Elementary Polyhedral Mereotopology

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Abstract. A *region-based* model of physical space is one in which the primitive spatial entities are regions, rather than points, and in which the primitive spatial relations take regions, rather than points, as their relata. Historically, the most intensively investigated region-based models are those whose primitive relations are topological in character; and the study of the topology of physical space from a region-based perspective has come to be called *mereotopology*. This paper concentrates on a mereotopological formalism originally introduced by Whitehead, which employs a single primitive binary relation $C(x, y)$ (read: "x is in contact with y"). Thus, in this formalism, all topological facts supervene on facts about contact. Because of its potential application to theories of qualitative spatial reasoning, Whitehead's primitive has recently been the subject of scrutiny from within the Artificial Intelligence community. Various results regarding the mereotopology of the Euclidean *plane* have been obtained, settling such issues as expressive power, axiomatization and the existence of alternative models. The contribution of the present paper is to extend some of these results to the mereotopology of three-dimensional Euclidean space. Specifically, we show that, in a first-order setting where variables range over tame subsets of $\mathbb{R}^3$, Whitehead's primitive is maximally expressive for topological relations; and we deduce a corollary constraining the possible region-based models of the space we inhabit.

Keywords: mereotopology, spatial reasoning, ontology of space

1. Introduction

A *region-based* model of physical space is one in which the primitive spatial entities are regions, rather than points, and in which the primitive spatial relations take regions, rather than points, as their relata. Historically, the most intensively investigated region-based models are those whose primitive relations are topological in character. Since the fundamental relation involving regions of space is the part-whole relation, and since the logic of the part-whole relation is known as mereology, the study of the topology of physical space from a region-based perspective has come to be called *mereotopology*.

Work on mereotopology originates with de Laguna [5] and Whitehead [20], and was given new impetus by Clarke [3] and [4]. (See also Simons [18] sec. 2.10). The most basic part of Whitehead's mereotopology employs a single primitive binary relation $C(x, y)$, which may be read "x is in contact with y"; and this primitive has formed the basis for many subsequent approaches. (Galton [7], Ch. 2 contains a useful survey.) Thus, on these approaches, all topological facts supervene on facts about contact. The purpose of this paper is to answer certain technical questions concerning the logical properties of any viable mereotopology based on Whitehead's primitive. Previous research in this area has yielded several results on *plane* mereotopology, settling such issues as alternative models (Pratt and Lemon [14]), expressive power (Papadimitriou, Suciu and Vianu [13]) and axiomatization (Pratt and Schoop [15]). The contribution of the present paper is to extend these investigations to the mereotopology of three-dimensional space.

The development of any mereotopology requires some assurance of conformity to the physical space we inhabit; and the most urgent methodological question concerns the source of this assurance. We proceed as follows. Denote by $\mathcal{L}$ the first-order language whose only non-logical primitive is the binary predicate C. Beginning with the familiar point-based Euclidean model of space as $\mathbb{R}^3$, we first select a collection of subsets of $\mathbb{R}^3$ to qualify as the domain of regions over which we quantify. Specifically, we confine our attention to a relatively restricted collection of regions: the set R of regular open polyhedra (defined below). Next, we interpret the non-logical primitive of $\mathcal{L}$ over this domain so as to reflect its intended meaning. Specifically, we stipulate that regular open polyhedra r and s are related by C just in case the closures of r and s have a point in common. Thus we may regard R as a structure (in the sense of model-theory) interpreting $\mathcal{L}$. This structure determines a truth-value for every sentence (closed formula) of $\mathcal{L}$, and the theory consisting of the set of true $\mathcal{L}$-sentences in R—standardly denoted $\mathrm{Th}(R)$—may reasonably be regarded as the mereotopology of the space we inhabit.

Two salient questions now arise regarding the language $\mathcal{L}$ and its polyhedral interpretation R. The first question is: how much expressive power does $\mathcal{L}$ give us? That is: can we express the kind of topological distinctions over R that will make it a representation language worth using? The second question is: what alternative region-based models of space make $\mathrm{Th}(R)$ true, and how they are related to R? Note that general models of this theory are simply structures over one (symmetric) binary relation—in effect, (undirected) graphs. Their elements are primitives and, unlike the regular open polyhedra in $\mathbb{R}^3$, need not owe their existence to sets of points or indeed to anything else. In this sense, such models are unimpeachably *region-based* models of space; at the same time, the fact that they make true the same $\mathcal{L}$-sentences as R gives us the right to call these structures models of physical space.

