conv is a unary predicate interpreted as the property of being "convex", $\leq$ is the binary predicate interpreted as the relation "part-of" and closer is the ternary predicate interpreted as the relation "closer-than". A region a is convex if, for every two points $p, q \in a$, a contains the line segment between p and q. For regions a, b and c, $\mathsf{closer}(a, b, c)$ if, for every pair of points $(p, q) \in a \times c$, there exists a pair of points $(r, s) \in a \times b$ such that $d(r, s) \leq d(p, q)$, where $d(p, q)$ denotes the distance between the points p and q.

Lemma 46 ([Dav06])**.** *Let $\mathcal{M}$ be $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ or $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$. Then, the property of being* convex *and the* part-of *relation are $\mathcal{L}_{closer}$-definable in $\mathcal{M}$.*

Let $\mathcal{M}$ be one of $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$. We define in $\mathcal{M}$ the set of hyperplanes in $\mathbb{R}^n$ using the $\mathcal{L}_{conv}$-formula:

$$\psi_/(x) := \mathsf{conv}(x) \land \mathsf{conv}(-x).$$

Clearly, the boundary of every $a \in \mathcal{M}$ with $\mathcal{M} \models \psi_/[a]$ is a hyperplane. Conversely, every rational hyperplane in $\mathbb{R}^n$ is the boundary of two regions in $\mathcal{M}$ each satisfying $\psi_/(x)$ in $\mathcal{M}$. Two hyperplanes are parallel and distinct, if they are the boundaries of two regions satisfying the $\mathcal{L}_{conv}$-formula:

$$\psi_{\|}(x,y) := \psi_{/}(x) \wedge \psi_{/}(y) \wedge x \leq y \wedge \neg y \leq x.$$

Lemma 47. *Let $\mathcal{M}$ be one of $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$. Then, the contact relation is $\mathcal{L}_{conv}$-definable in $\mathcal{M}$.*

Proof. (Taken from [Pra99, Theorem 5.1].) We define the contact relation using the formula:

$$\begin{aligned}\psi_C(x,y) \quad := \quad & \exists x' \exists y' (\mathsf{conv}(x') \wedge \mathsf{conv}(y') \wedge x' \leq x \wedge y' \leq y \rightarrow \\ & \qquad \forall z \forall z' \psi_{\|}(z,z') \rightarrow \neg(x' \leq z \wedge y' \leq -z')).\end{aligned}$$

Let $a, b \in \mathcal{M}$ be in contact. Since $\mathcal{M}$ is well-behaved, it is routine to show that there exist subregions a' and b' of a and b, respectively, which are convex and also in contact. No two convex regions which are in contact can be separated by two parallel rational hyperplanes. Conversely, if $a, b \in \mathcal{M}$ are disjoint, then every two convex subregions of a and b can be separated by two parallel rational hyperplanes. □

Showing that the contact relation is $\mathcal{L}_{conv}$-definable in $\mathsf{RC}(\mathbb{R}^n)$ requires a different approach.

Lemma 48. *The contact relation is $\mathcal{L}_{conv}$-definable in $\mathsf{RC}(\mathbb{R}^n)$.*

Proof. Adopting ideas from [Pra99, Dav06], we present a proof for $n = 2$, which can easily be generalised for $n \geq 1$. We identify every point p in $\mathbb{R}^2$ with the pairs of lines that intersect in p. A pair of regions represents a point if they satisfy the formula:

$$\psi_{\bullet}(x,y) := \psi_{/}(x) \wedge \psi_{/}(y) \wedge x \cdot y > 0 \wedge x \cdot (-y) > 0.$$

A point p in $\mathbb{R}^2$ which is represented by regions (a, b) lies in the interior of a convex region c exactly when a, b and c satisfy the formula:

$$\psi_{\bullet\circ}(x,y,z) := x \cdot z > 0 \wedge (-x) \cdot z > 0 \wedge y \cdot z > 0 \wedge (-y) \cdot z > 0.$$

Two regular closed sets a, b are in contact exactly when they have a point p in common, which is exactly when every open convex neighbourhood of p intersects a and b. Hence, the following formula defines the contact relation:

$$\psi_C(u,v) := \exists x \exists y(\psi_\bullet(x,y) \wedge \forall z(\mathsf{conv}(z) \wedge \psi_{\bullet\circ}(x,y,z) \to u \cdot z \neq 0 \wedge v \cdot z \neq 0)). \quad \Box$$

We summarise the above results in the following lemma. Recall from Section 2.3 that, for structures $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \leq^p_m \mathcal{N}$ and $\mathcal{M} \equiv^p_m \mathcal{N}$ denote that $T(\mathcal{M}) \leq^p_m T(\mathcal{N})$ and $T(\mathcal{M}) \equiv^p_m T(\mathcal{N})$, respectively.

Lemma 49. *Let $\mathcal{M}$ be one of the region algebras* $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}(\mathbb{R}^n)$, $\mathsf{RCP}_\mathbb{A}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^n)$. *Then:*

$$(\mathcal{M}, \mathsf{c}, \leq) \leq^p_m (\mathcal{M}, \leq, \mathsf{C}, \mathsf{c}) \equiv^p_m (\mathcal{M}, \mathsf{C}) \leq^p_m (\mathcal{M}, \mathsf{conv}, \leq) \equiv^p_m (\mathcal{M}, \mathsf{conv}, \mathsf{C}) \leq^p_m (\mathcal{M}, \mathsf{closer}).$$

Now we consider region algebras over $\mathbb{R}$. We show that the region algebras $\mathsf{RCS}(\mathbb{R})$, $\mathsf{RCP}(\mathbb{R})$, $\mathsf{RCP}_\mathbb{A}(\mathbb{R})$ and $\mathsf{RCP}_\mathbb{Q}(\mathbb{R})$ have the same $\mathcal{L}_\mathsf{C}$-theories which are $\mathcal{L}_\mathsf{C}$-definable in $\mathsf{RC}(\mathbb{R})$. We start by showing that the $\mathcal{L}_c$-theory of $\mathsf{RCP}(\mathbb{R})$ is $\mathcal{L}_c$-definable in $\mathsf{RC}(\mathbb{R})$. Since $(\mathsf{RCP}(\mathbb{R}), \mathsf{c}, \leq)$ is a substructure of $(\mathsf{RC}(\mathbb{R}), \mathsf{c}, \leq)$, we only have to provide an $\mathcal{L}_c$-formula $\psi_{RCP}(x)$ defining in $(\mathsf{RC}(\mathbb{R}), \mathsf{c}, \leq)$ the set $\mathsf{RCP}(\mathbb{R})$. Note that $\mathsf{RCP}(\mathbb{R})$ consists of the regular closed sets in $\mathbb{R}$ having finitely many components, which are exactly the regular closed sets in $\mathbb{R}$ having finitely many boundary points. We define the pairs $(a,b) \in \mathsf{RC}(\mathbb{R})$ such that a is a connected component of b using the formula:

$$\psi_{cc}(x,y) := \mathsf{c}(x) \wedge x \leq y \wedge \forall z(\mathsf{c}(z) \wedge z \leq y \wedge x \leq z \to z \leq x).$$

We identify a real numbers r with the regular closed sets having r as a unique endpoint. We define those sets using the formula:

$$\psi_{cut}(v) := \mathsf{c}(v) \wedge \mathsf{c}(-v) \wedge v \neq 0 \wedge v \neq 1.$$

The following formula defines the set of pairs of a real number and a connected neighborhood of that number:

$$\psi_\odot(v,u) := \mathsf{c}(u) \wedge \psi_{cut}(v) \wedge u \cdot v \neq 0 \wedge u \cdot (-v) \neq 0.$$

The formula $\psi_{EP}(v,x)$ defines the set of pairs of regions such that the first region represents an endpoint of a connected component of the second region:

$$\psi_{EP}(v, x) := \psi_{cut}(v) \wedge \exists y(\psi_{cc}(y, x) \wedge \mathsf{c}(y + v) \wedge \mathsf{C}(y, -v) \wedge (y \leq v \vee y \leq -v)).$$

The formula $\psi_{ISO\_EP}(x)$ defines the regions whose boundary points have no accumulation points:

$$\begin{aligned}\psi_{ISO\_EP}(x) \quad := \quad & \forall v\big(\psi_{cut}(v) \wedge \neg\psi_{EP}(v, x) \rightarrow \\ & \qquad \exists u\big(\psi_{\odot}(v, u) \wedge \forall v'(\psi_{EP}(v', x) \rightarrow \neg\psi_{\odot}(v', u))\big)\big).\end{aligned}$$

The following formula defines the regions whose boundary points are bounded:

$$\psi_{BND\_EP}(x) := \exists u\big(\mathsf{c}(u) \wedge \neg\mathsf{c}(-u) \wedge \forall l(\psi_{EP}(l, x) \rightarrow \psi_{\odot}(l, u))\big).$$

Finally, the formula $\psi_{RCP}(x)$ defines the set RCP.

$$\psi_{RCP}(x) := \psi_{ISO\_EP}(x) \wedge \psi_{BND\_EP}(x).$$

Lemma 50. *The $\mathcal{L}_c$-theory of* $\mathsf{RCP}(\mathbb{R})$ *is $\mathcal{L}_c$-definable in* $\mathsf{RC}(\mathbb{R})$.

Corollary 51. $(\mathsf{RCP}(\mathbb{R}), \mathsf{c}, \leq) \leq^p_m (\mathsf{RC}(\mathbb{R}), \mathsf{c}, \leq)$ *and* $(\mathsf{RCP}(\mathbb{R}), \mathsf{C})) \leq^p_m (\mathsf{RC}(\mathbb{R}), \mathsf{C})$.

Proof. Lemma 50 and Lemma 42. □

We will now show that

$$(\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCP}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCS}(\mathbb{R}), \mathsf{C}).$$

Note that $(\mathsf{RCP}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCS}(\mathbb{R}), \mathsf{C})$ trivially follows from $\mathsf{RCS}(\mathbb{R}) = \mathsf{RCP}(\mathbb{R})$. We need the following technical lemmas.

Definition 52. Let $\mathcal{X} = (X, \tau)$ be a topological space. Two n-tuples $\bar{a}$ and $\bar{b}$ of regular closed sets in $\mathcal{X}$ are *similarly situated*, denoted by $\bar{a} \sim \bar{b}$, if there is a homeomorphism $f : \mathcal{X} \to \mathcal{X}$ such that $f^+ : \bar{a} \mapsto \bar{b}$, where for $a \subseteq X$, $f^+(a) = \{f(x) \mid x \in a\}$.

Lemma 53. *Let $\mathcal{X} = (X, \tau)$, $\mathcal{Y} = (Y, \sigma)$ be topological spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a homeomorphism. Then $f^+ \restriction \mathsf{RC}(\mathcal{X})$ is a bijection from $\mathsf{RC}(\mathcal{X})$ to $\mathsf{RC}(\mathcal{Y})$. Moreover, for $a \subseteq X$, $f^+ \restriction cc_a$ is a bijection from cc_a to $cc_{f^+(a)}$, where cc_a and $cc_{f(a)}$ are the sets of connected components of a and $f^+(a)$, respectively.*

Proof. Since every homeomorphism $f : \mathcal{X} \to \mathcal{Y}$ is a bijective function preserving the set-theoretic operations and relations, the property of being connected and the closure and interior operations, we get that f is an isomorphism between the structures $(\wp(X), \mathsf{c}, \cdot^\circ, \cdot^-, \subseteq, \cap, \cup)$ and $(\wp(Y), \mathsf{c}, \cdot^\circ, \cdot^-, \subseteq, \cap, \cup)$. □

Lemma 54. *Every homeomorphism $f : \mathbb{R} \to \mathbb{R}$ induces a model isomorphism $f^+ : (\mathsf{RCP}(\mathbb{R}), \mathsf{C}) \to (\mathsf{RCP}(\mathbb{R}), \mathsf{C})$.*

Proof. As a direct consequence of Lemma 53, we get that $f^+ \restriction \mathsf{RCP}(\mathbb{R})$ is a bijection from $\mathsf{RCP}(\mathbb{R})$ to itself. Trivially,

$$\begin{aligned} \mathsf{C}(a,b) \iff & \qquad a \cap b \neq \emptyset \iff \\ & f^+(a) \cap f^+(b) \neq \emptyset \iff \qquad \mathsf{C}(f^+(a), f^+(b)). \end{aligned}$$ □

Definition 55. Let R be any subset of $\mathbb{R}$. By $\mathsf{RCP}_R(\mathbb{R})$ we denote the set of all elements in $\mathsf{RCP}(\mathbb{R})$ whose connected components have endpoints in R.

Lemma 56. *Let R be a dense subset of $\mathbb{R}$. Let $\bar{a}$ be an n-tuple of elements in $\mathsf{RCP}_R(\mathbb{R})$ and $b \in \mathsf{RCP}(\mathbb{R})$. Then there is an $a \in \mathsf{RCP}_R(\mathbb{R})$ such that $\bar{a}a \sim \bar{a}b$.*

Proof. Let $r_1, \ldots, r_k$ be the increasing linear ordering of the endpoints of the connected components of the regions $\bar{a}, b$. Setting $r_0 = q_0 = -\infty$ and $r_{k+1} = q_{k+1} = +\infty$, we define for $i = 1, \ldots, k$

$$q_i = \begin{cases} r_i & \text{if } r_i \in R \\ \text{any } q \in (q_{i-1}, r_i) \cap F & \text{if } r_i \in \mathbb{R} \setminus R. \end{cases}$$

It is always possible to select $q \in (q_{i-1}, r_i) \cap R$, since R is a dense linear subset of $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the homeomorphism that maps $[q_i, q_{i+1}]$ linearly to $[r_i, r_{i+1}]$, for $i = 0, \ldots, k$. Clearly, $f^+(b) \in \mathsf{RCP}_R(\mathbb{R})$ since, for every $r \in \{r_1, \ldots, r_k\}$, $f(r) \in R$. Finally, $f^+ : \bar{a} \mapsto \bar{a}$, since for every $r \in \{r_1, \ldots, r_k\} \cap R$, $f(r) = r$. □

Corollary 57. *Let R be dense subset of $\mathbb{R}$. Then for every $\mathcal{L}_\mathsf{C}$-formula $\psi(x, \bar{y})$, $\bar{a} \in \mathsf{RCP}_R(\mathbb{R}^n)$ and $b \in \mathsf{RCP}(\mathbb{R})$, if $\mathsf{RCP}(\mathbb{R}) \models \psi[b, \bar{a}]$, then $\mathsf{RCP}(\mathbb{R}) \models \psi[a, \bar{a}]$ for some $a \in \mathsf{RCP}_R(\mathbb{R})$.*

Proof. Lemma 54 and Lemma 56. □

Lemma 58. *[Tarski-Vaught Test] Let $\mathcal{A}$ be a substructure of $\mathcal{B}$. Then $\mathcal{A} \preceq \mathcal{B}$ if and only if for every formula $\varphi(x, \bar{y})$ and $\bar{a} \in A$, if there exists $b \in B$ such that $\mathcal{B} \models \varphi[b, \bar{a}]$, then there exists $a \in A$ such that $\mathcal{B} \models \varphi[a, \bar{a}]$.*

Proof. See e.g. Theorem 2.5.1 [HH93, p.55]. □

Lemma 59. $(\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}), \mathsf{C}) \preceq (\mathsf{RCP}(\mathbb{R}), \mathsf{C})$.

Proof. Lemma 58 and Corollary 57. □

As a result, we get the following lemma.

Lemma 60. *Let $\mathcal{M}$ and $\mathcal{N}$ be any of* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$*,* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$*,* $\mathsf{RCP}(\mathbb{R})$ *and* $\mathsf{RCS}(\mathbb{R})$*. Then* $(\mathcal{M}, \mathsf{C}) \equiv^p_m (\mathcal{N}, \mathsf{C}) \leq^p_m (\mathsf{RC}(\mathbb{R}), \mathsf{C})$.

Proof. Lemma 59 and Lemma 50. □

4.2 Axiomatisations

Let Σ be a collection of region algebras interpreting a logical signature σ. A way of describing the first-order σ-theory T of Σ is by axiomatizing it. I.e. to list a set of σ-sentences in T, called axioms, which logically imply all and only the σ-sentences in T. We recall the axiomatisations of few collections of region algebras, due to [DV06, DW05, Roe97], and a classic axiomatisation result about the collection of all set algebras, due to McKinsey and Tarski [MT44].

We start with the result by McKinsey and Tarski. Let $\sigma_{cl} = (\cup, \cap, {}^C, 0, 1, {}^-)$ be an extension of the signature of Boolean algebras with a unary functional symbol interpreted as the operation *topological closure*. We denote the first-order σ_{cl}-language by $\mathcal{L}_{cl}$. In addition to the axioms of Boolean algebra, every set algebra considered as a σ_{cl}-structure satisfies the following axioms:

$$
\begin{aligned}
\psi_{cl1} &:= \forall x (x \cap (x)^- = x); \\
\psi_{cl2} &:= \forall x ((x^-)^- = x^-); \\
\psi_{cl3} &:= \forall x \forall y ((x \cup y)^- = x^- \cup y^-); \\
\psi_{cl4} &:= 0^- = 0.
\end{aligned}
$$

Let Φ_{cl} be the set of axioms for Boolean algebras together with the formulas ψ_{cl1} – ψ_{cl4}, and let T_{cl} be the first-order σ_{cl}-theory of all set algebras. McKinsey and Tarski refer to the σ_{cl}-structures satisfying Φ_{cl} as *closure algebras*. Hence, every set algebra interpreting σ_{cl} is a closure algebra. McKinsey and Tarski proved the following representation theorem for closure algebras, which extensively uses Stone's representation theorem for Boolean algebras.

Theorem 61. *([MT44, Theorem 2.6]) Every closure algebra is isomorphic to a set algebra considered as a σ_{cl}-structure.*

As a result, of course, we get that:

Corollary 62. $\Phi_{cl} \models T_{cl}$.

We now turn to the more recent results by [DV06, DW05, Roe97]. Let $\sigma_{CA} = (+, \cdot, -, 0, 1, \mathsf{C})$ be the signature that extends the signature of Boolean algebras with a binary relational symbol interpreted as the *contact* relation, and let $\mathcal{L}_{CA}$ be the first-order language of σ_{CA}. Recall that every region algebra $\mathcal{M}$ over a topological space $\mathcal{X}$ forms a Boolean algebra. In addition to satisfying the axioms of Boolean algebra, $\mathcal{M}$ considered as a σ_{CA}-structure satisfies the following sentences:

$$\begin{aligned}
\psi_{CA1} &:= \forall x \forall y (\mathsf{C}(x,y) \to x \neq 0); \\
\psi_{CA2} &:= \forall x \forall y (\mathsf{C}(x,y) \to \mathsf{C}(y,x)); \\
\psi_{CA3} &:= \forall x \forall y \forall z (\mathsf{C}(x, y+z) \leftrightarrow \mathsf{C}(x,y) \vee \mathsf{C}(x,z)); \\
\psi_{CA4} &:= \forall x (x = 0 \vee \mathsf{C}(x,x)).
\end{aligned}$$

If $\mathcal{X}$ is a weakly-regular topological space and $\mathcal{M}$ is a dense region algebra over $\mathcal{X}$, then $\mathcal{M}$ also satisfies the *extensionality axiom* ψ_{ext}.

$$\psi_{ext} := \forall x (x = 0 \vee \exists y (y \neq 0 \wedge \neg \mathsf{C}(y, -x))) \tag{4.1}$$

If $\mathcal{X}$ is a compact Hausdorff topological space and $\mathcal{M}$ is a closed base for $\mathcal{X}$, then $\mathcal{M}$ also satisfies the *interpolation axiom* ψ_{int} (see [PH07, Lemma 2.151]).

$$\psi_{int} := \forall x \forall y (\neg \mathsf{C}(x, -y) \to \exists z (\neg \mathsf{C}(z, -y) \wedge \neg \mathsf{C}(x, -z))) \tag{4.2}$$

If $\mathcal{X}$ is a connected topological space, then $\mathcal{M}$ also satisfies the formula:

$$\psi_{conn} := \forall x (x = 0 \vee x = 1 \vee \mathsf{C}(x, -x)). \tag{4.3}$$

Let Φ_{CA} be the set of axioms for Boolean algebras together with the formulas ψ_{CA1} – ψ_{CA4}, let $\Phi_{ECA} := \Phi_{CA} \cup \{\psi_{ext}\}$ and let $\Phi_{NCA} := \Phi_{ECA} \cup \{\psi_{int}\}$. In [DV06] the structures satisfying Φ_{CA}, Φ_{ECA} and Φ_{NCA} are called *contact algebras*, *extensional contact algebras* and *normal contact algebras*, respectively. (Note that in [DW05] the structures satisfying Φ_{ECA} are called *Boolean contact algebras*.)

Let T_{CA} be the $\mathcal{L}_{CA}$-theory of the class Σ_{CA} of all region algebras; let T_{ECA} be the $\mathcal{L}_{CA}$-theory of the class Σ_{ECA} of dense region algebras over weakly-regular topological spaces; and let T_{NCA} be the $\mathcal{L}_{CA}$-theory of the class Σ_{NCA} of region algebras over compact Hausdorff topological spaces that are closed bases for their topologies.

We have the following representation theorems.

Theorem 63.

- *Every contact algebra is isomorphic to a region algebra in* Σ_{CA}.

- *Every extensional contact algebra is isomorphic to a region algebra in* Σ_{ECA}.

- *Every normal algebra is isomorphic to a region algebra in* Σ_{NCA}.

These results were obtained, respectively, in [DV06], [DW05] and [Roe97]. As a result we get:

Corollary 64.

- $\Phi_{CA} \models T_{CA}$;

- $\Phi_{ECA} \models T_{ECA}$;

- $\Phi_{NCA} \models T_{NCA}$.

All of the above axiomatisations are extensions of Stone's representation theorem for Boolean algebras. Since (atomless) Boolean algebras and complete (atomless) Boolean algebras have the same first-order σ_{BA}-theories, it is only natural to assume that region algebras and complete region algebras also have the same first-order σ_{CA}-theories. The same assumption can be made for the first-order σ_{cl}-theories of set algebras and complete set algebras. As we will show in the next section, both of these assumptions turn out to be false.

4.2.1 Complete Set and Region Algebras

We denote the classes of complete region algebras in Σ_{CA}, Σ_{ECA} and Σ_{NCA} by Σ_{CCA}, Σ_{CECA} and Σ_{CNCA}, respectively. We will now see that the $\mathcal{L}_{CA}$-theories of Σ_{CA}, Σ_{ECA} and Σ_{NCA} are different from the $\mathcal{L}_{CA}$-theories of Σ_{CCA}, Σ_{CECA} and

Σ_{CNCA}, respectively. We will provide a formula ψ_{CCA} satisfiable in every region algebra in Σ_{CCA}, but not satisfiable in the incomplete region algebra $\mathsf{RCP}(\mathbb{R}^2) \in \Sigma_{NCA}$. Since $\Sigma_{CNCA} \subseteq \Sigma_{CECA} \subseteq \Sigma_{CCA}$ and $\Sigma_{NCA} \subseteq \Sigma_{ECA} \subseteq \Sigma_{CA}$, ψ_{CCA} is in the $\mathcal{L}_{CA}$-theories of Σ_{CCA}, Σ_{CECA} and Σ_{CNCA}, and in none of the $\mathcal{L}_{CA}$-theories of Σ_{CA}, Σ_{ECA} and Σ_{NCA}. The results in this section were presented in [Nen09], and the idea behind them is due to my supervisor Ian Pratt-Hartman.

Note that a region algebra satisfying the $\mathcal{L}_{CA}$-formula

$$\psi_{triv} := \forall x(x = 0 \vee x = 1)$$

consists of at most two elements. We write $a \ll b$ as a shorthand for $\neg\mathsf{C}(a, -b)$. Note that, for regions a and b, $a \ll b$ exactly when a is contained in the interior of b.

Lemma 65. *Let $\mathcal{M}$ be a complete region algebra that satisfies the formulas ψ_{ext}, ψ_{int}, ψ_{conn} and $\neg\psi_{triv}$. Then, there exist regions a and $\{a_i\}_{i\in\omega}$ in $\mathcal{M}$ such that:*

$$\begin{array}{llll} i) & \mathsf{C}(a, -a); & ii) & a = \sum_{i\in\omega} a_i; \\ iii) & a_i \ll a_{i+1}; & iv) & a_i \ll a. \end{array}$$

Proof. Since $\mathcal{M} \models \neg\psi_{triv}$, there exists some $b \in \mathcal{M}$ such that $b \neq 0$ and $b \neq 1$. Because of $\mathcal{M} \models \psi_{ext}$, there exists some $a_0 \in \mathcal{M}$ such that $a_0 \ll b$ and $a_0 \neq 0$. Because $\mathcal{M} \models \psi_{int}$, there exists some $a_1 \in \mathcal{M}$ such that $a_0 \ll a_1 \ll b$. Again, since $\mathcal{M} \models \psi_{int}$, there exists some $a_2 \in \mathcal{M}$ such that $a_1 \ll a_2 \ll b$. Repeating this argument, we can construct a sequence $\{a_i\}_{i\in\omega}$ such that $a_0 \ll a_i \ll a_{i+1} \ll b$, for $i \in \omega$. Because $\mathcal{M}$ is complete, $a := \sum_{i\in\omega} a_i$ exists. We are only left to show that $\mathsf{C}(a, -a)$. We know that, $0 \neq a_0 \leq a \leq b \neq 1$, and since $\mathcal{M} \models \psi_{conn}$, we get that $\mathsf{C}(a, -a)$. □

Recall that, the $\mathcal{L}_{CA}$-formula $\psi_c(x)$ defines the property of being connected in every complete region algebra, and also in $\mathsf{RCP}(\mathbb{R}^2)$ (Lemma 41 and Corollary 42). Now, consider the $\mathcal{L}_{CA}$-formula:

$$\psi_{\odot}(x, y) := \mathsf{C}(x, y) \wedge (\forall z)(z \leq x \wedge \psi_c(z) \rightarrow \neg\mathsf{C}(z, y)).$$

Let $\mathcal{M}$ be a region algebra, and $a, b \in \mathcal{M}$ be such that $\mathcal{M} \models \psi_{\odot}[a, b]$. So, a is in contact with b, but no connected subregion of a is in contact with b. Clearly, if $\mathcal{M}$ respects components, this would only be possible if a has infinitely many components. Hence, $\psi_{\odot}(x,y)$ is not satisfiable over finitely-decomposable region algebras that respect components, and, in particular, over $\mathsf{RCP}(\mathbb{R}^2)$. The $\mathcal{L}_{CA}$-formula ψ_{CCA} is given by:

$$\psi_{CCA} := \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv} \rightarrow \exists x \exists y \psi_{\odot}(x,y).$$

Lemma 66. *If $\mathcal{M}$ is a complete region algebra, then $\mathcal{M} \models \psi_{CCA}$*

Proof. By Lemma 65 and $\mathcal{M} \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$, there exist regions a and $\{a_i\}_{i\in\omega}$ in $\mathcal{M}$ such that: $a = \sum_{i\in\omega} a_i$; $\mathsf{C}(a,-a)$; and $a_i \ll a_{i+1} \ll a$, for $i \in \omega$. Taking $a_{-1} = 0$, we define, for $i \in \omega$:

$$\begin{array}{llll} b_i = a_{2i} \cdot (-a_{2i-1}), & b = \sum_{i\in\omega} b_i, & b_{i-} = \sum_{j<i} b_j, & b_{i+} = \sum_{j>i} b_j, \\ c_i = a_{2i+1} \cdot (-a_{2i}), & c = \sum_{i\in\omega} c_i, & c_{i-} = \sum_{j<i} c_j, & c_{i+} = \sum_{j>i} c_j. \end{array}$$

Note that from $\mathcal{M}$ being a complete region algebra, it follows that b, c, b_{i-}, c_{i-}, b_{i+}, c_{i+} are all in $\mathcal{M}$.

We now show that, $\neg\mathsf{C}(b_i, b\cdot(-b_i))$, $i \in \omega$. From $b_{i-} \leq a_{2i-2} \ll a_{2i-1}$ and $b_i \leq -a_{2i-1}$ it follows that $\neg\mathsf{C}(b_i, b_{i-})$. From $b_{i+} \leq -a_{2i+1} \ll -a_{2i}$ and $b_i \leq a_{2i}$ it follows that $\neg\mathsf{C}(b_i, b_{i+})$. Now, since $b\cdot(-b_i) = b_{i-} + b_{i+}$, $\neg\mathsf{C}(b_i, b\cdot(-b_i))$. Similarly, $\neg\mathsf{C}(c_i, c\cdot(-c_i))$.

We now show for every connected subregion b' of b that $b' \ll a$. If b' is the empty region, this is immediate. Suppose that b' is nonempty. Since $0 < b' \leq b$, there exists some $i \in \omega$ such that $b' \cdot b_i \neq 0$. From b' being connected and $\neg\mathsf{C}(b_i, b\cdot(-b_i))$, it follows that $b' \leq b_i$. Hence, $b' \leq b_i \leq a_{2i} \ll a$. Similarly, for every connected c' of c is $c' \ll a$.

Finally, from $a = b + c$ and $\mathsf{C}(a,-a)$, it follows that either $\mathsf{C}(b,-a)$ or $\mathsf{C}(c,-a)$, but since no connected subregion of b or c is in contact with $-a$, we get that $\mathcal{M} \models \psi_{CCA}$. □

We show that there exists a region algebra in Σ_{NCA}, which does not satisfy ψ_{CCA}.

Lemma 67. $\mathsf{RCP}(\mathbb{R}^2) \not\models \psi_{CCA}$.

Proof. We already noted that $\mathsf{RCP}(\mathbb{R}^2) \not\models \exists x \exists y \psi_{\odot}(x,y)$, and it is routine to show that $\mathsf{RCP}(\mathbb{R}^2) \models \psi_{ext} \wedge \psi_{int} \wedge \psi_{conn} \wedge \neg\psi_{triv}$. □

We can now state the main result of this section.

Theorem 68.

- *The $\mathcal{L}_{CA}$-theories of Σ_{CA} and Σ_{CCA} are different.*
- *The $\mathcal{L}_{CA}$-theories of Σ_{ECA} and Σ_{CECA} are different.*
- *The $\mathcal{L}_{CA}$-theories of Σ_{NCA} and Σ_{CNCA} are different.*

Proof. Lemma 66 and Lemma 67. □

We can now deduce that $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$ have different $\mathcal{L}_{CA}$-theories. In fact, the following more general result holds.

Theorem 69. *Let $\mathcal{X}$ be a connected, compact and Hausdorff topological space, such that $\mathsf{RC}(\mathcal{X})$ is a non-trivial region algebra. Further, let $\mathcal{M}$ be a finitely decomposable region algebra over $\mathcal{X}$ which respects components. Then $\mathsf{RC}(\mathcal{X})$ and $\mathcal{M}$ have different $\mathcal{L}_{CA}$-theories.*

Although Σ_{CCA}, Σ_{CECA} and Σ_{CNCA} have different $\mathcal{L}_{CA}$-theories from their respective larger classes, in the next theorem we show that none of these classes is $\mathcal{L}_{CA}$-definable. In other words, there is no set of $\mathcal{L}_{CA}$-formulas T_{CCA} such that for every $\mathcal{L}_{CA}$-structure $\mathcal{M}$, $\mathcal{M} \models T_{CCA}$ if and only if $\mathcal{M} \in \Sigma_{CCA}$, and similarly for Σ_{CECA} and Σ_{CNCA}.

Theorem 70. *Σ_{CA}, Σ_{CECA} and Σ_{CNCA} are not $\mathcal{L}_{CA}$-definable.*

Proof. We make use of the Downward Löwenheim-Skolem theorem (see Theorem 13). Since $\Sigma_{CNCA} \subseteq \Sigma_{CECA} \subseteq \Sigma_{CCA}$, it suffices to show that there exists a region algebrain Σ_{CNCA} which is elementary equivalent to a structure in $\Sigma_{NCA} \setminus \Sigma_{CCA}$. Let $\mathcal{M} \in \Sigma_{CNCA}$ be the region algebra $\mathsf{RC}(\mathbb{R})$ considered as an $\mathcal{L}_{CA}$-structure. Choose Q to be the countable set of the elements of $\mathsf{RC}(\mathbb{R})$ that are closed intervals with rational endpoints. Since, $|Q| + |\mathcal{L}_{CA}| \leq \omega \leq |\mathsf{RC}(\mathbb{R})|$, by Theorem 13, it follows that there exists a *countable* elementary substructure $\mathcal{N}$ of $\mathcal{M}$ whose domain contains Q. We claim that $\mathcal{N}$ is not complete. Suppose it is. Every closed interval r in $\mathbb{R}$ is the least upper bound of all elements of Q that are contained in r, i.e. $r = \bigcup\{q \in Q \mid q \leq r\}$. Hence, since $\mathcal{N}$ is complete, every closed interval r in $\mathbb{R}$ will be in $\mathcal{N}$. But that contradicts the fact that $\mathcal{N}$ is countable. □

We extend Theorem 68 to set algebras. Let Σ_{CLA} be the class of all set algebras, and Σ_{CCLA} be the class of all complete set algebras. We will show that Σ_{CLA} and Σ_{CCLA} have different $\mathcal{L}_{cl}$-theories. For every set algebra $\mathcal{M}$, we denote the collection of regular closed sets in $\mathcal{M}$ by $\mathcal{M}_{RC}$.

