Chapter 1

WHAT IS SPATIAL LOGIC?

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By a *spatial logic*, we understand any formal language interpreted over a class of structures featuring geometrical entities and relations, broadly construed. The formal language in question may employ any logical syntax: that of first-order logic, or some fragment of first-order logic, or perhaps higher-order logic. The structures over which it is interpreted may inhabit any class of geometrical 'spaces': topological spaces, affine spaces, metric spaces, or perhaps a specific structure such as the projective plane or Euclidean 3-space. And the non-logical primitives of the language may be interpreted as any geometrical properties or relations defined over the relevant domains: topological connectedness of regions, parallelism of lines, or perhaps equidistance of two points from a third. What all these logics have in common is that the operative notion of validity depends on the underlying geometry of the structures over which their distinctively spatial primitives are interpreted. Spatial logic, then, is simply the study of the family of spatial logics, so conceived.

An analogy will help elucidate this rather austere-looking definition. From our stance, spatial logic parallels the more established area of temporal logic. A temporal logic is a formal language interpreted over some class of structures based on frameworks of temporal relations, broadly construed. The language in question, though usually some modal fragment of first- or higher-order logic, may in principle employ any logical syntax; the objects over which that syntax is interpreted may include points, paths, or extended intervals over any variety of partial orders; and the assumed partial order ultimately provides the interpretation for the distinctively temporal primitives of the formal language. What all temporal logics have in common, whether point- or interval-based, is that their operative notion of validity depends on the assumed properties of the underlying temporal flow. And what gives them their enduring appeal is the way in which the formal languages they employ balance expressive power against computational complexity. In this respect, temporal logic is the computationally motivated study of time.

Let us set the scene for the treatment of spatial logic in this book by examining some of the historical trends that have given rise to it. Classical geometry, the cultural model of deductive proof par excellence since Euclid's *Elements*, was finally analyzed in full mathematical precision in Hilbert's *Grundlagen der Geometrie* (Hilbert, 1909; see also Hilbert, 1950), when all its axioms, and possible variations on them, had become clear. Yet, despite its starkly abstract view of points, lines and planes, the *Grundlagen* is still couched not in a formal language, but rather in (lightly mathematicized) idiomatic German. Hilbert's Axiom of Parallels provides a good example:

> Let a be a line, and A a point not on a. Then, in the plane determined by a and A, there is at most one line which passes through A and does not meet a. (tr. from Hilbert, 1909, p. 20)

No attempt is made to tease out the implicit logical syntax of this language, or to analyze the underlying inference engine much beyond what Euclid had already done in his Common Notions. This is perhaps clearest in the case of Hilbert's final Axiom of Completeness:

> The elements (points, lines, planes) of the geometry form a system of objects which is not capable of any extension, subject to maintenance of all the preceding axioms. That is to say: it is not possible to add to the system of points, lines and planes another system of objects in such a way that, in the combined system, all [previous] axioms are satisfied. (*Ibid.*, p. 22.)

It was not until after the development of the apparatus of formal logic and model-theoretic semantics in the first half of the Twentieth Century that logicians were able to probe the precise inferential and expressive resources of geometry, in a second round of formalization culminating in Tarski's *Elementary Geometry* (Tarski, 1959).

Tarski's decisive contribution in his 1959 paper was not simply to force Hilbert's axioms into the regimented syntax of some formal language, but rather, to investigate what happens when that syntax is restricted. Specifically, Tarski employs a first-order logic, with variables ranging over points in the Euclidean plane, and with non-logical predicates standing for two primitive spatial relations: a ternary relation of 'betweenness' and a quaternary relation of 'equidistance'. The resulting language is sufficiently expressive to formulate much of Euclidean geometry—for example, Pythagoras' theorem, or the existence of the nine-point Feuerbach circle. The computational reward for this loss of expressive power is considerable. Tarski showed that the theory of elementary geometry is *decidable*: there is a mechanical procedure to determine, of any given sentence in the relevant language, whether that sentence is true under the advertised interpretation. By contrast, the second-order theory needed to express all of Hilbert's axioms is undecidable.

Tarski's discovery illustrates the most distinctive feature of logic in the wake of the model-theoretic revolution of the previous century: its fundamentally linguistic orientation. The model-theoretic approach to logic takes as its central concern the often intricate relationship between mathematical structures and languages which describe them. On this view, spatial logic, as defined above, becomes the study of the relationship between geometrical structures and the spatial languages which describe them. It is this preoccupation with language which divides spatial logic from geometry as traditionally conceived. More recently, of course, the enterprise of automating logical deduction using electronic computers has necessitated new levels of precision and sophistication in reasoning about the properties of formal languages and their relationship to their subject matter. In this setting, the issue of balancing the expressive power of a language against the computational complexity of performing deductions within it occupies centre-stage.