Thus, the first question concerns the usefulness of $\mathcal{L}$ as a spatial representation language, while the second—more philosophical in character—concerns the quest for alternative spatial ontologies. The present paper answers both questions, and explains the intimate relationship between them. Our answers are summarized in section 2, and proved in the following three sections. These three sections are technical in character, and readers uninterested in such details may skip them altogether. Section 6 concludes with a discussion of the significance of our findings, and lists some open problems. Throughout the paper, technical vocabulary has been kept to relatively familiar notions of undergraduate-level topology and model theory.

One final terminological remark. Whitehead refers to the relation denoted by C as *connection*, risking confusion with the mathematically well-established, and quite different, property of *connectedness.* We have resolved this terminological clash by substituting the word *contact* and its cognates for Whitehead's relation, and using the term *connected* in its usual topological sense. Nothing substantive should be read into this decision.

2. Summary of results

Let $\mathcal{L}$ be the first-order language over the signature consisting of the single binary relation C. In the sequel, we refer to $\mathcal{L}$-formulas simply as *formulas.* Our first task is to define our chosen domain of interpretation for $\mathcal{L}$.

DEFINITION 2.1. *A subset a of a topological space X is said to be* regular open *if it is equal to the interior of its closure.*

NOTATION 2.2. *If a is any subset of a topological space X, we denote the interior of a in X by a° and the closure of a in X by a^-, reserving the more usual notation $\bar{a}$ to indicate tuples. Thus, regular open sets satisfy the equation $a = (a^-)^\circ$. In addition, we write $\mathcal{F}(a)$ for the frontier of a—that is, the set $a^- \setminus a^\circ$.*

The following result is well-known (Koppelberg [10], p. 26):

LEMMA 2.3. *If X is a topological space, then the set $\mathrm{RO}(X)$ of regular open sets in X forms a Boolean algebra with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $a \cdot b = a \cap b$, $a + b = ((a \cup b)^-)^\circ$ and $-a = X \setminus a^-$.*

The Boolean algebra order $\leq$ on $\mathrm{RO}(X)$ is of course just the subset relation. For clarity, we write $a \leq b$ rather than $a \subseteq b$ if a and b are regular open sets.

As usual, we take a *plane* in $\mathbb{R}^3$ to be the set of points satisfying a non-degenerate linear equation. The two residual domains of a plane in $\mathbb{R}^3$ are clearly regular open sets, which we call *half-spaces.* We then define:

DEFINITION 2.4. *A* polyhedron *is a Boolean combination in* $\mathrm{RO}(\mathbb{R}^3)$ *of finitely many half-spaces. We denote the set of polyhedra by* R.

Thus, the polyhedra are, in effect, the regular open semi-linear subsets of $\mathbb{R}^3$. Polyhedra, on this definition, can be disconnected, and also unbounded. Furthermore, the empty set and the whole of $\mathbb{R}^3$ qualify as polyhedra.

We can regard R as an $\mathcal{L}$-structure by taking the extension of our binary relation C to be

$$C^R = \{\langle a, b\rangle \in R^2 |\ a^- \cap b^- \neq \emptyset\},$$

which conforms to the notion of *contact* which Whitehead denoted by the same symbol. So understood, any $\mathcal{L}$-sentence has a truth-value in R; and more generally, any n-place formula $\phi(\bar{x})$ defines an n-ary relation over R—namely, the set of all and only those n-tuples satisfying it.

REMARK 2.5. *Since* C^R *is a symmetric relation,* $R \models \forall x \forall y (C(x, y) \to C(y, x))$*; since* C^R *is not transitive,* $R \models \neg\forall x \forall y \forall z (C(x, y) \wedge C(y, z) \to C(x, z))$*; since* $\emptyset$ *is a polyhedron and every other polyhedron contacts itself,* $R \models \exists! x \forall y \neg C(x, y)$.

Section 4 contains many examples of relations defined by formulas over R. Since R is (almost) the only structure discussed in this paper, when a tuple $\bar{a}$ of polyhedra satisfies a formula $\phi(\bar{x})$ in R, we simply say that $\bar{a}$ *satisfies* $\phi(\bar{x})$, leaving the reference to R implicit.

Our first main result concerns the expressive power of $\mathcal{L}$. The following definition will prove useful.