Lemma 71. *Let $\mathcal{M}$ be a set algebra over a topological space $\mathcal{X}$. $\mathcal{M}_{RC}$ is a region algebra and, if $\mathcal{M}$ is a complete set algebra, then $\mathcal{M}_{RC}$ is a complete region algebra.*

Proof. Clearly, X and $\emptyset$ are in $\mathcal{M}_{RC}$. Also, if $a, b \in \mathcal{M}_{RC}$, then so are $a+b = a\cup b$, $a\cdot b = (a \cap b)^{\circ -}$ and $-a = (a^C)^-$. Further, if $\mathcal{M}$ is a complete set algebra and $a_i \in \mathcal{M}_{RC}$, for $i \in I$, then so are $\sum_{i\in I} a_i = (\bigcup_{i\in I} a_i^\circ)^-$ and $\prod_{i\in I} a_i = -\sum_{i\in I} -a_i$. □

It is an easy exercise to show that there exist an $\mathcal{L}_{cl} \to \mathcal{L}_{CA}$-interpretation of $\mathcal{M}_{RC}$ in $\mathcal{M}$. Hence we have the following:

Lemma 72. *The $\mathcal{L}_{CA}$-theory of $\mathcal{M}_{RC}$ is $\mathcal{L}_{cl}$-definable in $\mathcal{M}$ (see Definition 12).*

Corollary 73. *There exists an $\mathcal{L}_{cl}$-formula ψ_{CLA} such that, for every $\mathcal{M} \in \Sigma_{CLA}$:*

$$\mathcal{M} \models \psi_{CLA} \iff \mathcal{M}_{RC} \models \psi_{CCA}.$$

Lemma 74. *ψ_{CLA} is in the $\mathcal{L}_{cl}$-theory of Σ_{CCLA}.*

Proof. By Lemma 71, whenever $\mathcal{M}$ is a complete set algebra, $\mathcal{M}_{RC}$ is a complete region algebra. From Theorem 68 it then follows that $\mathcal{M}_{RC} \models \psi_{CCA}$. Finally, by Corollary 73, we get that $\mathcal{M} \models \psi_{CLA}$. □

However, there exists a set algebra which does not satisfy ψ_{CLA}.

Lemma 75. *There exists a set algebra $\mathcal{N}$ such that $\mathcal{N} \not\models \psi_{CLA}$.*

Proof. Let $\mathcal{N}$ be the Boolean subalgebra of $(\wp(\mathbb{R}^2), \cup, \cap, ^C, 0, 1)$ generated by $\mathsf{RCP}(\mathbb{R}^2)$. Clearly, $\mathcal{N}_{RC}$ and $\mathsf{RCP}(\mathbb{R}^2)$ coincide. By Lemma 67 it follows that $\mathcal{N}_{RC} \not\models \psi_{CCA}$. Hence, by Lemma 73, $\mathcal{N} \not\models \psi_{CLA}$. □

As a result we get that:

Theorem 76. *The $\mathcal{L}_{cl}$-theories of Σ_{CLA} and Σ_{CCLA} are different.*

Similar to the classes of complete contact algebras, one cannot prove a representation theorem for Σ_{CCLA}.

		Signatures		
		(C) $(\mathsf{c}, \leq)$ $(\mathsf{C}, \mathsf{c}, \leq)$	$(\mathsf{conv}, \leq)$ $(\mathsf{C}, \mathsf{conv})$	(closer)
	$\mathsf{RC}(\mathbb{R})$	Decidable, NONELEMENTARY		Δ^1_ω-complete
D	$\mathsf{RCS}(\mathbb{R})$			Δ^1_ω-complete
o	$\mathsf{RCP}(\mathbb{R})$	Decidable, NONELEMENTARY		Δ^1_ω-complete
m	$\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$			Δ^0_ω-complete
a	$\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$			Δ^0_ω-complete
i	$\mathsf{RC}(\mathbb{R}^n)$	Δ^1_ω-complete	Δ^1_ω-complete	Δ^1_ω-complete
n	$\mathsf{RCS}(\mathbb{R}^n)$	Δ^0_ω-hard	Δ^1_ω-complete	Δ^1_ω-complete
s	$\mathsf{RCP}(\mathbb{R}^n)$	Δ^0_ω-hard	Δ^1_ω-complete	Δ^1_ω-complete
	$\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$	Δ^0_ω-complete	Δ^0_ω-complete	Δ^0_ω-complete
	$\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$	Δ^0_ω-complete	Δ^0_ω-complete	Δ^0_ω-complete

Table 4.1: Complexity map of first-order Euclidean spatial logics.

Theorem 77. *Σ_{CCLA} is not $\mathcal{L}_{cl}$-definable.*

Proof. Identical to the proof of Theorem 70. □

4.3 Computability of Euclidean Spatial Logics

In [Grz51] Grzegorczyk showed that the first-order theories of a wide range of topological structures are computationally at least as hard as the first-order arithmetic of the natural numbers, and hence are undecidable. These include first-order theories of the complete region algebras over $\mathbb{R}^n$, $n > 1$. In this section, we examine the exact complexity of different first-order theories of region algebras over Euclidean spaces. First we show that the Euclidean region algebras over $\mathbb{R}$ have decidable first-order theories, and we establish a lower complexity bound for these theories. Further, we discuss the undecidability of the theories of the Euclidean region algebras over $\mathbb{R}^n$, $n > 1$, and establish tight upper complexity bounds for most of them. As a consequence of the tight complexity bounds, we derive a surprising model-theoretic result (Theorem 104). The results in this section appeared as [NPH10], and are summarised in Table 4.1.

4.3.1 Decidability over $\mathbb{R}$: Upper Bounds

We show that the $\mathcal{L}_{\mathsf{C}}$-theory of $\mathsf{RC}(\mathbb{R})$ is definable in the monadic second-order theory of $(\mathbb{Q}, <)$, which was shown to be decidable in [Rab69].

Denote by $\mathcal{L}_<$ the monadic second-order language of $(\mathbb{Q}, <)$. To establish an $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}_<$ interpretation Γ of $\mathsf{RC}(\mathbb{R})$ in $\mathbb{Q}$, we identify each region $a \in \mathsf{RC}(\mathbb{R})$ with the set of rational points contained in it, i.e. $a \cap \mathbb{Q}$. Formally, we define a function $f : \wp(\mathbb{R}) \to \wp(\mathbb{Q})$ by $f(a) = a \cap \mathbb{Q}$. We already showed in Lemma 40 that $f \restriction \mathsf{RC}(\mathbb{R})$ is injective, hence its inverse is well defined. To complete Γ, we introduce $\mathcal{L}_<$-formulas $\psi_{RC}(X)$ and $\psi_C(X, Y)$ defining, respectively, the f-images of the regions in $\mathsf{RC}(\mathbb{R})$, and the contact relation on $\mathsf{RC}(\mathbb{R})$.

Denote by τ the open sets in $\mathbb{R}$. It is readily checked that if two open subsets of $\mathbb{R}$ intersect, then they share a rational point. Similarly, if an open set and a regular closed set intersect, then they share a rational point. We now state these facts formally, and we will use them implicitly throughout this section.

Fact 78. *Let o_1, o_2 be open sets in $\mathbb{R}$, and $a \in \mathsf{RC}(\mathbb{R})$. Then $o_1 \cap a = \emptyset$ if and only if $f(o_1) \cap f(a) = \emptyset$, and $o_1 \cap o_2 = \emptyset$ if and only if $f(o_1) \cap f(o_2) = \emptyset$.*

The set of f-images of *intervals* is defined by the formula:

$$\psi_i(X) := \forall x \forall y (X(x) \wedge X(y) \wedge x < y \to \forall z (x < z \wedge z < y \to X(z))).$$

The set of f-images of *open intervals* is defined by the formula:

$$\psi_o(X) := \psi_i(X) \wedge \forall x (X(x) \to \exists y \exists z (X(y) \wedge X(z) \wedge y < x \wedge x < z)).$$

We encode real numbers using Dedekind cuts, i.e. we identify a real number r with the pair (L, R) of sets of rational numbers for which $L = \{q \in \mathbb{Q} \mid q \leq r\}$ and $R = \{q \in \mathbb{Q} \mid q \geq r\}$. We define the set of these pairs by the formula:

$$\psi_{\mathbb{R}}(L, R) := L \cup R = 1 \wedge L \neq 0 \wedge R \neq 0 \wedge \forall x \forall y (L(x) \wedge R(y) \to x \leq y).$$

We define the set of pairs (q, A), where q is a rational number which is isolated from the set of rational numbers A, using the formula:

$$\psi_{iso}(x, X) := \exists Y_1 (Y_1(x) \wedge \psi_o(Y_1) \wedge \forall Y_2 (\psi_o(Y_2) \wedge Y_2 \subseteq X \to Y_1 \cap Y_2 = 0)).$$

We can define the set of f-images of regular closed sets in $\mathbb{R}$ using the formula:

$$\psi_{RC}(X) := \forall x(X(x) \leftrightarrow \neg\psi_{iso}(x, X)).$$

Lemma 79. *Let $t \subseteq \mathbb{Q}$. Then $\mathbb{Q} \models \psi_{RC}[t]$ if and only if there exists $a \in \mathsf{RC}(\mathbb{R})$ such that $f(a) = t$.*

Proof. ($\Rightarrow$) Let $t \subseteq \mathbb{Q}$ be such that $\mathbb{Q} \models \psi_{RC}[t]$. We define

$$a = \bigcup \{o \mid \mathbb{Q} \models \psi_o[f(o)] \text{ and } f(o) \subseteq t\}.$$

Since a is open, a^- must be regular closed. We have to show that $f(a^-) = t$. For $q \in \mathbb{Q}$, we have that $q \notin t$ if and only if $\mathbb{Q} \models \psi_{\mathbb{Q}iso}[q, t]$. I.e., $q \notin t$ if and only if there exists a neighbourhood of q which is disjoint from the f-image of every open interval contained in t. Hence, $q \notin t$ if and only if q is isolated from a, which is exactly when $q \notin f(a^-)$.

($\Leftarrow$) Let $a \in \mathsf{RC}(\mathbb{R})$. We will show that $\mathbb{Q} \models \psi_{RC}[f(a)]$. For $q \in \mathbb{Q}$ we have that $q \in f(a)$ if and only if every neighbourhood of q intersects a°, which is exactly when every neighbourhood of q intersects some open interval contained in a°. Hence, $q \in f(a)$ if and only if $\mathbb{Q} \models \neg\psi_{\mathbb{Q}iso}[q, f(a)]$. □

We now show how to define the contact relation. First, we define the set of tuples (U, V, W) in $\wp(\mathbb{Q})$ such that (U, V) represents a point $r \in \mathbb{R}$ and W represents a connected open neighbourhood of r.

$$\psi_\odot(L, R, X) := \psi_\mathbb{R}(L, R) \wedge \psi_o(X) \wedge L \cap X \neq 0 \wedge R \cap X \neq 0$$

Then, we define the set of tuples (U, V, W) in $\wp(\mathbb{Q})$ such that (U, V) represents a point $r \in \mathbb{R}$ that is in the closure of W.

$$\psi_\in(L, R, X) := \psi_\mathbb{R}(L, R) \wedge \forall N(\psi_\odot(L, R, N) \to N \cap X \neq 0).$$

Finally, two regular closed sets are in contact exactly when their f-images satisfy the formula:

$$\psi_C(X, Y) := \exists L \exists R(\psi_\in(L, R, X) \wedge \psi_\in(L, R, Y)).$$

Lemma 80. *Let t and u be subsets of $\mathbb{Q}$ such that $\mathbb{Q} \models \psi_{RC}[t]$ and $\mathbb{Q} \models \psi_{RC}[u]$.*

Then, $\mathbb{Q} \models \psi_C[t, u]$ *if and only if* $\mathsf{C}(f^{-1}(t), f^{-1}(u))$.

Proof. $\mathsf{C}(f^{-1}(t), f^{-1}(u))$ if there exists a real number r contained in $f^{-1}(t)$ and $f^{-1}(u)$, which is exactly when there exists a real number r every neighbourhood o of which intersects both $f^{-1}(t)$ and $f^{-1}(u)$. Hence, $\mathsf{C}(f^{-1}(t), f^{-1}(u))$ if there exists a real number r such that for every neighbourhood o of r, $f(o)$ intersects t and u, which is exactly when $\mathbb{Q} \models \psi_C[t, u]$. □

Hence we have the following:

Lemma 81. *The* $\mathcal{L}_{\mathsf{C}}$*-theory of* $\mathsf{RC}(\mathbb{R})$ *is* $\mathcal{L}_<$*-definable in* $\mathbb{Q}$*.*

Proof. The formulas $\psi_{RC}(X)$ and $\psi_C(X, Y)$, together with the inverse function of $f \restriction \mathsf{RC}(\mathbb{R})$ constitute an $\mathcal{L}_{\mathsf{C}} \to \mathcal{L}_<$-interpretation of $\mathsf{RC}(\mathbb{R})$ in $\mathbb{Q}$. □

Theorem 82. *[Rab69] The monadic second-order theory of* $(\mathbb{Q}, <)$ *is decidable.*

Theorem 83. *The* $\mathcal{L}_{\mathsf{C}}$*-,* $\mathcal{L}_c$*- and* $\mathcal{L}_{conv}$*-theories of* $\mathsf{RC}(\mathbb{R})$*,* $\mathsf{RCS}(\mathbb{R})$*,* $\mathsf{RCP}(\mathbb{R})$*,* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$ *are decidable.*

Proof. A direct consequence of Theorem 82, Lemma 81, Lemma 60, Lemma 49 and the fact that a region in $\mathsf{RC}(\mathbb{R})$ is convex exactly when it is nonempty and connected. □

The decidability of the theories considered in this section was shown by reduction to the monadic second-order theory of $(\mathbb{Q}, <)$, which however is known to be non-elementary. In the following section, we show that the theories of interest are also non-elementary.

4.3.2 Decidability over $\mathbb{R}$: Lower Bounds

We now show that the $\mathcal{L}_c$-theory of $\mathsf{RCP}(\mathbb{R})$ is non-elementary by introducing a polynomial reduction from the weak-monadic second-order theory of one successor to the $\mathcal{L}_c$-theory of $\mathsf{RCP}(\mathbb{R})$. Let $\mathcal{L}_{S1S}$ denotes the monadic second-order language of the structure $(\mathbb{N}, S)$, where $S = \{(n, n+1) \mid n \in \mathbb{N}\}$ is the successor relation, and let $WS1S$ denotes the weak-monadic second-order theory of $(\mathbb{N}, S)$.[1] $WS1S$ was shown to be non-elementary by Meyer [Mey75].

[1]Weak-monadic second order logic is an extension of first-order logic that allows quantification over finite subsets of the domain (see Section 2.1).

In this section we use the term *regions* to denote the elements of RCP($\mathbb{R}$). For $n \in \mathbb{N}$, we encode the initial segment of natural numbers $N = \{0, \ldots, n\}$ of $\mathbb{N}$ by pairs of non-overlapping regions (a, b) in RCP($\mathbb{R}$) (later defined by the formula $\psi_{\vdash}(x, y)$) such that: the components of $a + b$ are bounded intervals; a is connected and nonempty; and all the components of b are on the same side of a (see Figure 4.2). A natural number $k \in N$ is encoded by the $(k+1)$th component of b closest to a. A (finite) subset of N is represented by the sum (in RCP($\mathbb{R}$)) of the representatives of its members. First, we introduce some

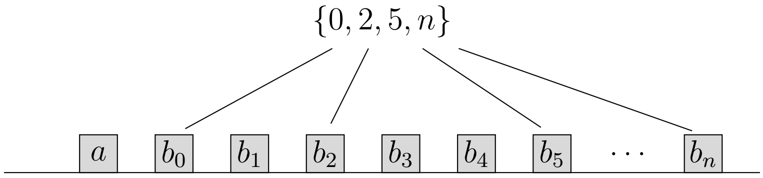


Figure 4.2: A pair of regions (a, b) in RCP($\mathbb{R}$) encoding the set $N = \{0, \ldots, n\}$. Every $k \in N$ is encoded by the component b_k of b. The set $\{0, 2, 5, n\}$ is encoded by the region $b_0 + b_2 + b_5 + b_n$.

simple formulas. The following formula defines the set of pairs of regions (a, b) such that a is a connected component of b:

$$\psi_{cc}(x, y) := \mathsf{c}(x) \wedge x \leq y \wedge \forall z(\mathsf{c}(z) \wedge x \leq z \wedge z \leq y \rightarrow z \leq x).$$

Consider the formula

$$\psi_{ord}(x, y, z) := \mathsf{c}(x) \wedge \mathsf{c}(y) \wedge \mathsf{c}(z) \wedge \neg \mathsf{c}(y+x) \wedge \neg \mathsf{c}(y+z) \wedge \forall t(\mathsf{c}(t) \wedge x+z \leq t \rightarrow y \leq t).$$

Regions a, b and c satisfy ψ_{ord} if they are pairwise disjoint, each of them is connected and the endpoints of b are between the endpoints of a and the endpoints of c (Figure 4.3).

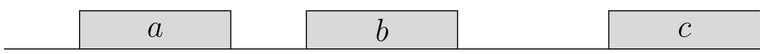


Figure 4.3: A tuple of regions (a, b, c) satisfying $\psi_{ord}(x, y, z)$ in RCP($\mathbb{R}$).

The formula

$$\psi_{\sqsubseteq}(x, y) := \forall x'(\psi_{cc}(x', x) \rightarrow \psi_{cc}(x', y)),$$

defines the pairs of regions (a, b) for which every component of a is a component of b. We define the pairs of regions encoding initial segments of $\mathbb{N}$ using the formula:

$$\begin{aligned}\psi_{\vdash}(x,y) \quad := \quad & \mathsf{c}(x) \wedge \neg \mathsf{c}(-x) \wedge x \cdot y = 0 \wedge \forall z(\psi_{cc}(z,y) \rightarrow \neg \mathsf{c}(-z)) \wedge \\ & \forall z \forall t(\psi_{cc}(z,y) \wedge \psi_{cc}(t,y) \rightarrow \neg \psi_{ord}(z,x,t));\end{aligned}$$

We need a way of extending initial segments of $\mathbb{N}$. We use the following formula to compare encodings of initial segments of $\mathbb{N}$.

$$\begin{aligned}\psi_{\preceq}(x,y,z) \quad := \quad & \psi_{\vdash}(x,y) \wedge \psi_{\vdash}(x,z) \wedge \psi_{\sqsubseteq}(y,z) \wedge \\ & \forall u \forall v(\psi_{cc}(u,y) \wedge \psi_{cc}(v,z) \wedge \psi_{ord}(x,v,u) \rightarrow \psi_{cc}(v,y)).\end{aligned}$$

This formula defines the set of tuples of regions (a, b, c) for which:

- (a, b) and (a, c) encode initial segments N and M of $\mathbb{N}$, with $N \subseteq M$;
- (a, b) and (a, c) are compatible, i.e. each $k \in N$ is represented in (a, b) and (a, c) by the same region. (See Figure 4.4)

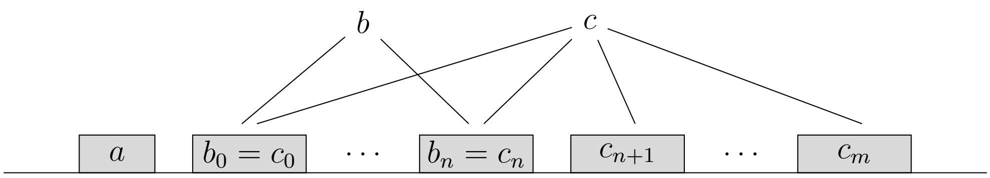


Figure 4.4: The pairs of regions (a, b) and (a, c) are compatible encodings of initial segments of $\mathbb{N}$. The tuple (a, b, c) satisfies the formula $\psi_{\preceq}(x, y, z)$.

We now define formally the encoding of natural numbers and finite sets of natural numbers.

Definition 84. Let (c, d) be a pair of regions satisfying $\psi_{\vdash}(x, y)$. Further, let $a_1, \ldots, a_s$ be regions such that for each a_i, the pair (a_i, d) satisfies $\psi_{cc}(x, y)$. Finally, let $b_1, \ldots, b_t$ be regions such that for each b_i, the pair (b_i, d) satisfies the formula $\psi_{\sqsubseteq}(x, y)$. We define functions f and g as follows:

$$\begin{aligned} f(c,d,a_i) &:= |\{a' \mid \mathsf{RCP}(\mathbb{R}) \models \psi_{cc}[a',d], \mathsf{RCP}(\mathbb{R}) \models \psi_{ord}[c,a',a_i]\}|, \\ g(c,d,b_j) &:= \{f(c,d,b') \mid \mathcal{M} \models \psi_{cc}[b',b_i]\}, \\ f(c,d,a_1,\ldots,a_s) &:= (f(c,d,a_1),\ldots,f(c,d,a_s)), \\ g(c,d,b_1,\ldots,b_t) &:= (f(c,d,b_1),\ldots,f(c,d,b_t)). \end{aligned}$$

So, given a pair (c, d) of regions which encodes an initial segment N of $\mathbb{N}$ (i.e. $\mathsf{RCP} \models \psi_{\vdash}[c, d]$), and given a component a of d, $f(c, d, a)$ returns the number of components of d that are between c and a. If $d_1, \ldots, d_s$ are components of d and $b = d_1 + \cdots + d_s$, then $g(c, d, b)$ returns the set of natural numbers encoded by $d_1, \ldots, d_s$. A number $k \in \mathbb{N}$ represented by a region a is contained in a finite $A \subseteq \mathbb{N}$ represented by a region b exactly when a is a connected component of b. Thus we define:

$$\psi_{\in}(x, y) := \psi_{cc}(x, y).$$

Let the pair of regions (c, d) represents an initial segment N of $\mathbb{N}$, and let the regions a and b represent in (c, d) some natural numbers $n_1, n_2 \in N$. Then $n_1 + 1 = n_2$ exactly when: a and b are distinct components of d; a is between c and b; and there are no components of d between a and b. This is exactly when (a, b, c, d) satisfy the formula

$$\begin{aligned} \psi_S(x,y,x_0,x_1) \;:=\; & \psi_{cc}(x,x_1) \wedge \psi_{cc}(y,x_1) \wedge \psi_{ord}(x_0,x,y) \wedge x \neq y \wedge \\ & \forall z(\psi_{cc}(z,x_1) \wedge \psi_{ord}(x_0,z,y) \wedge z \neq y \rightarrow \psi_{ord}(x_0,z,x)). \end{aligned}$$

For every $\mathcal{L}_{S1S}$-formula φ, we denote by $\delta(\varphi)$ the *quantifier depth* of φ. Let $\mathcal{L}^p_{S1S}$ be the set of $\mathcal{L}_{S1S}$-formulas which are in prenex normal form. We define a translation from $\mathcal{L}^p_{S1S}$ to $\mathcal{L}_c$.

Definition 85. For every $\mathcal{L}^p_{S1S}$-formula φ, we define an $\mathcal{L}_c$-formula φ_Γ inductively:

$$
\begin{aligned}
(x_n = x_m)_\Gamma &:= p_n = p_m,\\
(X_n = X_m)_\Gamma &:= q_n = q_m,\\
(S(x_n, x_m))_\Gamma &:= \psi_S(p_n, p_m, r, s_0),\\
(x_n \in X)_\Gamma &:= \psi_\in(p_n, q_m),\\
(\neg\psi)_\Gamma &:= \neg\psi_\Gamma,\\
(\psi' \wedge \psi'')_\Gamma &:= \psi'_\Gamma \wedge \psi''_\Gamma,\\
(\exists x_n \psi)_\Gamma &:= \exists p_n \exists s_{\delta(\psi)} \big(\psi_\preceq(r, s_{\delta(\psi)+1}, s_{\delta(\psi)}) \wedge \psi_{cc}(p_n, s_{\delta(\psi)}) \wedge \psi_\Gamma\big),\\
(\forall x_n \psi)_\Gamma &:= \forall p_n \forall s_{\delta(\psi)} \big(\psi_\preceq(r, s_{\delta(\psi)+1}, s_{\delta(\psi)}) \wedge \psi_{cc}(p_n, s_{\delta(\psi)}) \to \psi_\Gamma\big),\\
(\exists X_n \psi)_\Gamma &:= \exists q_n \exists s_{\delta(\psi)} \big(\psi_\preceq(r, s_{\delta(\psi)+1}, s_{\delta(\psi)}) \wedge \psi_\sqsubseteq(q_n, s_{\delta(\psi)}) \wedge \psi_\Gamma\big),\\
(\forall X_n \psi)_\Gamma &:= \forall q_n \forall s_{\delta(\psi)} \big(\psi_\preceq(r, s_{\delta(\psi)+1}, s_{\delta(\psi)}) \wedge \psi_\sqsubseteq(q_n, s_{\delta(\psi)}) \to \psi_\Gamma\big).
\end{aligned}
$$

Note that the mapping $(\cdot)_\Gamma : \mathcal{L}^p_{S1S} \to \mathcal{L}_c$ is polynomial-time computable, and we now show that it also preserves satisfiability.

Lemma 86. *For every:*

- *formula $\varphi \in \mathcal{L}^{PMon}_{S1S}$ with free variables among $(x_1, \ldots, x_s, X_1, \ldots, X_t)$;*
- *$c, d_{\delta(\varphi)} \in M$ such that $\mathcal{M} \models \psi_\vdash[c, d_{\delta(\varphi)}]$;*
- *$(\bar{a}, \bar{b}) = (a_1, \ldots, a_s, b_1, \ldots, b_t) \in M^{s+t}$ such that:*
  - *$\mathcal{M} \models \psi_\in[a_i, d_{\delta(\varphi)}]$, for $i = 1, \ldots, s$, and*
  - *$\mathcal{M} \models \psi_\sqsubseteq[b_i, d_{\delta(\varphi)}]$, for $i = 1, \ldots, t$;*

we have that:

$$\mathsf{RCP}(\mathbb{R}) \models \varphi_\Gamma[\bar{a}, \bar{b}, c, d_{\delta(\varphi)}] \iff \mathbb{N} \models \varphi[f(c, d_{\delta(\varphi)}, \bar{a}), g(c, d_{\delta(\varphi)}, \bar{b})].$$

Proof. We proceed by induction on the complexity of φ. In the case when φ is an atomic formula, the claim follows from the fact that f and g are injective, and from the properties of the formulas $\psi_S(x, y, x_0, x_1)$ and $\psi_\in(x, y)$. In the case when φ is of the form $\neg\psi$ or $(\psi_1 \wedge \psi_2)$, the claim is a direct consequence of the inductive hypothesis and the definition of $(\cdot)_\Gamma$. The only non-trivial cases are when φ is of the form $\exists x_n\psi$, $\forall x_n\psi$, $\exists X_n\psi$ or $\forall X_n\psi$. Since the proofs of the four cases are identical, we show only the case when φ is of the form $\exists x_n\psi$. We have that $\varphi_\Gamma = \exists p_n \exists s_{\delta(\psi)} \psi_\preceq(s_{\delta(\psi)+1}, s_{\delta(\psi)}) \wedge \psi_{cc}(p_n, s_{\delta(\psi)}) \wedge \psi_\Gamma$. So:

$$\begin{aligned}
&\mathsf{RCP}(\mathbb{R}) \models \varphi_\Gamma[\bar{a}, \bar{b}, c, d_{\delta(\varphi)}] \\
&\quad\iff \text{for some } a_{s+1}, d_{\delta(\psi)} \in \mathsf{RCP}(\mathbb{R}), \mathsf{RCP}(\mathbb{R}) \models \psi_\preceq[c, d_{\delta(\varphi)}, d_{\delta(\psi)}], \\
&\qquad\quad \mathsf{RCP}(\mathbb{R}) \models \psi_\in[a_{s+1}, d_{\delta(\psi)}] \text{ and } \mathsf{RCP}(\mathbb{R}) \models \psi_\Gamma[\bar{a}, a_{s+1}, \bar{b}, c, d_{\delta(\psi)}] \\
&\quad\iff \text{for some } a_{s+1}, d_{\delta(\psi)} \in \mathsf{RCP}(\mathbb{R}), \mathsf{RCP}(\mathbb{R}) \models \psi_\preceq[c, d_{\delta(\varphi)}, d_{\delta(\psi)}], \\
&\qquad\quad \mathsf{RCP}(\mathbb{R}) \models \psi_\in[a_{s+1}, d_{\delta(\psi)}] \text{ and } \mathbb{N} \models \psi[f(c, d_{\delta(\psi)}, \bar{a}, a_{s+1}), g(c, d_{\delta(\psi)}, \bar{b})] \\
&\quad\iff \text{for some } n_{a_{s+1}} \in \mathbb{N}, \mathbb{N} \models \psi[f(c, d_{\delta(\psi)}, \bar{a},), n_{a_{s+1}}, g(c, d_{\delta(\psi)}, \bar{b})] \\
&\quad\iff \mathbb{N} \models \varphi[f(\bar{a}), g(\bar{b})]. \qquad \square
\end{aligned}$$

As a direct consequence of Lemma 86, we get that:

Lemma 87. *For every sentence* $\varphi \in \mathcal{L}^p_{S1S}$

$$\mathbb{N} \models \varphi \iff \mathsf{RCP}(\mathbb{R}) \models \forall r \forall s_{\delta(\varphi)} \psi_\vdash(r, s_{\delta(\varphi)}) \to \varphi_\Gamma.$$

Theorem 88. *[Mey75] The monadic second-order theory of the structure* $(\mathbb{N}, S)$ *is non-elementary.*

Since the mapping $(\cdot)_\Gamma : \mathcal{L}^p_{S1S} \to \mathcal{L}_c$ is polynomial-time computable, we get the following corollary.

Corollary 89. *The* $\mathcal{L}_c$*-theory of* $\mathsf{RCP}(\mathbb{R})$ *is non-elementary.*

More generally:

Theorem 90. *The* $\mathcal{L}_c$*- and* $\mathcal{L}_\mathsf{C}$*-theories of* $\mathsf{RC}(\mathbb{R})$*,* $\mathsf{RCS}(\mathbb{R})$*,* $\mathsf{RCP}(\mathbb{R})$*,* $\mathsf{RCP}_\mathbb{A}(\mathbb{R})$ *and* $\mathsf{RCP}_\mathbb{Q}(\mathbb{R})$ *are non-elementary.*

Proof. Follows from Corollary 89, Lemma 49 and Lemma 60. □

In this section we showed that, although the topological theories of region algebras over $\mathbb{R}$ are decidable, they are non-elementary. In the following section, we show that the Euclidean spatial logics in higher dimensions are all undecidable.

4.3.3 Undecidability over $\mathbb{R}^n$: Lower Bounds

In this section we show that, for $n > 1$, the $\mathcal{L}_c$-theory of every region subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ that extends $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$ is Δ^0_ω-hard, and that the $\mathcal{L}_c$-theory of $\mathsf{RC}(\mathbb{R}^n)$ is Δ^1_ω-hard. We combine ideas from [Dav06] and [Grz51].

Let $\mathcal{M}$ be a region subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ that extends $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, for $n > 1$. In this section, we use the term regions to denote the members of $\mathcal{M}$. For every region a, denote by $|a|$ the number of connected components of a. We define the first-order theory of $(\mathbb{N}, +, \cdot)$ in the $\mathcal{L}_c$-theory of $\mathcal{M}$, by encoding every natural number k by the regions in $\mathcal{M}$ having k connected components. Every region with finitely many components encodes a natural number, so we need to define in $\mathcal{L}_c$ the set of all such regions. Before that, we provide an $\mathcal{L}_c$-formula $\psi_\sim(x, y)$ that is satisfied by the pairs of regions (a, b) in $\mathcal{M}$ having the same number of connected components.

