On the computational complexity of spatial logics with connectedness constraints

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Abstract. We investigate the computational complexity of spatial logics extended with the means to represent topological connectedness and restrict the number of connected components. In particular, we show that the connectedness constraints can increase complexity from NP to PSpace, ExpTime and, if component counting is allowed, to NExpTime.

1 Introduction

A subset of a topological space T is *connected* if it cannot be covered by the union of two disjoint non-empty open sets in T. Connectedness is known to be one of the most fundamental concepts of topology, and any textbook in the field contains a substantial chapter on connectedness. In spatial representation and reasoning in AI, the distinction between connected and disconnected regions is recognized as indispensable for various modelling and representation tasks; see, e.g., [1, 4]. (After all, a disconnected plot is usually only worth half the value of a connected plot.) In spite of this, so far only sporadic attempts have been made to investigate the computational complexity of spatial logics with connectedness constraints [3, 21, 23, 15].

In this paper, we consider extensions of standard spatial logics designed for qualitative spatial representation and reasoning (see, e.g., [18, 4] for recent surveys) with connectedness constraints such as 'region r is connected' (or $c(r)$, in symbols) and 'region r contains at most k connected components' (or $c^{\leq k}(r)$). Our main aim is to provide a systematic study of the impact of these constraints on the computational complexity of the satisfiability problem. We focus only on quantifier-free spatial logics because first-order qualitative theories of topological spaces are generally undecidable or non-recursively enumerable even without connectedness constraints [10, 7, 5, 12].

The weakest spatial formalisms for which the addition of connectedness constraints is of interest appear to be '9-intersections' and $\mathcal{RCC}$-8 [8, 16], where one can relate regions (regular closed sets) using binary predicates such as mereological $\mathsf{O}(r, s)$ ('regions r and s overlap') or mereotopological $\mathsf{EC}(r, s)$ ('regions r and s are externally connected'). However, as far as satisfiability is concerned, these logics cannot distinguish between arbitrary regions, connected regions, or regions with k connected components [17], primarily because no Boolean operators on regions are available in their languages. That is why the weakest spatial formalism, $\mathcal{B}$, considered in this paper consists of only Boolean region terms denoting

Boolean combinations of regions. $\mathcal{B}$ itself is also rather weak (in fact, reasoning in $\mathcal{B}$ coincides with Boolean reasoning about sets), but we show that its extensions $\mathcal{B}c$ and $\mathcal{B}cc$ with constraints $c(r)$ and $c^{\leq k}(r)$, respectively, are full-fledged topological logics with considerably more expressive power. Moreover—and this was quite an unexpected result for the authors—the computational complexity jumps from NP for $\mathcal{B}$ to ExpTime for $\mathcal{B}c$ and NExpTime for $\mathcal{B}cc$.

Another spatial logic we deal with in this paper is $\mathcal{BRCC}$-8 [23] which extends $\mathcal{RCC}$-8 with Boolean region terms. An equivalent formalism was also considered in the framework of Boolean contact algebras by extending the Boolean algebra of regular closed (or open) sets with Whitehead's 'extensive connection' predicate $C(r, s)$; see [22, 6]. Here we denote this logic by $\mathcal{C}$ (in order to unify the two lines of research). As shown in [23], $\mathcal{C}$ is still NP-complete. We prove, however, that its extensions $\mathcal{C}c$ and $\mathcal{C}cc$ with constraints of the form $c(r)$ and $c^{\leq k}(r)$ are also ExpTime-complete and NExpTime-complete, respectively. Our maximal spatial logic has its roots in the seminal paper by McKinsey and Tarski [13]. Following the modal logic tradition, we call it $\mathcal{S}4_u$ ($\mathcal{S}4$ with the universal modality). In contrast to $\mathcal{B}$ and $\mathcal{C}$, $\mathcal{S}4_u$ is PSpace-complete. Its extensions $\mathcal{S}4_uc$ and $\mathcal{S}4_ucc$, however, turn out to be ExpTime-complete and NExpTime-complete again.

Thus, the addition of connectedness constraints to standard spatial logics with Boolean region terms leads to considerably more expressive languages of higher computational complexity. However, this increase in complexity is 'stable:' the extensions $\mathcal{B}c$ and $\mathcal{S}4_uc$ of such different formalisms as $\mathcal{B}$ and $\mathcal{S}4_u$ are of the same complexity. Another interesting result is that by restricting these languages to formulas with just one connectedness constraint of the form $c(r)$, we obtain logics that are still in PSpace, but two such constraints lead to ExpTime-hardness. In fact, if the connectedness predicate is applied only to regions $r_1, \dots, r_n$ that are known to be pairwise disjoint, then it does not matter how many times this predicate occurs in the formula: satisfiability is still in PSpace.

The first main ingredient of our proofs is representation theorems allowing us to work with Aleksandrov topological spaces rather than arbitrary ones. Such spaces can be represented by Kripke frames with quasi-ordered accessibility relations. Topological connectedness in these frames corresponds to the graph-theoretic connectedness in the (non-directed) graphs induced by the accessibility relations. Based on this observation, one can prove the upper bounds in a more or less standard way using known techniques from modal and description logic. The lower bounds are much more involved and unexpected. They can be regarded as the main contribution of this paper. We give extended sketches of proofs of these results in Section 4 below; detailed proofs can be found in the full version of the paper at `www.dcs.bbk.ac.uk/~roman`. In Section 5 we discuss, among other things, the computational behaviour of our spatial logics interpreted over Euclidean spaces $\mathbb{R}^n$, although here we do not have tight complexity bounds yet.

2 Topological logics

All our spatial logics are interpreted over *topological spaces*. Given such a space T and a set $X \subseteq T$, we denote by X° the *interior* of X in T and by X^- its *closure*. As usual in spatial KR&R, by a *region* of T we understand any *regular closed* subset of T, i.e., any $X \subseteq T$ with $X = X^{\circ -}$. Denote by $\boldsymbol{RC}(T)$ the set of all regular closed subsets of T. It is known that $\boldsymbol{RC}(T)$ is a Boolean algebra with top and bottom elements given by T and $\emptyset$, Boolean operations $\cdot, -$ given by $X \cdot Y = (X \cap Y)^{\circ -}$ and $-X = (\overline{X})^{\circ -}$, and Boolean order $\leq$ by the relation $\subseteq$. Let $\mathcal{R} = \{r_i \mid i < \omega\}$ be a set of *region variables*. A *regular topological model over* T is a pair $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$, where $\cdot^{\mathfrak{M}}$ is a map from $\mathcal{R}$ to $\boldsymbol{RC}(T)$. Our minimal spatial logic, called $\mathcal{B}$, is defined as follows. The set of $\mathcal{B}$-*terms* is given by:

$$\tau \quad ::= \quad r_i \quad | \quad -\tau \quad | \quad \tau_1 \cdot \tau_2.$$

We abbreviate $-((-\tau_1) \cdot (-\tau_2))$ by $\tau_1 + \tau_2$, $r_0 \cdot (-r_0)$ by $\mathbf{0}$, and $-\mathbf{0}$ by $\mathbf{1}$. The set of $\mathcal{B}$-*formulas* is defined by:

$$\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2.$$

Given a model $\mathfrak{M}$, the *extension* $\tau^{\mathfrak{M}}$ of a $\mathcal{B}$-term τ in $\mathfrak{M}$ is defined inductively by the equations $(-\tau)^{\mathfrak{M}} = (\overline{\tau^{\mathfrak{M}}})^{\circ -}$ and $(\tau_1 \cdot \tau_2)^{\mathfrak{M}} = (\tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}})^{\circ -}$, where $\overline{X} = T \setminus X$. The *truth-relation* for $\mathcal{B}$-formulas is defined by setting $\mathfrak{M} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathfrak{M}} = \tau_2^{\mathfrak{M}}$, and interpreting the Boolean connectives $\neg$ and $\wedge$ in the standard way. We say that a formula φ is *satisfiable* (over a topological space T) if $\mathfrak{M} \models \varphi$, for some model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$. Topologically, the logic $\mathcal{B}$ is quite poor: every satisfiable $\mathcal{B}$-formula φ is satisfied in a discrete topological space. In fact, it as expressive as the modal logic $\mathcal{S}5$, with $\tau = \mathbf{1}$ playing the role of the $\mathcal{S}5$-box.