DEFINITION 2.6. *If* X *is a topological space and* a, b *are subsets of* X*, we say that* a *and* b *are* similarly situated *in* X*—written* $a \sim b$*—if there is a homeomorphism from* X *onto itself mapping* a *to* b*. This notion extends to tuples (and tuples of tuples) in the obvious way.*

Similarly situated (tuples of) sets may be regarded as topologically indistinguishable. Notice that similar situation is a much stronger relation than merely being homeomorphic. For instance, the open cube, given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| < 1,\ |x_2| < 1,\ |x_3| < 1\}$ is homeomorphic to the whole of $\mathbb{R}^3$, but the two sets are certainly not similarly situated in $\mathbb{R}^3$.

It follows from results established below that two tuples of polyhedra which are similarly situated in $\mathbb{R}^3$ satisfy exactly the same formulas. Conversely, we might wonder if non-similarly situated tuples of polyhedra satisfy different formulas. Accordingly, we define:

DEFINITION 2.7. *A formula is* topologically complete *if all tuples of polyhedra satisfying it are similarly situated in the space* $\mathbb{R}^3$.

THEOREM 2.8. *Every tuple of polyhedra satisfies some topologically complete formula.*

Thus, the language $\mathcal{L}$ is maximally expressive among topological languages interpreted over R, in that every finite configuration of polyhedra can be completely topologically characterized by some formula.

Our second main result concerns alternative models of the theory of R. Let us say that a *rational plane* is the set of points satisfying a non-degenerate linear equation with rational coefficients, and let us call the two residual domains of a rational plane in $\mathbb{R}^3$ *rational half-spaces.* We then define:

DEFINITION 2.9. *A* rational polyhedron *is a Boolean combination in* $\mathrm{RO}(\mathbb{R}^3)$ *of finitely many rational half-spaces. We denote the set of rational polyhedra by* Q.

Again, we can regard Q as a structure interpreting $\mathcal{L}$, by setting:

$$C^Q = \{\langle a, b\rangle \in Q^2 |\; a^- \cap b^- \neq \emptyset\},$$

exactly as for R. Thus, Q is by definition a substructure of R. It follows from results established below that Q is in fact an *elementary* submodel of R and hence that $\mathrm{Th}(Q) = \mathrm{Th}(R)$. In fact, Q has a special status among models of its theory. Recall that, in model theory, a structure $\mathfrak{A}$ is said to be *prime* if, for any structure $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$. Prime models of theories are unique up to isomorphism, and are considered the simplest models of their theories. The second result proved below is:

THEOREM 2.10. *The structure Q is prime.*

Thus, any alternative, region-based model of space must either falsify some $\mathcal{L}$-sentence true in R, or else must contain a copy of Q as an elementary submodel.

3. The domain of polyhedra

The aim of this (long) section is to prove some technical results about the set R of polyhedra. In particular, we define the notion of an A-cell partition of $\mathbb{R}^3$—in effect, a kind of triangulation of space by polyhedra—and we show in Lemma 3.38 that A-cell partitions may be characterized up to similar situation in combinatorial terms. Readers may wish to read only the material marked as 'definition' or 'notation' as well as Lemma 3.38 itself. All other material is ancillary.

3.1. Closed space

For technical reasons, it will be useful to work not in the topological space $\mathbb{R}^3$, but rather, in its one-point (Alexandroff) compactification $\hat{\mathbb{R}}^3$ (sometimes written S^3). Formally, $\hat{\mathbb{R}}^3$ is the set $\mathbb{R}^3 \cup \{\infty\}$, where $\infty \notin \mathbb{R}^3$; and the open sets of $\hat{\mathbb{R}}^3$ are simply the collection:

$$\{X | X \subseteq \mathbb{R}^3 \text{ is open}\} \cup \{(\mathbb{R}^3 \setminus Y) \cup \{\infty\} | Y \subseteq \mathbb{R}^3 \text{ is compact}\}.$$

The point ∞ is called the *point at infinity*. We take planes in $\hat{\mathbb{R}}^3$ to be the sets $\pi \cup \{\infty\}$ where π is any plane in $\mathbb{R}^3$. It is easy to see that planes in $\hat{\mathbb{R}}^3$ again have two half-spaces as their residual domains, which are elements of $\text{RO}(\hat{\mathbb{R}}^3)$. Accordingly, we define

DEFINITION 3.1. *A* closed-space polyhedron *is a Boolean combination in* $\text{RO}(\hat{\mathbb{R}}^3)$ *of finitely many half-spaces. We denote the set of closed-space polyhedra by* $\hat{R}$.