As before, the $\mathcal{L}_c$-formula

$$\psi_{cc}(x, y) := \mathsf{c}(x) \wedge x \leq y \wedge \forall z(\mathsf{c}(z) \wedge x \leq z \wedge z \leq y \rightarrow z \leq x)$$

defines the set of pairs of regions (a, b) in $\mathcal{M}$ such that a is a connected component of b.

For regions a and a' in $\mathcal{M}$, we say that a' is a *shrinking* of a if and only if every connected component of a' is contained in a connected component of a and every connected component of a contains exactly one connected component of a'. The $\mathcal{L}_c$-formula

$$\psi_{shrink}(x, y) := x \leq y \wedge \forall y'(\psi_{cc}(y', y) \rightarrow \psi_{cc}(x \cdot y', x))$$

defines the pairs of regions (a', a) such that a' is a shrinking of a. Clearly, if a region is a shrinking of another region, then the two regions have the same number of components.

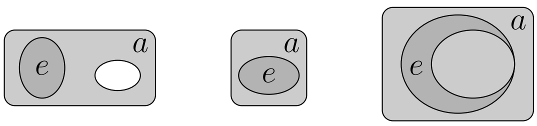


Figure 4.5: Two regions a and e such that e is a shrinking of a.

We say that a region c is a *wrapping* of the regions a and b if and only if a

and b are contained in c, and every connected component of c contains exactly one connected component of both a and b. Evidently, the formula

$$\psi_{d\sim}(x,y) := \exists z(x+y \leq z \wedge \forall z'(\psi_{cc}(z',z) \rightarrow \psi_{cc}(x \cdot z', x) \wedge \psi_{cc}(y \cdot z', y))$$

defines the pairs of regions (a,b) for which there exists a region c which is a wrapping of a and b. Again, if two regions a and b have a common wrapping, then they have the same number of components.

In the following lemma we show that the binary relation $|a| = |b|$ can be expressed in terms of "shrinking" and "wrapping".

Lemma 91. *Let $a, b \in \mathcal{M}$. Then a and b have the same number of components exactly when there exist regions a', b' and c such that a' and b' are shrinkings of a and b, and c is a wrapping of a' and b'.*

Proof. ($\Leftarrow$) Since the regions a' and b' have a common wrapping, $|a'| = |b'|$. Further, since a' and b' are shrinkings of a and b, $|a| = |a'|$ and $|b| = |b'|$. Hence, $|a| = |b|$.

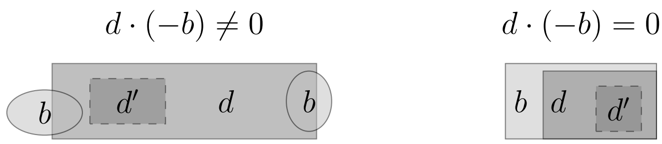


Figure 4.6: Shrinking a component d of b to a closed n-dimensional rational hypercube that is either disjoint from b or contained in the interior of a component of b.

($\Rightarrow$) Denote by C the set of closed n-dimensional rational hypercubes. Note that C is dense in $\mathcal{M}$. Let a and b have the same number of components. We show how to construct shrinkings of a and b, whose components are pairwise disjoint regions in C.

Consider a component d of a that has a nonempty interior. If $d \cdot (-b)$ is nonempty, choose d' to be a region in C that is contained in the interior of $d \cdot (-b)$. Otherwise, take d' to be a region in C that is contained in the interior of d. Hence, d' is either disjoint from b, or contained in the interior of a component of b. Define $a' = \sum \{d' \mid d \text{ a component of } a\}$. If e is a component of b that has a nonempty interior, then $e \cdot (-a')$ is nonempty. Indeed, suppose that e is part of a'. Since e is connected, it is part of a component d of a'. As we previously observed, d must also be part of e, and in fact contained in the interior of e. So, $d = e$ and $d \subseteq e^\circ = d^\circ$, which contradicts the choice of d. Hence, for every component e of b we can choose a region e' in C contained in $e \cdot (-a')$. Define $b' = \{e' \mid e \text{ is a component of } b\}$. Let η be a countable cardinal number (possibly infinite), and let $\{a_i\}_{i<\eta}$ and $\{b_i\}_{i<\eta}$ be the components of a' and b' that have nonempty interiors. Set $c_0 = a' + b'$. Note that the complement of c_0 is connected so that we can connected the components a_0 and b_0 with regular closed "rod" d_0 maintaining the connectivity of the complement of $c_1 = c_0 + d_0$. Hence, we can connect the components a_1 and b_1 in the complement of c_1 with a regular closed "rod" d_1 maintaining the connectivity of the complement of $c_2 = c_1 + d_1$. In general, for each $i < \eta$, we connect a_i with b_i in the complement of c_i with a regular closed d_i so that $c_{i+1} = d_i + c_i$ has a connected complement. Finally, we take $c = \sum \{c_i\}_{i<\eta}$. Clearly, c is a wrapping of a and b. □

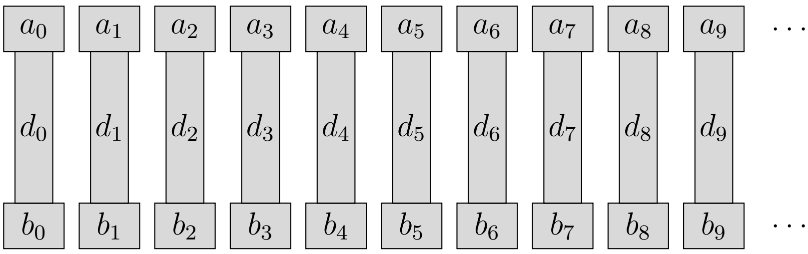


Figure 4.7: Connecting up the components of a and b.

As a result, we get that two regions have the same number of components if and only if they satisfy the formula

$$\psi_{\sim}(x, y) := \exists x' \exists y' (\psi_{shrink}(x', x) \wedge \psi_{shrink}(y', y) \wedge \psi_{d\sim}(x', y'))$$

It is easy to see that the formula

$$\psi_S(x, y) := \exists x' (\psi_{cc}(x', x) \wedge \psi_{\sim}(x \cdot -x', y))$$

defines the pairs of regions (a, b) for which $|a| = |b| + 1$ (taking $\aleph_0 + 1 = \aleph_0$). Hence, the formula

$$\psi_{fin}(x) := \neg \psi_S(x, x)$$

defines the regions having finitely many connected components. Since $\mathcal{M}$ extends $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, we get the following lemma.

Lemma 92. *The function* $f_\Gamma : \psi_{fin}(\mathcal{M}) \to \mathbb{N}$ *defined by*

$$f_\Gamma(a) = |a|$$

is surjective.

Two regions have disjoint components if they satisfy the $\mathcal{L}_c$-formula:

$$\psi_{\neg C}(x, y) := \forall x' \forall y' (\psi_{cc}(x, x') \land \psi_{cc}(y, y') \to \neg \mathsf{c}(x' + y')).$$

We define the arithmetic operations on natural numbers by the formulas:

$$\psi_+(x, y, z) := \exists x' \exists y' (\psi_\sim(x, x') \land \psi_\sim(y, y') \land \psi_{\neg C}(x', y') \land \psi_\sim(x' + y', z));$$
$$\psi_\times(x, y, z) := \exists u \exists v [\psi_{shrink}(u, z) \land u \leq v \land \psi_\sim(v, y) \land \forall t (\psi_{cc}(t, v) \to \psi_\sim(t \cdot u, x))].$$

Lemma 93. *Let* a*,* b *and* c *be regions in* $\mathcal{M}$ *having finitely many components. Then:*

$$\begin{aligned} \mathcal{M} \models \psi_+[a, b, c] &\iff |a| + |b| = |c|; \\ \mathcal{M} \models \psi_\times[a, b, c] &\iff |a| \cdot |b| = |c|. \end{aligned}$$

Proof. The only difficulty is to show that if $|a| \cdot |b| = |c|$, then $\mathcal{M} \models \psi_\times[a, b, c]$. We proceed as in Lemma 91, to show that there exists a shrinking c' of c whose connected components are closed n-dimensional hypercubes. It is routine to show by induction on the number of components of b that there exist connected regions $\{d_i\}_{i<|b|}$, each containing exactly $|a|$ components of c' and $\neg\mathsf{C}(d_i, \sum_{j \neq i} d_j)$. Thus, $\mathcal{M} \models \psi_\times[a, b, c]$. □

We have shown that:

Lemma 94. *There exists an interpretation of* Δ^0_ω *in the* $\mathcal{L}_c$*-theory of* $\mathcal{M}$*.*

As a result we get that:

Theorem 95. *Let* σ *be one of the signatures* (C)*,* $(\mathsf{c}, \leq)$*,* $(\mathsf{conv}, \leq)$ *and* (closer)*, and let* $\mathcal{M}$ *be any of* $\mathsf{RC}(\mathbb{R}^n)$*,* $\mathsf{RCS}(\mathbb{R}^n)$*,* $\mathsf{RCP}(\mathbb{R}^n)$*,* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$*,* $n > 1$*. Then the first-order* σ*-theory of* $\mathcal{M}$ *is* Δ^0_ω*-hard.*

Proof. Lemma 94, Lemma 49 and Lemma 17. □

We now show that the first-order theory of $(\mathbb{N}, \wp(\mathbb{N}), +, \cdot, \in)$ is definable in the $\mathcal{L}_c$-theory of $\mathsf{RC}(\mathbb{R}^n)$. We identify every set $A \subseteq \mathbb{N}$ with a pair of regions (a, b) in $\mathsf{RC}(\mathbb{R}^n)$ such that, for every $k \in \mathbb{N}$, $k \in A$ if and only if there exists a connected component a' of a with $|a' \cdot b| = k$. We define the set of such pairs using the formula

$$\psi_{set}(x, y) := \forall x'(\psi_{cc}(x', x) \rightarrow \psi_{fin}(x' \cdot y)).$$

The membership relation can then defined using the formula

$$\psi_{\in}(x, y, z) := \exists x'(\psi_{cc}(x', x) \wedge \psi_{\sim}(z, x' \cdot y)).$$

Let the regions a, b and c in $\mathsf{RC}(\mathbb{R}^n)$ be such that c represents a natural number k and (a, b) represents a set of natural numbers A. Then (a, b, c) satisfies $\psi_{\in}(x, y, z)$ if and only if $k \in A$. Two pairs of regions (a, b) and (a', b') in $\mathsf{RC}(\mathbb{R}^n)$ represent the same set of natural numbers if and only if (a, b, a', b') satisfy the formula:

$$\psi_{set\sim}(x, y, x', y') := \forall z(\psi_{\in}(x, y, z) \leftrightarrow \psi_{\in}(x', y', z)).$$

We have to define a surjective map $\psi_{set}(\mathsf{RC}(\mathbb{R}^n)) \rightarrow \wp(\mathbb{N})$.

Lemma 96. *The function* $f'_\Gamma : \psi_{set}(\mathcal{M}) \rightarrow \wp(\mathbb{N})$ *is surjective, where*

$$f'_\Gamma(a, b) = \{|a' \cdot b| : a' \textit{ is a connected component of } a\}.$$

The encodings of natural numbers and sets of natural numbers are compatible in the following sense.

Lemma 97. *Let* $a, b, c \in \mathsf{RC}(\mathbb{R}^n)$*. If* $\mathsf{RC}(\mathbb{R}^n) \models \psi_{fin}[a]$ *and* $\mathcal{M} \models \psi_{set}[b, c]$*, then*

$$\mathcal{M} \models \psi_{\in}[a, b, c] \iff f(a) \in f'(b, c).$$

We have shown the following lemma.

Lemma 98. *The first-order theory of* $(\mathbb{N}, \wp(\mathbb{N}), +, \cdot, \in)$ *is* $\mathcal{L}_c$*-definable in* $\mathsf{RC}(\mathbb{R}^n)$*,* $n > 1$.

As a result we also have that:

Theorem 99. *Let σ be one of the signatures* (C), $(\mathsf{c}, \leq)$, $(\mathsf{conv}, \leq)$ *and* (closer). *Then the first-order σ-theory of* $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$, *is Δ^1_ω-hard.*

Proof. Lemma 98, Lemma 49 and Lemma 18. □

The first-order theories of some of the structures considered in Theorem 95 and Theorem 99 are known to have even higher computational complexities. In [Dav06] it was shown that if $\sigma = (\mathsf{conv}, \leq)$ or $\sigma = (\mathsf{closer})$, then the first-order σ-theories of $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, are Δ^1_ω-hard. In contrast with the results that we established in Section 4.3.1, it was also shown in [Dav06] that the (closer)-theories of $\mathsf{RC}(\mathbb{R})$, $\mathsf{RCS}(\mathbb{R})$ and $\mathsf{RCP}(\mathbb{R})$ are Δ^1_ω-hard, and that the (closer)-theories of $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$ are Δ^0_ω-hard. These results are summarised in the following lemmas.

Lemma 100. *[Dav06, Section 5]*

- *The $\mathcal{L}_{closer}$-theories of* $\mathsf{RC}(\mathbb{R})$, $\mathsf{RCS}(\mathbb{R})$ *and* $\mathsf{RCP}(\mathbb{R})$ *are Δ^1_ω-hard.*
- *The $\mathcal{L}_{closer}$-theories of* $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R})$ *and* $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R})$ *are Δ^0_ω-hard.*

Theorem 101. *[Dav06, Lemma 8] The $\mathcal{L}_{conv}$- and $\mathcal{L}_{closer}$-theories of* $\mathsf{RC}(\mathbb{R}^n)$, $\mathsf{RCS}(\mathbb{R}^n)$ *and* $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, *are Δ^1_ω-hard.*

In this section we discussed lower complexity bounds for a number of Euclidean spatial logics. In the following section we show for all but two of these logics that the established lower complexity bounds are tight.

4.3.4 Undecidability over $\mathbb{R}^n$: Upper Bounds

In this section we establish upper complexity bounds for a number of Euclidean spatial logics. We start with the theories of region algebras of semi-linear regions.

Semi-Linear Regions

Recall from Section 2.4 the signature $\sigma_F^+ = (<, +, \cdot, 0, 1, \pi, [\,], N)$. We show that for $n > 0$ the first-order theories of the structures $(\mathsf{RCP}(\mathbb{R}^n), \mathsf{closer})$, $(\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n), \mathsf{closer})$ and $(\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n), \mathsf{closer})$ are definable in the first-order theories of the structures $(\mathbb{R}, \sigma_F^+)$, $(\mathbb{A}, \sigma_F^+)$ and $(\mathbb{Q}, \sigma_F^+)$, respectively. We already saw in Section 2.4 that $(\mathbb{R}, \sigma_F^+)$ is in Δ_ω^1 and that $(\mathbb{A}, \sigma_F^+)$ and $(\mathbb{Q}, \sigma_F^+)$ are in Δ_ω^0. Throughout this section R denotes any of the fields $\mathbb{R}$, $\mathbb{A}$ and $\mathbb{Q}$, $\mathcal{R}$ denotes the structure (R, σ_F^+) and $\mathsf{RCP}_{\mathbb{R}}(\mathbb{R}^n)$ denotes $\mathsf{RCP}(\mathbb{R}^n)$.

Recall from Section 2.6 that the regions in $\mathsf{RCP}_R(\mathbb{R}^n)$ are exactly the sums of finitely many products of finitely many half-spaces whose boundaries are $n-1$-dimensional hyperplanes definable by polynomials in $R[X_1, \ldots, X_n]$. Observe that:

$$\mathsf{RCP}_R(\mathbb{R}^n) = \left\{ \sum_{i=1}^{m} \prod_{j=1}^{m} \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \middle| \sum_{k=1}^{n} t_k^{i,j} x_k + t_{n+1}^{i,j} \leq 0 \right\} \middle| \; i, j = 1, \ldots, m; t_1^{i,j}, \ldots, t_{n+1}^{i,j} \in R \right\}.$$

So, for every sequence of sequences of half-spaces in $\mathsf{RCP}_R(\mathbb{R}^n)$

$$s = ((a_{1,1}, \ldots, a_{1,m}), \ldots, (a_{m,1}, \ldots, a_{m,m})),$$

there exists a unique region $a \in \mathsf{RCP}_R(\mathbb{R}^n)$ such that $a = \sum_{i=1}^{m} \prod_{j=1}^{m} a_{i,j}$. Conversely, every region $a \in \mathsf{RCP}_R(\mathbb{R}^n)$ is represented by some (in fact infinitely many) sequences of that form. This allows us to identify the regions in $\mathsf{RCP}_R(\mathbb{R}^n)$ with sequences of sequences of half-spaces.

We now define an interpretation of the structure $(\mathsf{RCP}_R(\mathbb{R}^n), \mathsf{closer})$ in the structure $\mathcal{R} = (R, \sigma_F^+)$. We introduce first-order σ_F^+-formulas defining in $\mathcal{R}$ different n-dimensional entities related to the regions in $\mathsf{RCP}_R(\mathbb{R}^n)$. Every point in R^n is identified by the sequence of its coordinates:

$$\psi_{\bullet}(x) := \pi(x) \wedge x[0] = n.$$

We encode every half-space A by the sequences of the $n+1$ coefficients of a linear polynomial inequality defining A:

$$\psi_{\setminus}(x) := \pi(x) \wedge x[0] = n + 1.$$

Basic polytopes in $\mathsf{RCP}_R(\mathbb{R}^n)$ are defined by the formula:

$$\psi_{bptope}(x) := \pi(x) \wedge \forall i \big(N(i) \wedge 0 < i \wedge i \leq x[0] \rightarrow \psi_{\setminus}(x[i]) \big).$$

Polytopes in $\mathsf{RCP}_R(\mathbb{R}^n)$ are then defined by the formula:

$$\psi_{ptope}(x) := \pi(x) \wedge \forall i\big(N(i) \wedge 0 < i \wedge i \leq x[0] \rightarrow \psi_{bptope}(x[i])\big).$$

We now define membership relations. Whether a point is contained in a half-space is determined by the formula:

$$\psi_{\in\backslash}(x, y) := \sum_{i=1}^{n}(x[i] \cdot y[i]) + y[n+1] \leq 0.$$

Whether a point is contained in the interior of a half-space is determined by the formula:

$$\psi_{\in^\circ\backslash}(x, y) := \sum_{i=1}^{n}(x[i] \cdot y[i]) + y[n+1] < 0.$$

Whether a point is contained in the interior of the product of finitely many half-spaces is determined by the formula:

$$\psi_{\in^\circ bptope}(x, y) := 0 < y[0] \wedge \forall i\big(N(i) \wedge 0 < i \wedge i \leq y[0] \rightarrow \psi_{\in^\circ\backslash}(x, y[i])\big).$$

A point in R^n is contained the product of finitely many half-spaces $a_1, \ldots, a_s$ in $\mathsf{RCP}_R(\mathbb{R}^n)$ $(s \in \mathbb{N})$, if it is contained in $a_1 \cdots a_s$ and $a_1 \cdots a_s$ has a nonempty interior. This is determined by the formula:

$$\psi_{\in bptope}(x, y) := \exists z(\psi_{\in^\circ}(z, y)) \wedge \forall i\big(N(i) \wedge 0 < i \wedge i \leq y[0] \rightarrow \psi_{\in\backslash}(x, y[i])\big).$$

A point is contained in the sum of finitely many basic polytopes, if it is contained in at least one of them:

$$\psi_{\in ptope}(x, y) := \exists i\big(N(i) \wedge 0 < i \wedge i \leq y[0] \wedge \psi_{\bullet\in bptope}(x, y[i])\big).$$

Because of Lemma 40, the following formula defines the "part-of" relation in $\mathsf{RCP}_R(\mathbb{R}^n)$:

$$\psi_{\leq}(x, y) := \forall z(\psi_{\bullet}(z) \wedge \psi_{\in ptope}(z, x) \rightarrow \psi_{\in ptope}(y, x)).$$

Two points in R^n represent the same polytope if they satisfy the formula:

$$\psi_{ptope\sim}(x, y) := \psi_{\leq}(x, y) \wedge \psi_{\leq}(y, x).$$

The following formula determines when a point in R^n is contained in the interior of a region in $\mathsf{RCP}_R(\mathbb{R}^n)$:

$$\psi_{\in^\circ ptope}(x, y) := \exists z(\psi_{bptope}(z) \wedge \psi_{\leq}(z, y) \wedge \psi_{\in^\circ bptope}(x, z)).$$

For $a_1, a_2, b_1, b_2 \in R^n$, a_1 is closer to a_2 than b_1 is to b_2 if (a_1, a_2, b_1, b_2) satisfy the formula:

$$\psi_{\bullet closer}(p, q, r, s) := \sum_{i=1}^{n} \big((p[i] - q[i]) \cdot (p[i] - q[i])\big) \leq \sum_{i=1}^{n} \big((r[i] - s[i]) \cdot (r[i] - s[i])\big).$$

Finally, employing Lemma 40 we define the "closer-than" relation using the formula:

$$\begin{aligned}\psi_{closer}(x, y, z) \quad := \quad & \forall p \forall q \big(\psi_{\in^\circ ptope}(p, x) \wedge \psi_{\in^\circ ptope}(q, z) \rightarrow \\ & \qquad \exists r \exists s \big(\psi_{\in^\circ ptope}(r, x) \wedge \psi_{\in^\circ ptope}(s, y) \wedge \psi_{\bullet closer}(r, s, p, q)\big)\big).\end{aligned}$$

We can now prove the following lemma.

Lemma 102. *For $n > 0$, the first-order theory of $(\mathsf{RCP}_R(\mathbb{R}^n), \mathsf{closer})$ is definable in the first-order theory of $\mathcal{R}$.*

Proof. Consider the interpretation defined by:

1. the formulas $\psi_{ptope}(x)$ and $\psi_{ptope\sim}(x, y)$;

2. the formula $\psi_{closer}(x, y, z)$ corresponding to the "closer-than" relation;

3. the surjective map $f : \psi_{ptope}(\mathcal{R}) \rightarrow \mathsf{RCP}_R(\mathbb{R}^n)$ defined by:

$$f(a) = \sum_{i=1}^{a[0]} \prod_{j=1}^{a[i][0]} \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \Big| \sum_{k=1}^{n} a[i][j][k] \cdot x_k + a[i][j][n+1] \leq 0\Big\}. \quad \square$$

Hence we have established the following upper complexity bounds.

Theorem 103. *Let σ be one of the signatures (C), $(\mathsf{conv}, \leq)$ and (closer), and $n > 0$. Then:*

- $(\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n), \sigma)$ *is in Δ^0_ω (Lemma 102, Lemma 16 and Lemma 19);*

- $(\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n), \sigma)$ *is in Δ^0_ω (Lemma 102, Lemma 16 and Lemma 20);*

- $(\mathsf{RCP}(\mathbb{R}^n), \sigma)$ *is in* Δ^1_ω *(Lemma 102, Lemma 16 and Lemma 21).*

We obtain a surprising model-theoretic result from the established complexity bounds. Pratt [Pra99] observed that the first-order σ_{conv}-theories of $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^2)$ are different. The observation is based on a simple geometrical figure allowing the construction, in $\mathsf{RCP}(\mathbb{R}^2)$ of square roots of arbitrary lengths. Because all real numbers constructable in this way are algebraic, one might be tempted to think that the first-order σ_{conv}-theories of $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$ are the same. This, however, turns out to be false, because, as we just showed, the two theories have different complexities.

Theorem 104. *The structures* $(\mathsf{RCP}(\mathbb{R}^n), \mathsf{conv}, \leq)$ *and* $(\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n), \mathsf{conv}, \leq)$ *are not elementary equivalent, for* $n \geq 2$.

Semi-Algebraic Regions

In this section we show that the first-order theories of $(\mathsf{RCS}(\mathbb{R}^n), \mathsf{closer})$, for $n > 0$, are definable in the first-order theory of $\mathcal{R} = (\mathbb{R}, <, +, \cdot, 0, 1, \pi, [\,], N)$. To do so, we define in $\mathcal{R}$ the sets of semi-algebraic sets in $\mathbb{R}^n$, and by the means of n-balls we define those which are regular-closed. We fix $n > 0$.

We start by defining in $\mathcal{R}$ the set of semi-algebraic sets in $\mathbb{R}^n$. Recall from Section 2.6 that the semi-algebraic subsets of $\mathbb{R}^n$ are those of the form:

$$\bigcup_{i=1}^{s}\bigcap_{j=1}^{r_i}\{x \in \mathbb{R}^n | f_{i,j} *_{i,j} 0\},$$

where $f_{i,j}$ is a polynomial in $\mathbb{R}[X_1, \ldots, X_n]$ and $*_{i,j}$ is one of $<$ and $=$, for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

We identify each point in $\mathbb{R}^n$ with the n-tuple of its coordinates. I.e. :

$$\psi_{\bullet}(x) := \pi(x) \wedge x[0] = n.$$

We now define some operations on sequences of real numbers. The formulas

$$\begin{aligned}\psi_{\sharp\bar{r}}(x, y) \quad := \quad & \pi(x) \wedge \exists z\big(\pi(z) \wedge y = z[z[0]] \wedge x[0] = z[0] \wedge x[1] = z[1] \wedge \\ & \forall t(N(t) \wedge 1 \leq t \wedge t < x[0] \rightarrow z[t+1] = z[t] \sharp x[t+1])\big),\end{aligned}$$

where $\sharp$ is either $+$ or $\cdot$, define the sum and product of the elements of a sequence. The formulas

$$\begin{aligned}\psi_{\bar{r}\sharp\bar{r}}(x,y,z) \quad := \quad & \pi(x) \wedge \pi(y) \wedge \pi(z) \wedge x[0] = y[0] \wedge y[0] = z[0] \wedge \\ & \forall t(N(t) \wedge 1 \leq t \wedge t \leq x[0] \to z[t] = x[t]\sharp y[t]),\end{aligned}$$

where $\sharp$ is either $+$ or $\cdot$ define the element-wise sum and product of sequences. Exponentiation can be defined by the formula

$$\begin{aligned}\psi_{x^n}(x,y,z) \quad := \quad & N(y) \wedge \exists t\big(\pi(t) \wedge t[0] = y+1 \wedge t[1] = 1 \wedge z = t[t[0]] \wedge \\ & \forall s(N(s) \wedge 1 \leq s \wedge s \leq y \to t[s+1] = t[s] \cdot x)\big).\end{aligned}$$

We identify a polynomial equation/inequality with the a sequence of real numbers each encoding one of the terms. For example the equation $\sqrt{2}x_1^5 + ex_1^3x_4^7 + 5 = 0$ and the inequality $\sqrt{2}x_1^5 + ex_1^3x_4^7 + 5 < 0$ are encoded by the sequences

$$\begin{aligned}&(0, ((\sqrt{2}, ((1,5))), (e, ((1,3),(4,7))), (5, ())));\\ &(1, ((\sqrt{2}, ((1,5))), (e, ((1,3),(4,7))), (5, ()))).\end{aligned}$$

We define a *factor* to be the part of a term without the coefficient. We encode a factor as a sequence of pairs of natural numbers such that a pair (i,p) represents the variable x_i raised to the power of p. We define the set of factors using the formula:

$$\begin{aligned}\psi_{factor}(x) \quad := \quad & \pi(x) \wedge \forall y(N(y) \wedge 1 \leq y \wedge y \leq x[0] \to \\ & \pi(x[y]) \wedge x[y][0] = 2 \wedge N(x[y][2])) \wedge \\ & N(x[y][1]) \wedge 1 \leq x[y][1] \wedge x[y][1] \leq n.\end{aligned}$$

A term is then just a pair of a real number (a coefficient) and a factor:

$$\psi_{term}(x) := \pi(x) \wedge x[0] = 2 \wedge \psi_{factor}(x[2]).$$

Polynomials, polynomial equations and polynomial inequalities are defined by:

$$\begin{aligned}
\psi_{Poly}(x) &:= \pi(x) \wedge \forall y(N(y) \wedge 1 \leq y \wedge y \leq x[0] \rightarrow \psi_{term}(x[y]));\\
\psi_{Poly=}(x) &:= \pi(x) \wedge x[0] = 2 \wedge x[1] = 0 \wedge \psi_{Poly}(x[2]);\\
\psi_{Poly<}(x) &:= \pi(x) \wedge x[0] = 2 \wedge x[1] = 1 \wedge \psi_{Poly}(x[2]).
\end{aligned}$$

The semi-algebraic sets are identified with the sequence of polynomial equations and inequalities that define them:

$$\psi_{SAS}(x) := \pi(x) \wedge \forall y(N(y) \wedge 1 \leq y \wedge y \leq x[0] \rightarrow \psi_{Poly=}(x) \vee \psi_{Poly<}(x)).$$

In few steps we define the relation $f(x) = y$, for $f \in \mathbb{R}[X_1, \ldots, X_n]$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

$$\begin{aligned}
\psi_{factorVal}(x, y, z) &:= \exists t(\pi(t) \wedge t[0] = x[0] \wedge z = t[t[0]] \wedge\\
&\qquad \forall s(N(s) \wedge 1 \leq s \wedge s \leq t[0] \rightarrow\\
&\qquad\qquad \psi_{x^n}(y[x[s][1]], x[s][2], t[s])));\\
\psi_{termVal}(x, y, z) &:= \exists z'(\psi_{factorVal}(x[2], y, z') \wedge z = z' \cdot x[1]);
\end{aligned}$$

$$\begin{aligned}
\psi_{PolyVal}(x, y, z) &:= \exists t(\pi(t) \wedge t[0] = x[0] \wedge \psi_{+\bar{r}}(t, z)\\
&\qquad \forall s(N(s) \wedge 1 \leq s \wedge s \leq t[0] \rightarrow \psi_{termVal}(x[s], y, t[s])).
\end{aligned}$$

A point p is contained in a semi-algebraic set A if and only if p and A satisfy the formula:

$$\begin{aligned}
\psi_{\bullet \in SAS}(x, y) &:= \exists t\big(N(t) \wedge 1 \leq t \wedge t \leq y[0] \wedge\\
&\qquad \big((\psi_{Poly=}(y[t]) \wedge \psi_{PolyVal}(y[t], x, 0)) \vee\\
&\qquad\qquad (\psi_{Poly<}(y[t]) \wedge \exists s(s < 0 \wedge \psi_{PolyVal}(y[t], x, s)))\big)\big).
\end{aligned}$$

To recognise the semi-algebraic sets which are regular closed, we impose additional conditions stated in terms of open n-balls. We identify an n-ball with the $n+1$ coefficients of its equation ($\sum_{i=1}^{n}(x_i - a_i)^2 < a_{n+1}$).

$$\psi_{\bigcirc}(x) := \pi(x) \wedge x[0] = n + 1.$$

Whether a a point lies in the interior of a ball is determined by the formula

$$\psi_{\bullet\in\bigcirc}(x,y) := \sum_{i=1}^{n}(x[i]-y[i])^2 < x[n+1],$$

and whether a point is isolated from a semi-algebraic set is determined by the formula

$$\psi_{\odot}(x,y) \quad := \quad \exists z(\psi_{\bigcirc}(z) \wedge \psi_{\bullet\in\bigcirc}(x,z) \wedge \forall t(\psi_{\bullet}(t) \wedge \psi_{\bullet\in\bigcirc}(t,z) \rightarrow \neg\psi_{\bullet\in SAS}(t,y))).$$

Consider the function $f^n : \wp(\mathbb{R}^n) \rightarrow \wp(\mathbb{Q}^n)$ given by:

$$f^n(a) := a \cap \mathbb{Q}^n.$$

A set of rational points is an f^n-image of a regular closed set if it contains exactly the rational points that are dense in it, i.e. exactly the points which cannot be isolated from it by an open ball. This is defined by the formula:

$$\psi_{RCS}(x) := \psi_{SAS}(x) \wedge \forall y(\psi_{\bullet\in\bigcirc}(y,x) \leftrightarrow \neg\psi_{\odot}(y,x)).$$

Two regular closed semi-algebraic sets are the same if they satisfy:

$$\psi_{RCS\sim}(x,y) := \forall t(\psi_{\bullet}(t) \rightarrow (\psi_{\bullet\in SAS}(t,x) \leftrightarrow \psi_{\bullet\in SAS}(t,y))).$$

The formula $\psi_{closer}(x,y,z)$ is a direct translation of the definition of the relation "closer-than".

$$\begin{aligned}\psi_{closer}(x,y,z) \quad := \quad & \forall p\forall q(\psi_{\bullet\in SAS}(p,x) \wedge \psi_{\bullet\in SAS}(q,z) \rightarrow \\ & \exists r\exists s(\psi_{\bullet\in SAS}(r,x) \wedge \psi_{\bullet\in SAS}(s,y) \wedge \\ & \sum_{i=1}^{n}(r[i]-s[i])^2 \leq \sum_{i=1}^{n}(p[i]-q[i])^2)).\end{aligned}$$

We can now show that:

Lemma 105. *For $n > 0$, the first-order theory of* $(\mathsf{RCS}(\mathbb{R}^n), \mathsf{closer})$ *is first-order definable in $\mathcal{R}$.*

Proof. Consider the interpretation defined by:

1. the formulas $\psi_{RCS}(x)$ and $\psi_{RCS\sim}(x,y)$;

2. the formula $\psi_{closer}(x, y, z)$ corresponding to the “closer-than" relation;
3. the surjective map $f : \psi_{RCS}(\mathcal{R}) \to \mathsf{RCS}(\mathbb{R}^n)$ defined by:

$$f(a) = \{(k[1], \ldots, k[n]) \mid \mathcal{R} \models \psi_{\bullet}[k], \mathcal{R} \models \psi_{\bullet \in SAS}[k, a]\} . \qquad \square$$

We have established the following upper complexity bounds.