The logic $\mathcal{C}$ extends $\mathcal{B}$ with the binary *contact* relation C due to Whitehead [22]. Specifically, $\mathcal{C}$-*formulas* are defined in the same way as the $\mathcal{B}$-formulas, except that we have the additional clause

$$\varphi \quad ::= \quad \dots \quad | \quad C(\tau_1, \tau_2) \quad | \quad \dots,$$

where τ_1 and τ_2 are $\mathcal{B}$-terms. The intended meaning of $C(\tau_1, \tau_2)$ is as expected: $\mathfrak{M} \models C(\tau_1, \tau_2)$ iff $\tau_2^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}} \neq \emptyset$, that is τ_1 is in contact with τ_2 in $\mathfrak{M}$. (It is to be noted that we may have $\mathfrak{M} \models \big(\tau_1 \cdot \tau_2 = \mathbf{0}\big) \wedge C(\tau_1, \tau_2)$.) Unlike $\mathcal{B}$, the logic $\mathcal{C}$ can express a number of important topological relationships between regions, e.g., all the $\mathcal{RCC}$-8 relations.

Finally, we define the well-known modal logic $\mathcal{S}4_u$ which can be regarded as a spatial logic in view of the topological interpretation of $\mathcal{S}4$ due to McKinsey and Tarski [13]. As $\mathcal{S}4_u$ is expressive enough to define the property of being regular closed, we take a new set $\mathcal{V} = \{v_i \mid i < \omega\}$ of *set variables* and interpret them by *arbitrary* sets of topological spaces. The $\mathcal{S}4_u$-*terms* are given by

$$\tau \quad ::= \quad v_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \cap \tau_2 \quad | \quad \tau^{\circ}.$$

We abbreviate $\overline{(\overline{\tau}^{\circ})}$ by τ^{-}, $\overline{(\overline{\tau_1} \cap \overline{\tau_2})}$ by $\tau_1 \cup \tau_2$, $v_0 \cap \overline{v_0}$ by $\mathbf{0}$, and $\overline{\mathbf{0}}$ by $\mathbf{1}$. The $\mathcal{S}4_u$-*formulas* are defined in the same way as $\mathcal{B}$-formulas.

In a *topological model* $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$ for $\mathcal{S}4_u$, $\cdot^{\mathfrak{M}}$ is a map from $\mathcal{V}$ to 2^T. The *extension* $\tau^{\mathfrak{M}}$ of a term τ in $\mathfrak{M}$ is defined inductively by the equations:

$$(\overline{\tau})^{\mathfrak{M}} = \overline{(\tau^{\mathfrak{M}})}, \qquad (\tau_1 \cap \tau_2)^{\mathfrak{M}} = \tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}}, \qquad (\tau^{\circ})^{\mathfrak{M}} = (\tau^{\mathfrak{M}})^{\circ}.$$

And the truth-relation for $\mathcal{S}4_u$-formulas is defined in the same way as for $\mathcal{B}$-formulas. Note that both $\mathcal{B}$ and $\mathcal{C}$ can be regarded as proper fragments of $\mathcal{S}4_u$.

3 Topological logics with connectedness

Recall that a topological space T is *connected* just in case it is not the union of two non-empty, disjoint, open sets; a subset $X \subseteq T$ is *connected in* T just in case either it is empty, or the topological space X (with the subspace topology) is connected. If $X \subseteq T$, a maximal connected subset of X is called a (*connected*) *component* of X. Every set X has at least one component, and a set is connected just in case it has at most one component. The $\mathcal{S}4_u$-formula

$$(v_1 \neq \mathbf{0}) \ \wedge \ (v_2 \neq \mathbf{0}) \ \wedge \ (v_1 \cup v_2 = \mathbf{1}) \ \wedge \ (v_1^- \cap v_2 = \mathbf{0}) \ \wedge \ (v_1 \cap v_2^- = \mathbf{0})$$

is satisfiable in a topological space T iff T is not connected; it was used in [21] to axiomatize the logic (in the standard language of $\mathcal{S}4_u$) of connected spaces.

We now extend the logics $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}4_u$ with the connectedness predicate $c(\cdot)$ and denote the resulting languages by $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{S}4_u c$, respectively. Their formulas are defined as before, except that we now have the additional clause:

$$\varphi \quad ::= \quad \ldots \quad | \quad c(\tau) \quad | \quad \ldots .$$

The meaning of $c(\tau)$ in a model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$ is as follows: $\mathfrak{M} \models c(\tau)$ iff $\tau^{\mathfrak{M}}$ is connected in T. For example, most textbooks on general topology prove the following facts: (i) the union of two intersecting, connected sets is connected; (ii) any set sandwiched between a connected set and its closure is itself connected. These facts are expressible as the following $\mathcal{S}4_u c$-validities:

$$c(v_1) \ \wedge \ c(v_2) \ \wedge \ (v_1 \cap v_2 \neq \mathbf{0}) \ \rightarrow \ c(v_1 \cup v_2),$$
$$c(v_1) \ \wedge \ (v_1 \subseteq v_2) \ \wedge \ (v_2 \subseteq v_1^-) \ \rightarrow \ c(v_2)$$

One can increase the expressive power of the connectedness predicate $c(\tau)$ by generalizing it to the 'counting' predicates $c^{\leq k}(\tau)$, $1 \leq k < \omega$, which state that τ has at most k connected components. We denote the languages with such predicates by $\mathcal{B}cc$, $\mathcal{C}cc$ and $\mathcal{S}4_u cc$. Their formulas are defined in the same way as before, except that we have the additional clause, where $1 \leq k < \omega$:

$$\varphi \quad ::= \quad \ldots \quad | \quad c^{\leq k}(\tau) \quad | \quad \ldots .$$

The meaning of $c^{\leq k}(\tau)$ is as follows: $\mathfrak{M} \models c^{\leq k}(\tau)$ iff $\tau^{\mathfrak{M}}$ has at most k components in T. We write $\neg c^{\leq k}(\tau)$ as $c^{\geq k+1}(\tau)$ and abbreviate $c^{\leq 1}(\tau)$ by $c(\tau)$. Thus, we may regard $\mathcal{S}4_u c$ as a sub-language of $\mathcal{S}4_u cc$. The numerical superscripts k in $c^{\leq k}$ are assumed to be coded in *binary*.

Note that for each $\mathcal{S}4_u cc$-formula φ one can construct an equi-satisfiable $\mathcal{S}4_u c$-formula φ' using the observation that $c^{\leq k}(\tau)$ can be replaced by (1) if it occurs positively in φ and by (2) if the occurrence is negative, where

$$\big(\tau = \bigcup_{1 \leq i \leq k} v_i\big) \ \wedge \ \bigwedge_{1 \leq i \leq k} c(v_i), \tag{1}$$

$$\left(\tau = \bigcup_{1 \le i \le k+1} v_i\right) \quad \wedge \quad \bigwedge_{1 \le i \le k+1} (v_i \neq \mathbf{0}) \quad \wedge \quad \bigwedge_{1 \le i < j \le k+1} (\tau \cap v_i^- \cap v_j^- = \mathbf{0}) \qquad (2)$$

with fresh $v_1, \dots, v_k$. Note, however, that these $\mathcal{S}4_u c$-formulas are exponentially larger than the literals they replace.