We can broaden our perspective by considering two further examples of spatial logics in addition to Tarski's Elementary Geometry. To motivate our second example, recall that, in Elementary Geometry, all variables are taken to range over *points* in the Euclidean plane. This allows for quantification over geometrical figures defined by a fixed number of points, such as line segments, triangles, circles , and so on, but not over spatial constellations defined by point-sets of arbitrary finite size, such as polygons, let alone those defined by infinite sets of points, such as, for example, arbitrary connected regions. The question therefore arises as to what happens when these restrictions are lifted. In fact, Tarski himself had already investigated such a language in his *Geometry of Solids* (Tarski, 1956). This system employs the syntax of *second*-order logic, with the object variables ranging not over points, but instead over certain 'regions' in three-dimensional Euclidean space (hence, the set-variables range over sets of regions). The regions in question—Tarski called them *solids*—are the regular closed subsets of $\mathbb{R}^3$, namely, those subsets of $\mathbb{R}^3$ equal to the closure of their interior. Tarski's language features two non-logical predicates: one standing for the the binary relation of *parthood*, the other for the unary property of *being spherical*. Again, Tarski establishes a remarkable fact about the relationship between the formal language and the structure it is interpreted over: the resulting theory can be axiomatized completely (in a second-order sense), and moreover is *categorical*: all models of this theory are isomorphic to the standard interpretation on the reals. This sort of axiomatization in very powerful logical languages has found many successors, e.g., in qualitative axiomatizations of physics.

For our third example of a spatial logic, we turn to topology. While Euclidean geometry is associated with rigid transformations like translations, rotations, and inversions, the mathematicians creating topology in the early decades of the 20th Century focused on much coarser transformations deforming shapes up to tearing and knotting. Subsequent to its invention, topology, too, became an object of logical study, and yet again, Tarski's work proved seminal. Tarski observed that topology has small decidable fragments which could be brought to light by treating the topological interior operation as a *modal operator* (McKinsey and Tarski, 1944). The connection to the other spatial logics discussed above becomes apparent if we subject McKinsey and Tarski's original modal language to some essentially cosmetic reformulation. The variables of this language are taken to range over arbitrary subsets of any fixed topological space. These variables may be combined to form complex terms by means of function-symbols denoting various set-theoretic operations (union, intersection and complement), and topological operations (interior and closure); such terms denote subsets of the topological space over which they are interpreted. With terms constructed in this way, the language then features equality as its only predicate. Here we have extreme poverty of expressive resources: primitive function-symbols expressing only set-theoretic and topological operations, no non-logical predicates, and no quantifiers. But there is again a computational reward: the satisfiability problem for this logic is decidable in polynomial space. While too inexpressive to represent much of topology, this language has had profound repercussions in other areas, in particular in the universal algebra of Boolean algebras with added operators, and much contemporary modal logic. Furthermore, it has also been the inspiration for much recent work on topological logics, many of them equipped with more elaborate syntax and richer topological primitives, as the reader of this book will soon discover.

With these examples to guide us, let us return to the abstract characterization of spatial logic with which we began. Spatial logics arise by making a number of design choices, along three principal dimensions. The first concerns the collection of geometrical entities which make up our interpretations: points, lines, regions (of various kinds), and so on. In Tarski's (plane) Elementary Geometry, variables range over the collection of points in the Euclidean plane; likewise, in his Geometry of Solids, variables range over the collection of regular closed subsets of $\mathbb{R}^3$; and in his modal topological language, variables range over the collection of all subsets of some topological space. The second principal dimension concerns the choice of primitive relations and operations over these entities to interpret the non-logical primitives of our language. This choice of primitives of course reflects the level of spatial structure the particular logic is concerned with—metric, affine, projective, or topological; but even within these broad divisions, there is room for almost endless variation. The third principal dimension concerns the purely logical resources at our disposal. We have already seen that these can be set at many levels: from weak 'constraint' languages through to richer first-order languages or even higher-order formalisms which include the resources of set theory. Needless to say, none of the choices along these principal dimensions is intrinsically right or wrong: they simply parametrize the family of available spatial logics.