Theorem 106. *Let σ be one of the signatures* (C)*,* $(\mathsf{conv}, \leq)$ *and* (closer)*. Then the first-order σ-theory* $\mathsf{RCS}(\mathbb{R}^n)$ *is in Δ^1_ω, $n > 0$.*

Proof. Follows from Lemma 105, Lemma 16 and Lemma 21. $\square$

Non-tame Regions

In this section we show that for $n > 0$ the first-order theory of the structure $(\mathsf{RC}(\mathbb{R}^n), \mathsf{closer})$ is second-order definable in $\mathcal{Q} = (\mathbb{Q}, <, +, \cdot, 0, 1, \pi, [\,], N)$, which we already saw in Section 2.4 to be in Δ^1_ω.

Recall that a subset of $\mathbb{R}^n$ is regular closed if and only if it contains exactly the points that are dense in it. So, we can identify each region in $\mathsf{RC}(\mathbb{R}^n)$ with the set of rational points that it contains. A point is “dense in” (as opposed to “isolated from”) a set if no open neighbourhood of the point is disjoint with the set. Again, we make use of the mapping $f^n : \wp(\mathbb{R}^n) \to \wp(\mathbb{Q}^n)$, defined by:

$$f^n(a) := a \cap \mathbb{Q}^n.$$

As usual, we identify each rational point with its coordinates.

$$\psi_{\bullet}(x) := \pi(x) \wedge x[0] = n.$$

We identify each rational balls in $\mathbb{R}^n$ with the $n + 1$ coefficients of its equation $(\sum_{i=1}^{n}(x_i - a_i)^2 < a_{n+1})$.

$$\psi_{\bigcirc}(x) := \pi(x) \wedge x[0] = n + 1.$$

We can determine when a rational point lies in the interior of a ball using the formula

$$\psi_{\bullet \in \bigcirc}(x, y) := \sum_{i=1}^{n}(x[i] - y[i])^2 < x[n + 1].$$

Also, we determine when rational point being isolated from a set using the formula

$$\psi_{\odot}(x, X) \;:=\; \exists z(\psi_{\bigcirc}(z) \wedge \psi_{\bullet\in\bigcirc}(x, z) \wedge \forall t(\neg(\psi_{\bullet\in\bigcirc}(t, z) \wedge X(t)))).$$

A set of rational points is an f^n-image of a regular closed set if it contains exactly the rational points that are dense in it. This is determined by the formula:

$$\psi_{RC}(X) \;:=\; \forall x(X(x) \rightarrow \psi_{\bullet}(x)) \wedge \forall x(X(x) \leftrightarrow \neg\psi_{\odot}(x, X)).$$

Two regular closed sets are equal if and only if they contain the same set of rational points. This is determined by the formula:

$$\psi_{RC\sim}(X, Y) := \forall x(X(x) \leftrightarrow Y(x)).$$

We determine whether point is contained in a regular closed set using the formula:

$$\psi_{\in\circ}(x, y) := \exists y(\psi_{\bigcirc}(y) \wedge \psi_{\bullet\in\bigcirc}(x, y) \wedge \forall t(\psi_{\bullet\in\bigcirc}(t, z) \rightarrow t \in X)).$$

Let $a_1, a_2, b_1, b_2 \in \mathbb{Q}^n$. Then a_1 is closer to a_2 than b_1 is to b_2 if and only if the encodings of a_1, a_2, b_1 and b_2 satisfy the formula:

$$\psi_{\bullet closer}(x, y, z, t) := \sum_{i=1}^{n}(x[i] - y[i])^2 \leq \sum_{i=1}^{n}(z[i] - t[i])^2.$$

Because of Lemma 40, we can define the ternary relation "closer-than" using the formula:

$$\psi_{closer}(X, Y, Z) \;:=\; \forall x \forall z\big(\psi_{\bullet}(x) \wedge \psi_{\bullet}(z) \wedge \psi_{\in\circ}(x, X) \wedge \psi_{\in\circ}(z, Z) \rightarrow \exists x' \exists y(\psi_{\bullet}(x') \wedge \psi_{\bullet}(y) \wedge \psi_{\in\circ}(x', X) \wedge \psi_{\in\circ}(y, Y) \wedge \psi_{\bullet closer}(x', y, x, z))\big).$$

Putting this all together, we get the following lemma:

Lemma 107. *For $n > 0$, the first-order theory of* $(\mathsf{RC}(\mathbb{R}^n), \mathsf{closer})$ *is second-order definable in $\mathcal{Q}$.*

Proof. Consider the interpretation defined by:

1. the formulas $\psi_{RC}(X)$ and $\psi_{RC\sim}(X,Y)$;
2. the formula $\psi_{closer}(X,Y,Z)$ corresponding to the relation "closer-than";
3. the inverse of f^n as a surjective map. □

As a corollary we get the following upper complexity bounds:

Theorem 108. *Let* σ *be one of* (C)*,* $(\mathsf{conv}, \leq)$ *and* (closer)*. Then the first-order theory of* $(\mathsf{RC}(\mathbb{R}^n), \sigma)$ *is in* Δ^1_ω*.*

Proof. Follows from Lemma 105, Lemma 16 and Lemma 19. □

4.4 Conclusion

In this chapter we considered first-order spatial logics. We discussed axiomatisations of theories of set algebras and region algebras over different topological spaces. We showed that the theories of the corresponding complete set algebras and region algebras are different (Theorem 68, Theorem 76) which, of course, raises the problem of finding axiomatisations of these theories. Additionally, we examined the computational properties of various Euclidean spatial logics. We showed that the region algebras over $\mathbb{R}$ have decidable topological theories (Theorem 83), which are all non-elementary (Theorem 90). In higher dimensions, all Euclidean region algebras have undecidable theories, which are sufficiently expressive to encode first-order arithmetic (Theorem 95) and, in some cases, second-order arithmetic as well (Theorem 99, Theorem 101). We established the exact complexity bounds for all but two of these theories (Theorem 103, Theorem 106 and Theorem 108), which helped us to derive a surprising model-theoretic result about the region algebras of polytopes and algebraic polytopes (Theorem 104).

The study of spatial logics over arbitrary topological spaces is of great theoretical interest. However, from practical point of view, the most interesting spatial logics are the Euclidean spatial logics. As we established in this chapter, the only decidable Euclidean spatial logics are those over $\mathbb{R}$, which are arguably suitable for reasoning about regions occupied by physical objects. The undecidability of Euclidean spatial logics in higher dimensions motivates the pursuit of decidable fragments of these logics. In the following chapter we study the computational properties of the spatial logics that have gained most of the attention of the AI research community—the quantifier-free Euclidean spatial logics.

Chapter 5

Quantifier-Free Euclidean Spatial Logics

From an AI perspective, the most interesting spatial logics are the ones that can be used in practice. Given that most reasonable first-order spatial logics are undecidable, a natural thing to do is to examine the computational properties of spatial logics with simpler logical syntax. In this chapter we focus on quantifier-free spatial logics featuring topological and Boolean primitives.

Due to the limited expressiveness of quantifier-free logics, the choice of different signatures leads to essentially different quantifier-free logics. Consider for example the signature $\sigma_C = (\mathsf{C})$ consisting only of the binary relational symbol interpreted as the contact relation (two regions are in contact if they share a point). As we discussed in Section 4.1, the Boolean primitives and the property of being connected are definable in the first-order σ_C-logic of most reasonable region algebras. The situation with the corresponding quantifier-free logic is completely different—not only that connectedness and the Boolean operations and relations are not definable, but one can hardly define any sensible relation other than the contact relation, its complement and the property of being a non-trivial region (different from the empty set and the whole space). This naturally leads to a significant increase in the number of "non-equivalent" topological signatures for quantifier-free spatial logics.

In this chapter we study the expressiveness and the computational properties of various quantifier-free Euclidean spatial logics. We start by introducing the logics and discussing some of their properties. We then show that the Euclidean spatial logics that can express some notion of connectedness are sensitive to regions with infinitely many components. And we finish by showing that the satisfiability problem for each of these logics is undecidable. The results in Section 5.2, Section 5.3 and Section 5.4 are joint work with Roman Kontchakov, Ian Pratt-Hartmann and Michael Zakharyaschev, and were presented in [KNPHZ11a, KNPHZ11b]. The undecidability result in Section 5.5 is due to Ian Pratt-Hartmann and is still unpublished.

5.1 Languages and Expressiveness

We consider quantifier-free languages featuring Boolean primitives, as defined in Table 2.1, together with some of the following topological primitives: the property $\mathsf{c}(x)$ of being (topologically) connected, the property $\mathsf{c}^\circ(x)$ of having a connected interior and the Whitehead's contact relation $\mathsf{C}(x,y)$ comprising the pairs of intersecting regions. We interpret these languages over Euclidean region algebras, and, in particular, the region algebra $\mathsf{RC}(\mathbb{R}^n)$ together with its *tame* Boolean subalgebra $\mathsf{RCP}(\mathbb{R}^n)$. (It is not necessary to consider the Euclidean region algebras $\mathsf{RCS}(\mathbb{R}^n)$, $\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$ and $\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$, since they all satisfy the same topological quantifier-free formulas as $\mathsf{RCP}(\mathbb{R}^n)$). Most quantifier-free spatial languages considered in the literature are completely insensitive to Euclidean interpretations. However, languages featuring both Boolean and connectedness primitives, can distinguish between region algebras over different lower-dimensional Euclidean spaces, and are also sensitive to the presence of non-tame regions when interpreted over region algebras over $\mathbb{R}$ and $\mathbb{R}^2$ [KPHZ10]. The satisfiability problems of the resulting Euclidean spatial logics were known to have high computational complexity (ExpTime-hard [KPHZ10]), but it was not known whether they were even decidable. We look into the exact complexities of these Euclidean spatial logics, and we also address the expressiveness of the languages with Boolean and connectedness predicates in higher-dimensional Euclidean spaces. First, however, we will discuss these logics in the context of other quantifier-free spatial logics considered in the literature.

The best studied quantifier-free spatial logic $\mathcal{RCC}8$ is based on the eight jointly exhaustive and pairwise disjoint binary topological relations EQ, EC, DC, PO, TPP, NTPP, TPPi and NTPPi, which are defined in Table 5.1 and illustrated in Figure 3.1. A fragment of $\mathcal{RCC}8$ that is insensitive to how the boundaries of the regions are related is $\mathcal{RCC}5$. $\mathcal{RCC}5$ is based on the relations EQ, DC $\cup$ EC, PO, TPP $\cup$ NTPP and TPPi $\cup$ NTPPi. $\mathcal{RCC}8$ was initially introduced in [EF91], and once it was presented to the AI community in [RCC92], it became the subject of intense scientific research. Notably, it was shown that when interpreted over Euclidean spaces, $\mathcal{RCC}8$ is so inexpressive that it is insensitive not only to the type of Euclidean regions that its variables are mapped to, but also to the dimension of the space hosting those regions. In particular, it was shown in [Ren98] that every satisfiable $\mathcal{RCC}8$-formula is also satisfiable in $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 1$. Formally, taking Σ to be the collection of all region algebras, we have:

Name	Symbol	Definition
Equal	$\mathsf{EQ}(a, b)$	$a = b$
Disconnected	$\mathsf{DC}(a, b)$	$a \cup b \neq \emptyset$, $a \cap b = \emptyset$
External Contact	$\mathsf{EC}(a, b)$	$a \cap b \neq \emptyset$, $a^\circ \cap b^\circ = \emptyset$
Partially Overlapping	$\mathsf{PO}(a, b)$	$a^\circ \cap b^\circ \neq \emptyset$, $a \not\subseteq b$, $b \not\subseteq a$
Tangential Proper Part	$\mathsf{TPP}(a, b)$	$a \subseteq b$, $a \not\subseteq b^\circ$
Inverse Tangential Proper Part	$\mathsf{TPPi}(a, b)$	$b \subseteq a$, $b \not\subseteq a^\circ$
Non-Tangential Proper Part	$\mathsf{NTPP}(a, b)$	$a \subseteq b^\circ$, $a \cup b \neq \emptyset$
Inverse Non-Tangential Proper Part	$\mathsf{NTPPi}(a, b)$	$b \subseteq a^\circ$, $a \cup b \neq \emptyset$

Table 5.1: The primitives in $\mathcal{RCC}8$.

Theorem 109. *[Ren98] Sat*$(\mathcal{RCC}8, \Sigma)$ $=$ *Sat*$(\mathcal{RCC}8, \mathsf{RC}(\mathbb{R}^n))$, $(n \geq 1)$, *and Sat*$(\mathcal{RCC}8, \Sigma) =$ *Sat*$(\mathcal{RCC}8, \mathsf{RCP}(\mathbb{R}^n))$, $(n \geq 3)$.

Regarding computability, it is known that satisfiability of $\mathcal{RCC}8$, and even $\mathcal{RCC}5$, are NP-complete.

Theorem 110. *[RN97] The membership problems for Sat*$(\mathcal{RCC}8, \Sigma)$ *and Sat*$(\mathcal{RCC}5, \Sigma)$ *are* NP-*complete.*

A maximal tractable fragment of $\mathcal{RCC}8$ was identified in [RN97], and the maximal tractable fragments of $\mathcal{RCC}5$ were completely classified in [JD97].

The limited expressiveness of $\mathcal{RCC}8$ stimulated the investigation of various extensions of the language. One way of extending $\mathcal{RCC}8$ is to include in the language symbols for expressing the Boolean primitives: sum $(+)$; product $(\cdot)$;

complement ($-$); the empty region (0); and the region occupying the whole space (1). The resulting language was originally introduced in [WZ00] under the name $\mathcal{BRCC}8$. It was observed in [WZ00] that $\mathcal{BRCC}8$ is sensitive to connected spaces. In particular, the formula

$$x + y = 1 \wedge \mathsf{DC}(x, y) \wedge x \neq 0 \wedge y \neq 0$$

is satisfiable in the region algebra of a topological space if and only if the space is disconnected. An equally expressive version of $\mathcal{BRCC}8$ is the quantifier-free language $\mathcal{C}$ of the signature $(+, \cdot, -, 0, 1, \mathsf{C})$, where C is the contact relation. $\mathcal{C}$ was introduced in [KPHWZ08a], where it was noted that the $\mathcal{RCC}8$ relations are $\mathcal{C}$-definable (see Table 5.2).

$\mathsf{EQ}(x, y)$	$\iff$	$x = y$
$\mathsf{DC}(x, y)$	$\iff$	$\neg\mathsf{C}(x, y)$
$\mathsf{EC}(x, y)$	$\iff$	$x \cdot y = 0 \wedge \mathsf{C}(x, y)$
$\mathsf{PO}(x, y)$	$\iff$	$x \cdot y \neq 0 \wedge x \not\leq y \wedge y \not\leq x$
$\mathsf{TPP}(x, y)$	$\iff$	$x \leq y \wedge y \not\leq x \wedge \mathsf{C}(x, -y)$
$\mathsf{NTPP}(x, y)$	$\iff$	$\neg\mathsf{C}(x, -y) \wedge y \not\leq x$

Table 5.2: Defining the $\mathcal{RCC}8$ relations in $\mathcal{C}$.

In [WZ00] it was shown that checking the satisfiability of $\mathcal{C}$-formulas in region algebras over arbitrary topological spaces is NP-complete (the same complexity as that of $\mathcal{RCC}8$). However, checking the satisfiability of $\mathcal{C}$-formulas in dense region algebras over $\mathbb{R}^n$, $n > 0$, is PSPACE-complete. More generaly, taking Σ_c to be the collection of region algebras over connected topological spaces, we have the following:

Theorem 111. *[WZ00] Sat*$(\mathcal{C}, \Sigma)$ *is* NP*-complete. If* $\mathcal{M}$ *is a dense region algebra over* $\mathbb{R}^n$ *(*$n > 0$*), then Sat*$(\mathcal{C}, \Sigma_c)$ *and Sat*$(\mathcal{C}, \mathcal{M})$ *coincide and are both* PSPACE*-complete.*

It is natural to expect a topological language to be able to express the property of being *connected*. However, neither $\mathcal{RCC}8$, nor its extension $\mathcal{BRCC}8$ have this ability. In addition to the property of being *connected*, denoted by $\mathsf{c}(a)$, one can also consider the less standard property of being *interior-connected*, denoted by $\mathsf{c}^{\circ}(a)$. A region is *interior-connected*, if it has a connected interior. Clearly, every interior-connected region is also connected. (See Figure 5.1.)

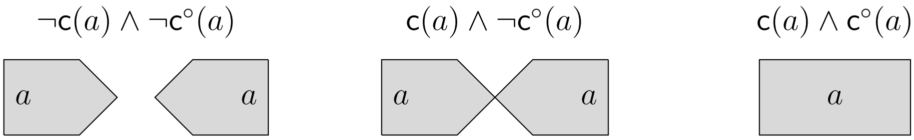


Figure 5.1: Examples of disconnected, connected and interior-connected planar regions.

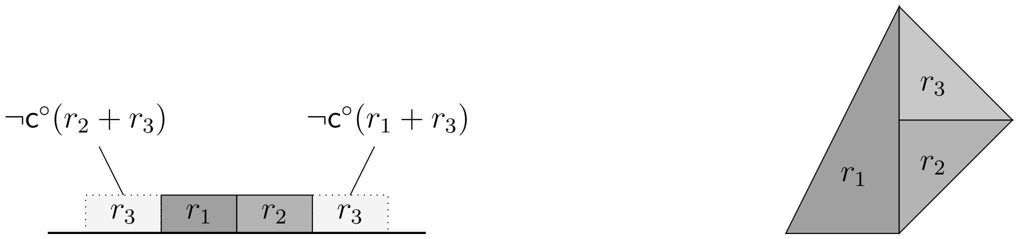
Various spatial logics with connectedness were studied in the literature [PH02, KPHWZ08b, KPHWZ08a, KPHZ10, KPHWZ10, KNPHZ11a, KNPHZ11b]. From these we consider the logics of the six languages $\mathcal{RCC}8\mathsf{c}$, $\mathcal{RCC}8\mathsf{c}^\circ$, $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$, $\mathcal{B}\mathsf{c}$ and $\mathcal{B}\mathsf{c}^\circ$ ([KPHWZ08b, KPHWZ08a]) when interpreted over the Euclidean region algebras $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 1$. The languages $\mathcal{RCC}8\mathsf{c}$ and $\mathcal{RCC}8\mathsf{c}^\circ$ extend $\mathcal{RCC}8$ with predicates for, respectively, the property of being connected and the property of being interior-connected. The languages $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$, $\mathcal{B}\mathsf{c}$ and $\mathcal{B}\mathsf{c}^\circ$ are defined analogously, by taking $\mathcal{B}$ to be the quantifier-free languages for Boolean algebras. Considering $\mathcal{B}\mathsf{c}$ and $\mathcal{B}\mathsf{c}^\circ$ allows us to examine the computability and expressiveness of (interior-)connectedness in the absence of other topological primitives.

Enhancing the three basic languages $\mathcal{RCC}8$, $\mathcal{B}$ and $\mathcal{C}$ with the ability to express (interior-)connectedness substantially increases their expressiveness. Consider for example the $\mathcal{B}\mathsf{c}^\circ$-formula $\mathsf{G_m}(x_1, \ldots, x_m)$ given by:

$$\bigwedge_{i=1}^{m}\big(\mathsf{c}^\circ(x_i) \wedge (x_i > 0)\big) \wedge \bigwedge_{1\leq i<j\leq m} \big(\mathsf{c}^\circ(x_i + x_j)\big) \wedge (x_i \cdot x_j = 0)\big).$$

As noted in [KNPHZ11a], the formula $\mathsf{G_3}(x_1, x_2, x_3)$ is not satisfiable in $\mathsf{RC}(\mathbb{R})$, and hence also in $\mathsf{RCP}(\mathbb{R})$, but is satisfiable in $\mathsf{RCP}(\mathbb{R}^n)$, and hence also in $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$. Indeed, as shown in Figure 5.2a, no three non-overlapping closed intervals have pairwise connected unions, but, as shown in Figure 5.2b, there are connected polygons with that property.

Similarly, the formula $\mathsf{G_5}(x_1, \ldots, x_5)$ is not satisfiable in $\mathsf{RC}(\mathbb{R}^2)$, (and hence also in $\mathsf{RCP}(\mathbb{R}^2)$), but is satisfiable in $\mathsf{RCP}(\mathbb{R}^n)$, (and hence also in $\mathsf{RC}(\mathbb{R}^n)$), $n > 2$. To see the former, suppose that $\mathsf{G_5}(\bar{x})$ is satisfiable by a tuple of planar regions $(r_1, \ldots, r_5)$. Choose points p_i in the interiors of r_i, $i = 1, \ldots, 5$. By $\mathsf{c}^\circ(r_i + r_j)$, $1 \leq i < j \leq 5$, there exist Jordan arcs $\alpha_{i,j} \subseteq (r_i + r_j)^\circ$ connecting p_i and p_j. Moreover, these arcs can be selected to be pairwise disjoint (except at



(a) No three non-overlapping intervals form connected sums with each other.

(b) $\mathsf{G}_3(x_1, x_2, x_3)$ is easily satisfiable by regions in $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$.

Figure 5.2: The formula $\mathsf{G}_3(x_1, x_2, x_3)$ satisfiable in $\mathbb{R}^2$, but not in $\mathbb{R}$.

their endpoints), and hence establishing a drawing of the non-planar graph K_5. (Figure 5.3 depicts four planar regions that satisfy $\mathsf{G}_4(\bar{x})$, and hence containing an embedding of the graph K_4 in the plane.) On the other hand, since every graph, and in particular K_5, can be embedded in $\mathbb{R}^n$, $n > 2$, the formula $\mathsf{G}_5(\bar{x})$ is easily satisfiable in $\mathsf{RCP}(\mathbb{R}^n)$.

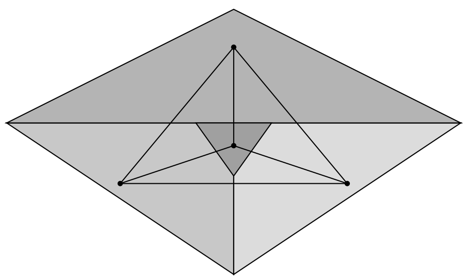


Figure 5.3: Four regions satisfying the formula $\mathsf{G}_4(\bar{x})$ and containing an embedding of the graph K_4.

Another manifestation of the enhanced expressiveness of the languages featuring (interior-)connectedness is their ability to sense the presence of non-tame regions—i.e. regions having infinitely many connected components or regions not having the *curve-selection* property (see Section 2.5). Consider, for example the $\mathcal{B}c^\circ$-formula $\mathsf{wiggly}(x_1, x_2, x_3)$ given by:

$$\bigwedge_{i=1}^{3} c^\circ(x_1 + x_2 + x_3) \wedge \neg c^\circ(x_1 + x_2) \wedge \neg c^\circ(x_1 + x_3) \text{ [PH07, p.23]}.$$

The formula "says" that there exist three interior-connected regions whose sum is interior-connected, but the sum of the first region with any of the other two is not interior-connected. It was shown in [PH07, Lemma 2.56] that no regions in $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, satisfy $\mathsf{wiggly}(x_1, x_2, x_3)$. On the other hand, as shown in Figure 5.4, there exist regions in $r_1, r_2, r_3 \in \mathsf{RC}(\mathbb{R}^2)$ that satisfy $\mathsf{wiggly}(x_1, x_2, x_3)$. Taking $r_i^{n+2} = r_i \times \mathbb{R}^n$, $i = 1, 2, 3$, we get a satisfying tuple of $\mathsf{wiggly}(x_1, x_2, x_3)$ in $\mathsf{RC}(\mathbb{R}^{n+2})$.

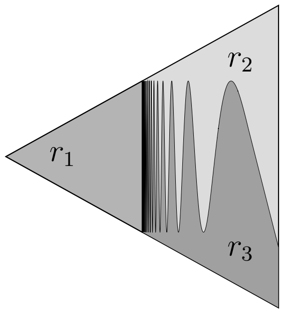


Figure 5.4: Three planar regions that satisfy the formula $\mathsf{wiggly}(x_1, x_2, x_3)$.

Consider now the region algebras $\mathsf{RC}(\mathbb{R})$ and $\mathsf{RCP}(\mathbb{R})$. Let $\psi^1_\omega(\bar{x})$ be the following $\mathcal{RCC}8$c-formula:

$$\mathsf{c}(x) \wedge \mathsf{EC}(x, y_1) \wedge \mathsf{EC}(x, y_2) \wedge \mathsf{EC}(x, y_3) \wedge \mathsf{DC}(y_1, y_3) \wedge \mathsf{DC}(y_1, y_3) \wedge \mathsf{EC}(y_1, y_2).$$

An easy inspection shows that no tuple of finitely-decomposable regions in $\mathsf{RC}(\mathbb{R})$ satisfies $\psi^1_\omega(\bar{x})$. However, as can be seen in Figure 5.5, $\psi^1_\omega(\bar{x})$ is satisfiable in $\mathsf{RC}(\mathbb{R})$ by regions some of which have infinitely many components.

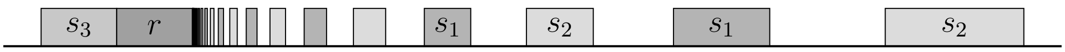


Figure 5.5: Regions in $\mathsf{RC}(\mathbb{R})$ satisfying $\psi^1_\omega(\bar{x})$

It is significantly more challenging to show sensitivity of the languages $\mathcal{B}$c and $\mathcal{C}$c to tameness in $\mathbb{R}^n$, $n > 1$. In Section 5.2 we show that there are $\mathcal{B}$c- and $\mathcal{C}$c-formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, but only by tuples featuring regions with infinitely many components. In Section 5.3 we show the same for the languages $\mathcal{B}$c° and $\mathcal{C}$c° in the Euclidean plane. Before we do so, we show how to define in $\mathcal{B}$c and $\mathcal{B}$c° the complement of the *contact* relation for a large class of regions in $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$.

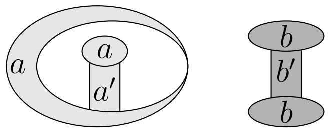


Figure 5.6: Regions (a, b, a', b') satisfying ψ_{DC2} in $\mathcal{M}$.

5.1.1 Separating Regions Using Connectedness

Recall that the language $\mathcal{B}\mathsf{c}$ is a (genuine) restriction of the language $\mathcal{C}\mathsf{c}$ whose only topological primitive (c) is interpreted as the property of being connected. Although $\mathcal{C}\mathsf{c}$ is strictly more expressive than $\mathcal{B}\mathsf{c}$, most of the established (positive) expressiveness results for $\mathcal{C}\mathsf{c}$ (in particular when interpreted in Euclidean spaces of dimension greater than 1) can be extended for $\mathcal{B}\mathsf{c}$. For this section we fix $\mathcal{M}$ to be one of the region algebras $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, and we use the term *regions* to refer to the elements of $\mathcal{M}$. We identify $\mathcal{B}\mathsf{c}$-formulas $\psi(x, y, \bar{z})$ which are satisfiable only by tuples $(a, b, \bar{c})$ in $\mathcal{M}$ for which a and b are disjoint. Moreover, we would like these formulas to be satisfiable by as many pairs of disjoint regions as possible. The results in this section apeared in [KNPHZ11a].

The formulas that we consider are all based on the following fact about connected sets.

Lemma 112. *Let $\mathcal{X}$ be a topological space. Then two connected subsets of $\mathcal{X}$ are in contact if and only if their union is connected.*

Hence, the formula

$$\psi_{\mathsf{DC1}}(x, y) := x = 0 \lor y = 0 \lor \mathsf{c}(x) \land \mathsf{c}(y) \land \neg\mathsf{c}(x + y)$$

defines the DC relation for the connected regions in $\mathcal{M}$.

This can easily be generalised for regions contained in disjoint connected regions. Evidently, the formula

$$\psi_{\mathsf{DC2}}(x, y, x', y') := \psi_{\mathsf{DC1}}(x + x', y + y')$$

is satisfiable by tuples of regions (a, b, a', b') for which a and b are disjoint (see Figure 5.6).

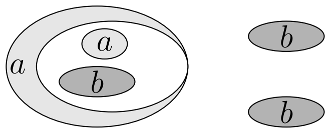


Figure 5.7: A pair of regions (a, b) that cannot satisfy ψ_{DC2} in $\mathcal{M}$.

Note that $\psi_{\mathsf{DC2}}(x, y, \bar{z})$ is satisfied only by tuples $(a, b, \bar{c})$ such that a is contained in an *interior-component* of $-b$ and vice versa. As depicted in Figure 5.7, if a and b are two disjoint regions such that “pieces” of b lie in different interior-components of $-a$, then there would exist no region b' such that $b + b'$ is both connected and disjoint from a. Although the interior of $-a$ is disconnected, we can easily partition a into regions a_1 and a_2 whose complements are interior connected. We can partition b into regions b_1 and b_2 in a similar manner. Then, applying $\psi_{\mathsf{DC2}}(x, y, x', y')$, we can ensure that a_i and b_j, $i, j = 1, 2$, are disjoint, which would be sufficient for a and b to be disjoint as well. Consider the formula

$$\psi_{\mathsf{DC3}}(x, y, \bar{z}) \quad := \quad x = x_1 + x_2 \wedge y = y_1 + y_2 \wedge \bigwedge_{i,j=1,2} \psi_{\mathsf{DC2}}(x_i, y_j, x_{i,j}, y_{i,j}).$$

Evidently, if a tuple of regions $(a, b, \bar{c})$ satisfies $\psi_{\mathsf{DC3}}(x, y, \bar{z})$, then a and b are

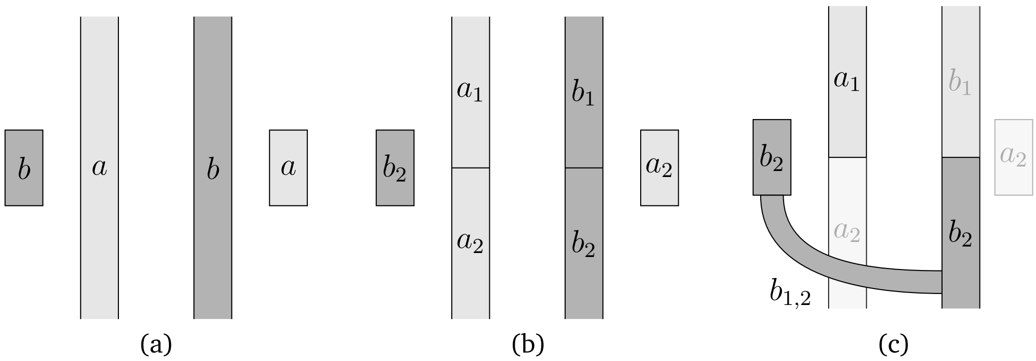


Figure 5.8: Partitioning disjoint regions into regions with interior-connected complements.

disjoint. Consider now the disjoint regions a and b depicted in Figure 5.8a. Both $-a$ and $-b$ have disconnected interiors, however, there are regions $\bar{c}$ such that $(a, b, \bar{c})$ satisfies ψ_{DC3}. To see that, partition a and b into regions a_1, a_2, b_1

and b_2 as shown in Figure 5.8b. Clearly, $\mathcal{M} \models \psi_{\mathsf{DC2}}[a_1, b_1, 0, 0]$, and as shown in Figure 5.8c, there exists a region $b_{1,2}$ such that $\mathcal{M} \models \psi_{\mathsf{DC2}}[a_1, b_2, 0, b_{1,2}]$.