4 Computational complexity

There are two known complexity results for the spatial logics with connectedness constraints introduced above. According to [15], satisfiability of $\mathcal{S}4_u cc$-formulas is NExpTime-complete, which gives the NExpTime upper bound for all of these logics. On the other hand, it follows from [23] that $\mathcal{C}c$ is PSpace-hard (more precisely, satisfiability of $\mathcal{C}$-formulas in *connected* spaces is PSpace-complete).

We begin by showing that, as far as satisfiability is concerned, we can restrict attention to topological spaces of a special kind. Recall that a topological space is called an *Aleksandrov space* if arbitrary (not only finite) intersections of open sets are open. Aleksandrov spaces can be characterized in terms of *Kripke frames* $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and R is a transitive and reflexive relation (i.e., a *quasi-order*) on W. Every such $\mathfrak{F}$ induces the interior operator $\cdot^{\circ}_{\mathfrak{F}}$ on W:

$$X^{\circ}_{\mathfrak{F}} \;=\; \{x \in X \mid \forall y \in W\ (xRy \to y \in X)\}, \quad \text{for every } X \subseteq W.$$

It is well-known [2] that the resulting topological space is Aleksandrov and, conversely, every Aleksandrov space is induced by a quasi-order. Topological models over Aleksandrov spaces will be called *Aleksandrov models*. Note that the Aleksandrov space induced by $\mathfrak{F} = (W, R)$ is connected iff $\mathfrak{F}$ is *connected* as a non-directed graph, that is, between any two points $x, y \in W$ there is a path along the relation $R \cup R^{-1}$, where R^{-1} is the inverse of R. This observation is used implicitly throughout this paper. It is shown in [15] that $\mathcal{S}4_u cc$ is complete w.r.t. finite Aleksandrov models; this is a consequence of the following lemma.

Lemma 1 ([13, 15]). (i) *For every $\mathcal{S}4_u cc$-formula φ and every $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$ there exist an Aleksandrov model $\mathfrak{A} = (T_A, \cdot^{\mathfrak{A}})$ with $|T_A| \le 2^{|\varphi|}$ and a continuous function $f\colon T \to T_A$ such that, for every sub-term τ of φ, $\tau^{\mathfrak{A}} = f(\tau^{\mathfrak{M}})$.*

(ii) *Every $\mathcal{S}4_u cc$-formula φ can be transformed (in* LogSpace*) into an $\mathcal{S}4_u cc$-formula φ' such that it has no negative occurrences of $c^{\le k}(\tau)$, $|\varphi'|$ is polynomial in $|\varphi|$, and both φ and φ' are satisfiable over the same topological spaces.*

According to the next lemma, satisfiable $\mathcal{C}cc$-formulas can be satisfied in Aleksandrov models based on partial orders (W, R) *of depth* 1, i.e., R is the reflexive closure of a subset of $W_1 \times W_0$, where W_i is the set of points *of depth i*; see Fig. 1. Such frames and models are called *quasi-saws* and *quasi-saw models*.

Lemma 2. *For every finite Aleksandrov model $\mathfrak{A} = (T_A, \cdot^{\mathfrak{A}})$, with T_A induced by (W, R_A), there is a quasi-saw model $\mathfrak{B} = (T_B, \cdot^{\mathfrak{B}})$ such that T_B is induced by (W, R_B) with $R_B \subseteq R_A$ and, for every $\mathcal{B}$-term τ,* (i) *$\tau^{\mathfrak{B}} = \tau^{\mathfrak{A}}$, and* (ii) *$\tau$ has the same number of components in $\mathfrak{A}$ and $\mathfrak{B}$.*

Proof. Let W_0 be the set of points from final clusters in (W, R_A), i.e., $W_0 = \{v \in W \mid vR_A u \text{ implies } uR_A v, \text{ for all } u \in W\}$. In every final cluster $C \subseteq W_0$

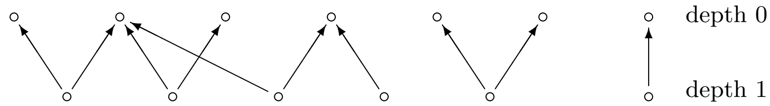

Fig. 1. Quasi-saw.

with $|C| \geq 2$ we select a point and denote by U the set of all selected points. Then we set $V_0 = W_0 \setminus U$ and $V_1 = W \setminus V_0$, and define R_B to be the reflexive closure of $R_A \cap (V_1 \times V_0)$. Clearly, (W, R_B) is a quasi-saw, with V_0 and V_1 being the sets of points of depth 0 and 1, respectively. For each variable r_i, let $r_i^{\mathfrak{B}} = r_i^{\mathfrak{A}}$. Claims (i) and (ii) are proved by induction on the construction of τ. ❑

4.1 Upper complexity bounds

We first prove the ExpTime-upper bound for $\mathcal{S}4_u c$ and a PSpace-upper bound for certain fragments of $\mathcal{S}4_u c$. To start with, we transform a given $\mathcal{S}4_u c$-formula φ into negation normal form (NNF$^+$) in the following way. First, we push negation $\neg$ inward to atoms $\tau_1 = \tau_2$ and $c(\tau)$, then use (2), for $k = 1$, to get rid of negative occurrences of $c(\tau)$, and finally replace each $c(\tau)$ with $(c(\tau) \wedge (\tau \neq \mathbf{0})) \vee (\tau = \mathbf{0})$, and each $(\tau_1 = \tau_2)$ with $(\tau_1 \cap \overline{\tau}_2 = \mathbf{0}) \wedge (\overline{\tau}_1 \cap \tau_2 = \mathbf{0})$.

Every $\mathcal{S}4_u c$-formula φ in NNF$^+$ is clearly equivalent to a disjunction $\bigvee \Psi_\varphi$, where each $\psi \in \Psi_\varphi$ is a conjunction of the form

$$\psi \quad = \quad \bigwedge_{i=1}^{l} \big(\rho_i = \mathbf{0}\big) \;\wedge\; \bigwedge_{i=1}^{m} \big(\tau_i \neq \mathbf{0}\big) \;\wedge\; \bigwedge_{i=1}^{k} \big(c(\sigma_i) \wedge (\sigma_i \neq \mathbf{0})\big) \qquad (3)$$

such that each atom of φ occurs either positively or negatively in ψ. For any such conjunction, it is decidable in polynomial time (in $|\varphi|$) whether it is in Ψ_φ.

Theorem 1. *Satisfiability of $\mathcal{S}4_u c$-formulas is in* ExpTime.

Proof. The proof is by reduction to the satisfiability problem for propositional dynamic logic ($\mathcal{PDL}$) with converse and nominals, which is known to be ExpTime-complete [9, Section 7.3]. Let ψ be as in (3). Take two atomic programs α and β and, for each σ_i, a nominal ℓ_i. For a term τ, denote by $\tau^\dagger$ the $\mathcal{PDL}$-formula obtained by replacing in τ, recursively, each sub-term ϑ° with $[\alpha^*]\vartheta$. Thus α^* simulates the $\mathcal{S}4$-accessibility relation, and the universal box will be simulated by $[\gamma]$, where $\gamma = (\beta \cup \beta^- \cup \alpha \cup \alpha^-)^*$. Consider now the formula ψ'

$$\bigwedge_{i=1}^{l} [\gamma]\neg\rho_i^\dagger \;\wedge\; \bigwedge_{i=1}^{m} \langle\gamma\rangle\tau_i^\dagger \;\wedge\; \bigwedge_{i=1}^{k} \Big(\langle\gamma\rangle(\ell_i \wedge \sigma_i^\dagger) \;\wedge\; [\gamma](\sigma_i^\dagger \to \langle(\alpha \cup \alpha^-; \sigma_i^\dagger?)^*\rangle\ell_i)\Big).$$

It is not hard to see that ψ' is satisfiable iff ψ is satisfiable: the first conjunct of ψ' states that all ρ_i are empty, the second that all τ_i are non-empty, the third states that each σ_i holds at a point where ℓ_i holds and that from each σ_i-point there is a path (along $\alpha \cup \alpha^-$) to ℓ_i which lies entirely within σ_i. ❑

Denote by $\mathcal{S}4_u c^1$ the set of $\mathcal{S}4_u c$-formulas in NNF$^+$ with *at most one* occurrence of an atom of the form $c(\tau)$.