Classification of geometrical languages in terms of the range of the spatial primitives they feature of course recalls the long-standing classification of 'geometries', broadly conceived, given by Klein's *Erlanger Programm* (Klein, 1893b; see also Klein, 1893a). And indeed the most sophisticated accounts of expressive power of such languages today are couched in terms of invariance relations between models (isomorphism, bisimulation, and the like), much in the same spirit. However, the logical approach opens up many new possibilities in this regard, such as, for instance, a new sort of invariance between topological patterns, much coarser even than topological homeomorphism, viz. modal bisimulation. This is topology taken to the extreme, but there are interesting interpretations in terms of model comparison games—a style of thinking which might have appealed to the founders of geometry, given Brouwer's early use of games in defining the notion of topological dimension (Brouwer, 1913, p. 148).

Once we have fixed a spatial logic, four salient issues present themselves. First, how can we characterize its valid formulas? Second, what is its expressive power? Third, what is its computational complexity? And fourth, what alternative interpretations does it have? We briefly consider each of these in turn. The first issue is so familiar as to require little explanation. Given a formal language interpreted over a certain class of geometrical structures, it typically makes sense to ask (depending on details of syntax) which sentences of that language are true in all structures of that class. Mostly, these characterizations are couched in the form of a list of axioms and (finitary) rules-of-inference. However, there are cases where additional machinery is required, for example, when the set of validities is not recursively enumerable, or where explicit proof systems are required to provide geometrical 'constructions' in Euclid's sense.

Second, we have already noted that current treatments of expressive power in logic are derived from the geometrical notions of invariance relations across models, setting the level of semantic resolution beyond which the given language cannot probe. Examples of such invariance relations are potential isomorphism for first-order logic, or bisimulation for modal languages; but there are many more. Within given models, such relations specialize to notions of automorphism or internal bisimulation—a viewpoint which is actually somewhat closer to the mathematician's usual way of thinking about 'symmetries' of a spatial structure. Weyl at one point observed that point tuples in Euclidean space which are related by an automorphism must satisfy the same geometrical formulas, and raised the converse question of whether sharing the same properties in some given logical language implies automorphism invariance (Weyl, 1949, p. 73). Indeed, invariance is not just descriptive weakness, but also the source of information flow across situations! Logical model theory has a host of sophisticated results concerning invariance. In particular, invariance relations can be fine-tuned in terms of games, such as Ehrenfeucht-Fraïssé games matching first-order logic. Given a notion of invariance, the model theory of definability can start, and indeed, many results about expressive power of spatial languages can be found in the chapters to follow.

Third, complexity-theoretic analyses of logical systems typically focus on two problems: model-checking (determining whether a given formula is true in a *given* interpretation) and satisfiability checking (determining whether a given formula is true in *some* interpretation *or other*). Model-checking has been little-explored in the context of spatial logics; satisfiability checking, by contrast, has received much more attention. Most first- (or higher-) order spatial logics interpreted over familiar spatial domains are undecidable; therefore, this issue is obviously of greatest interest when dealing with spatial logics with more limited expressive power. A striking example is provided by spatial logics interpreted over the regular closed sets of arbitrary topological spaces whose language involves just Boolean connectives (no quantifiers) and whose spatial primitives represent various topological relations and functions. The satisfiability problem for such logics is generally decidable, and its complexity has been determined for a range of cases. In this light, spatial logics actually do pose an interesting challenge which is not yet well-understood. The general methodology in logic design has been to find expressive yet decidable formalisms, cleverly steering a middle course between the opposing evils of expressive poverty and undecidability. However, methods of analysis which work with general models are often powerless when confronted with languages interpreted over specific structures, as is generally the case with spatial logics. Sometimes, the spatial models over which one is working themselves support decidability for rich languages—witness again Elementary Geometry, where it is the structure of Euclidean space that drives the quantifier elimination procedure establishing decidability. We are still far from understanding the precise balance between all these triggers of higher or lower complexity in spatial logics.