Can we always do that? I.e. if a and b are two disjoint regions, can we always find regions $\bar{c}$ such that $\mathcal{M} \models \psi_{\mathsf{DC3}}[a, b, \bar{c}]$? The answer is no. Consider, for example, the disjoint regions a and b depicted in Figure 5.9a. For every region b', if $b + b'$ is connected, then it separates the components of a. Hence, there are no regions a' and b' such that (a, b, a', b') satisfies $\psi_{\mathsf{DC2}}(x, y, x', y')$ in $\mathcal{M}$. Arguing in a similar way, it is not difficult to see that for the disjoint regions a and b depicted in Figure 5.9b there are no regions $\bar{c}$ such that $(a, b, \bar{c})$ satisfy ψ_{DC3}.

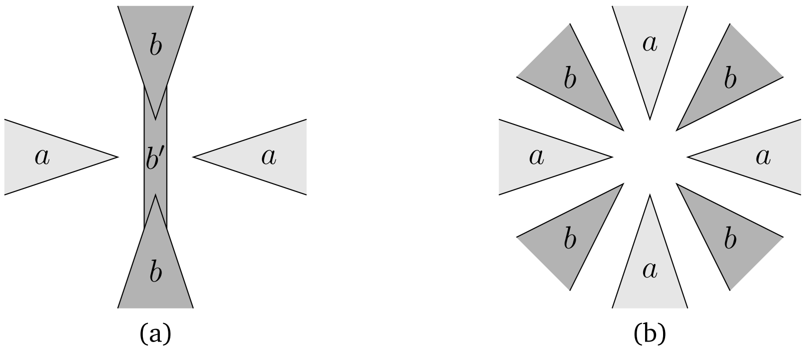


Figure 5.9: Examples of regions that cannot satisfy ψ_{DC2} and ψ_{DC3} in $\mathcal{M}$.

In this section we showed how to force non-contact constraints in $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, in the language $\mathcal{B}c$. The results are based on the fact that two connected regions are disjoint if and only if their sum is disconnected (Lemma 112). Observe that if we replace in Lemma 112 "connected" with "interior-connected", the result will no longer be true, due to the fact that there are connected regions which are not interior-connected (see Figure 5.1). However, in the following section we show that interior-connectedness can be used to force non-contact constraints in the Euclidean plane.

5.1.2 Separating Regions Using Interior-Connectedness

The aim of this section is to define in the language $\mathcal{B}c^{\circ}$ the *non-contact* relation for a large collection of regions in $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$. For the following results we extensively use planarity arguments and, hence, they fail for higher-dimensional Euclidean spaces. Recall that a *Jordan arc* in a topological space $\mathcal{X}$

is an injective continuous function from the unit interval $[0,1]$ to $\mathcal{X}$. A *Jordan curve* in $\mathcal{X}$ is an injective continuous function from the unit circle (the points in the Euclidean plane satisfying the equation $x^2 + y^2 = 1$) to $\mathcal{X}$. If α is a Jordan arc, and p and q are points on α such that q occurs after p, we denote by $\alpha[p,q]$ the segment of α from p to q, i.e. the restriction of α to the interval $[\alpha^{-1}(p), \alpha^{-1}(q)]$. The results in this section apeared in [KNPHZ11a].

To avoid notational clutter we use the same symbols to represent the variables of $\mathcal{B}\mathsf{c}^\circ$-formulas and the regions that those variables get assigned to. For instance, consider the formula $\mathsf{disc}(a) := \mathsf{c}^\circ(a) \wedge \mathsf{c}^\circ(-a)$, whose only variable is a. We can then say that if a polygon a satisfies $\mathsf{disc}(a)$ (i.e. $\mathsf{RCP}(\mathbb{R}^2) \models \mathsf{disc}[a]$), then a and its complement have connected interiors.

Consider the formula $\mathsf{frame}^\circ(a_0, \ldots, a_{n-1})$ given by:

$$\bigwedge_{0 \le i < n} \big(\mathsf{c}^\circ(a_i) \wedge \mathsf{c}^\circ(a_i + a_{\lfloor i+1 \rfloor}) \wedge a_i \neq 0\big) \wedge \bigwedge_{j-i>1} a_i \cdot a_j = 0,$$

where $\lfloor k \rfloor$ denotes $k \mod n$. This formula allows us to construct Jordan curves in the plane, in the following sense.

Lemma 113. *Let $n \geq 3$, and suppose $\mathsf{frame}^\circ(a_0, \ldots, a_{n-1})$. Then there exist Jordan arcs α_0, …, α_{n-1} such that $\alpha_0 \ldots \alpha_{n-1}$ is a Jordan curve lying in the interior of $a_0 + \cdots + a_{n-1}$, and $\alpha_i \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$, for all i, $0 \leq i < n$.*

Proof. For all i ($0 \leq i < n$), pick $p'_i \in a^\circ_i$, and pick a Jordan arc $\alpha'_i \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$ from p_i to $p_{\lfloor i+1 \rfloor}$. For all i ($2 \leq i \leq n$), let $p_{\lfloor i \rfloor}$ be the first point of α_{i-1} lying on $\alpha_{\lfloor i \rfloor}$, and let p''_1 be the first point of α'_0 lying on α'_1. For all i ($2 \leq i < n$), let $\alpha_i = \alpha'_i[p_i, p_{i+1}]$, let $\alpha''_1 = \alpha'_1[p''_1, p_2]$, and let α''_0 denote the section of α'_0 (in the appropriate direction) from p_0 to p''_1. Now let p_1 be the first point of α''_0 lying on α''_1, let $\alpha_0 = \alpha''_0[p_0, p_1]$, and let $\alpha_1 = \alpha''_1[p_1, p_2]$. It is routine to verify that the arcs α_0, …, α_{n-1} have the required properties. □

We say that a region r is *quasi-bounded* if either r or $-r$ is bounded. We can now prove the following.

Lemma 114. *There exists a $\mathcal{B}\mathsf{c}^\circ$-formula $\psi^\circ_{DC1}(r, s, \bar{v})$ with the following properties: (i) $\psi^\circ_{DC1}(r, s, \bar{v})$ entails $\neg C(r, s)$ over $\mathsf{RC}(\mathbb{R}^2)$; (ii) if the regions r and s can be separated by a Jordan curve, then there exist polygons $\bar{v}$ such that $\psi^\circ_{DC1}(\tau_1, \tau_2, \bar{v})$; (iii) if r, s are disjoint polygons such that r is quasi-bounded and $\mathbb{R}^2 \setminus (r + s)$ is connected, then there exist polygons $\bar{v}$ such that $\psi^\circ_{DC1}(\tau_1, \tau_2, \bar{v})$.*

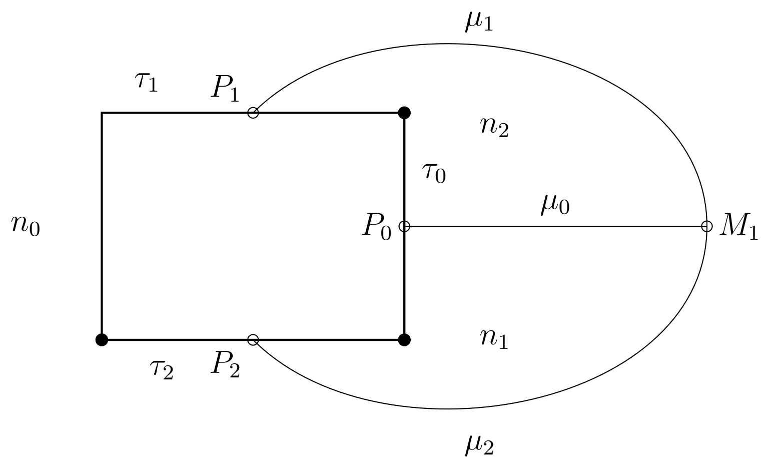


Figure 5.10: A Jordan curve $\Gamma = \tau_0\tau_1\tau_2$ separating m_1 from m_2 (Lemma 114).

Proof. Let $\bar{v}$ be the tuple of variables $(t_0, \ldots, t_5, m_1, m_2)$, and let $\psi^{\circ}_{DC1}(r, s, \bar{v})$ be the formula

$$\mathsf{frame}^{\circ}(t_0, \ldots, t_5) \wedge r \leq m_1 \wedge s \leq m_2 \wedge m_1 \cdot m_2 = 0 \wedge \\ (t_0 + \cdots + t_5) \cdot (m_1 + m_2) = 0 \wedge \bigwedge_{\substack{i=1,3,5 \\ j=1,2}} \mathsf{c}^{\circ}(t_i + m_j).$$

Property (*i*) follows by a simple planarity argument. By $\mathsf{frame}^{\circ}(t_0, \ldots, t_5)$ and Lemma 113, let α_i, for $0 \leq i \leq 5$, be such that $\Gamma = \alpha_0 \cdots \alpha_5$ is a Jordan curve included in $(t_0 + \cdots + t_5)^{\circ}$. Further, let $\tau_i = \alpha_{2i}\alpha_{2i+1}$, $0 \leq i \leq 2$ (Figure 5.10). Note that all points in a_{2i+1}, $0 \leq i \leq 2$, that are on Γ are on τ_i. By $\mathsf{c}^{\circ}(t_{2i+1} + m_1)$, $0 \leq i \leq 2$, let $\mu_i \subseteq (m_1 + t_{2i+1})^{\circ}$ be a Jordan arc with endpoints $M_1 \in m_1^{\circ}$ and $T_i \in \tau_i \cap t^{\circ}_{2i+1}$. We may assume that these arcs intersect only at their common endpoint M_1, so that they divide the residual domain of Γ which contains M_1 into three sub-domains n_i, for $0 \leq i \leq 2$, with $\delta(n_i) \cap \tau_i = \emptyset$. The existence of a point $M_2 \in m_2$ in any n_i, $0 \leq i \leq 2$, will contradict $\mathsf{c}^{\circ}(t_{2i+1} + m_2)$. Indeed, suppose that there is a point $M_2 \in n_2^{\circ}$. By $\mathsf{c}^{\circ}(t_{2i+1} + m_2)$, there exists an arc $\nu_2 \subseteq (t_5 + m_2)^{\circ}$ connecting T_2 and M_2. Since the two points lie in different residual domains of $\delta(n_2)$, ν_2 must intersect $\delta(n_2)$. Because $\delta(n_2)$ is a subset of $\mu_0 \cup \mu_1 \cup \tau_0 \cup \tau_1 = \mu_0 \cup \mu_1 \cup \alpha_0\alpha_1 \cup \alpha_2\alpha_3$, it is contained in $(m_1 + t_1)^{\circ} \cup (m_1 + t_3)^{\circ} \cup (t_0 + t_1)^{\circ} \cup (t_1 + t_2)^{\circ} \cup (t_2 + t_3)^{\circ} \cup (t_3 + t_4)^{\circ}$. Hence, by the non-overlapping constraints, $\delta(n_2)$ must be disjoint from $\nu_2 \subseteq (t_5 + m_2)^{\circ}$. So, m_2 must be contained entirely in the residual domain of Γ not containing M_1. Similarly, all points in m_1 must lie in the residual domain of Γ containing M_1. It follows that m_1 and m_2 are disjoint, and by $r \leq m_1$ and $s \leq m_2$, that r and s are disjoint as well. For Property (*ii*), let Γ be a Jordan curve separating r and s. Now thicken Γ to form an annular element of RCP($\mathbb{R}^2$), still disjoint from r and s, and divide this annulus into the five regions $t_0, \ldots, t_5$ as shown (up to similar situation) in Figure 5.11. Choose m_1 and m_2 to be the connected components of $-(t_0 + \cdots + t_5)$ containing r and s, respectively. For Property (*iii*), it is routine using Lemma 27-Lemma 31 to show that there exists a piecewise linear Jordan curve Γ in $\mathbb{R}^2 \setminus (r + s)$ separating r and s. □

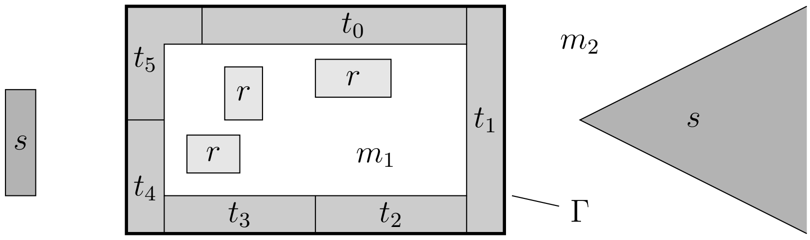


Figure 5.11: Separating a polygon from a disjoint quasi-bounded polygon by an annulus.

Lemma 115. *There exists a* $\mathcal{B}c^\circ$*-formula* $\psi^\circ_{DC2}(r, s, \bar{v})$ *with the following properties:* (*i*) $\psi^\circ_{DC2}(r, s, \bar{v})$ *entails* $\neg C(r, s)$ *over* RC($\mathbb{R}^2$)*;* (*ii*) *if* r*,* s *are disjoint quasi-bounded polygons, then there exist polygons* $\bar{v}$ *such that* $\psi^\circ_{DC2}(\tau_1, \tau_2, \bar{v})$*.*

Proof. Let $\psi^\circ_{DC2}(r, s, \bar{v})$ be the formula

$$r = r_1 + r_2 \wedge s = s_1 + s_2 \wedge \bigwedge_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2}} \psi^\circ_{DC1}(r_i, s_j, \bar{u}_{i,j}),$$

where ψ°_{DC1} is the formula given in Lemma 114. Property (*i*) is then immediate. For Property (*ii*), it is routine to show that there exist polygons r_1, r_2 such that $r = r_1 + r_2$ and $\mathbb{R}^2 \setminus r_i$ is connected for $i = 1, 2$; let s_1, s_2 be chosen analogously. Then for all i $(1 \leq i \leq 2)$ and j $(1 \leq j \leq 2)$ we have that $r_i \cap s_j = \emptyset$ and, by Lemma 27, that $\mathbb{R}^2 \setminus (r_i + s_j)$ connected. By Lemma 114, let $\bar{u}_{i,j}$ be such that $\psi^\circ_{DC1}(r_i, s_j, \bar{u}_{i,j})$. □

5.2 Infinitely Many Components with Connectedness

We will now show that the languages $\mathcal{C}$c, $\mathcal{C}$c$^\circ$ and $\mathcal{B}$c are sensitive to tameness in a large class of topological spaces. We will introduce a $\mathcal{C}$c-formula ψ_∞ such that ψ_∞ is satisfiable over RC($\mathbb{R}^n$), $n \geq 2$, but any tuple of regions in a region algebra over a unicoherent topological space that satisfies it, features regions with infinitely many components. We then transform ψ_∞ to $\mathcal{C}$c$^\circ$- and $\mathcal{B}$c-formulas having the same property. The results in this section apeared in [KNPHZ11a, KNPHZ11b] and are joined work of the authors. As before, we use the same symbols to denote the variables of $\mathcal{C}$c-, $\mathcal{C}$c$^\circ$- and $\mathcal{B}$c-formulas and the regions that these variables get assigned to.

We present the $\mathcal{C}$c-formula ψ_∞ in groups of conjuncts. First we take four regions in RC($\mathcal{X}$) $d_0, \ldots, d_3$ that partition $\mathcal{X}$:

$$d_0 + d_1 + d_2 + d_3 = 1 \wedge \bigwedge_{0 \leq i < j < 4} d_i \cdot d_j = 0. \tag{5.1}$$

We also require non-empty subregions a_i of d_i $(0 \leq i < 3)$, and a non-empty region t:

$$\bigwedge_{0 \leq i < j < 4} \big(0 < a_i \wedge a_i \leq d_i\big) \;\wedge\; t \neq 0. \tag{5.2}$$

The configuration of regions we have in mind is depicted in Figure 5.12, where the d_i are arranged so that the components of d_1 are "wrapped around" the components of d_0, the components of d_2 are "wrapped around" the components of d_1, and so on. The region t passes through every component of the d_i while avoiding the regions a_i. To enforce a configuration of this sort, we need the following three formulas, for $0 \leq i \leq 3$:

$$\mathsf{c}(a_i + d_{\lfloor i+1 \rfloor} + t), \tag{5.3}$$

$$\neg\mathsf{C}(a_i, d_{\lfloor i+1 \rfloor} \cdot (-a_{\lfloor i+1 \rfloor})) \;\wedge\; \neg\mathsf{C}(a_i, t), \tag{5.4}$$

$$\neg\mathsf{C}(d_i, d_{\lfloor i+2 \rfloor}), \tag{5.5}$$

where $\lfloor k \rfloor = k \mod 4$. Formulas (5.3) and (5.4) ensure that each component of a_i is in contact with $a_{\lfloor i+1 \rfloor}$, while (5.5) ensures that no component of d_i can touch any component of $d_{\lfloor i+2 \rfloor}$.

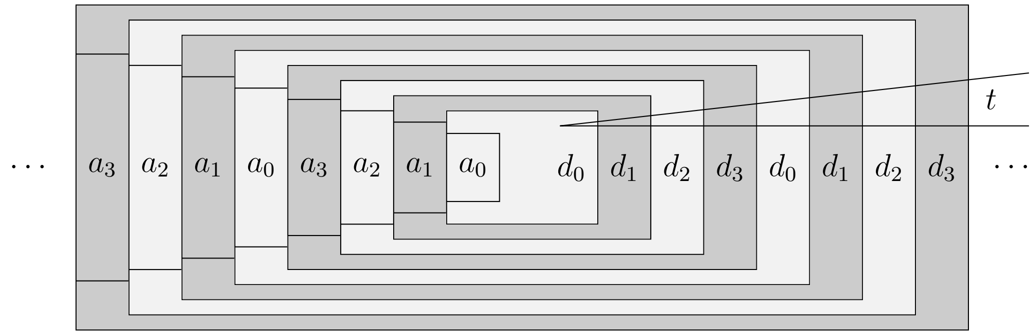


Figure 5.12: Planar regions satisfying ψ_∞.

Denote by ψ_∞ the conjunction of the constraints (5.1)-(5.1). As shown in Figure 5.12, ψ_∞ is satisfiable in RC($\mathbb{R}^2$). By taking $t^{n+2} = t \times \mathbb{R}^n$ and similarly for a_i^{n+2} and d_i^{n+2}, $(0 \le i < 3)$, we get a satisfying tuple for ψ_∞ in RC($\mathbb{R}^{n+2}$).

Note that the regions d_i in Figure 5.12 have infinitely many components. We will now show that in locally-connected unicoherent spaces, ψ_∞ is only satisfiable by tuples featuring regions having infinitely many components.

Theorem 116. *Let $\mathcal{X}$ be a locally connected unicoherent space. If ψ_∞ is satisfiable by a tuple $\bar{b}$ of regions in* RC($\mathcal{X}$)*, then some of the regions in $\bar{b}$ have infinitely many components.*

Proof. Let $b_i = d_i \cdot (-a_i)$. We construct a sequence of components X_i of $d_{\lfloor i \rfloor}$ and open sets V_i connecting X_i to X_{i+1} (Figure 5.13). By the first conjunct of (5.2), let X_0 be a component of d_0 containing points in a_0. Suppose X_i has been constructed, for $i \ge 0$. By (5.3) and (5.4), there exists a point $q \in X_i \cap a_{\lfloor i+1 \rfloor}$. Since $q \notin b_{\lfloor i+1 \rfloor} \cup d_{\lfloor i+2 \rfloor} \cup d_{\lfloor i+3 \rfloor}$, and because $\mathcal{X}$ is locally connected, there exists a connected neighbourhood V_i of q such that $V_i \cap (b_{\lfloor i+1 \rfloor} \cup d_{\lfloor i+2 \rfloor} \cup d_{\lfloor i+3 \rfloor}) = \emptyset$, and so, by (5.1), $V_i \subseteq d_{\lfloor i \rfloor} + a_{\lfloor i+1 \rfloor}$. Further, since $q \in a_{\lfloor i+1 \rfloor}$, $V_i \cap a^\circ_{\lfloor i+1 \rfloor} \neq \emptyset$. Take X'_{i+1} to be a component of $a_{\lfloor i+1 \rfloor}$ that intersects V_i and X_{i+1} the component of $d_{\lfloor i+1 \rfloor}$ containing X'_{i+1}.

To see that the X_i are distinct, let S_{i+1} and R_{i+1} be the components of $-X_{i+1}$ containing X_i and X_{i+2}, respectively. It suffices to show $S_{i+1} \subseteq S^\circ{}_{i+2}$. Note that the connected set V_i must intersect $\delta(S_{i+1})$. Evidently, $\delta(S_{i+1}) \subseteq X_{i+1} \subseteq d_{\lfloor i+1 \rfloor}$. Also, $\delta(S_{i+1}) \subseteq -X_{i+1}$; hence, by (5.1) and (5.5), $\delta(S_{i+1}) \subseteq d_i \cup d_{\lfloor i+2 \rfloor}$. By Lemma 34, $\delta(S_{i+1})$ is connected, and therefore, by (5.5), is entirely contained

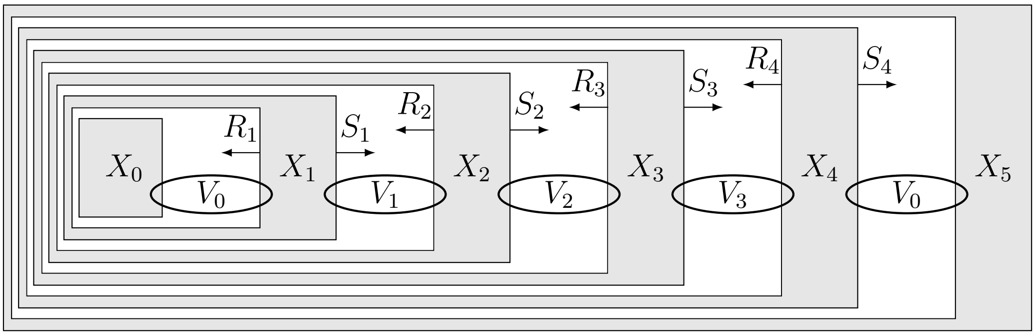


Figure 5.13: The sequence $\{X_i, V_i\}_{i\geq 0}$ generated by ψ_∞. (S_{i+1} and R_{i+1} are the 'holes' of X_{i+1} containing X_i and X_{i+2}.

either in $d_{\lfloor i \rfloor}$ or in $d_{\lfloor i+2 \rfloor}$. Since $V_i \cap \delta(S_{i+1}) \neq \emptyset$ and $V_i \cap d_{\lfloor i+2 \rfloor} = \emptyset$, we have $\delta(S_{i+1}) \not\subseteq d_{\lfloor i+2 \rfloor}$, so $\delta(S_{i+1}) \subseteq d_i$. Similarly, $\delta(R_{i+1}) \subseteq d_{i+2}$. By (5.5), then, $\delta(S_{i+1}) \cap \delta R_{i+1} = \emptyset$, and since S_{i+1} and R_{i+1} are components of the same set, they are disjoint. Hence, $S_{i+1} \subseteq (-R_{i+1})^\circ$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^\circ$. So, S_{i+1} lies in the interior of a component of $-X_{i+2}$, and since $\delta(S_{i+1}) \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be S_{i+2}. □

We extend this result to the language $\mathcal{C}c^\circ$. All occurrences of c in ψ_∞ have positive polarity. Let ψ°_∞ be the result of replacing them with the predicate c°. In the configuration of Figure 5.12, all connected regions mentioned in ψ_∞ are in fact interior-connected; hence ψ°_∞ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$. Since interior-connectedness implies connectedness, ψ°_∞ entails ψ_∞ in a common extension of $\mathcal{C}c^\circ$ and $\mathcal{C}c$. Hence:

Theorem 117. *The $\mathcal{C}c^\circ$-formula ψ°_∞ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. If $\mathcal{X}$ is a locally connected unicoherent space and ψ°_∞ is satisfiable by a tuple $\bar{b}$ of regions in $\mathsf{RC}(\mathcal{X})$, then some of the regions in $\bar{b}$ have infinitely many components.*

To extend Theorem 116 to the language $\mathcal{B}c$, notice that all occurrences of C in ψ_∞ are negative. We shall eliminate these using only the predicate c. We use the techniques developed in Section 5.1.1. We replace $\neg\mathsf{C}(a_i, t)$ with $\psi_{DC2}(a_i, t, a_0 + a_1 + a_2 + a_3, 0)$, which is clearly satisfiable by the regions on Figure 5.12. Further, we replace $\neg\mathsf{C}(a_i, b_{\lfloor i+1 \rfloor})$ with $\psi^c_{DC2}(a_i, b_{\lfloor i+1 \rfloor}, s, t)$. As shown in Figure 5.14, there exists a region s satisfying this formula. Instead of dealing

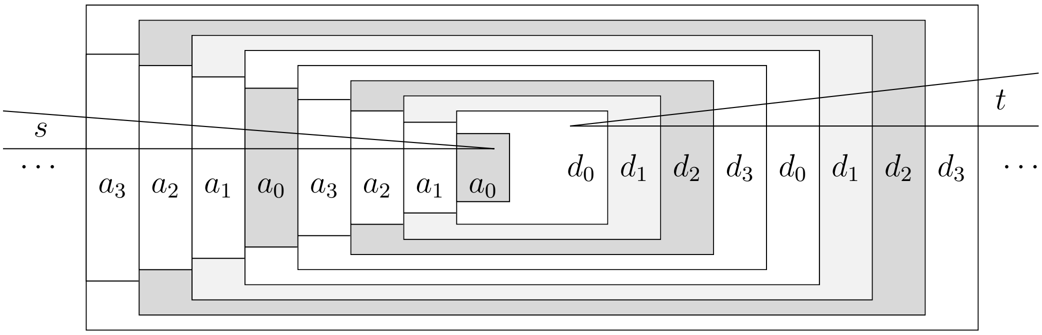


Figure 5.14: Satisfying $\psi_{DC2}(a_0, s, b_1, t)$ and $\psi_{DC2}(a_0, s, b_2, t)$.

with $\neg\mathsf{C}(d_i, d_{i+2})$, we consider the equivalent:

$$\neg\mathsf{C}(a_i, b_{\lfloor i+2\rfloor}) \wedge \neg\mathsf{C}(b_i, a_{\lfloor i+2\rfloor}) \wedge \neg\mathsf{C}(a_i, a_{\lfloor i+2\rfloor}) \wedge \neg\mathsf{C}(b_i, b_{\lfloor i+2\rfloor}).$$

We replace $\neg\mathsf{C}(a_i, b_{\lfloor i+2\rfloor})$ by $\psi^c_{DC2}(a_i, b_{\lfloor i+2\rfloor}, s, t)$, which is satisfiable by the regions depicted in Figure 5.14. We ignore $\neg\mathsf{C}(b_i, a_{\lfloor i+2\rfloor})$, because it is logically equivalent to $\neg\mathsf{C}(a_i, b_{\lfloor i+2\rfloor})$, for different values of i. We replace $\neg\mathsf{C}(a_i, a_{\lfloor i+2\rfloor})$ by $\psi^c_{DC2}(a_i, a_{\lfloor i+2\rfloor}, a'_i, a'_{\lfloor i+2\rfloor})$, which is satisfiable by the regions depicted on Figure 5.15. The fourth conjunct is then treated symmetrically. Transforming ψ_∞

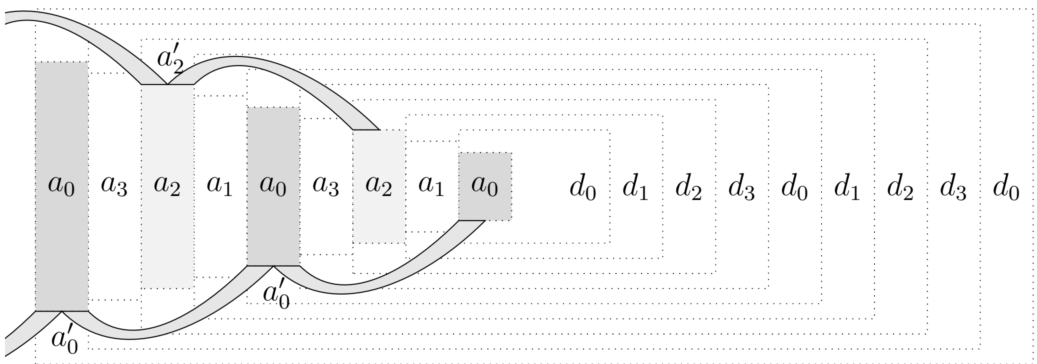


Figure 5.15: Satisfying $\psi^c_{DC2}(a_0, a'_0, a_2, a'_2)$.

in the way just described, we obtain a $\mathcal{B}$c-formula ψ^c_∞, which implies ψ_∞ (in the language $\mathcal{C}$c) and which is satisfiable by the arrangement of $\mathsf{RC}(\mathbb{R}^n)$. Hence, we obtain the following:

Theorem 118. *The $\mathcal{B}$c-formula ψ^c_∞ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$. If $\mathcal{X}$ is a locally connected unicoherent space and ψ^c_∞ is satisfiable by a tuple $\bar{b}$ of regions in*

$\mathsf{RC}(\mathcal{X})$, *then some of the regions in* $\bar{b}$ *have infinitely many components.*

To show that in the region algebras $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$, the languages $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$ and $\mathcal{B}\mathsf{c}$ are sensitive to regions with infinitely many components we extensively used the fact that $\mathbb{R}^n$ is a unicoherent topological space. In the following section, we show that the language $\mathcal{B}\mathsf{c}^\circ$ is sensitive to regions in $\mathsf{RC}(\mathbb{R}^2)$ with infinitely many components—a result entirely based on planarity arguments.

5.3 Infinitely Many Components with Interior Connectedness

We show that there exists a $\mathcal{B}\mathsf{c}^\circ$-formula which is satisfiable in $\mathsf{RC}(\mathbb{R}^2)$, but only by tuples containing regions with infinitely many components. The result is based on the Jordan curve theorem (Lemma 28), which asserts that every Jordan curve Γ in the Euclidean plane (image of an injective continuous map of the unit circle into $\mathbb{R}^2$) separates $\mathbb{R}^2$ into two connected sets having Γ as their common boundary. The results in this section apeared in [KNPHZ11a, KNPHZ11b] and are joined work of the authors. The techniques used are similar to the ones used in [KPHZ10] for showing the corresponding results for the languages $\mathcal{C}\mathsf{c}$ and $\mathcal{B}\mathsf{c}$. We start by demonstrating a technique that will be used repeatedly in the course of the main proof.

Consider the formula $\mathsf{stack}^\circ(a_1, \ldots, a_n)$ given by:

$$\bigwedge_{1 \le i < n} (\mathsf{c}^\circ(a_i + \cdots + a_n) \wedge a_i \cdot a_{i+1} = 0) \wedge \bigwedge_{j-i>1} \neg\mathsf{C}(a_i, a_j),$$

This formula allows us to construct in the Euclidean plane sequences of arcs in the following sense.