Theorem 2. *Satisfiability of $\mathcal{S}4_u c^1$-formulas is in* PSPACE.

Proof. We sketch a nondeterministic PSPACE algorithm. Let φ be in NNF$^+$. Guess a ψ of the form (3) and check whether it is in Ψ_φ. Now check whether ψ is satisfiable: if ψ does not contain a conjunct of the form $c(\sigma) \wedge (\sigma \neq \mathbf{0})$, then a standard satisfiability checking algorithm for $\mathcal{S}4_u$ is applied. If it contains $c(\sigma) \wedge (\sigma \neq \mathbf{0})$, then the algorithm proceeds as follows. Let $\tau_0 = \bigcap_{i=1}^{l} \rho_i$. Set $\mathbf{B} = \{\overline{\tau_0}^\circ\} \cup \{\tau, \overline{\tau} \mid \tau \in \mathit{term}(\psi)\}$, where $\mathit{term}(\varphi)$ is the set of all sub-terms of ψ. A subset $\mathfrak{t}$ of $\mathbf{B}$ is called a *type for* ψ if $\overline{\tau_0}^\circ \in \mathfrak{t}$ and $\tau \in \mathfrak{t}$ iff $\overline{\tau} \notin \mathfrak{t}$, for all $\overline{\tau} \in \mathbf{B}$.

Now, guess a type $\mathfrak{t}_\sigma$ containing σ and start $m+1$ $\mathcal{S}4$-tableau procedures with inputs $\tau_1 \cap \overline{\tau_0}^\circ$, $\tau_2 \cap \overline{\tau_0}^\circ, \dots, \tau_m \cap \overline{\tau_0}^\circ$, and $\bigcap \mathfrak{t}_\sigma \cap \overline{\tau_0}^\circ$ in the usual way expanding branch-by-branch, recovering the space once branches are checked. We may as well assume that the nodes of these tableaux are types. Suppose $\mathfrak{t}$ is a type occurring in one of them. If $\sigma \in \mathfrak{t}$, it suffices to check that $\mathfrak{t}$ can be connected by a path of $\leq 2^{|\psi|}$ points in σ to $\mathfrak{t}_\sigma$. To complete the proof we present a subroutine which, given types $\mathfrak{t}_0, \mathfrak{t}_1 \ni \sigma$ and $d \geq 0$, checks, in PSPACE, whether $\mathfrak{t}_0$ and $\mathfrak{t}_1$ can be connected by a path of $\leq 2^d$ points in σ to $\mathfrak{t}_\sigma$.
Subroutine: If $d = 0$, we check that $\mathfrak{t}_0$ and $\mathfrak{t}_1$ can be made accessible one direction or the other. If $d > 0$, we guess a type $\mathfrak{t}$ with $\sigma \in \mathfrak{t}$ that represents the half-way point between $\mathfrak{t}_0$ and $\mathfrak{t}_1$. First we check that $\mathfrak{t}$ is an allowable type by constructing an $\mathcal{S}4$-tableau with root $\mathfrak{t}$. The tableau can be discarded after completion: although it may contain types $\mathfrak{t}'$ with $\sigma \in \mathfrak{t}'$, these type can never threaten the connectedness of σ, since they are all accessible from the root $\mathfrak{t}$ of the tableau (the $\mathcal{S}4$ accessibility relation is transitive!), and so are connected to both $\mathfrak{t}_0$ and $\mathfrak{t}_1$ anyway. Then the subroutine calls itself recursively with parameters $(\mathfrak{t}_0, \mathfrak{t}, d-1)$ and $(\mathfrak{t}, \mathfrak{t}_1, d-1)$. Completing this recursive procedure requires at most d items to be placed on the stack. ❑

Observe that the argument above shows that satisfiability of formulas φ in NNF$^+$ with conjuncts $\bigwedge_{i=1}^{k} c(\tau_i)$ such that $(\tau_i^- \cap \tau_j^- = \mathbf{0})$, $i \neq j$, are conjuncts of φ, is decidable in PSPACE as well.

4.2 Lower complexity bounds

We first prove the matching lower bound for $\mathcal{C}c$. Observe that when constructing a model for an $\mathcal{S}4_u c^1$-formula with one positive occurrence of $c(\tau)$, we can check 'connectivity' of two τ-points by an (exponentially long) path using a PSPACE-algorithm because it is not necessary to keep in memory all the points on the path. However, if two statements $c(\tau_1)$ and $c(\tau_2)$ have to be satisfied, then, while connecting two τ_1-points using a path, one has to check whether the τ_2-points on that path can be connected by a path, which, in turn, can contain another τ_1-point, and so on. The crucial idea in the proof below is simulating infinite binary (*non-transitive*) trees using quasi-saws. Roughly, the construction is as follows. We start by representing the root v_0 of the tree as a point also denoted by v_0 (see Fig. 2), which is forced to be connected to an auxiliary point z by means of some $c(\tau_0)$. On the connecting path from v_0 to z we represent the two successors v_1 and v_2 of the root, which are forced to be connected in their turn to z by some other $c(\tau_1)$. On each of the two connecting paths, we again

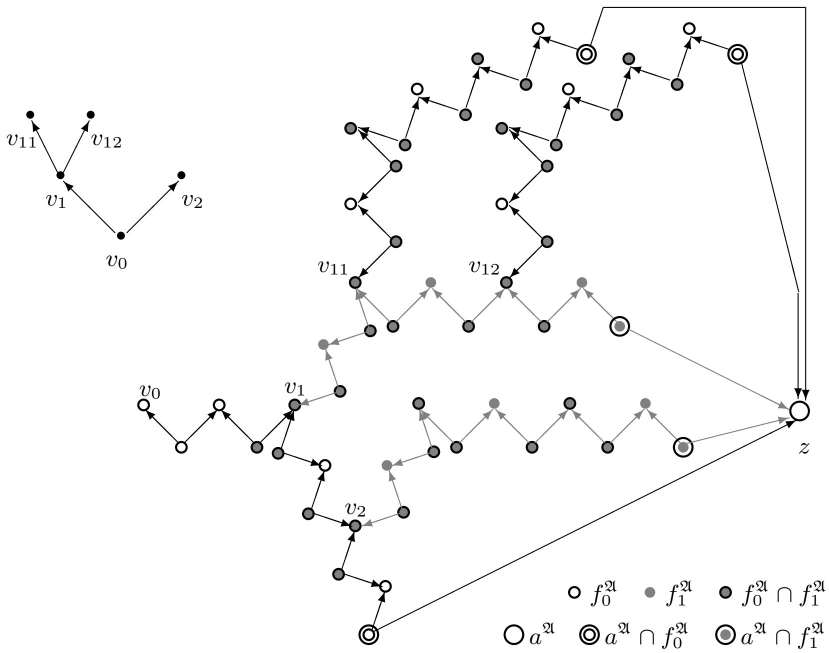

Fig. 2. First 4 steps of encoding the full binary tree using 7-saws.

take two points representing the successors of v_1 and v_2, respectively. We treat these four points in the same way as v_0, reusing $c(\tau_0)$, and proceed *ad infinitum* alternating between τ_0 and τ_1 when forcing the paths which generate the required successors. Of course, we also have to pass certain information from a node to its two successors (say, if $\Diamond\psi$ holds in the node, then ψ holds in one of its successors). Such information can be propagated along connected regions. Note now that all points are connected to z. To distinguish between the information we have to pass from distinct nodes of even (respectively, odd) level to their successors, we have to use *two* connectedness formulas of the form $c(f_i + a)$, $i = 0, 1$, in such a way that the f_i points form initial segments of the paths to z and a contains z. The f_i-segments are then used locally to pass information from a node to its successors without conflict. We now present the reduction in more detail.