Fourth, and most speculatively, we have the issue of alternative interpretations. Tarski's Geometry of Solids possesses, as we have seen, just one model up to isomorphism, but most spatial logics have many models. To some extent, this is just the expression of a familiar phenomenon in logic, and mathematics generally. Some theories, such as group theory or the the theory of affine spaces, are designed to have many models, and the more of these there are, the greater their range of applicability. Other theories were intended to describe one particular structure, such as the natural numbers, Euclidean space, or most imperialistically of all, the set-theoretic universe. Geometry provided early examples of how theories originally conceived as characterizations of specific structures could turn out to have alternative models. This issue is brought to the fore in the subject of spatial logic, where the formal systems under investigation expressly invite the search for alternative interpretations and thus alternative ways of conceptualizing space. Even bolder views were ventured by Beth in the 1950s, who claimed that it was geometry's move from one unique Space to a plurality of 'spaces' that underlay the system-based methodology of modern science and the fall of Aristotelian *a priori* dogmatism (Beth, 1959, Sec. 21). Be that as it may, the present authors agree that spatial logic can have philosophical repercussions beyond its narrower technical confines.

More prosaically, much of the renewed interest in spatial logic in recent years has come from computer science. We identify three examples of this trend. The first comes from artificial intelligence, where attempts have recently been made to develop logics of *qualitative spatial reasoning.* The motivation is as follows: numerical co-ordinate descriptions of the objects which surround us are hard to acquire, inherently error-prone, and probably unnecessary for most everyday tasks we want to perform (or want a machine to perform); therefore—so goes the argument—reasoning with purely qualitative descriptions of those objects' spatial configurations is closer to human reasoning and thus will lead to more efficient and effective AI. But *which* qualitative spatial terms, exactly, should we reason with? Ready-made tools from geometry or topology will not do: we have to devise new logics for ourselves. Many of these logics are discussed in this book.

The second example comes from the theory of spatial databases. In computer applications, spatial data is frequently stored in the form of polygons (or polyhedra)—in effect, sets of points definable by Boolean combinations of linear inequalities. These sets can be finitely represented, and their well-behaved character makes them particularly amenable to computer processing. But in fact there is no need to set our expressive sights so low; for polygons and polyhedra are a special case of the more general class of *semi-algebraic* sets, that is, those sets of points definable by Boolean combinations of polynomial inequalities. Within mathematics, semi-algebraic sets form the basis for real algebraic geometry; within computer science, they have given rise to the discipline of *constraint databases*. In a constraint database, spatial data is stored in the form of first-order formulas in the language of fields. The key fact here is the quantifier-elimination theorem for the theory of the reals. This result allows constraint databases to be accessed effectively using queries which are likewise written as first-order formulas over an appropriate vocabulary. The relevant chapter in this book explores some of the intricate logical issues that arise from this approach to spatial data.

Our third example comes from image processing, where it is convenient to describe objects as sets of vectors that can be 'added' (taking all linear sums) or 'subtracted' (taking all linear differences). By variously combining these 'Minkowski operations', certain useful processing tasks can be performed, as, for example, when one set of vectors, representing an 'eraser', is used to 'clean up' the boundary of another, representing a perceived object. *Mathematical morphology* is a theory of subsets of vector spaces with the two operations of addition and subtraction at its core; the properties of these operations are generalized in abstract algebraic and category-theoretic ways. Looking at space in this way brings to light a surprising amount of new structure. This theory was not developed within mathematical logic; but the relevant chapter in this book will show how logical patterns do arise, involving both modal and first-order languages, while the calculus of valid principles shows surprising analogies with logical systems proposed in recent decades for very different purposes, such as linear logics of computational resource management. Again, we see how new choices of spatial objects and spatial structures lead to new mathematics—and there is no reason to think that this creative process has yet run dry.

Finally, let us remove a possible misunderstanding, again taking a cue from the history of geometry. Our presentation may have made it look as if there is a vast collection of different spatial logics, each a world unto itself in terms of objects, primitive relations, and logical strength. But one of the most striking discoveries in the foundations of geometry in the 19th Century, prominently displayed in Hilbert's *Grundlagen*, was the fact that very different-looking theories can turn out to be related at a deeper level of analysis. Inspiring examples are the embeddings of non-Euclidean logics into Euclidean ones given by Klein and Poincaré. Likewise, spatial logics show inter-connections which may be brought out by various means: semantic model transformations, direct linguistic translations, and so on. Even though little is known about the precise links between most known systems, we emphasize this point as a reassuring thought about the coherence of the field.

This concludes the editors' thoughts about the general setting for this book, while providing a way of positioning specific chapters. But of course, the real content is in the chapters themselves, which do much more than fit editorial preconceptions. Each tells a story about a particular approach to spatial logic. The chapters have been arranged in the following thread, though they can be read in other orders as well.