Lemma 119. *Let* $a_1, \ldots, a_n$ *be regions in* $\mathsf{RC}(\mathbb{R}^2)$ *satisfying* $\mathsf{stack}^\circ(a_1, \ldots, a_n)$*, for* $n > 1$*. Then every point* $p_1 \in {a^\circ}_1$ *can be connected to every point* $p_n \in {a^\circ}_n$ *by a Jordan arc* $\alpha = \alpha_1 \cdots \alpha_{n-1}$ *such that for all* i $(1 \le i < n)$*, each segment* $\alpha_i \subseteq (a_i + a_{i+1})^\circ$ *is a non-degenerate Jordan arc starting at some point* $p_i \in {a^\circ}_i$*.*

Proof. By $\mathsf{c}^\circ(a_1 + \cdots + a_n)$, let $\alpha'_1 \subseteq (a_1 + \cdots + a_n)^\circ$ be a Jordan arc connecting p_1 to p_n (Figure 5.16). By the non-contact constraints, α'_1 has to contain points in ${a^\circ}_2$. Let p'_2 be one such point. For $2 \le i < n$ we suppose $\alpha_1, \ldots, \alpha_{i-2}$, α'_{i-1}

and p'_i to have been defined, and proceed as follows. By $\mathsf{c}^\circ(a_i + \cdots + a_n)$, let $\alpha''_i \subseteq (a_i + \cdots + a_n)^\circ$ be a Jordan arc connecting p'_i to p_n. By the non-contact constraints, α''_i can intersect $\alpha_1 \cdots \alpha_{i-2}\alpha'_{i-1}$ only in its final segment α'_{i-1}. Let p_{i-1} be the first point of α'_{i-1} lying on α'_i; let α_{i-1} be the initial segment of α'_{i-1} ending at p_{i-1}; and let α'_i be the final segment of α''_i starting at p_{i-1}. It remains only to define α_{n-1}, and to this end, we simply set $\alpha_{n-1} := \alpha'_{n-1}$. To see that p_i, $2 \leq i < n$, are as required, note that $p_i \in \alpha_i \cap \alpha_{i-1}$. By the disjoint constraints p_i must be in a_i. If p_i was in $\delta(a_i)$, it would also have to be in $\delta(a_{i-1})$ and $\delta(a_{i+1})$, which is forbidden by the disjoint constraints. Hence $p_i \in a^\circ{}_i$, $1 \leq i \leq n$. Given $a_i \cdot a_{i+1} = 0$, $1 \leq i < n$, this also guarantees that the arcs α_i are non-degenerate. □

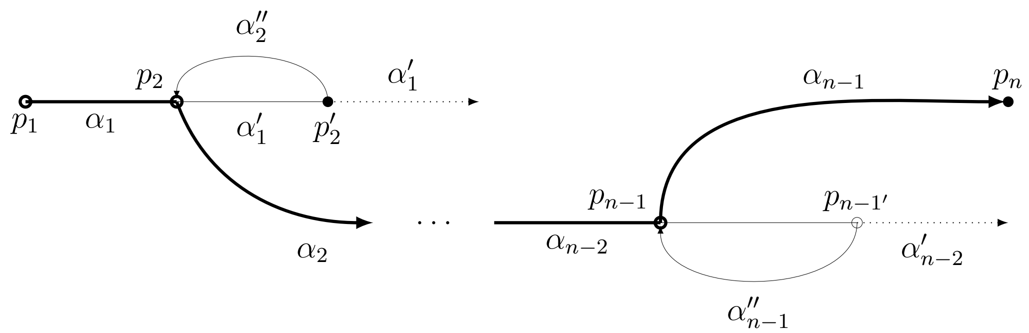


Figure 5.16: The constraint $\mathsf{stack}^\circ(a_1, \ldots, a_n)$ ensures the existence of a Jordan arc $\alpha = \alpha_1 \cdots \alpha_{n-1}$ which connects a point $p_1 \in a_1^\circ$ to a point $p_n \in a_n^\circ$.

Now that we can construct Jordan curves and sequences of Jordan arcs using the $\mathcal{B}\mathsf{c}^\circ$-formulas frame° (see Lemma 113) and stack°, respectively, we are ready to show that the language $\mathcal{B}\mathsf{c}^\circ$ is sensitive to regions with infinitely many components when interpreted in $\mathbb{R}^2$.

Theorem 120. *There is a $\mathcal{B}\mathsf{c}^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but only by regions with infinitely many components.*

Proof. We first write a $\mathcal{C}\mathsf{c}^\circ$-formula, ψ^*_∞ with the required properties, and then using the results established in Section 5.1.2 we show that all occurrences of C can be eliminated.

Let s, s', a, a', b, b', $a_{i,j}$ and $b_{i,j}$ $(0 \leq i < 2,\ 1 \leq j \leq 3)$ be variables. The

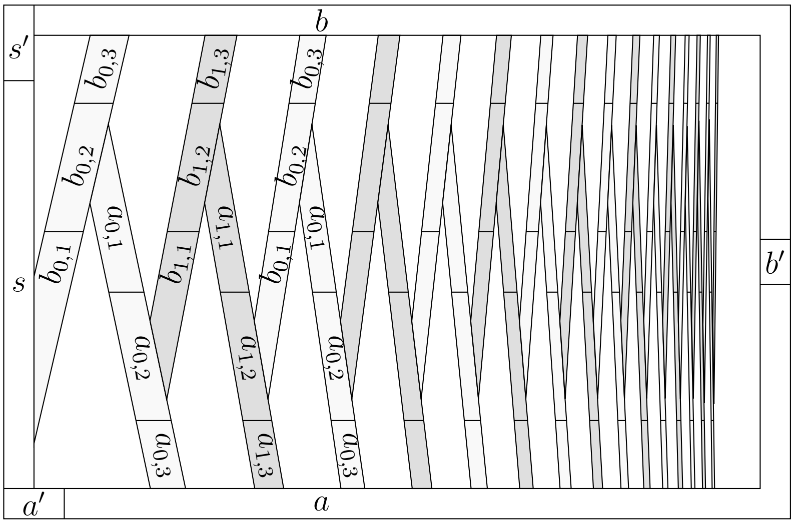


Figure 5.17: A tuple of regions satisfying (5.6)–(5.9): the pattern of components of the $a_{i,j}$ and $b_{i,j}$ repeats forever.

constraints

$$\mathsf{frame}^\circ(s, s', b, b', a, a') \tag{5.6}$$

$$\mathsf{stack}^\circ(s, b_{i,1}, b_{i,2}, b_{i,3}, b) \tag{5.7}$$

$$\mathsf{stack}^\circ(b_{\lfloor i-1 \rfloor,2}, a_{i,1}, a_{i,2}, a_{i,3}, a) \tag{5.8}$$

$$\mathsf{stack}^\circ(a_{\lfloor i-1 \rfloor,2}, b_{i,1}, b_{i,2}, b_{i,3}, b) \tag{5.9}$$

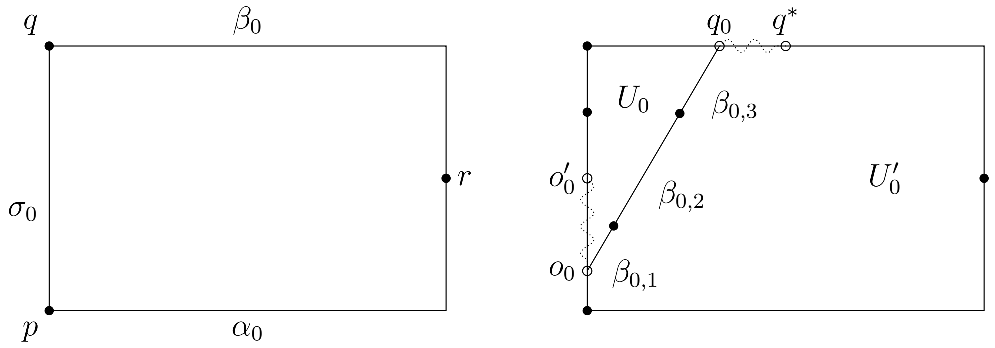
are evidently satisfied by the arrangement of Figure 5.17.

Let ψ^*_∞ be the conjunction of (5.6)–(5.9) as well as all conjuncts

$$r \cdot r' = 0, \tag{5.10}$$

where r and r' are any two distinct regions depicted in Figure 5.17. Note that the regions $a_{i,j}$ and $b_{i,j}$ have infinitely many connected components. We will now show that this is true for every satisfying tuple of ψ^*_∞.

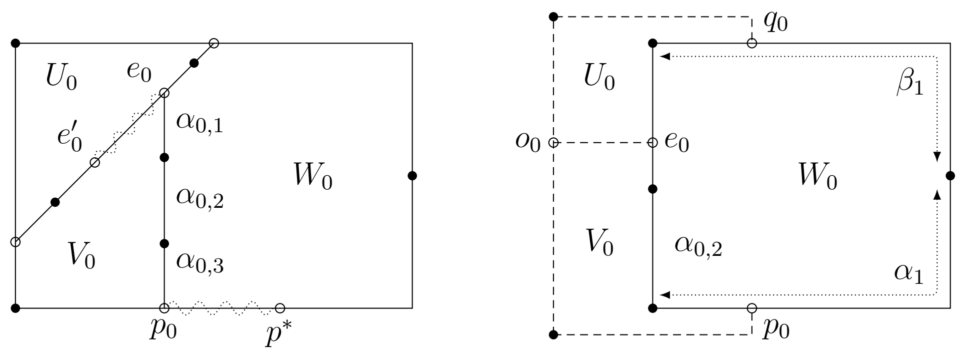
By (5.6), we can use Lemma 113 to construct a Jordan curve $\Gamma = \sigma\sigma'\beta\beta'\alpha\alpha'$ whose segments are Jordan arcs lying in the respective sets $(s + s')^\circ$, $(s' + b)^\circ$, $(b+b')^\circ$, $(b'+a)^\circ$, $(a+a')^\circ$, $(a'+s)^\circ$. Further, let $\sigma_0 = \sigma\sigma'$, $\beta_0 = \beta\beta'$ and $\alpha_0 = \alpha\alpha'$ (Figure 5.18a). Note that all points in s, a and b that are on Γ are on σ_0, α_0 and β_0, respectively. Let $o'_0 \in \sigma_0 \cap s^\circ$, and let $q^* \in \beta_0 \cap b^\circ$. By (5.7) and Lemma 119



(a) The arcs α_0, β_0 and σ_0.

(b) The regions U_0 and U'_0.

Figure 5.18: Establishing infinite sequences of arcs I.



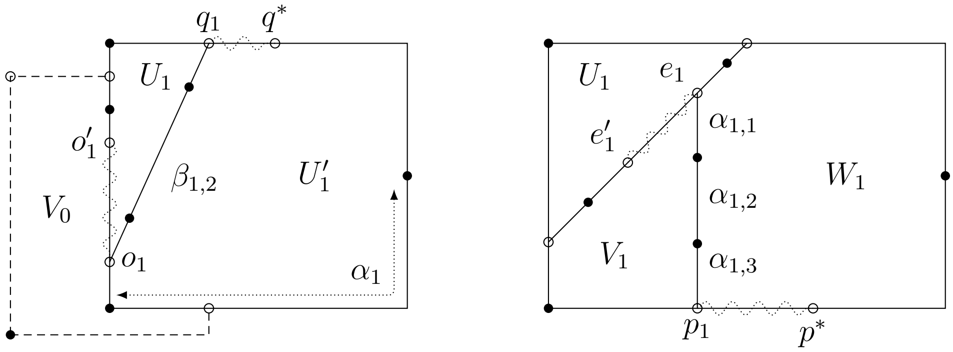
(a) The regions V_0 and W_0.

(b) The regions redrawn.

Figure 5.19: Establishing infinite sequences of arcs II.

we can connect o'_0 to q^* by a Jordan arc $\beta'_{0,1}\beta_{0,2}\beta'_{0,3}$ whose segments lie in the respective sets $(s + b_{0,1})^\circ$, $(b_{0,1} + b_{0,2} + b_{0,3})^\circ$ and $(b + b_{0,3})^\circ$ (Figure 5.18b). Let o_0 be the last point on $\beta'_{0,1}$ that is on σ_0 and let $\beta_{0,1}$ be the final segment of $\beta'_{0,1}$ starting at o_0. Similarly, let q_0 be the first point on $\beta'_{0,3}$ that is on β_0 and let $\beta_{0,3}$ be the initial segment of $\beta'_{0,3}$ ending at q_0. Hence, the arc $\beta_{0,1}\beta_{0,2}\beta_{0,3}$ divides one of the regions bounded by Γ into two sub-regions. We denote the sub-region whose boundary is disjoint from α_0 by U_0, and the other sub-region we denote by U'_0. Let $\beta_1 := \beta_{0,3}\beta_0[q_0, r] \subseteq (b + b_{0,3} + b_{1,3})^\circ$.

We will now construct a cross-cut $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ in U'_0. Let $e'_0 \in \beta_{0,2} \cap b^\circ_{0,2}$ and $p^* \in \alpha_0 \cap a^\circ$. By (5.8) and Lemma 119 we can connect e'_0 to p^* by a



(a) The regions U_1 and U'_1.

(b) The regions V_1 and W_1.

Figure 5.20: Establishing infinite sequences of arcs III.

Jordan arc $\alpha'_{0,1}\alpha_{0,2}\alpha'_{0,3}$ whose segments lie in the respective sets $(b_{0,2} + a_{0,1})^\circ$, $(a_{0,1} + a_{0,2} + a_{0,3})^\circ$ and $(a + a_{0,3})^\circ$ (Figure 5.19a). Let e_0 be the last point on $\alpha'_{0,1}$ that is on $\beta_{0,2}$ and let $\alpha_{0,1}$ be the final segment of $\alpha'_{0,1}$ starting at e_0. Similarly, let p_0 be the first point on $\alpha'_{0,3}$ that is on α_0 and let $\alpha_{0,3}$ be the initial segment of $\alpha'_{0,3}$ ending at p_0. By (5.10), $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ does not intersect the boundaries of U_0 and U'_0 except at its endpoints, and hence it is a cross-cut in one of these regions. Moreover, that region has to be U'_0 since the boundary of U_0 is disjoint from α_0. So, $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ divides U'_0 into two sub-regions. We denote the sub-region whose boundary contains β_1 by W_0, and the other sub-region we denote by V_0. Let $\alpha_1 := \alpha_{0,3}\alpha_0[p_0, r]$ (Fig 5.19b). Note that $\alpha_1 \subseteq (a + a_{0,3} + a_{1,3})^\circ$.

We can now forget about the region U_0, and start constructing a cross-cut $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ in W_0. As before, let $\beta'_{1,1}\beta_{1,2}\beta'_{1,3}$ be a Jordan arc connecting a point $o'_1 \in \alpha_{0,2} \cap a^\circ{}_{0,2}$ to a point $q^* \in \beta_1 \cap b^\circ{}_i$ such that its segments are contained in the respective sets $(a_{0,2} + b_{1,1})^\circ$, $(b_{1,1} + b_{1,2} + b_{1,3})^\circ$ and $(b + b_{1,3})^\circ$. As before, we choose $\beta_{1,1} \subseteq \beta'_{1,1}$ and $\beta_{1,3} \subseteq \beta'_{1,3}$ so that the Jordan arc $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ with its endpoints removed is disjoint from the boundaries of V_0 and W_0. Hence $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ has to be a cross-cut in V_0 or W_0, and since the boundary of V_0 is disjoint from β_1 it has to be a cross-cut in W_0 (Figure 5.20a). So, $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ separates W_0 into two regions U_1 and U'_1 so that the boundary of U_1 is disjoint from α_1. Let $\beta_2 := \beta_{1,3}\beta_1[q_1, r] \subseteq (b + b_{0,3} + b_{1,3})^\circ$. Now, we can ignore the region V_0, and reasoning as before we can construct a cross-cut $\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}$ in U'_1 dividing it into two sub-regions V_1 and W_1.

Evidently, this process continues forever. Now, note that by construction

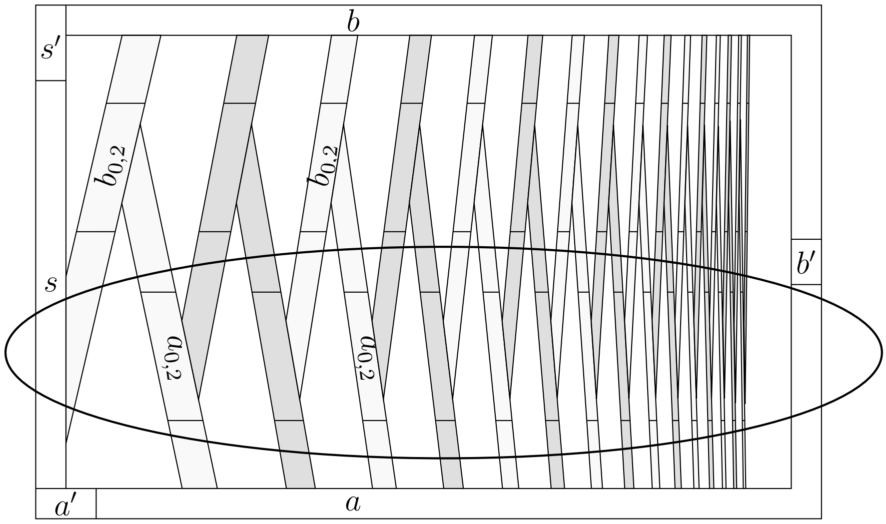


Figure 5.21: Separating $a_{0,2}$ from $b_{0,2}$ by a Jordan curve.

and (5.10), W_{2i} contains in its interior $\beta_{2i+1,2}$ together with the connected component c of $b_{1,2}$ which contains $\beta_{2i+1,2}$. On the other hand, W_{2i+2} is disjoint from c, and since $W_i \subseteq W_j$, $i > j$, $b_{1,2}$ has to have infinitely many connected components.

So far we know that the $\mathcal{C}$c°-formula ψ^*_∞ forces infinitely many components. Now we replace every conjunct in ψ^*_∞ of the form $\neg\mathsf{C}(r,s)$ by $\psi^\circ_{DC1}(r,s,\bar{v})$, where $\bar{v}$ are fresh variables each time. By Lemma 114, the resulting formula ψ_∞ entails ψ^*_∞, so we only have to show that it is still satisfiable. By Lemma 114 (*ii*), it suffices to separate by Jordan curves every two regions in Figure 5.17 that are required to be disjoint. It is shown in Figure 5.21 that there exists a curve which separates the regions $b_{0,2}$ and $a_{0,2}$. All other non-contact constraints are treated analogously. □

Since $\mathsf{RCP}(\mathbb{R}^2)$ contains only regions with finitely many components, we get that the region algebras $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$ satisfy different $\mathcal{B}$c°-formulas.

Corollary 121. *Sat*$(\mathcal{B}\mathsf{c}^\circ, \mathsf{RC}(\mathbb{R}^2)) \neq$ *Sat*$(\mathcal{B}\mathsf{c}^\circ, \mathsf{RCP}(\mathbb{R}^2))$.

In this section we showed that in the Euclidean plane the languages $\mathcal{B}$c° and $\mathcal{C}$c° are sensitive to regions having infinitely many components. We applied a simple but powerful tool for constructing sequences of arcs lying in certain Euclidean regions. In the following section we use a generalised version of this technique to show that the satisfiability problems for the languages $\mathcal{C}$c, $\mathcal{C}$c°, $\mathcal{B}$c and $\mathcal{B}c^\circ$ are undecidable when interpreted over the region algebras $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RC}(\mathbb{R}^2)$.

5.4 Undecidability: The Plane Case

In this section we show that the problems $\mathsf{Sat}(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and $\mathsf{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$, for $\mathcal{L}$ any of the languages $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$ are undecidable. We do so via a reduction from the *Post correspondence problem* (PCP). The result is based on planarity arguments that fail in Euclidean spaces of different dimensions. The proof is involved and lengthy, and will be presented only as a sketch. It requires some technical results, which will be presented in full. The results in this section apeared in [KNPHZ11a, KNPHZ11b] and are joined work of the authors.

Our first task is to find a $\mathcal{C}c$-formula stack (similar to the $\mathcal{B}c^\circ$-formula stack° from the previous section) that will help us to construct sequences of Jordan arcs lying in certain regions. For that we need to introduce the notion of a *3-region*. A *3-region* is a triple $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ of elements of $\mathsf{RC}(\mathbb{R}^2)$ such that $0 \neq \ddot{a} \ll \dot{a} \ll a$, where $r \ll s$ abbreviates $\neg\mathsf{C}(r, -s)$. It helps to think of $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ as consisting of a kernel, $\ddot{a}$, encased in two protective layers of shell. As a simple example, consider the sequence of 3-regions $\mathfrak{a}_1, \dots, \mathfrak{a}_5$ depicted in Figure 5.22, where the inner-most regions form a sequence of externally touching polygons.

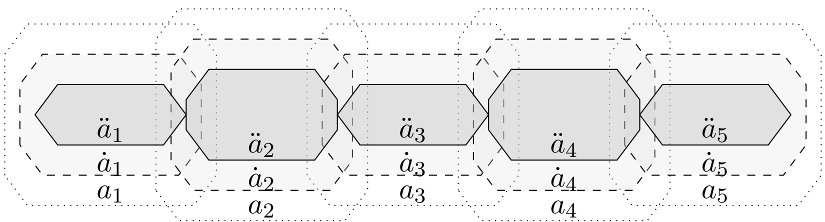


Figure 5.22: A chain of 3-regions satisfying $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_5)$.

We define the $\mathcal{C}c$-formula $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$, by:

$$\bigwedge_{1 \leq i \leq n} \mathsf{c}(\dot{a}_i + \ddot{a}_{i+1} + \cdots + \ddot{a}_n) \wedge \bigwedge_{j-i>1} \neg\mathsf{C}(a_i, a_j).$$

It is readily seen that the 3-regions $\mathfrak{a}_1, \dots, \mathfrak{a}_5$ satisfy the above formula. Note that if $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_{n+1})$, then also $\mathsf{stack}(\mathfrak{a}_2, \dots, \mathfrak{a}_{n+1})$ and $\mathsf{stack}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$. The

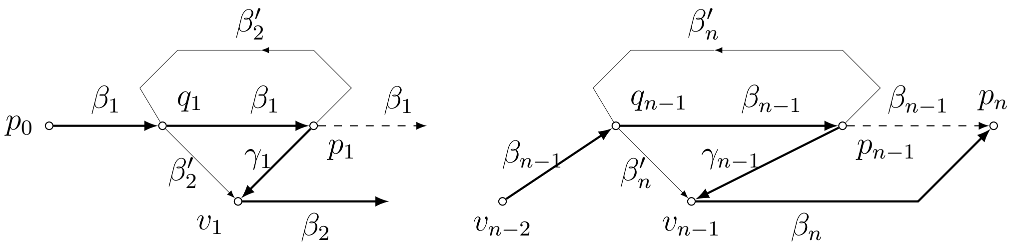


Figure 5.23: Proof of Lemma 122.

following lemma is crucial for the results in this section.

Lemma 122. *Let* $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ *be 3-regions satisfying* $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, *for* $n \geq 3$. *Then, for every point* $p_0 \in \dot{a}_1$ *and every point* $p_n \in \ddot{a}_n$, *there exist points* $p_1, \ldots, p_{n-1}$ *and Jordan arcs* $\alpha_1, \ldots, \alpha_n$ *such that*:

(*i*) $\alpha = \alpha_1 \cdots \alpha_n$ *is a Jordan arc from* p_0 *to* p_n;

(*ii*) *for all* i $(0 \leq i < n)$, $p_i \in \dot{a}_{i+1} \cap \alpha_i$; *and*

(*iii*) *for all* i $(1 \leq i \leq n)$, $\alpha_i \subseteq a_i$.

Proof. Since $\dot{a}_1 + \ddot{a}_2 + \cdots + \ddot{a}_n$ is a connected subset of $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)^\circ$, let β_1 be a Jordan arc connecting p_0 to p_n in $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)^\circ$. Since a_1 is disjoint from all the a_i except a_2, let p_1 be the first point of β_1 lying in $\dot{a}_2$, so $\beta_1[p_0, p_1] \subseteq a^\circ{}_1 \cup \{p_1\}$, i.e., the arc $\beta_1[p_0, p_1]$ is either included in $a^\circ{}_1$, or is an end-cut of $a^\circ{}_1$. (We do not rule out $p_0 = p_1$.) Similarly, let β'_2 be a Jordan arc connecting p_1 to p_n in $(a_2 + \dot{a}_3 + \cdots + \dot{a}_n)^\circ$, and let q_1 be the last point of β'_2 lying on $\beta_1[p_0, p_1]$. If $q_1 = p_1$, then set $v_1 = p_1$, $\alpha_1 = \beta_1[p_0, p_1]$, and $\beta_2 = \beta'_2$. so that the endpoints of β_2 are v_1 and p_n. Otherwise, we have $q_1 \in a^\circ{}_1$. We can now construct an arc $\gamma_1 \subseteq a^\circ{}_1 \cup \{p_1\}$ from p_1 to a point v_1 on $\beta'_2[q_1, p_n]$, such that γ_1 intersects $\beta_1[p_0, p_1]$ and $\beta'_2[q_1, p_n]$ only at its endpoints, p_1 and v_1 (upper diagram in Figure 5.23). Let $\alpha_1 = \beta_1[p_0, p_1]\gamma_1$, and let $\beta_2 = \beta'_2[v_1, p_n]$.

Since β_2 contains a point $p_2 \in \dot{a}_3$, we may iterate this procedure, obtaining $\alpha_2, \alpha_3, \ldots \alpha_{n-1}, \beta_n$. We remark that α_i and α_{i+1} have a single point of contact by construction, while α_i and α_j $(i < j-1)$ are disjoint by the constraint $\neg\mathsf{C}(a_i, a_j)$. Finally, we let $\alpha_n = \beta_n$ (lower diagram in Figure 5.23). □

We can add a 'switch' w to the formula $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$, in the following sense. If w is a region variable, consider the formula $\mathsf{stack}_w(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$

$$\neg\mathsf{C}(w \cdot \dot{a}_1, (-w) \cdot \dot{a}_1) \;\wedge\; \mathsf{stack}((-w) \cdot \mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n),$$

where $w \cdot \mathfrak{a}$ denotes the 3-region $(w \cdot a, w \cdot \dot{a}, w \cdot \ddot{a})$. The first conjunct of $\mathsf{stack}_w(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ ensures that any component of $\dot{a}_1$ is either included in w or included in $-w$. The second conjunct then has the same effect as $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ for those components of $\dot{a}_1$ included in $-w$. That is, if $p \in \dot{a}_1 \cdot (-w)$, we can find an arc $\alpha_1 \cdots \alpha_n$ starting at p, with the properties of Lemma 122. However, if $p \in \dot{a} \cdot w$, no such arc need exist. Thus, w functions so as to 'de-activate' the formula $\mathsf{stack}_w(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ for any component of $\dot{a}_1$ included in it.

As a further application of Lemma 122, consider the formula $\mathsf{frame}(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)$ given by:

$$\mathsf{stack}(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}) \wedge \neg\mathsf{C}(a_n, a_1 + \ldots + a_{n-2}) \wedge \\ \mathsf{c}(\dot{a}_n) \wedge \dot{a}_0 \cdot \dot{a}_n \neq 0 \wedge \ddot{a}_{n-1} \cdot \dot{a}_n \neq 0. \quad (5.11)$$

This formula allows us to construct Jordan curves in the plane, in the following

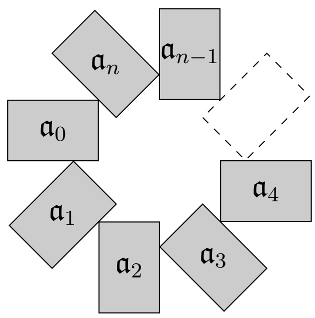


Figure 5.24: 3-regions satisfying the formula $\mathsf{frame}(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)$.

sense:

Lemma 123. *Let* $n \geq 3$*, and suppose* $\mathsf{frame}(\mathfrak{a}_0, \ldots, \mathfrak{a}_n)$*. Then there exist Jordan arcs* $\gamma_0, \ldots, \gamma_n$ *such that* $\gamma_0 \ldots \gamma_n$ *is a Jordan curve, and* $\gamma_i \subseteq a_i$*, for all* i*,* $0 \leq i \leq n$*.*

Proof. By $\mathsf{stack}(\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1})$, let $\alpha_0, \ldots, \alpha_{n-1}$ be Jordan arcs in the respective regions $a_0, \ldots, a_{n-1}$ such that, $\alpha = \alpha_0 \cdots \alpha_{n-1}$ is a Jordan arc connecting a point $p' \in \dot{a}_0 \cdot \dot{a}_n$ to a point $q' \in \ddot{a}_{n-1} \cdot \dot{a}_n$ (see Figure 5.25). Because $\dot{a}_n$ is a connected subset of the interior of a_n, let $\alpha_n \subseteq a^\circ{}_n$ be an arc connecting p' and q'. Note that α_n does not intersect α_i, for $1 \leq i < n-1$. Let p be the last point on α_0 that is on α_n (possibly p'), and q be the first point on α_{n-1} that is on α_n (possibly q'). Let γ_0 be the final segment of α_0 starting at p. Let $\gamma_i := \alpha_i$, for $1 \leq i \leq n-2$. Let γ_{n-1} be the initial segment of α_{n-1} ending at q. Finally, take γ_n to be the segment of α_n between p and q, reversed if necessary. Evidently, the arcs γ_i, $0 \leq i \leq n$, are as required. □

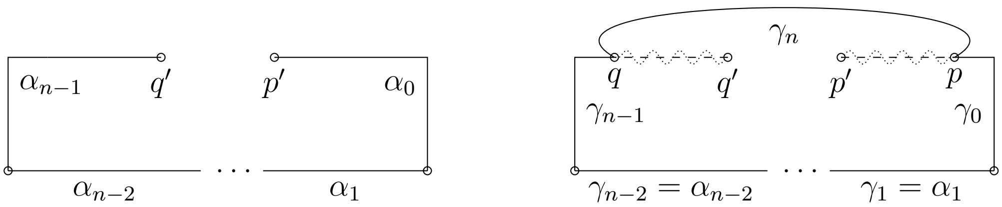


Figure 5.25: Establishing a Jordan curve.

Our final technical preliminary is a simple device for labelling the components of regions in order to encode certain information.

Lemma 124. *Suppose r, t_1, ..., t_ℓ are regions such that*

$$(r \leq t_1 + \cdots + t_\ell) \wedge \bigwedge_{1 \leq i < j \leq \ell} \neg \mathsf{C}(r \cdot t_i, r \cdot t_j), \tag{5.12}$$

and let X be a connected subset of r. Then X is included in exactly one of the t_i, $1 \leq i \leq \ell$.

Proof. If $X \cap t_1$ and $X \cap t_2$ are non-empty, then $X \cap t_1$ and $X \cap (t_2 + \cdots + t_\ell)$ partition X into non-empty, non-intersecting sets, closed in X. □

When (5.12) holds, we may think of the regions $t_1, \ldots, t_\ell$ as 'labels' for any connected $X \subseteq r$—and, in particular, for any Jordan arc $\alpha \subseteq r$. Hence, any sequence $\alpha_1, \ldots, \alpha_n$ of such arcs encodes a word over the alphabet $\{t_1, \ldots, t_\ell\}$.

We now turn to the proof of the main theorem of this section. First we give a detailed sketch of the proof for the language $\mathcal{C}$c, and then, using the techniques described in Section 5.1, we extend it to the languages $\mathcal{C}$c°, $\mathcal{B}$c and $\mathcal{B}$c°.

Theorem 125. *For $\mathcal{L} \in \{\mathcal{B}c^\circ, \mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$, $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete.*

Proof. Let a PCP-instance $\mathbf{w} = (\{0,1\}, T, \mathsf{w}_1, \mathsf{w}_2)$ be given, where T is a finite alphabet, and $\mathsf{w}_i \colon T^* \to \{0,1\}^*$ a word-morphism ($i = 1, 2$). We call the elements of T *tiles*, and, for each tile t, we call $\mathsf{w}_1(t)$ the *lower word* of t, and $\mathsf{w}_2(t)$ the *upper word* of t. So the question is whether there is a sequence of tiles such that the concatenation of their upper words is the same as the concatenation of their lower words. We only consider PCP-instances whose upper and lower words are non-empty—a restriction that simplifies the encoding, but does not affect the undecidability of the problem.

We define a formula $\varphi_{\mathbf{w}}$ consisting of a large conjunction of $\mathcal{C}$c-literals, which, for ease of understanding, we introduce in groups. Whenever conjuncts are introduced, it can be readily checked that—provided $\mathbf{w}$ is positive—they are satisfiable by elements of $\mathsf{RCP}(\mathbb{R}^2)$. (Figures 5.26, 5.28, 5.35 and 5.36 depict *part* of a satisfying assignment; this drawing is additionally useful as an aid to intuition throughout the course of the proof.) The main object of the proof is to show that, conversely, if $\varphi_{\mathbf{w}}$ is satisfied by any tuple in $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is positive. Thus, the following are equivalent:

1. $\mathbf{w}$ is positive;
2. $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$;
3. $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$.

This establishes the r.e.-hardness of $\mathrm{Sat}(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and $\mathrm{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ for $\mathcal{L} = \mathcal{C}$c; we then extend the result to the languages $\mathcal{B}$c, $\mathcal{C}$c$^\circ$ and $\mathcal{B}$c$^\circ$ using the techniques established in Section 5.1.

The proof proceeds in four stages.