Theorem 3. *Satisfiability of $\mathcal{C}c$-formulas is* ExpTime*-hard.*

Proof. The proof is by reduction of the following problem. Denote by $\mathcal{D}_2^f$ the bimodal logic (with $\Box_1$ and $\Box_2$) determined by Kripke models based on the full infinite binary tree $\mathfrak{G} = (V, R_1, R_2)$ with *functional* accessibility relations R_1 and R_2. Consider the *global consequence relation* $\models_2^f$ defined as follows: $\chi \models_2^f \psi$ iff $\mathfrak{K} \models \chi$ implies $\mathfrak{K} \models \psi$, for every Kripke model $\mathfrak{K}$ based on $\mathfrak{G}$. Using standard modal logic technique one can show ExpTime-hardness of this global consequence relation. We construct a $\mathcal{C}c$-formula $\Phi(\chi, \psi)$, for any $\mathcal{D}_2^f$-formulas χ, ψ, such that (i) $|\Phi(\chi, \psi)|$ is polynomial in $|\chi| + |\psi|$ and (ii) $\Phi(\chi, \psi)$ is satisfiable iff $\chi \not\models_2^f \psi$. While constructing $\Phi(\chi, \psi)$, we will assume that $\mathfrak{A}$ is a quasi-saw model induced by (W, R) and W_0 is the set of points of depth 0 in (W, R).

Let $sub(\chi, \psi)$ be the closure under single negation of the set of subformulas of χ, ψ. For each $\varphi \in sub(\chi, \psi)$ we take a fresh variable q_φ, and for $\Box_i\varphi \in sub(\chi, \psi)$ and $j = 0, 1$, we fix fresh variables $m^{i,j}_\varphi$ and $m^{i,j}_{\neg\varphi}$. We also need fresh variables s^i_j, for $j = 0, 1$ and $0 \leq i \leq 6$. Let $d = s^0_0 + s^0_1$. Intuitively, d simulates the domain of the binary tree, where s^0_0 and s^0_1 stand for nodes with even and, respectively, odd distance from the root. Suppose that the following $\mathcal{C}c$-formulas hold in $\mathfrak{A}$

$$(s^6_0 = s^6_1) \quad \wedge \quad (s^6_0 \neq \mathbf{0}) \quad \wedge \quad c(f_0 + s^6_0) \quad \wedge \quad c(f_1 + s^6_1), \tag{4}$$

$$\bigwedge_{0 \leq k < k' \leq 6} (s^k_j \cdot s^{k'}_j = \mathbf{0}) \quad \wedge \quad \bigwedge_{\substack{0 \leq k < k' \leq 6 \\ |k-k'|>1}} \neg C(s^k_j, s^{k'}_j), \tag{5}$$

where $f_j = s^0_j + s^1_j + s^2_j + s^3_j + s^4_j + s^5_j$, for $j = 0, 1$. (Note that s^6_0 and s^6_1 play the role of a in the explanation above; see Fig. 2.) It follows that, for $j = 0, 1$, if there is a point $x_0 \in (s^0_j)^{\mathfrak{A}} \cap W_0$ then there is a (not necessarily unique) sequence of points x_1, x_2, x_3, x_4, x_5 from the same connected component of $f^{\mathfrak{A}}_j$ such that $x_i \in (s^i_j)^{\mathfrak{A}} \cap W_0$, $1 \leq i \leq 5$. Points x_2 and x_4 will be used to construct similar sequences for the two successors of the node represented by x_0: if (4)–(5) and

$$s^{2i}_0 \ \leq \ s^0_1 \qquad \text{and} \qquad s^{2i}_1 \ \leq \ s^0_0, \qquad \text{for } i = 1, 2, \tag{6}$$

hold in $\mathfrak{A}$ and $x_0 \in (s^0_j)^{\mathfrak{A}} \cap W_0$, then one can recover from $\mathfrak{A}$ the infinite binary tree with the root at x_0. The formula

$$(q_{\neg\psi} \cdot s^0_0 \neq \mathbf{0}) \quad \wedge \quad (d \ \leq \ q_\chi) \tag{7}$$

ensures then that there is $x_0 \in (s^0_j)^{\mathfrak{A}} \cap W_0$, the root of the tree, in which ψ holds, and χ holds everywhere in the tree, while the formulas

$$d \cdot q_{\neg\varphi} = d \cdot (-q_\varphi), \qquad\qquad d \cdot q_{\varphi_1 \wedge \varphi_2} = d \cdot (q_{\varphi_1} \cdot q_{\varphi_2}), \tag{8}$$

for all $\neg\varphi, \varphi_1 \wedge \varphi_2 \in sub(\chi, \psi)$, capture the meaning of the Boolean connectives from $sub(\chi, \psi)$ relativized to d. The formulas

$$\neg C(f_j \cdot m^{i,j}_\varphi, f_j \cdot m^{i,j}_{\neg\varphi}), \tag{9}$$

$$(s^0_j \cdot q_{\Box_i\varphi} \leq m^{i,j}_\varphi) \quad \wedge \quad (m^{i,j}_\varphi \cdot s^{2i}_j \ \leq \ q_\varphi), \tag{10}$$

$$(s^0_j \cdot q_{\neg\Box_i\varphi} \leq m^{i,j}_{\neg\varphi}) \quad \wedge \quad (m^{i,j}_{\neg\varphi} \cdot s^{2i}_j \ \leq \ q_{\neg\varphi},), \tag{11}$$

for all $\Box_i\varphi \in sub(\chi, \psi)$ and $j = 0, 1$, are used to propagate information regarding $\Box_i\varphi$ along the connected components of f_j using the markers $m^{i,j}_\varphi$ and $m^{i,j}_{\neg\varphi}$.

We define $\Phi(\chi, \psi)$ to be the conjunction of all the above formulas. Clearly, $|\Phi(\chi, \psi)|$ is polynomial in $|\chi| + |\psi|$ and contains only two occurrences of the connectedness predicate in (4).

Conversely, suppose that $\mathfrak{K}$ is a model for $\mathcal{D}^f_2$ based on the full infinite binary tree $\mathfrak{G} = (V, R_1, R_2)$ with root v_0 and such that $\mathfrak{K} \models \chi$ and $\mathfrak{K}, v_0 \not\models \psi$. We construct a quasi-saw model $\mathfrak{A}$ satisfying $\Phi(\chi, \psi)$ by induction (as in Fig. 2) using infinitely many copies of the 7-saw shown in Fig. 3. For each node v of $\mathfrak{G}$, we take a fresh 7-saw $\mathfrak{S}^v = (S^v, R^v)$, where $S^v = \{y_i^v, z_i^v, u^v \mid 0 \le i \le 5\}$, $z_i^v R^v y_i^v$, $z_i^v R^v y_{i+1}^v$, for $0 \le i \le 5$, and $z_5^v R^v u^v$, and identify the following points: $y_2^v = y_0^{v_1}$, $y_4^v = y_0^{v_2}$, $u^{v_1} = u^{v_2} = u^v$, if v_1 and v_2 are the R_1- and R_2-successors of v. The assignment is left to the reader.

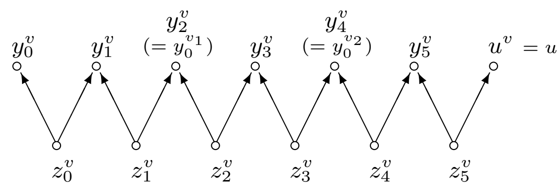

Fig. 3. A 7-saw for v.