Of course, $\hat{\mathbb{R}}^3$ is not directly visualizable. However, its two-dimensional analogue, the one-point compactification of $\mathbb{R}^2$ (denoted $\hat{\mathbb{R}}^2$) is: $\hat{\mathbb{R}}^2$ can be homeomorphically projected onto the surface of a sphere. This analogy may help to make some of the ensuing lemmas more intuitive.

In the sequel, $\tilde{\mathbb{R}}^3$ stands ambiguously for either $\mathbb{R}^3$ or $\hat{\mathbb{R}}^3$; similarly, $\tilde{R}$ stands ambiguously for either R or $\hat{R}$. Disambiguation is assumed to be uniform in a given context.

NOTATION 3.2. *If* $a \in \text{RO}(\mathbb{R}^3)$, *define*

$$\hat{a} = \begin{cases} a \cup \{\infty\} & \textit{if } -a \textit{ is bounded} \\ a & \textit{otherwise.} \end{cases}$$

The proof of the following lemma is routine, though tedious.

LEMMA 3.3. *The function* $a \mapsto \hat{a}$ *is a Boolean algebra isomorphism from* $\mathrm{RO}(\mathbb{R}^3)$ *to* $\mathrm{RO}(\hat{\mathbb{R}}^3)$*; furthermore, its restriction to* R *is a Boolean algebra isomorphism from* R *to* $\hat{R}$.

It is obvious that $a \in \mathrm{RO}(\mathbb{R}^3)$ is connected in $\mathbb{R}^3$ if and only if $\hat{a}$ is connected in $\hat{\mathbb{R}}^3$.

NOTATION 3.4. *For any point* $p \in \mathbb{R}^3$*, and any* $r > 0$*, we take* $B(p, r)$ *to denote the* ball $\{q \in \mathbb{R}^3 | d(q, p) < r\}$. *In addition, we take* O *to denote the point* $(0, 0, 0)$.

DEFINITION 3.5. *In the topological space* $\tilde{\mathbb{R}}^3$*, a* disc *is a subset of* $\tilde{\mathbb{R}}^3$ *similarly situated to* $\{(x_1, x_2, 0) \in \mathbb{R}^3 | x_1^2 + x_2^2 < 1\}$*; and a* ball *is a subset of* $\tilde{\mathbb{R}}^3$ *similarly situated to* $B(O, 1)$. *A* closed disc *is the closure of a disc; a* closed ball *is the closure of a ball; and a* sphere *is the frontier of a ball.*

Of course, the above definitions are relative to the choice of space $\tilde{\mathbb{R}}^3 = \mathbb{R}^3$ or $\tilde{\mathbb{R}}^3 = \hat{\mathbb{R}}^3$. However, when using these terms in the sequel, we may suppress mention of the topological space when this is clear from the context. As usual, a *Jordan arc* in a topological space X is a homeomorphism from the interval $[0, 1]$ to a subset of X; and a *Jordan curve* in X is a homeomorphism from the unit circle S^1 to a subset of X.

3.2. Balls in closed space

The main results of section 3.2 are Theorems 3.11 and 3.14.

To motivate the following lemma, we present a two-dimensional analogue. Let $a = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 < 1\}$. Thus, $a \in \mathrm{RO}(\hat{\mathbb{R}}^2)$; in fact, $-a = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 > 1\} \cup \{\infty\}$. By considering the usual stereographic projection of $\hat{\mathbb{R}}^2$ onto the sphere, it is immediate that that a is similarly situated to $-a$ in the space $\hat{\mathbb{R}}^2$. Furthermore, this observation generalizes to higher dimensions. In particular we have:

LEMMA 3.6. *Let* $a \in \mathrm{RO}(\hat{\mathbb{R}}^3)$ *be a ball in* $\hat{\mathbb{R}}^3$. *Then so is* $-a$.

Proof. Since for any homeomorphism $\alpha : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ and any $a \in \mathrm{RO}(\hat{\mathbb{R}}^3)$, we have $\alpha(-a) = -\alpha(a)$, it suffices to prove that $-B(O, 1)$ is a ball. But the inversion $\beta : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ defined, using spherical-polar coordinates, by

$$\begin{aligned} \beta((r, \theta, \phi)) &= (1/r, \theta, \phi) \text{ if } r > 0 \\ \beta(O) &= \infty \\ \beta(\infty) &= O \end{aligned}$$

is visibly a homeomorphism taking $B(O,1)$ to $-B(O,1)$, whence $-B(O,1)$ is a ball.