Stage 1. In the first stage, we define the 'frame' of the rest of the construction. What we have in mind is the 3-regions $\mathfrak{s}_0, \ldots, \mathfrak{s}_9$, $\mathfrak{s}'_8, \ldots, \mathfrak{s}'_1$, $\mathfrak{d}_0, \ldots, \mathfrak{d}_6$ as depicted in Figure 5.26. We force a similar configuration using the formulas:

$$\mathsf{frame}(\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_8, \mathfrak{s}_9, \mathfrak{s}'_8, \ldots, \mathfrak{s}'_1), \tag{5.13}$$

$$(s_0 \leq \dot{d}_0) \wedge (s_9 \leq \ddot{d}_6), \tag{5.14}$$

$$\mathsf{stack}(\mathfrak{d}_0, \ldots, \mathfrak{d}_6). \tag{5.15}$$

together with a non-contact constraint

$$\neg \mathsf{C}(r, r') \tag{5.16}$$

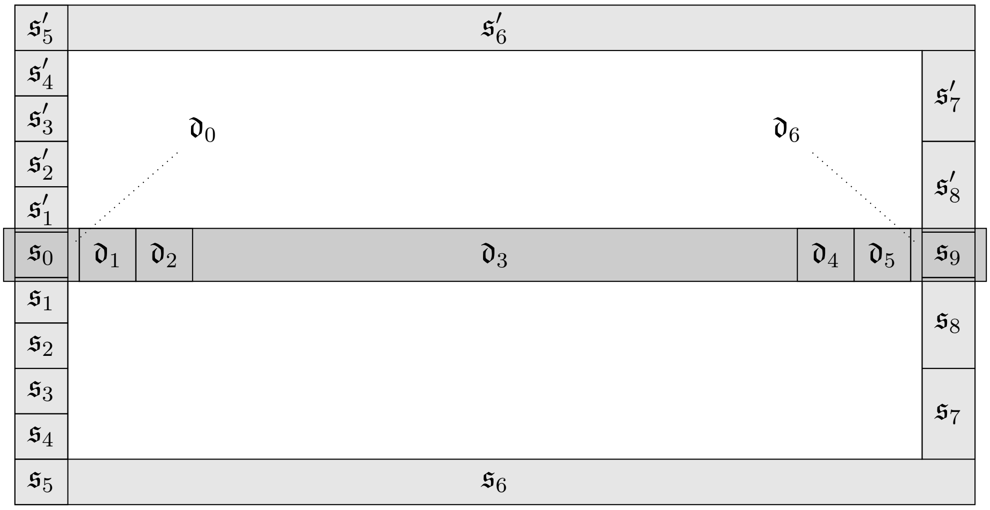


Figure 5.26: A tuple of 3-regions satisfying (5.13)–(5.15).

for every two of the depicted 3-regions $\mathfrak{r}$ and $\mathfrak{r}'$ that are not drawn as being in contact. Recall that r is the outer-most region of a 3-region $\mathfrak{r}$, while we depict $\mathfrak{r}$ by drawing its inner-most region $\ddot{r}$. Thus, for example, (5.16) includes $\neg\mathsf{C}(s_0, d_1)$, but not $\neg\mathsf{C}(s_0, d_0)$ of $\neg\mathsf{C}(d_0, d_1)$.

Now suppose $\mathfrak{s}_0, \ldots, \mathfrak{s}_9$, $\mathfrak{s}'_8, \ldots, \mathfrak{s}'_1$, $\mathfrak{d}_0, \ldots, \mathfrak{d}_6$ is a collection of arbitrary 3-regions in the Euclidean plane satisfying (5.13)–(5.16). Using Lemma 123 one can show that there exist Jordan arcs $\gamma_0, \ldots, \gamma_9$, $\gamma'_8, \ldots, \gamma'_1$ contained in the respective regions $s_0, \ldots, s_9, s'_8, \ldots, s'_1$, such that $\Gamma = \gamma_0 \cdots \gamma_9 \cdot \gamma'_8 \cdots \gamma'_1$ is a Jordan curve. Using Lemma 122, one can also show that there exist Jordan arcs $\chi_1 \subseteq (d_0 + d_1)$, $\chi_2 \subseteq d_2 + d_3 + d_4$ and $\chi_3 \subseteq d_5 + d_6$ such that $\chi_1\chi_2\chi_3$ is a Jordan arc that is a chord in Γ with endpoints on γ_0 and γ_9. This configuration of arcs is depicted in Figure 5.27, where $\gamma_{i,j}$ denotes the Jordan arc $\gamma_i \ldots \gamma_j$, and similarly for $\gamma'_{i,j}$. Note that these Jordan arcs separate the Euclidean space into three open sets exactly one of which is unbounded. Figure 5.27 depicts only one of three possible cases—the one when χ_2 is part of the boundary of the two bounded sets. The choice of the unbounded region does not affect the encoding of the PCP and from now on will be disregarded. We refer to the open set whose boundary contains the arcs γ_6 and χ_2 as 'lower window', and, similarly, we refer to the open set whose boundary contains the arcs γ'_6 and χ_2 as 'upper window'.

Stage 2. In this stage, we we construct two sequences of arcs, $\{\alpha_i\}$ and $\{\beta_i\}$ of

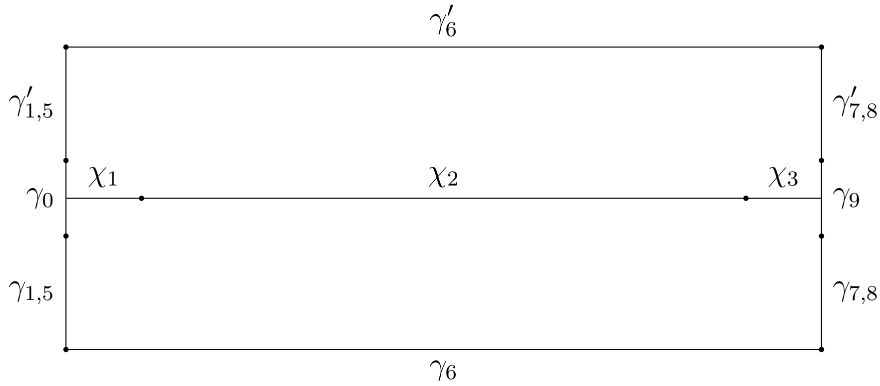


Figure 5.27: The arcs $\gamma_0, \ldots, \gamma_9$, $\gamma'_1, \ldots, \gamma'_8$ and χ_1, χ_2, χ_3.

indeterminate length $n \geq 1$, such that the arcs α_i lie in the lower window. In the sequel, we write $\lfloor k \rfloor$ to denote k modulo 3. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}_{i,j}$ and $\mathfrak{b}_{i,j}$ ($0 \leq i < 3$, $1 \leq j \leq 6$) be 3-regions, and let z be a region satisfying the formulas:

$$(s_6 \leq \ddot{a}) \wedge (s'_6 \leq \ddot{b}) \wedge (s_3 \leq \dot{a}_{0,3}), \tag{5.17}$$

$$\mathsf{stack}_z(\mathfrak{a}_{\lfloor i-1 \rfloor,3}, \mathfrak{b}_{i,1}, \ldots, \mathfrak{b}_{i,6}, \mathfrak{b}), \tag{5.18}$$

$$\mathsf{stack}(\mathfrak{b}_{i,3}, \mathfrak{a}_{i,1}, \ldots, \mathfrak{a}_{i,6}, \mathfrak{a}). \tag{5.19}$$

The arrangement of polygonal 3-regions depicted in Figure 5.28 (with z assigned appropriately) is one such satisfying assignment. Again we add non-contact constraints $\neg C(r, s)$ for every two 3-regions $\mathfrak{r}$ and $\mathfrak{s}$ in Figure 5.26 or Figure 5.28 not depicted as being in contact on either of the two figures.

Using Lemma 122 and arguing as in the proof of Theorem 120, one can show that there exist non-empty sequences of arcs $\{\eta_i\}$ and $\{\zeta_i\}$ arranged as shown in Figure 5.29 and such that $\zeta_i \subseteq a_{\lfloor i-1 \rfloor,3} + b_{\lfloor i \rfloor,1} + \cdots + b_{\lfloor i \rfloor,6} + b$ and $\eta_i \subseteq b_{\lfloor i \rfloor,3} + a_{i,1} + \cdots + a_{\lfloor i \rfloor,6} + a$. To ensure that the two sequences are indeed non-empty, we add the constraint

$$\neg\mathsf{C}(s_3, z). \tag{5.20}$$

We also need to ensure that the sequences are finite, i.e. the 'switch' z in (5.18) does 'fire' at some point. To do so, we add the constraint

$$\mathsf{c}(b_{0,5} + d_3), \tag{5.21}$$

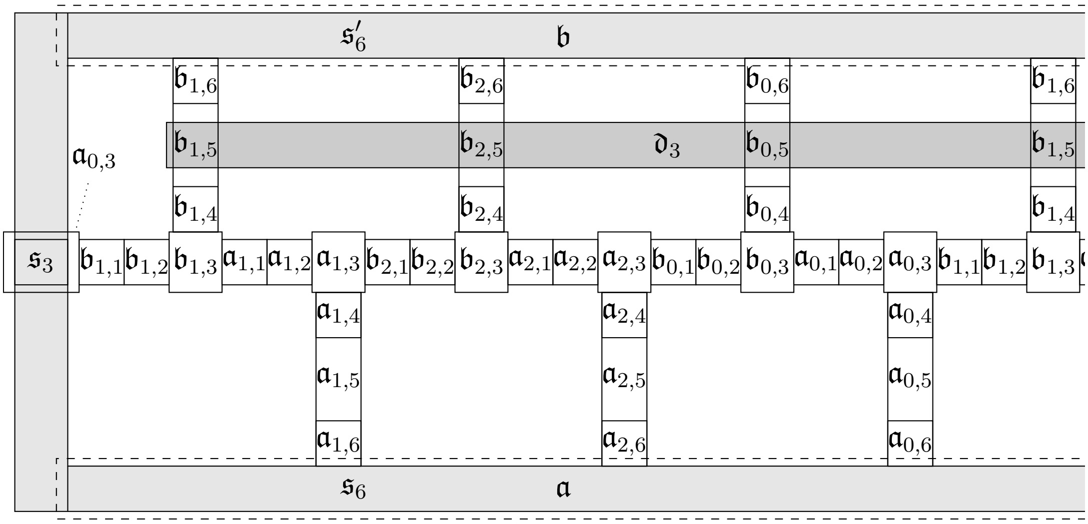


Figure 5.28: A tuple of 3-regions satisfying (5.17)–(5.19). The arrangement of components of the $\mathfrak{a}_{i,j}$ and $\mathfrak{b}_{i,j}$ repeats an indeterminate number of times. The 3-regions $\mathfrak{a}$, $\mathfrak{b}$ and one component of $\mathfrak{a}_{0,3}$ are shown in dotted lines. The 3-regions $\mathfrak{s}_3$, $\mathfrak{s}_6$, $\mathfrak{s}'_6$ and $\mathfrak{d}_3$ are as in Figure 5.26, but not drawn to scale.

which guarantees that η_i and χ_2 lie in the same residual domain of Γ. Due to the non-contact constraints in (5.16), we get that each η_i intersects χ_2. Further, if we assume that the η_i are infinitely many, we will get an infinite sequence of points $P_i \in b_{\lfloor i \rfloor,5}$ $(i \in \omega)$ occurring consequently on the Jordan arc χ_2, which is a compact set. Any such sequence must have an accumulation point on χ_2, and it will be a common point for the three disjoint regions $b_{0,5}$, $b_{1,5}$ and $b_{2,5}$—a contradiction. Hence the sequences $\{\eta_i\}$ and $\{\zeta_i\}$ are indeed finite, and let n be their length.

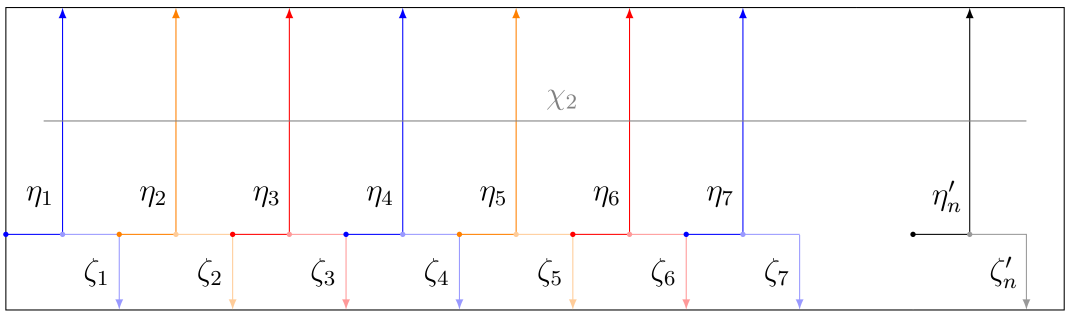


Figure 5.29: The arcs η_i and ζ_i.

Now we 're-package' the arcs $\{\zeta_i\}$ and $\{\eta_i\}$, as illustrated in Figure 5.30.

Further, defining, for $0 \leq i < 3$,

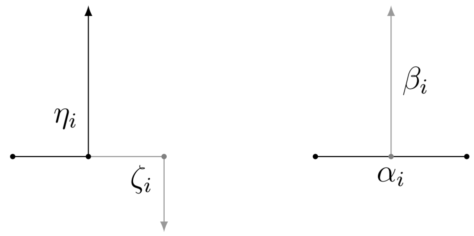


Figure 5.30: 'Re-packaging' of ζ_i and η_i into α_i and β_i: before and after.

$$\begin{aligned} a_i &= a_{1-i,3} + b_{i,1} + \cdots + b_{i,4} + a_{i,1} + \cdots + a_{i,4} \\ b_i &= b_{i,2} + \cdots + b_{i,5}, \end{aligned}$$

we get for $1 \leq i \leq n$ that

$$\begin{aligned} \alpha_i &\subseteq a_{\lfloor i \rfloor} \\ \beta_i &\subseteq b_{\lfloor i \rfloor}. \end{aligned}$$

Note that the arcs α_i are located entirely in the 'lower window', and that each arc β_i connects α_i to a point on γ'_6.

We repeat the same with the 'upper' and 'lower' windows exchanged. We add to $\varphi_{\mathbf{w}}$ constraints for regions $\mathfrak{a}'_{i,j}, \mathfrak{b}'_{i,j}, a'_i, b'_i$ that establish sequences of arcs $\{\zeta'_i\}$, $\{\eta'_i\}$, $(1 \leq i \leq n')$ arranged as shown in Figure 5.31. Again, we 're-package' these Jordan arcs into the Jordan arcs $\{\alpha'_i\}$ and $\{\beta'_i\}$ so that

$$\begin{aligned} \alpha'_i &\subseteq a'_{\lfloor i \rfloor} \\ \beta'_i &\subseteq b'_{\lfloor i \rfloor} \end{aligned}$$

for $1 \leq i \leq n'$. Now, the arcs α'_i are located entirely in the 'upper window', and each arc β'_i connects α'_i to a point on γ_6.

Stage 3. In this stage we extend $\varphi_{\mathbf{w}}$ to ensure that the Jordan arcs β_i and β'_i establishes a 1-1 mapping between the Jordan arcs α_i to α'_i, and in particular that $n = n'$.

Recall that α_n and $\alpha'_{n'}$ contain some points in z. Ensuring that z is connected and contained in the interior of a region z^*, we can join these points by a Jordan

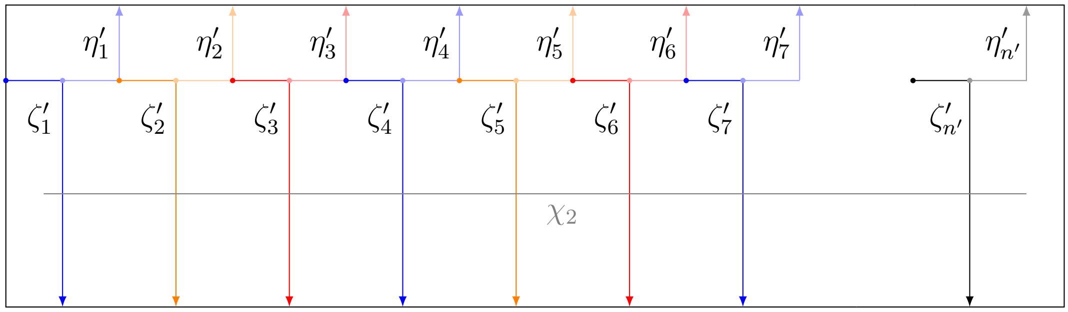


Figure 5.31: The arcs η'_i and ζ'_i.

arc β^* lying in the interior of z^*. Restricting z^* to be disjoint from the regions $b_{i,j}$ and $b'_{i,j}$ for i $(0 \leq i < 3)$ and j $(1 \leq i \leq 6)$ we ensure that β^* is (essentially) as shown in Figure 5.32.

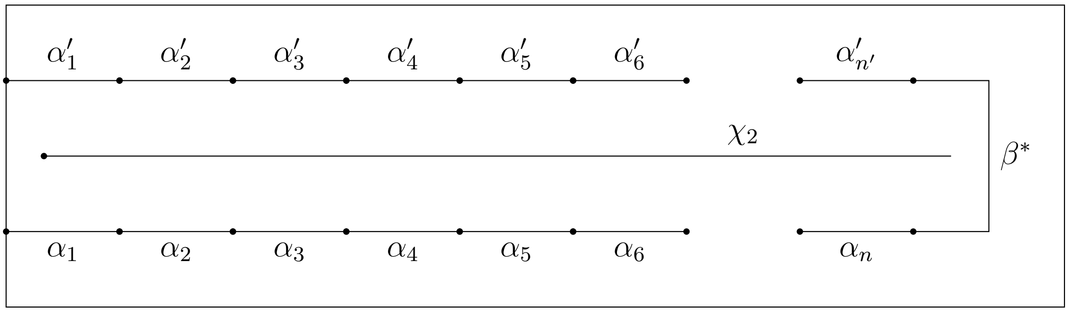


Figure 5.32: The arc β^*.

Further, we add to φ_{w} the constraints

$$\neg\mathsf{C}(a'_i, b_j) \qquad \neg\mathsf{C}(a_i, b'_j) \qquad \neg\mathsf{C}(b_i, b'_j)$$

for $0 \leq i < 3$, $0 \leq j < 3$, $i \neq j$. One can show by induction that each β_i $(1 \leq i \leq n)$ crosses α_i and α'_i $(1 \leq i \leq n')$, and, similarly, that each β'_i $(1 \leq i \leq n')$ crosses α_i and α'_i $(1 \leq i \leq n')$. Hence, the arcs β_i establish a 1–1 correspondence between the arcs α_i and α'_i as depicted in Figure 5.33. Of course, this implies that $n = n'$.

The conjuncts of φ_{w} introduced so far are the same for each instance of the PCP. The part of φ_{w} that is specific for each instance of the PCP are introduced in the following stage. We use as a running example the instance of the PCP v, which is given in Table 5.3. The 'tile' representation of v and one of its solutions are depicted in Figure 5.34. The formula φ_{w}, however, is presented in

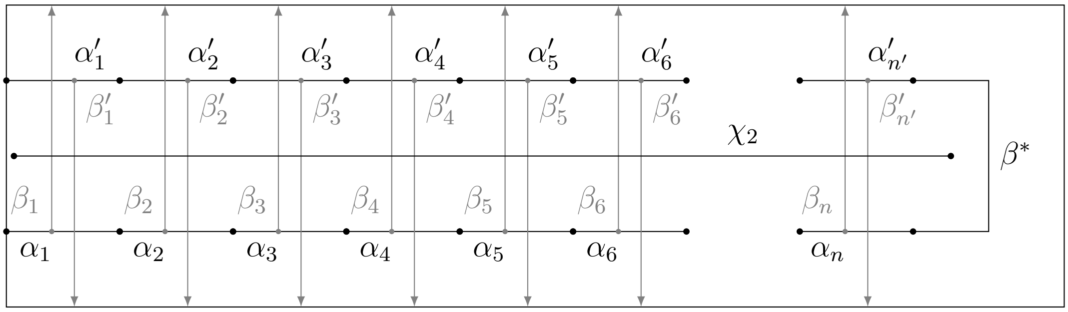


Figure 5.33: The 1–1 correspondence between the α_i and the α'_i established by the β_i and the β'_i.

S	=	$\{0,1\}$	$w_1(1)$	=	0	$w_1(2)$	=	01	$w_1(3)$	=	110
T	=	$\{1,2,3\}$	$w_2(1)$	=	100	$w_2(2)$	=	00	$w_2(3)$	=	11

Table 5.3: A running example of the PCP denoted by $\mathbf{v}$.

the general case.

Figure 5.34: The PCP-instance $\mathbf{v}$ (left) together with one of its solutions (right).

Stage 4. In this stage, we 'label' the arcs $\beta_1, \ldots, \beta_n$, with regions representing the elements of $\{0,1\}$, defining in this way a word σ over $\{0,1\}^n$. Further, we label the arcs $\alpha_1, \ldots, \alpha_n$ defining a word τ over the alphabet T of length m, and, symmetrically, we label the arcs $\alpha'_1, \ldots, \alpha'_n$ defining a word τ' over T of length m'. We then synchronise the labellings to ensure that $\tau = \tau'$ and that $\mathsf{w}_1(\tau) = \mathsf{w}_2(\tau') = \sigma$. What we have in mind is depicted in Figures 5.35 and 5.36.

We label the arcs β_i with regions l_0 and l_1 representing, respectively, the letters 0 and 1, by direct application of the technique shown in Lemma 124. If $\varphi_{\mathbf{w}}$ is satisfied, then each component of b_0, b_1 and b_2, and hence each of the arcs $\beta_1, \ldots, \beta_n$, will be contained in exactly one of the regions l_0 and l_1. Hence, the arcs $\beta_1, \ldots, \beta_n$ will define a word $\sigma \in \{0,1\}^n$.

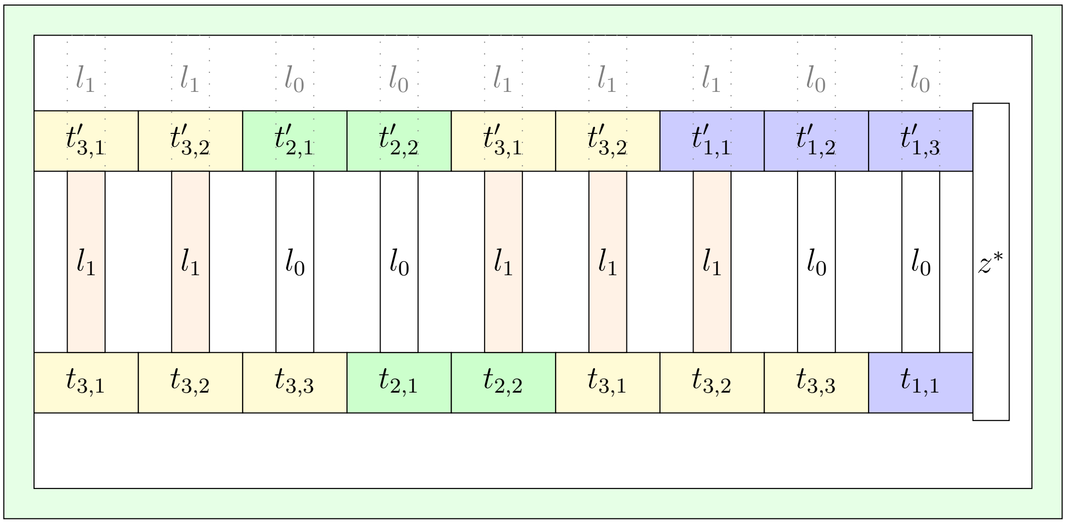


Figure 5.35: Satisfying φ_{v}: the regions $t_{i,j}, t'_{i,j}, l_0, l_1$.

Consider now the arcs $\alpha_1, \ldots, \alpha_n$. Take $T = \{t_1, \ldots, t_\ell\}$, and define for j $(1 \leq j \leq \ell)$: $\sigma_j := \mathsf{w}_1(t_j)$; $\sigma'_j := \mathsf{w}_2(t_j)$; $u(j) := |\sigma_j|$ and $u'(j) := |\sigma'_j|$. We introduce regions $t_{j,k}$ $(1 \leq j \leq \ell,\ 1 \leq k \leq u(j))$, where $t_{j,k}$ represents the kth letter in the word σ_j, and we call these regions 'position labels'. Using Lemma 124, we add to φ_{w} constraints ensuring that each components of a_i $(0 \leq i \leq 2)$ is 'labelled with' one of the 'position labels'. This ensures that each of the arcs $\alpha_1, \ldots, \alpha_n$ is labelled with exactly one of the $t_{j,k}$. Further, we add non-contact constraints to ensure that the labels are organised into blocks, $E_1, \ldots, E_m$ such that, the sequence of labels each block E_h reads $t_{j,1}, \ldots, t_{j,u(j)}$, for some j $(1 \leq j \leq \ell)$.

Symmetrically, we label the arcs α'_i $(1 \leq i \leq n)$ with 'position labels' $\{t'_{j,k} \mid 1 \leq j \leq \ell, 1 \leq k \leq u'(j)\}$, again ensuring that these labels are organised into m' contiguous blocks, $E'_1, \ldots, E'_{m'}$ such that in the hth block, E'_h, the sequence of labels reads $t'_{j,1}, \ldots, t'_{j,u'(j)}$, for some j $(1 \leq j \leq \ell)$.

We now need to synchronise the three labellings to ensure that $\sigma = \mathsf{w}_1(\tau) = \mathsf{w}_1(\tau')$ and that $\tau = \tau'$. The first task is straight forward—we just add to φ_{w} the constraints:

$$\neg\mathsf{C}(l_h, t_{j,k}) \qquad \text{the } k\text{'th letter of } \sigma_j \text{ is not } h$$
$$\neg\mathsf{C}(l_h, t'_{j,k}) \qquad \text{the } k\text{'th letter of } \sigma'_j \text{ is not } h.$$

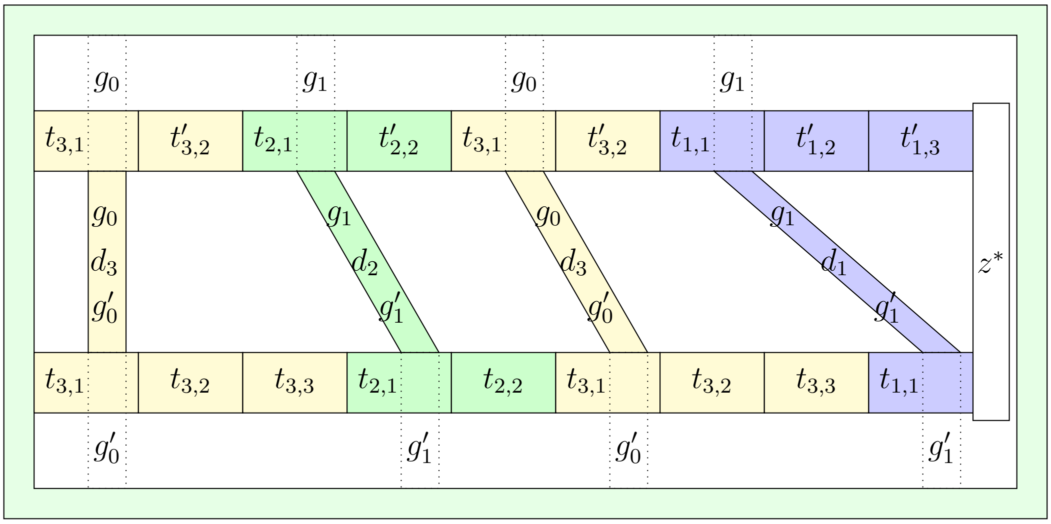


Figure 5.36: Satisfying $\varphi_{\mathbf{v}}$: the regions $g_0, g'_0, g_1, g'_1, d_1, \ldots, d_l$.

To show that $\tau = \tau'$, we use establish a 1-1 correspondence between the initial 'position labels' of the blocks $E_1, \ldots, E_m$ and the initial 'position labels' of the blocks $E'_1, \ldots, E'_{m'}$. To do so we introduce regions g_i, g'_i $(0 \leq i \leq 1)$. Using Lemma 122, we extend $\varphi_{\mathbf{w}}$ to ensure that g_i connect the components of $t_{j,1}$ to s'_6 and, symmetrically, that g'_i connect the components of $t'_{j,1}$ to s_6 (see Figure 5.36). By adding non-contact constraints for the regions not shown as being in contact in Figure 5.36, we guarantee that $|\tau| = |\tau'|$. Further, we label the regions g_0 and g_1 with 'pattern labels' $d_1, \ldots, d_\ell$, where each d_k represents the tile t_k $(1 \leq k \leq \ell)$. Finally, by adding the constraints:

$$\neg\mathsf{C}(t_{j,k}, d_{j'}) \qquad (j \neq j')$$
$$\neg\mathsf{C}(t'_{j,k}, d_{j'}) \qquad (j \neq j')$$

where $1 \leq j \leq \ell$, $1 \leq k \leq u(j)$ and $1 \leq j' \leq \ell$, we make sure that blocks E_j and E'_j represent the same letter of T, which secures $\tau = \tau'$. Hence, if $\varphi_{\mathbf{w}}$ is satisfiable in $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is a positive instance of the PCP. On the other hand, as shown for $\mathbf{v}$ in Figures 5.26, 5.28, 5.35 and 5.36, if an instance $\mathbf{w}$ of the PCP has a solution, one can satisfy the formula $\varphi_{\mathbf{w}}$ by regions in $\mathsf{RCP}(\mathbb{R}^2)$.

We have established the r.e.-hardness of the problems $\mathsf{Sat}(\mathcal{C}\mathsf{c}, \mathsf{RC}(\mathbb{R}^2))$ and $\mathsf{Sat}(\mathcal{C}\mathsf{c}, \mathsf{RCP}(\mathbb{R}^2))$. We must now extend these results to the other languages considered here. We deal with the languages $\mathcal{C}\mathsf{c}^\circ$ and $\mathcal{B}\mathsf{c}$ as in Section 5.1. Let $\varphi^\circ{}_{\mathbf{w}}$ be the $\mathcal{C}\mathsf{c}^\circ$ formula obtained by replacing all occurrences of c in $\varphi_{\mathbf{w}}$

with c°. Since all occurrences of c in $\varphi_{\mathbf{w}}$ are positive, ${\varphi^{\circ}}_{\mathbf{w}}$ entails $\varphi_{\mathbf{w}}$. On the other hand, the connected regions satisfying $\varphi_{\mathbf{w}}$ can always be selected to be interior-connected, and thus satisfy ${\varphi^{\circ}}_{\mathbf{w}}$ as well.

For the language $\mathcal{B}$c, observe that, as in Section 5.1, all conjuncts of $\varphi_{\mathbf{w}}$ featuring the predicate C are *negative*, including those needed for the definition of a 3-region. Recall from Section 5.1.1 that

$$\varphi_{DC2}(r, s, r', s') := \mathsf{c}(r + r') \wedge \mathsf{c}(s + s') \wedge \neg\mathsf{c}((r + r') + (s + s')),$$

and consider the effect of replacing any literal $\neg\mathsf{C}(r, s)$ from (5.16) with the $\mathcal{B}$c-formula $\varphi_{DC2}(r, s, r', s')$ where r' and s' are fresh variables, and let the formula obtained be ψ. It is easy to see that ψ entails $\varphi_{\mathbf{w}}$; hence if ψ is satisfiable, then $\mathbf{w}$ is a positive instance of the PCP. To see that ψ is satisfiable, consider the satisfying tuple of $\varphi_{\mathbf{w}}$. Note that if $\mathfrak{r}$ and $\mathfrak{s}$ are 3-regions whose outer-most elements r and s are disjoint (for example: $\mathfrak{r} = \mathfrak{a}_{0,1}$, $\mathfrak{s} = \mathfrak{a}_{0,3}$), then r and s have finitely many connected components each being a disc-homeomorph. Hence, it is easy to find r' and s' in RCP($\mathbb{R}^2$) satisfying the corresponding formula $\varphi_{DC2}(r, s, r', s')$. Figure 5.37 represents the situation in full generality. We may therefore assume, that all such literals involving C have been eliminated from $\varphi_{\mathbf{w}}$.