Note that y_1^v and y_3^v are required to make the set of points in $f_j^{\mathfrak{A}}$ representing a node v of $\mathfrak{G}$ disconnected from the subset of $f_j^{\mathfrak{A}}$ representing another node v' of $\mathfrak{G}$ and thus satisfy (9); y_5^v are required to satisfy the last conjunct of (5). ❑

We now consider the lower complexity bound for $\mathcal{C}$ with constraints on the number of connected components.

Theorem 4. *Satisfiability of $\mathcal{C}cc$-formulas is* NExpTime*-hard.*

Proof. The proof is by reduction of the NExpTime-complete $2^d \times 2^d$ *tiling problem*: Given $d < \omega$, a finite set $\mathcal{T}$ of *tile types*—i.e., 4-tuples of colours $T = (\mathit{left}(T), \mathit{right}(T), \mathit{up}(T), \mathit{down}(T))$—and a $T_0 \in \mathcal{T}$, decide whether $\mathcal{T}$ can tile the $2^d \times 2^d$ grid in such a way that T_0 is placed onto $(0,0)$. In other words, the problem is to decide whether there is a function f from $\{(i,j) \mid i,j < 2^d\}$ to $\mathcal{T}$ such that $\mathit{up}(f(i,j)) = \mathit{down}(f(i,j+1))$, for all $i < 2^d$, $j < 2^d - 1$, $\mathit{right}(f(i,j)) = \mathit{left}(f(i+1,j))$, for all $i < 2^d - 1$, $j < 2^d$, $f(0,0) = T_0$. We construct a $\mathcal{C}cc$-formula $\varphi_{\mathcal{T},d}$ such that (i) $|\varphi_{\mathcal{T},d}|$ is polynomial in $|\mathcal{T}|$ and d and (ii) $\varphi_{\mathcal{T},d}$ is satisfiable iff $\mathcal{T}$ tiles the $2^d \times 2^d$ grid, with T_0 being placed onto $(0,0)$. While constructing $\varphi_{\mathcal{T},d}$, we will assume that $\mathfrak{A}$ is a quasi-saw model induced by (W,R) and W_0 is the set of points of depth 0 in (W,R).

We partition all points of W_0 with the help of a pair of variable triples B_X^0, B_X^1, B_X^2 and B_Y^0, B_Y^1, B_Y^2. Suppose that the formulas, for $0 \le \ell < 3$,

$$B_X^0 + B_X^1 + B_X^2 = \mathbf{1}, \qquad B_X^\ell \cdot B_X^{\ell \oplus_3 1} = \mathbf{0} \tag{12}$$

and their Y-counterparts hold in $\mathfrak{A}$, where $\oplus_3$ denotes addition modulo 3. Then every point in W_0 is in exactly one of the $(B_X^\ell)^{\mathfrak{A}}$ and exactly one of the $(B_Y^\ell)^{\mathfrak{A}}$.

To encode coordinates of the tiles in binary, we take a pair of variables X_j and Y_j, for each $d \ge j \ge 1$. For $0 \le n < 2^d$, let n_X be the $\mathcal{B}$-term $X'_d \cdot X'_{d-1} \cdot \dots \cdot X'_1$, where $X'_j = X_j$ if the jth bit in the binary representation of n is 1, and $X'_j = -X_j$ otherwise. For a point $u \in W_0$, we denote by $X(u)$ the binary d-bit number n, called the *X-value* of u, such that $u \in n_X^{\mathfrak{A}}$; the jth bit of $X(u)$ is denoted by $X_j(u)$. The term n_Y, the *Y-value* $Y(u)$ of u and its jth bit $Y_j(u)$ are defined analogously. For a point u of depth 0 we write $\mathsf{coor}(u)$ for $(X(u), Y(u))$. We will use the variables X_i and Y_j to generate the $2^d \times 2^d$ grid, which consists of pairs (i_X, j_Y), for $0 \leq i, j < 2^d$. Consider the following formulas, for $0 \leq \ell < 3$,

$$\neg C(X_k \cdot B^\ell_X,\ (-X_k) \cdot B^\ell_X), \qquad d \geq k \geq 1, \tag{13}$$

$$\neg C(X_j \cdot (-X_k) \cdot B^\ell_X,\ (-X_j) \cdot B^{\ell \oplus_3 1}_X), \qquad d \geq j > k \geq 1, \tag{14}$$

$$\neg C((-X_j) \cdot (-X_k) \cdot B^\ell_X,\ X_j \cdot B^{\ell \oplus_3 1}_X), \qquad d \geq j > k \geq 1, \tag{15}$$

$$\neg C((-X_k) \cdot X_{k-1} \cdot \ \cdots \ \cdot X_1 \cdot B^\ell_X,\ \ (-X_k) \cdot B^{\ell \oplus_3 1}_X), \qquad d \geq k > 1, \tag{16}$$

$$\neg C((-X_k) \cdot X_{k-1} \cdot \ \cdots \ \cdot X_1 \cdot B^\ell_X,\ \ X_i \cdot B^{\ell \oplus_3 1}_X), \qquad d \geq k > i \geq 1, \tag{17}$$

$$\neg C(X_d \cdot \ \cdots \ \cdot X_1,\ (-X_d) \cdot \ \cdots \ \cdot (-X_1)), \tag{18}$$

the Y-counterparts of (13)–(18), and the following, for $d \geq j, k \geq 1$,

$$\neg C(X_j \cdot Y_k,\ (-X_j) \cdot (-Y_k)), \qquad \neg C((-X_j) \cdot Y_k,\ X_j \cdot (-Y_k)). \tag{19}$$

Given a point $v \in W_0$, denote by $\mathsf{4\text{-}nb}(v)$ the set which consists of $\mathsf{coor}(v)$ and its (at most four) neighbours in the $2^d \times 2^d$ grid. Suppose that $\mathfrak{A}$ satisfies all the formulas above. If $u, v \in W_0$ and zRu and zRv, for some $z \in W$, then $\mathsf{coor}(u) \in \mathsf{4\text{-}nb}(v)$. Moreover, (i) $X(v) = X(u) = n$ iff u and v are in the same component of $n^{\mathfrak{A}}_X$, and (ii) for each $m = -1, 0, 1$, $X(v) = X(u) + m$ iff $u \in (B^\ell_X)^{\mathfrak{A}}$ and $v \in (B^{\ell \oplus_3 m}_X)^{\mathfrak{A}}$, for $\ell = 0, 1, 2$ (in particular, $X(u) = X(v)$ iff u and v are in the same connected component of $(B^\ell_X)^{\mathfrak{A}}$). Likewise for Y in place of X.