Elements of $\hat{R}$ which are balls in $\hat{\mathbb{R}}^3$ admit of a particularly simple characterization, which we now proceed to develop. First, some general preliminary results.

LEMMA 3.7. *Let a be a regular open set in some topological space. Then* $\mathcal{F}(a) = \mathcal{F}(-a)$.

Proof. Trivial.

DEFINITION 3.8. *If* $a \in \mathrm{RO}(\tilde{\mathbb{R}}^3)$*, we say that a is* cc *if a is connected and nonempty with connected nonempty complement.*

We remark that no conflict can arise in the above definition for elements which are in both $\mathrm{RO}(\mathbb{R}^3)$ and $\mathrm{RO}(\hat{\mathbb{R}}^3)$. The following result is less trivial than it looks.

LEMMA 3.9. *Let* $a \in \mathrm{RO}(\tilde{\mathbb{R}}^3)$ *be cc. Then* $\mathcal{F}(a)$ *is connected.*

Proof. Newman [12], p. 137 derives the following corollary of Alexander's lemma for $\tilde{\mathbb{R}}^n$ $(n \geq 2)$: if F_1 and F_2 are non-intersecting closed sets in $\tilde{\mathbb{R}}^n$, and points p and q are connected in the complement of F_1 and also in the complement of F_2, then they are connected in the complement of $F_1 \cup F_2$. Suppose now that $a \in \mathrm{RO}(\tilde{\mathbb{R}}^3)$ is cc but that the closed set $\mathcal{F}(a)$ is not connected. Let F_1 and F_2 be closed sets partitioning $\mathcal{F}(a)$, and let $p \in a$, $q \in -a$. Thus, p and q are not connected in $\tilde{\mathbb{R}}^3 \setminus \mathcal{F}(a)$. Since a is cc, it is easy to see that the conditions of Newman's corollary are fulfilled (with $n = 3$, of course), so that p and q are connected in $\tilde{\mathbb{R}}^3 \setminus (F_1 \cup F_2)$. But this is absurd given that $F_1 \cup F_2 = \mathcal{F}(a)$.

Note that Lemma 3.9 fails if $\tilde{\mathbb{R}}^3$ is replaced by some arbitrary topological space. The surface of a torus is a counterexample: it is easy to find two complementary cc-elements in this space whose common frontier is not connected.

As usual, we take K_5 to denote the pentagram, that is, the complete graph with five vertices. It turns out that K_5 provides a simple means to identify spheres using the language $\mathcal{L}$ (though it is certainly not the only method of doing so). Recall that a *2-manifold* is a Hausdorff space locally homeomorphic at every point to a disc, and that a *surface* is a connected 2-manifold. It is easy to see that the frontier of a cc element of $\hat{R}$ need not be a surface. However, we have the following result.

LEMMA 3.10. *Let* $a \in \hat{R}$ *such that a is cc and* $\mathcal{F}(a)$ *is not a surface. Then* K_5 *can be embedded in* $\mathcal{F}(a)$.

Proof. Decompose $\mathcal{F}(a)$ into a finite collection of triangles, i.e., a triangulation. (Note that a triangle in closed space may have ∞ as a vertex.) Given the construction of $\hat{R}$, this is clearly possible. Call any point where $\mathcal{F}(a)$ is not locally homeomorphic to a disc a *bad point*; and call any edge of the triangulation all of whose points are bad a *bad edge*. Obviously, any bad point either occurs on a bad edge or else is an isolated bad point at a vertex of the triangulation.

If there is a bad edge, then more than two triangles must share this edge, and the embedding of K_5 in $\mathcal{F}(a)$ proceeds as shown in figure 1a. Assume, then, that there are no bad edges, but that some vertex p of the triangulation is an isolated bad point. Call two triangles with p as a vertex *neighbours* if they share an edge having p as a vertex. Since all edges are good, these triangles can clearly be arranged into disjoint cycles such that each triangle belongs to the same cycle as its two neighbours. Choose one such cycle. By applying a homeomorphism if necessary, we may assume that this triangle-cycle forms a cone with vertex p as shown in figure 1b. Since there are only finitely many triangles in the triangulation, we can ensure that we choose a triangle-cycle such that the points inside the tip of the cone either all belong to a or all belong to $-a$. Let b be either a or $-a$ depending on which of these possibilities is realized. Note that, since a is cc, so is b.