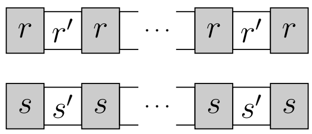


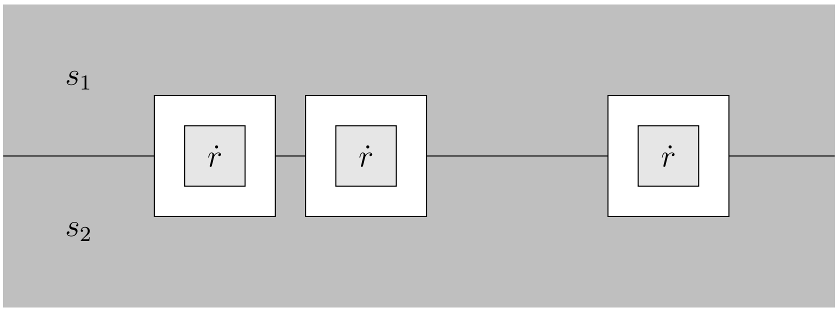
Figure 5.37: Satisfying $\varphi_{DC2}(r, s, r', s')$

We also have to show that we can replace the *implicit* non-contact constraints that come with the use of 3-regions by suitable $\mathcal{B}$c-formulas. For example, a 3-region variable $\mathfrak{r}$ involves the implicit constraints $\neg\mathsf{C}(\ddot{r}, -\dot{r})$ and $\neg\mathsf{C}(\dot{r}, -r)$. Since the two conjuncts are identical in form, we only show how to deal with $\neg\mathsf{C}(\dot{r}, -r)$. Because the complement of $-r$ is in general not connected, a direct use of φ_{DC2} will result in a formula which is not satisfiable. Instead, we represent $-r$ as the sum of two regions s_1 and s_2 with connected complements, and then proceed as before. In particular, we replace $\neg\mathsf{C}(\dot{r}, -r)$

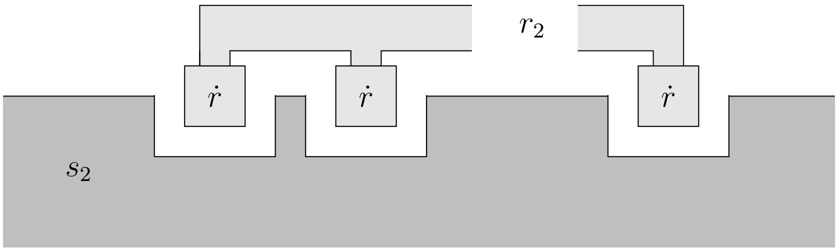
by:

$$-r = s_1 + s_2 \wedge \varphi_{DC2}(\dot{r}, s_1, r_1, s_1) \wedge \varphi_{DC2}(\dot{r}, s_2, r_2, s_2).$$

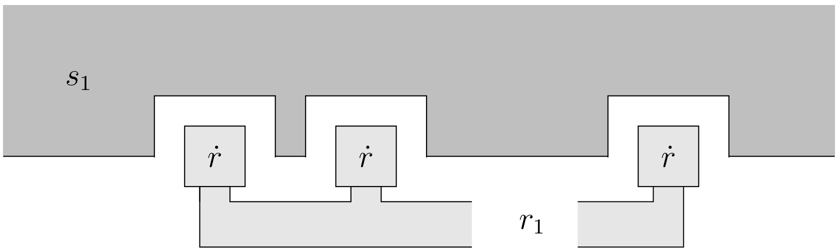
For $i = 1, 2$, $\dot{r} + r_i$ is a connected region that is disjoint from s_i. So, $\dot{r}$ is disjoint from s_1 and s_2, and hence disjoint from their sum $-r := s_1 + s_2$. Figure 5.38 shows regions s_i, r_i, for $i = 1, 2$, which satisfy the above formula. Let $\psi_{\mathbf{w}}$ be



(a) The region $-r$ is the sum of s_1 and s_2.



(b) The mutually disjoint connected regions $\dot{r} + r_2$ and s_2.



(c) The mutually disjoint connected regions $\dot{r} + r_1$ and s_1.

Figure 5.38: Eliminating the conjuncts of the form $\neg\mathsf{C}(-r, \dot{r})$.

the result of replacing all the conjuncts (explicit or implicit) containing the predicate C, as just described. We have thus shown that, if $\psi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is positive, and that, if $\mathbf{w}$ is positive, then $\psi_{\mathbf{w}}$ is satisfiable over

$\mathsf{RCP}(\mathbb{R}^2)$.

The final case we must deal with is that of $\mathcal{B}c^\circ$. We use the r.e.-hardness results already established for $\mathcal{C}c^\circ$, and proceed, as before, to eliminate occurrences of C. Since all the polygons in the tuple satisfying $\varphi^\circ{}_{\mathbf{w}}$ are quasi-bounded, we can eliminate all occurrences of C from $\varphi^\circ{}_{\mathbf{w}}$ using Lemma 114 (*iii*). This completes the proof of Theorem 125. □

In this section we showed that the satisfiability problem for the four languages considered by us are undecidable when interpreted over the region algebras $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$. To do so, we adapted the techniques used in Section 5.3 for forcing regions with $\aleph_0$ components in the Euclidean plane. In the following section we show how the techniques used in Section 5.2 for recognising regions in $\mathbb{R}^n$ with $\aleph_0$ components can be used to establish undecidability of the satisfiability problem for the corresponding languages when interpreted over the region algebras $\mathsf{RCP}(\mathbb{R}^n)$.

5.5 Undecidability: The Polyhedral Case

In this section we further investigate the configurations of regions in region algebras over unicoherent topological spaces that can be forced using quantifier-free topological languages. In particular, we show that the graphs of the components of certain partitions of such region algebras are trees. This fact proved significant, for it was used by Ian Pratt-Hartmann to show the undecidability of the satisfiability problems for the languages $\mathcal{C}c$, $\mathcal{B}c$ and $\mathcal{C}c^\circ$ when interpreted over $\mathsf{RCP}(\mathbb{R}^n)$ $(n > 1)$. The undecidability result is still unpublished, however, the authors of [KNPHZ11a] are currently preparing a journal paper including it. For completeness, we provide a rough sketch of its proof, emphasising how the tree component structure of the partitions is used. The actual encoding of the PCP is similar to the one presented in Section 5.4 and will be described here only on intuitive level.

As we already mentioned, the undecidability of $\mathsf{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$, for $\mathcal{L} \in \{\mathcal{C}c, \mathcal{B}c, \mathcal{C}c^\circ\}$, is shown by a reduction of the Post correspondence problem (PCP). The encoding of a solution of a PCP-instance hinges on the notion of a *sub-cyclic partition*, some of whose properties were already implicitly studied in Section 5.2. A *sub-cyclic (k-)partition* in a region algebra $\mathcal{M}$ is a tuple of regions $\bar{r} = (r_0, \ldots, r_{k-1})$ in $\mathcal{M}$ satisfying the $\mathcal{C}c$-formula scycle given by:

$$r_0 + \cdots + r_{k-1} = 1 \quad \wedge \bigwedge_{0 \le i,j < k} r_i \cdot r_j = 0 \quad \wedge \bigwedge_{\substack{\lfloor j-i \rfloor > 1 \\ \lfloor i-j \rfloor > 1}} \neg \mathsf{C}(r_i, r_j),$$

where $\lfloor i \rfloor$ denotes $i \pmod k$. Note that if $\bar{r}$ satisfies scycle, then every rotation $\bar{s}$ of $\bar{r}$ will also satisfy scycle. (A sequence $\bar{s}$ is a rotation of another sequence $\bar{r}$ if for some j and every $0 \le i < k$, $s_i = r_{\lfloor i+j \rfloor}$.) For every sequence of regions $\bar{r} = (r_0, \ldots, r_{k-1})$ in a region algebra $\mathcal{M}$, the *component graph* of $\bar{r}$ is defined as the graph $H(\bar{r}) = (V, E)$, where V is the set of the connected components of the regions r_i, and E is the restriction of the contact relation to V^2, i.e. for $r, s \in V$, $(r, s) \in E \iff \mathsf{C}(r, s) \iff r \cap s \neq \emptyset$. Although in general the component graph of a sub-cyclic partition may have arbitrary structure, it turns out that in finitely decomposable region algebras over unicoherent spaces it is always a trees (connected graphs with no simple cycles). Before we prove this crucial property, we make some further observations.

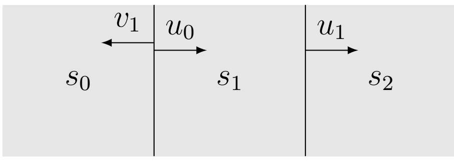


Figure 5.39: A simple path in the component graph of a sub-cyclic partition.

Lemma 126. *Let $\mathcal{M}$ be a finitely-decomposable region algebra over a unicoherent topological space, and let $\bar{r}$ be a sub-cyclic k-partition with component graph $H(\bar{r})$. Further, let $s_0 s_1 s_2$ be a simple path in $H(\bar{r})$. If u_i, for $i = 0, 1$, is the component of $-s_i$ containing s_{i+1}, then $u_1 \ll u_0$, i.e. $\neg \mathsf{C}(u_1, -u_0)$.*

Proof. Take v_1 to be the component of $-s_1$ containing s_0 (see Figure 5.39). It suffices to show that v_1 and u_1 are different components of $-s_1$. Indeed, if this is the case, we will then have that $u_1 \ll -v_1 \le -s_0$. I.e. u_1 will be contained in the interior of $-s_0$. Since u_1 is connected, it has to be contained in the interior of a component of $-s_0$, and since $u_1 \cap s_1 \neq \emptyset$ and $s_1 \le u_0$, that component has to be u_0. Hence $u_1 \ll u_0$.

To see that $v_1 \neq u_1$, we show that their boundaries are disjoint by making use of the fact that the space is unicoherent. Without loss of generality, we may assume that s_1 is a component of r_1. We know from Lemma 33 that the boundary of v_1, $\delta(v_1)$, is connected, and because $\bar{r}$ is a sub-cyclic partition, $\delta(v_1) \subseteq r_0 + r_2$. Hence, by $\neg\mathsf{C}(r_0, r_2)$, $\delta(v_1)$ is contained in exactly one of the regions r_0 and r_2. Now, since $\delta(v_1)$ is connected, $\delta(v_1)$ has to be contained in a component of r_0 or r_2 and it has to be disjoint from all the other components of these two regions. Hence, $\delta(v_1)$ must be contained in s_0. Similarly, $\delta(u_1)$ must be contained in s_2. If we assume that $\delta(u_1)$ and $\delta(v_1)$ have a point in common we will get that s_0 and s_2 are in contact. This, however, contradicts the two possible cases for s_0 and s_2: when they are (different) components of one of the regions r_0 and r_2; and when one of them is a component of r_0 and the other a component of r_2. Hence, $v_1 \neq u_1$, and as a result $u_1 \ll u_0$. □

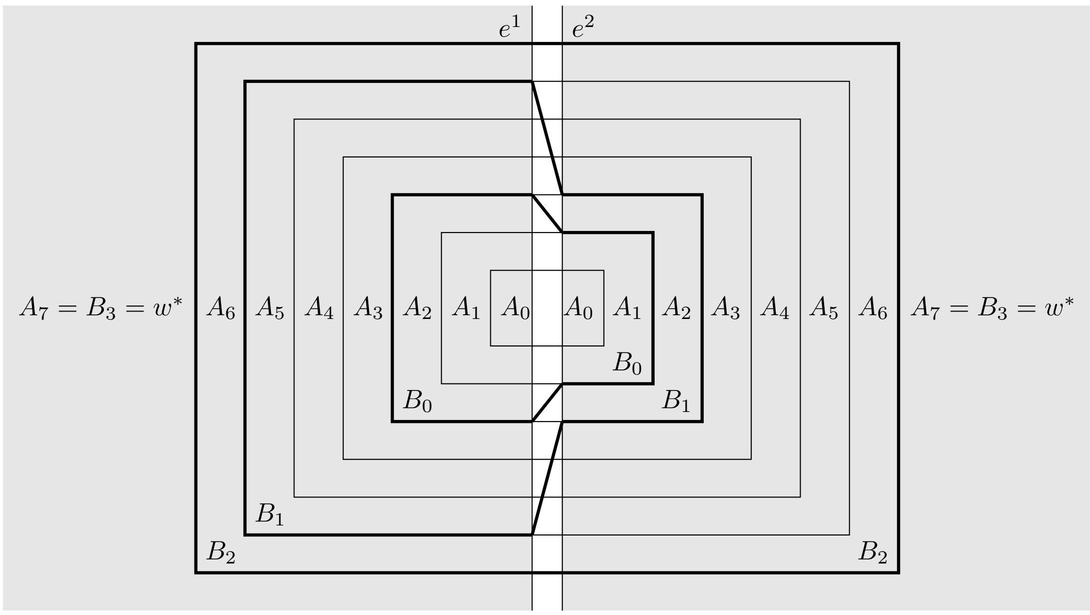


Figure 5.40: Two simple paths $\bar{A} = \{A_0, \ldots, A_7\}$ and $\bar{B} = \{B_0, \ldots, B_3\}$ in the component graphs of the sub-cyclic partitions $\bar{r}$ and $\bar{s}$. The partition $\bar{A}$ refines the partition $\bar{B}$ within each of the regions e_1 and e_2. The two refinements are independent, which allows the encoding of the tiles of arbitrary PCP-instances.

Lemma 127. *Let $\mathcal{X}$ be a unicoherent topological space, $\mathcal{M}$ a finitely-decomposable region algebra over $\mathcal{X}$, and $\bar{r} = (r_0, \ldots, r_{k-1})$ be a sub-cyclic partition in $\mathcal{M}$. Then the component graph $H(\bar{r})$ is a tree.*

Proof. First, since $\mathcal{M}$ is connected and finitely decomposable, $H(\bar{r})$ must be finite and connected. Assume that $H(\bar{r})$ is not a tree. Then there has to be a simple cycle $s_0, \ldots, s_{m-1}$, for some $m > 2$. Take u_i $(0 \leq i < m)$ to be the component of $-s_i$ containing $s_{\lfloor i+1 \rfloor}$, where $\lfloor i \rfloor$ now denotes $i \pmod m$. By Lemma 126, we have that $u_i \ll u_{\lfloor i-1 \rfloor}$, for $0 \leq i < m$. Since $\ll$ is transitive, we get that $u_0 \ll u_0$, which in connected spaces implies that u_0 is either 0 or 1. This, however, is not the case, because, for example, s_0 and s_2 are disjoint and non-empty. □

Lemma 127 plays a key role in establishing the undecidability of $\mathsf{Sat}(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$, for $n > 1$ and $\mathcal{L} \in \{\mathcal{C}c, \mathcal{C}c^\circ, \mathcal{B}c\}$. In the rest of the section we will show how it can be used to encode any solution to a positive PCP-instance by regions in $\mathsf{RCP}(\mathbb{R}^n)$. For the rest of the section we fix $n > 1$ and we use the term *region* to denote the elements of $\mathsf{RCP}(\mathbb{R}^n)$.

Consider two sub-cyclic partitions $\bar{r} = (r_0, \ldots, r_3)$ and $\bar{s} = (s_0, \ldots, s_3)$:

$$\mathsf{scycle}(r_0, r_1, r_2, r_3) \qquad\qquad \mathsf{scycle}(s_0, s_1, s_2, s_3),$$

and two non-empty regions e_1 and e_2 (see Figure 5.40). For $k = 1, 2$ define $e_k \cdot \bar{r} := (e_k \cdot r_0, \ldots, e_k \cdot r_3)$ and $e_k \cdot \bar{s} := (e_k \cdot s_0, \ldots, e_k \cdot s_3)$. Clearly, both the regions in $e_k \cdot \bar{r}$ and the regions in $e_k \cdot \bar{s}$ partition the region e_k. One can find a $\mathcal{C}c$-formula which ensure that the partition $e_k \cdot \bar{r}$ is a refinement of the partion $e_k \cdot \bar{s}$, i.e. each component of a region in $e_k \cdot \bar{r}$ is contained in a component of a region in $e_k \cdot \bar{s}$.

Just as we did in Section 5.2, we force every component of $r_i \cdot e_k$, for $0 \leq i < 4$ and $k = 1, 2$, either to be in contact with a component of $r_{\lfloor i+1 \rfloor} \cdot e_k$, or to be contained in a fixed component w^* of one of the regions in $\bar{r}$. Further, we can ensure that there exists a component A_0 of r_0 that intersects both e_1 and e_2. Hence, we can construct two sequences of regions

$$\bar{A}_1 = (A_0 = A_0^1, \ldots, A_{m(1)}^1 = w^*)$$
$$\bar{A}_2 = (A_0 = A_0^2, \ldots, A_{m(2)}^2 = w^*),$$

where A_i^k is a component of $r_{\lfloor i \rfloor}$, for $0 \leq i < m(k)$, and $A_i^k \cdot e^k$ is in contact with $A_{i+1}^k \cdot e^k$, for $0 \leq i < m(k) - 1$. By construction, however, the sequences $\bar{A}_1$ and $\bar{A}_2$ are simple paths in the component graph $H(\bar{r})$, and by Lemma 127 the two sequences must coincide. Hence, we may skip the superscripts and write:

$$\bar{A} = \{A_0, \ldots, A_m = w^*\}.$$

Now consider the sub-cyclic partition $\bar{s}$. Recall that each component of a region in $e_k \cdot \bar{r}$ is contained in a component of a region in $e_k \cdot \bar{s}$. However, if t is a component of a region in $\bar{r}$, then $t \cdot e_1$ and $t \cdot e_2$ will in general be contained in different components of different regions in $\bar{s}$. So, every $A_i \cdot e_k$, for $(0 \leq i < m)$ and $k = 1, 2$, is contained in some component $\hat{B}_i^k$ of a region in $\bar{s}$. Thus we have the following two sequences

$$(\hat{B}_0^1, \ldots, \hat{B}_m^1)$$
$$(\hat{B}_0^2, \ldots, \hat{B}_m^2)$$

of components of regions in $\bar{s}$. By removing the neighbouring duplicates, we obtain the sequences

$$\bar{B}^1 = (B_0^1, \ldots, B_{\ell(1)}^1)$$
$$\bar{B}^2 = (B_0^2, \ldots, B_{\ell(2)}^2),$$

where $\ell(1) < m$ and $\ell(2) < m$. Adding further constraints, one can show that B_i^k is a component of s_i, for $k = 1, 2$ and $(0 \leq i < \ell(k))$. As a consequence, we get that $\bar{B}^1$ and $\bar{B}^2$ are simple paths in $H(\bar{s})$ starting and ending at the same nodes. Again, by Lemma 127, these paths must coincide, and we can therefore ignore the k-superscripts and write:

$$\bar{B} = \{B_0, \ldots, B_\ell\}.$$

It is now only a matter of 'labelling' the components of the regions in $\bar{r}$, $\bar{s}$, $e_1 \cdot \bar{r}$ and $e_2 \cdot \bar{r}$ to encode a solution of a PCP-instance. Fix a PCP-instance $\mathbf{w} = (\{0, 1\}, T, \mathsf{w}_1, \mathsf{w}_2)$, where $T = \{t_0, \ldots, t_p\}$ is a finite alphabet, and $\mathsf{w}_i \colon T^* \to \{0, 1\}^*$ a word-morphism ($i = 1, 2$). We introduce regions ℓ_0 and ℓ_1 representing the letters 0 and 1, and 'label' with them the components of the regions in $\bar{r}$. Further, we introduce regions t_i $(0 \leq i \leq p)$, called *pattern labels*, and we use them to 'label' the components of the regions in $\bar{s}$. In this way, the sequences $(A_0, \ldots, A_m)$ and $(B_0, \ldots, B_\ell)$ encode the two words $v \in \{0, 1\}^{m+1}$

and $\tau \in T^{\ell+1}$, respectively. What we are left to show is that $\mathsf{w}_1(\tau) = \upsilon$ and $\mathsf{w}_2(\tau) = \upsilon$.

For each $k = 1, 2$, we introduce regions $t^k_{i,j}$, for $(0 \leq i \leq r)$ and $(0 \leq j \leq |\mathsf{w}_k(i)|)$, called *position labels*, and we use them to label the components of the regions in $e_k \cdot \bar{r}$. Further, we ensure that if a region $B \in \bar{B}$ is labelled with a pattern label t_i, $(0 \leq i \leq p)$, then, for $k = 1, 2$, the subsequence $(e_k \cdot A_{q(0)}, \ldots, e_k \cdot A_{q(|t_i-1|)})$ of $\bar{A}$ contained in $e_k \cdot B$ encodes the word $(t^k_{i,0}, \ldots, t^k_{i,|t_i|-1})$. By making certain that every position label is correctly labeled with one of the regions ℓ_0 and ℓ_1, we ensure that $\mathsf{w}_1(\tau) = \upsilon$ and $\mathsf{w}_2(\tau) = \upsilon$. Hence, for every PCP-instance $\mathbf{w}$ there exists a $\mathcal{C}$c-formula that is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$ exactly when $\mathbf{w}$ has a solution. Using the same techniques as those in Section 5.2, we can show that this also holds for the languages $\mathcal{B}$c and $\mathcal{C}$c$^\circ$. As a result we get the following theorem.

Theorem 128. *Let $\mathcal{L}$ be one of the languages $\mathcal{C}$c, $\mathcal{B}$c and $\mathcal{C}$c$^\circ$, and $n > 1$. Then Sat$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ is r.e.-complete.*

5.6 Conclusion

In this chapter we considered quantifier-free languages for qualitative spatial reasoning. In particular, we focused on the languages $\mathcal{C}$c, $\mathcal{C}$c$^\circ$, $\mathcal{B}$c and $\mathcal{B}$c$^\circ$, which feature symbols for connectedness predicates and Boolean operations. It was known from previous studies that these languages are sensitive to lower-dimensional Euclidean interpretations, i.e. for each of these languages $\mathcal{L}$, the region algebras $\mathsf{RC}(\mathbb{R})$, $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RC}(\mathbb{R}^3)$ satisfy different $\mathcal{L}$-formulas [KPHZ10]. It was also known that the satisfiability problems for these languages when interpreted over Euclidean region algebras of dimension at least two are ExpTime-hard, with certain exceptions for the language $\mathcal{B}$c$^\circ$ [KPHZ10]. Using the fact that Euclidean spaces are unicoherent, we showed that the languages $\mathcal{C}$c, $\mathcal{C}$c$^\circ$ and $\mathcal{B}$c contain formulas which are satisfiable in the region algebras $\mathsf{RC}(\mathbb{R}^n)$, for $(n > 1)$, but only by tuples containing regions with infinitely many connected components. We also argued that using similar techniques one can show the undecidability of the satisfiability problem for the languages $\mathcal{C}$c, $\mathcal{C}$c$^\circ$ and $\mathcal{B}$c when interpreted over the region algebras $\mathsf{RCP}(\mathbb{R}^n)$, for $(n \geq 2)$. Applying very different techniques based on planarity arguments, we further showed that the four languages contain formulas that are satisfiable in $\mathsf{RC}(\mathbb{R}^2)$, but only by tuples containing regions with infinitely many components. Using the same techniques, we also showed that the satisfiability problems for the four languages when interpreted over $\mathsf{RC}(\mathbb{R}^2)$ or $\mathsf{RCP}(\mathbb{R}^2)$ are undecidable.

The above undecidability results are in stark contrast with the relatively low computational complexity of other studied quantifier-free Euclidean spatial logics. For example, the satisfiability problems for the languages $\mathcal{RCC}8$, $\mathcal{RCC}8\mathsf{c}$ and $\mathcal{C}$ with respect to Euclidean interpretations are all decidable—NP-complete in the case of $\mathcal{RCC}8$ and $\mathcal{RCC}8\mathsf{c}$ and PSPACE-complete in the case of $\mathcal{C}$—and that in spite of the fact that in the languages $\mathcal{RCC}8\mathsf{c}$ and $\mathcal{C}$ one can express connectedness constraints and Boolean constraints, respectively. It is only when both these types of constraints are present in a language that its computational complexity increases dramatically. This leaves us with the major challenge of identifying languages featuring both connectedness and Boolean constraints, but having decidable satisfiability problem with respect to Euclidean interpretations. A natural strategy for achieving this is to restrict the interaction between Boolean and connectedness constraints in each of the languages $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$, $\mathcal{B}\mathsf{c}$ and $\mathcal{B}\mathsf{c}^\circ$. Another open problem is to determine the computational properties of the languages $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$ and $\mathcal{B}\mathsf{c}$ when interpreted over the region algebras $\mathsf{RC}(\mathbb{R}^n)$, for $(n \geq 3)$.

Chapter 6

Conclusion

This thesis has investigated certain formal systems, called *spatial logics*, for reasoning about regions in space. Spatial logics are of particular interest to the AI community as a means to enable an intelligent agent to represent and reason about spatial knowledge. From practical point of view, the most interesting spatial logics are those for reasoning about regions in Euclidean space; we call these *Euclidean spatial logics*. The collection of regions is assumed to be a Boolean algebra of arbitrary or regular closed subsets of some topological space, and is called respectively a *set algebra* or a *region algebra*; we add the word *complete* in case the Boolean algebra is complete.

It has long been understood that most interesting first-order spatial logics have undecidable satisfiability problems, and much early research consequently focused on their axiomatic characterisations and model-theoretic properties. However, various mathematical problems remained open. Some quantifier-free Euclidean spatial logics, such as $\mathcal{RCC}8$, $\mathcal{RCC}8\text{c}$ and $\mathcal{BRCC}8$ were known to have relatively low computational complexities (NP and PSPACE). However, quantifier-free Euclidean spatial logics with both connectedness and Boolean primitives were known to have significantly higher computational complexity (at least EXPTIME-hard), and they were not even known to be decidable. The aims of this study were to survey the latest results on the model-theoretic and computational properties of spatial logics, and to solve some of the problems which were left open in the literature.

The present study has made several contributions. The first is presented in Section 4.2.1, and concerns the first-order theories of complete region algebras in the language $\mathcal{L}_\text{C}$, the language of Boolean algebras augmented with

Whitehead's contact relation. $\mathcal{L}_{\mathsf{C}}$-structures satisfying certain axioms are known in the literature as (*Boolean*) *contact algebras*. We showed that the $\mathcal{L}_{\mathsf{C}}$-theory of complete region algebras is different from the $\mathcal{L}_{\mathsf{C}}$-theory of all region algebras (Theorem 68). A similar result was obtained for complete region algebras over different collections of topological spaces. The result was extended to the first-order theory of complete set algebras in the language $\mathcal{L}_{cl}$, the language of Boolean algebras augmented with Kuratowski's closure operation. $\mathcal{L}_{cl}$-structures satisfying certain axioms were introduced by McKinsey and Tarski under the name of *closure algebras*.

The second major contribution of this study is presented in Section 4.3, and concerns the computability of first-order spatial logics. The first-order theories of many higher-dimensional Euclidean spatial logics had been known to be undecidable. This research presented here, however, shows that the topological theories of region algebras over the real line are all decidable (Section 4.3.1), but non-elementary (Section 4.3.2). Another finding, which is presented in Section 4.3.3, is the improved lower complexity bounds on the topological theories of the region algebras $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$. Further, a number of upper complexity bounds were obtained for the (undecidable) Euclidean spatial logics, yielding tight complexity bounds for all but two these logics (see Section 4.3.4). These complexity results imply the surprising model-theoretic result that the region algebra of polytopes ($\mathsf{RCP}(\mathbb{R}^n)$) and the region algebra of algebraic polytopes ($\mathsf{RCP}_{\mathbb{A}}(\mathbb{R}^n)$) have different $\mathcal{L}_{conv}$-theories, where $\mathcal{L}_{conv}$ is the language of Boolean algebras augmented with the property of being convex.

The third major contribution of this study is presented in Section 5, and concerns the expressiveness and computability of quantifier-free spatial logics. This research focuses on languages with Boolean primitives and connectedness predicates interpreted over region algebras over Euclidean spaces. One of the most significant findings is that there exist formulas in the languages $\mathcal{C}c$, $\mathcal{B}c$ and $\mathcal{C}c^\circ$ that are satisfiable in $\mathsf{RC}(\mathbb{R}^n)$, $n > 1$, but only by tuples containing regions with infinitely many components. The methods that were used in establishing this result were applied in an unpublished work of Pratt-Hartmann to show that the satisfiability problem for any of the languages $\mathcal{C}c$, $\mathcal{B}c$ and $\mathcal{C}c^\circ$ over the polygonal region algebras $\mathsf{RCP}(\mathbb{R}^n)$, $n > 1$, is undecidable. Both these results rely on the fact that Euclidean spaces are *unicoherent*. Another finding, based this time on planarity arguments, is that the language $\mathcal{B}c^\circ$ also contains a formula that is satisfiable in $\mathsf{RC}(\mathbb{R}^2)$, but only by tuples containing regions with infinitely many components. Using a similar construction, it was further shown that the satisfiability problem for each of the languages $\mathcal{C}\mathsf{c}$, $\mathcal{B}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$ and $\mathcal{B}\mathsf{c}^\circ$, when interpreted over $\mathsf{RC}(\mathbb{R}^2)$ or $\mathsf{RCP}(\mathbb{R}^2)$, is undecidable.

Future Work

Various questions remain open, and we conclude with a survey of these. The $\mathcal{L}_\mathsf{C}$-theories of region algebras over different classes of topological spaces have been recently axiomatised [Roe97, DW05, DV06]. The finding that these theories are different from the $\mathcal{L}_\mathsf{C}$-theories of the complete region algebras over the respective topological spaces (see Section 4.2.1) raises the problem of obtaining axiomatisations of the latter theories. Similarly, the problem of obtaining an axiomatisation of the $\mathcal{L}_{cl}$-theory of the class of complete set algebras is now also open.

Another question motivated by the current study concerns the degree of undecidability of first-order theories of region algebras. Although this study has filled in many of the gaps left in the literature, it was unable to establish tight complexity bounds for the $\mathcal{L}_\mathsf{C}$-theories of the region algebras $\mathsf{RCP}(\mathbb{R}^n)$ and $\mathsf{RCS}(\mathbb{R}^n)$, $n > 2$, and established only Δ^0_ω-hardness and membership in Δ^1_ω. For $n = 2$, the corresponding problem was resolved using the previously obtained result that the $\mathcal{L}_\mathsf{C}$-structures $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RCS}(\mathbb{R}^2)$ are elementary equivalent to the $\mathcal{L}_\mathsf{C}$-structure $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^2)$, and hence have theories that are Δ^0_ω-complete. Whether a similar result can be obtained for $n > 2$ is an open problem.

Further model-theoretic research is suggested by the finding that $\mathsf{RCP}(\mathbb{R}^n)$ and $\mathsf{RCP}_\mathbb{A}(\mathbb{R}^n)$ have different $\mathcal{L}_{conv}$-theories. It is known that these two theories are different from the $\mathcal{L}_{conv}$-theory of $\mathsf{RCP}_\mathbb{Q}(\mathbb{R}^n)$, which has been recently axiomatised for $n = 2$ [Try10]. As discussed in Section 3.2, it is suspected that the axiomatisation can be extended to the $\mathcal{L}_{conv}$-theory of $\mathsf{RCP}_\mathbb{A}(\mathbb{R}^n)$, but since it relies on the fact that the structure is countable, the techniques employed cannot be extended to the $\mathcal{L}_{conv}$-theory of $\mathsf{RCP}(\mathbb{R}^n)$.

Another problem that remains unsolved is to determine the computational properties of the languages $\mathcal{C}\mathsf{c}$, $\mathcal{C}\mathsf{c}^\circ$ and $\mathcal{B}\mathsf{c}$ when interpreted over the region algebras $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 3$. The natural strategy for doing so is to try to extend the decidability or undecidability results for other quantifier-free topological languages when interpreted over $\mathbb{R}^n$, $n \geq 3$. Furthermore, the undecidability of the satisfiability problem of quantifier-free topological languages interpreted over Euclidean spaces poses the major challenge of identifying fragments of these languages featuring connectedness and Boolean constraints, and having decidable satisfiability problems. Since the languages are quantifier-free, the only possible strategy for achieving this is to restrict the interaction between Boolean and connectedness constraints in each of the languages $\mathcal{C}c$, $\mathcal{C}c^{\circ}$, $\mathcal{B}c$ and $\mathcal{B}c^{\circ}$. It is our hope that detailed analysis of the undecidability proofs of Chapter 5 might provide some clues as to the most appropriate kinds of restrictions to examine.

List of Symbols

$\mathsf{RCP}_{\mathbb{Q}}(\mathbb{R}^n)$	regular-closed rational polytopes in $\mathbb{R}^n$, page 39
$\mathsf{RCS}(\mathbb{R}^n)$	regular-closed semi-algebraic sets in $\mathbb{R}^n$, page 38
$\mathbb{R}^+$	positive real numbers, page 34
$\mathcal{L}^2_\sigma$	second-order language of a signature σ, page 21
$\mathcal{L}^2_{\mathcal{M}}$	second-order language of $\mathcal{M}$, page 21
$T_2(\mathcal{M})$	second-order theory of a structure $\mathcal{M}$, page 22

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