Suppose now that the following formulas are true in $\mathfrak{A}$ as well:

$$0_X \cdot 0_Y \neq \mathbf{0},\ (2^d - 1)_X \cdot (2^d - 1)_Y \neq \mathbf{0},\ c(0_X + (2^d - 1)_Y),\ c((2^d - 1)_X + 0_Y). \tag{20}$$

These constraints guarantee that in the connected set $(0_X + (2^d - 1)_Y)^{\mathfrak{A}}$ there are points $u_{(0,i)}$ and $u_{(i,2^d-1)}$, $0 \leq i < 2^d$, such that $\mathsf{coor}(u_{(0,i)}) = (0, i)$ and $\mathsf{coor}(u_{(i,2^d-1)}) = (i, 2^d - 1)$. Similarly for the connected set $((2^d - 1)_X + 0_Y)^{\mathfrak{A}}$. This gives us the border of the $2^d \times 2^d$ grid we are after. And the constraints

$$c((-X_1) + 0_Y), \quad c(X_1 + 0_Y), \quad c(0_X + (-Y_1)), \quad c(0_X + Y_1) \tag{21}$$

ensure that we can find inner points of the grid. It is to be noted, however, that in general $u \neq v$ even if $\mathsf{coor}(u) = \mathsf{coor}(v)$. In other words, the constructed points do not necessarily form a proper $2^d \times 2^d$ grid. Let

$$\mathbf{b} = \big(X_1 \cdot (-Y_1)\big) + \big((-X_1) \cdot Y_1\big) \quad \text{and} \quad \mathbf{w} = \big((-X_1) \cdot (-Y_1)\big) + \big(X_1 \cdot Y_1\big).$$

Points in $\mathbf{b}^{\mathfrak{A}}$ and $\mathbf{w}^{\mathfrak{A}}$ can be thought of as *black* and *white* squares of a chessboard. Observe that if $u, v \in \mathbf{b}^{\mathfrak{A}} \cap W_0$ and $\mathsf{coor}(u) \neq \mathsf{coor}(v)$ then u and v cannot belong to the same component of $\mathbf{b}^{\mathfrak{A}}$. Thus, there are at least 2^{d-1} components in both $\mathbf{b}^{\mathfrak{A}}$ and $\mathbf{w}^{\mathfrak{A}}$. Our next constraints

$$c^{\leq 2^{d-1}}(\mathbf{b}), \qquad c^{\leq 2^{d-1}}(\mathbf{w}) \tag{22}$$

say that $\mathbf{b}^{\mathfrak{A}}$ and $\mathbf{w}^{\mathfrak{A}}$ have precisely 2^{d-1} components. In particular, if $u, v \in W_0$ belong to the same component of $\mathbf{b}^{\mathfrak{A}}$ then $\mathsf{coor}(u) = \mathsf{coor}(v)$. This gives a proper $2^d \times 2^d$ grid on which we encode the tiling conditions. The formulas

$$\textstyle\sum_{T \in \mathcal{T}} T = \mathbf{1} \qquad \text{and} \qquad T \cdot T' = \mathbf{0}, \quad \text{for } T \neq T', \tag{23}$$

$$\neg C(B_X^\ell \cdot B_Y^{\ell'} \cdot T,\ B_X^\ell \cdot B_Y^{\ell'} \cdot T'), \quad \text{for } \ell, \ell' = 0, 1, 2 \quad \text{and} \quad T \neq T', \tag{24}$$

say that every point in W_0 is covered by precisely one tile and that all points in the same component of $(B_X^\ell \cdot B_Y^{\ell'})^{\mathfrak{A}}$ are covered by the same tile. That the colours of adjacent tiles match is ensured by

$$\neg C(B_X^\ell \cdot T,\ B_X^{\ell \oplus_3 1} \cdot T'), \qquad \text{for } T, T' \in \mathcal{T} \text{ with } \mathit{right}(T) \neq \mathit{left}(T'), \tag{25}$$

$$\neg C(B_Y^\ell \cdot T,\ B_Y^{\ell \oplus_3 1} \cdot T'), \qquad \text{for } T, T' \in \mathcal{T} \text{ with } \mathit{top}(T) \neq \mathit{bot}(T'). \tag{26}$$

Finally, we have to say that $(0, 0)$ is covered with T_0:

$$0_X \cdot 0_Y \leq T_0. \tag{27}$$

One can check that the conjunction $\varphi_{\mathcal{T},d}$ of these $\mathcal{C}cc$-formulas is as required. ❑

The ExpTime and NExpTime lower bounds for $\mathcal{B}c$ and $\mathcal{B}cc$ will be proved by reduction of satisfiability for $\mathcal{C}c$ and $\mathcal{C}cc$, respectively; that is, by eliminating occurrences of the predicate C in $\mathcal{C}c$- and $\mathcal{C}cc$-formulas. Clearly, two *connected* closed sets are in contact iff their union is connected; in other words, the formula $c(\tau_1) \wedge c(\tau_2) \rightarrow \big(C(\tau_1, \tau_2) \ \leftrightarrow \ c(\tau_1 + \tau_2)\big)$ is a $\mathcal{C}cc$-validity. However, this 'reduction' of C to c cannot be directly applied to our formulas since the arguments of the contact predicates in them are not necessarily connected. The next three lemmas show how to overcome this problem.

We write $\varphi[\psi]^+$ (or $\varphi[\psi]^-$) to indicate that φ contains a positive (respectively, negative) occurrence of ψ; then $\varphi[\chi]^+$ (or $\varphi[\chi]^-$) denotes the result of replacing this occurrence of ψ in φ by χ.

Lemma 3. *Let $\varphi[C(\tau_1, \tau_2)]^+$ be a $\mathcal{C}cc$-formula, and t, t_1, t_2 fresh variables. Then φ is equisatisfiable with the formula*

$$\varphi^* \quad = \quad \varphi[t = \mathbf{0}]^+ \quad \wedge \quad \big((t = \mathbf{0}) \quad \rightarrow \quad c(t_1 + t_2) \ \wedge \bigwedge_{i=1,2} (t_i \leq \tau_i) \wedge c(t_i)\big).$$

Proof. It is easy to see that $\models \varphi^* \rightarrow \varphi$. On the other hand, every model of φ can be turned into a model of φ^* by changing the extensions of t, t_1, t_2. ❑

Suppose X is a topological space, and S a regular closed subset of X. Then S is itself a topological space (with the subspace topology), which has its own regular closed algebra: $\boldsymbol{RC}(S) = \{S \cdot R \mid R \in \boldsymbol{RC}(X)\}$. Denoting the Boolean operations in $\boldsymbol{RC}(S)$ by $\cdot_S$ and $-_S$, etc., we have, for any $R_1, R_2 \in \boldsymbol{RC}(S)$: (i) $R_1 \cdot_S R_2 = R_1 \cdot R_2$; (ii) $-_S(R_1) = S \cdot (-R_1)$, (iii) $\mathbf{1}_S = S$ and $\mathbf{0}_S = \mathbf{0}$. For a formula φ and a variable s, define $\varphi_{|s}$ to be the result of replacing every maximal term τ occurring in φ by the term $s \cdot \tau$. For any model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$, define $\mathfrak{M}_{|s}$ to be the model over the topological space $s^{\mathfrak{M}}$ (with the subspace topology) obtained by setting $r^{\mathfrak{M}_{|s}} = (r \cdot s)^{\mathfrak{M}}$ for all variables r.

Lemma 4. *For any $\mathcal{C}cc$-formula, $\mathfrak{M} \models \varphi_{|s}$ iff $\mathfrak{M}_{|s} \models \varphi$.*

Proof. One can show by induction that $(s \cdot \tau)^{\mathfrak{M}} = \tau^{\mathfrak{M}_{|s}}$, for any $\mathcal{B}$-term τ. ❑

Lemma 5. *Let* $\varphi[C(\tau_1,\tau_2)]^-$ *be a* $\mathcal{C}cc$*-formula, and* s,t,t_1,t_2 *fresh variables. Then* φ *is equisatisfiable with the formula*

$$\varphi^* \;=\; \big(\varphi[t \neq \mathbf{0}]^-\big)_{|s} \;\wedge\; \big((t \cdot s = \mathbf{0}) \;\rightarrow\; \neg c(t_1+t_2) \;\wedge\; \bigwedge_{i=1,2} c(t_i) \wedge (\tau_i \cdot s \leq t_i)\big).$$