We start in Tarski's geometrical spirit, with first-order languages. In Chapter **??**, Pratt-Hartmann considers first-order topological languages interpreted over low-dimensional Euclidean spaces, applying techniques from logical model theory to analyze expressive power and axiomatizability. In Chapter **??**, Bennett and Düntsch study both first-order and weaker modal topological languages over a large class of topological spaces, emphasizing basic decidable structures of wide use in AI and beyond. Renz and Nebel take this even further in Chapter **??**, with syntactically highly restricted constraint languages for spatial structures, allowing for great computational efficiency.

From fragments of first-order languages, there is a natural transition to modal logics for topology, continuing the tradition started by Tarski and others in the 1930s. Chapter **??** by Bezhanishvili and van Benthem tells the story of modern modal approaches to topology (and a few other spatial structures), emphasizing the main axiomatic and semantic techniques developed in modern modal logic. This theme is then continued in Chapter **??** by Moss, Parikh and Steinsvold, who explore the other logical tradition of thinking about topology, viz. as an account of information structure. Next, Chapter **??** by Balbiani, Goranko, Kellerman and Vakarelov takes the modal viewpoint to the study of affine and metric geometry, moving up to first-order languages where needed. In particular, completeness theorems turn out to be related to the basic geometrical issue of coordinatization. Finally, Chapter **??** by Vickers takes the epistemic view of topology to the higher mathematical level of topos theory, merging spatial logic and epistemic logic with category theory and type theory.

Just as in science generally, so too in spatial logic, space enters into natural combinations with other fundamental notions. One obvious case is the combination of space and time, which is unavoidable in many practical computational settings, and of course, also, in the foundations of physics. Chapter **??** by Kontchakov, Kurucz, Wolter and Zakharyaschev studies temporal logics with added affine and metric modalities, using sophisticated techniques from current research on the complexity of combined modal logics. A special case of this type of combination is found in Chapter **??** by Kremer and Mints, who add a dynamic temporal operator of one-step system evolution to modal logics of topology, and show that this simple move provides significant results like the Poincaré recurrence theorem. Finally, Chapter **??** by Andréka, Madarász and Németi goes far beyond simple modal languages of space-time, and develops both the special and the general theory of relativity on a first-order basis, continuing Tarski's program for geometry to obtain striking new foundational results which are at the same time conceptually enlightening.

The next group of chapters represent a counterpoint to the 'logical' investigations so far, reporting further mathematical and computational advances. Chapter **??** by Smyth and Webster explores the extent to which topological ideas can be developed in discrete spaces, moving closer to the discrete topologies used in modern mathematics, pattern recognition, and image processing. Chapter **??** by Geerts and Kuipers describes the use of algebraic constraints for spatial databases to describe regions in Euclidean space, reminding us of the great tradition of analytic geometry which also underlies the coordinatizations employed by Tarski, and by several authors in our book. Chapter **??** by Bloch, Heijmans and Ronse develops the theory of mathematical morphology, both on concrete vector spaces and in algebraic abstraction, and introducing, at the end, logical formalisms based on them.

Beyond these technical subjects, our book still has a coda. We have indicated already that spatial logic also has a broader conceptual aspect. Chapter **??** by Varzi is an extensive discussion of spatial structure in the philosophical tradition, both ancient and modern, using logical tools to develop philosophical conceptions.

Despite the wealth of topics in our fifteen chapters, this book also set itself definite limits. First, we have not even exhausted the mathematical connection, witness the long-standing historical interest in 'diagrammatic reasoning' spawned by Euclid's Elements, and reinforced by modern research on graphical representation of information and associated styles of inference. There are deep issues here about the connection between symbolic and visual paradigms, bypassed in our cheerfully technical account of 'spatial logics'. We acknowledge them; but they are beyond the scope of this book. Likewise, many further varieties of spatial representation and spatial reasoning occur in disciplines like linguistics and psychology, and many more patterns await formal logical study. In addition, cognitive neuro-science tells us about the often surprising interplay between visual, diagrammatic, and more symbolically oriented parts of the brain in any reasoning task. Again, we think this is a fascinating theme, and we trust that many interesting interactions with the spatial logics of this book will one day come to light. But we have chosen the current set of chapters for their coherence in topic and methodology, and frankly also, their mathematical quality. We see the broader area of spatial reasoning; we recognize its relevance to the contents of this book; and exclusion does not imply disrespect. Broader texts on spatial reasoning should, and no doubt will, appear. But, in putting together this tighter book, the editors have stuck to what they see as the basic axiom of 'social geometry': *Always leave room for others.*

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