Proof. Evidently, $\bigwedge_{i=1,2}(c(t_i) \wedge (\tau_i \cdot s \leq t_i)) \wedge \neg c(t_1+t_2) \rightarrow \neg C(\tau_1 \cdot s, \tau_2 \cdot s)$ is a $\mathcal{C}cc$-validity. So any model $\mathfrak{A}$ of φ^* is a model of $(\varphi[C(\tau_1,\tau_2)]^-)_{|s}$, whence, by Lemma 4, $\mathfrak{A}_{|s} \models \varphi[C(\tau_1,\tau_2)]^-$. Conversely, suppose $\mathfrak{A} \models \varphi[C(\tau_1,\tau_2)]^-$, for a quasi-saw model $\mathfrak{A}$ induced by (W,R). Let W_i, $(i=0,1)$ be the set of points of depth i in (W,R). Without loss of generality, we may assume that every point in W_0 has an R-predecessor in W_1. If $\mathfrak{A} \models C(\tau_1,\tau_2)$, let $\mathfrak{A}^*$ be exactly like $\mathfrak{A}$ except that $s^{\mathfrak{A}^*}$ and $t^{\mathfrak{A}^*}$ are both the whole space. Then $\mathfrak{A}^* \models \varphi^*$. On the other hand, if $\mathfrak{A} \not\models C(\tau_1,\tau_2)$, we add, for $i=1,2$, an extra point u_i to W to connect up the points in $\tau_i^{\mathfrak{A}}$. Formally, let $W^* = W \cup \{u_1,u_2\}$, where $u_1,u_2 \notin W$, and let R^* be the reflexive closure of the union of R and $\{(z,u_i) \mid z \in \tau_i^{\mathfrak{A}} \cap W_1\}$, for $i=1,2$. Clearly, W is a regular closed subset of the topological space (W^*,R^*). Now define the interpretation $\mathfrak{A}^*$ over (W^*,R^*) by setting $s^{\mathfrak{A}^*}=W$, $t^{\mathfrak{A}^*}=\emptyset$, $t_i^{\mathfrak{A}^*}=\tau_i^{\mathfrak{A}} \cup \{u_i\}$ $(i=1,2)$, and $r^{\mathfrak{A}^*}=r^{\mathfrak{A}}$ for all other variables r. Thus, $\mathfrak{A}=\mathfrak{A}^*_{|s}$, whence, by Lemma 4, $\mathfrak{A}^* \models (\varphi[C(\tau_1,\tau_2)]^-)_{|s}$, and so $\mathfrak{A}^* \models (\varphi[t \neq \mathbf{0}]^-)_{|s}$. By construction, $\mathfrak{A}^* \models \bigwedge_{i=1,2}(c(t_i) \wedge (\tau_i \cdot s \leq t_i)) \wedge \neg c(t_1+t_2)$. Thus, $\mathfrak{A}^* \models \varphi^*$. ❑

It follows from these lemmas that the satisfiability problem for $\mathcal{C}c$ (and $\mathcal{C}cc$) is reducible to the satisfiability problem for $\mathcal{B}c$ ($\mathcal{B}cc$, respectively). For, by repeated application of Lemmas 3 and 5, successive occurrences of C in a $\mathcal{C}c$- or $\mathcal{C}cc$-formula may be equisatisfiably eliminated, using only logarithmic space.

As a consequence, by Theorems 3 and 4, we obtain:

Theorem 5. *Satisfiability of* $\mathcal{B}c$*- and* $\mathcal{B}cc$*-formulas is, respectively,* ExpTime*- and* NExpTime*-complete.*

We remark in passing that the full reduction is not required for Theorem 5. For the proofs of Theorems 3 and 4 in fact rely on formulas in which conjuncts $C(\tau_1,\tau_2)$ occur only in negative contexts (and thus Lemma 5 is enough).

5 Discussion and further work

In this paper, we have reported on the computational complexity of the satisfiability problems for the spatial logics $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}4_u$ extended with connectedness constraints. All these logics feature variables which range over subsets of topological spaces: regular subsets in the case of logics based on $\mathcal{B}$ and $\mathcal{C}$, and arbitrary subsets in the case of $\mathcal{S}4_u$. However, topological spaces form an extremely general category: and it is natural to ask what happens when we restrict consideration to particular classes of topological spaces. Most saliently of all: what happens when these logics are interpreted over the *specific* topological spaces $\mathbb{R}^2$ or $\mathbb{R}^3$?

Without the ability to express connectedness, topological spatial logics are almost completely insensitive to the underlying topology. Thus, a $\mathcal{B}$-formula is satisfiable over $\mathbb{R}^n$, for any fixed n, iff it is satisfiable (over some space); a $\mathcal{C}$-formula is satisfiable over $\mathbb{R}^n$, for any fixed n, iff it is satisfiable over a connected space [23]; and an $\mathcal{S}4_u$-formula is satisfiable over $\mathbb{R}^n$, for any fixed n, iff it is satisfiable over a connected, dense-in-itself, separable metric space [21]. Adding connectedness constraints to these logics changes the situation radically, however. As a simple illustration, consider the $\mathcal{B}c$ formula

$$\bigwedge_{1 \leq i \leq 3} c(r_i) \quad \wedge \bigwedge_{1 \leq i < j \leq 3} \big(r_i \cdot r_j \neq \mathbf{0}\big) \quad \wedge \quad \big(r_1 \cdot r_2 \cdot r_3 = \mathbf{0}\big),$$

which states that there are three pairwise overlapping, connected regions whose common part has an empty interior. Since connected subsets of $\mathbb{R}$ are intervals, this formula is not satisfiable over $\mathbb{R}$; yet it is satisfiable over $\mathbb{R}^n$, for any $n > 1$. Or again, it can be shown (see [14], p. 137) that the $\mathcal{S}4_u c$-formula

$$(v_1 \cap v_2 = \mathbf{0}) \;\wedge\; \bigwedge_{i=1,2} ((v_i^- \subseteq v_i) \;\wedge\; c(\overline{v_i})) \quad \wedge \quad \neg c(\overline{v_1} \cap \overline{v_2})$$

is not satisfiable over $\mathbb{R}^n$ (for any n); yet it is easily seen to be satisfiable over other manifolds (even of dimension 1!).

What can we say about the complexity of determining satisfiability over these spaces? In the one-dimensional case, matching complexity bounds are available.

Theorem 6. *Satisfiability of $\mathcal{S}4_u cc$-formulas in topological models based on $\mathbb{R}$ is* PSPACE*-complete.*

Proof. The proof is by reduction to the propositional temporal logic of the real line, for which satisfiability is known to be PSPACE-complete [19]. Since, for $\mathcal{C}$-formulas, satisfiability over connected spaces implies satisfiability over $\mathbb{R}$, it follows from [23] that this bound is tight. ❑

For $n > 1$, the work reported here yields lower-bound information for satisfiability over $\mathbb{R}^n$:

Theorem 7. *Satisfiability of $\mathcal{C}c$- and $\mathcal{C}cc$-formulas in topological models based on $\mathbb{R}^n$, for each $n > 1$, is* EXPTIME*- and* NEXPTIME*-hard, respectively.*

Proof. The proof is based on the fact that the models constructed in the proofs of Theorem 3 and 4 can be turned into models over $\mathbb{R}^2$, and so over any $\mathbb{R}^n$, for $n \geq 2$. ❑

It follows that the EXPTIME and NEXPTIME lower bounds hold for satisfiability of $\mathcal{S}4_u c$- and $\mathcal{S}4_u cc$-formulas over $\mathbb{R}^n$, respectively.

We mention that, when variables are restricted to range over closed disc-homeomorphs in $\mathbb{R}^2$, then the problem of determining the satisfiability of $\mathcal{RCC}$-8-constraints is known to be in NP [20]—a very surprising result, since the smallest satisfying drawings may involve exponentially many intersection points [11]. At present, no upper complexity bounds for the logics $\mathcal{B}c$, $\mathcal{B}cc$, $\mathcal{C}c$, $\mathcal{C}cc$, $\mathcal{S}4_u c$, interpreted over Euclidean spaces of fixed dimension greater than 1 are known.

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