Let $c \in \hat{R}$ be a small element representing the tip of the cone, indicated by the light dotted lines in figure 1b. Now, removing c from b visibly does not disconnect b, so that $b \cdot -c$ is connected; moreover, c shares some face with $-b$, so that $c + -b = -(b \cdot -c)$ is also connected. Thus, $b \cdot -c$ is cc, whence, by Lemma 3.9, $\mathcal{F}(b \cdot -c)$ is connected. Moreover, since p is bad, there must be at least two triangle-cycles with p as vertex; whence $p \in \mathcal{F}(b \cdot -c)$. Thus we may choose a point q on the base rim of c and connect it to p by a Jordan arc α in $\mathcal{F}(b)$ such that the locus of α is disjoint from $\mathcal{F}(c)$ except for its endpoints, as shown in figure 1c. The embedding of K_5 in $\mathcal{F}(b) = \mathcal{F}(a)$ then proceeds as depicted.

THEOREM 3.11. *For all $a \in \hat{R}$, a is a ball in $\hat{\mathbb{R}}^3$ if and only if a is cc and K_5 cannot be embedded in $\mathcal{F}(a)$.*

Proof. The only-if direction is well-known (see, e.g. Wilson [21], p. 23). So suppose that $a \in \hat{R}$ is cc and that K_5 cannot be embedded in $\mathcal{F}(a)$. By Lemma 3.10, $\mathcal{F}(a)$ is a surface. Moreover, $\mathcal{F}(a)$ is compact. By the classification theorem for compact surfaces (Massey [11], p. 10), $\mathcal{F}(a)$ is either (i) a sphere, or (ii) the connected sum of finitely many tori, or (iii) the connected sum of finitely many projective planes. But cases (ii) and (iii) are ruled out by the fact that K_5 cannot be embedded

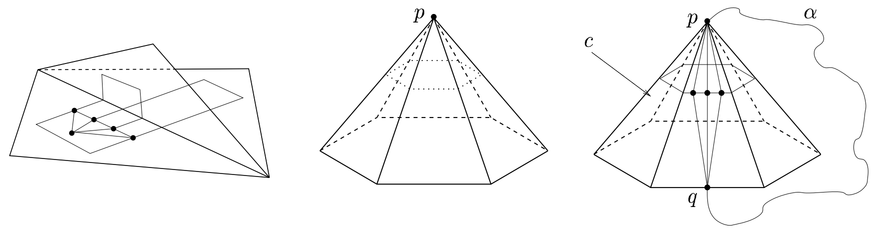

Figure 1. Embedding K_5 in non-surfaces (Proof of Lemma 3.10).

in $\mathcal{F}(a)$ (indeed, case (iii) is impossible anyway). Therefore $\mathcal{F}(a)$ is a sphere, whence a is a ball, by Lemma 3.6.

Of course, Theorem 3.11 depends on the existence of triangulations of $\mathcal{F}(a)$, and fails for arbitrary $a \in \mathrm{RO}(\hat{\mathbb{R}}^3)$.

COROLLARY 3.12. *For all bounded $a \in R$, the following are equivalent: (i) a is a ball in $\mathbb{R}^3$; (ii) $\mathcal{F}(a)$ is a sphere in $\mathbb{R}^3$; (iii) a is cc and K_5 cannot be embedded in $\mathcal{F}(a)$.*

In the sequel, we shall be concerned with assemblies of balls in $\hat{\mathbb{R}}^3$ which are themselves balls in $\hat{\mathbb{R}}^3$. The following preliminary result will be useful.

LEMMA 3.13. *Let a and b be regular open sets in some topological space, and let $c = -(a + b)$. Then $\mathcal{F}(a) \cap (a + b) = \mathcal{F}(a) \setminus \mathcal{F}(c)$.*

Proof. Since a and c are disjoint, $c \leq -a = X \setminus a^-$. Hence, $\mathcal{F}(a) \cap c = \emptyset$. Moreover, $c^- = \mathcal{F}(c) \cup c$. Thus,

$$\mathcal{F}(a) \setminus \mathcal{F}(c) = \mathcal{F}(a) \setminus c^- = \mathcal{F}(a) \cap -c = \mathcal{F}(a) \cap (a + b).$$

Now to the result about assemblies of balls. Consider the two hemispheres $a = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 < 1,\ x_1 < 0\}$ and $b = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 < 1,\ x_1 > 0\}$. It is obvious that a and b are balls, that $a + b = B(O, 1)$ is also a ball, and moreover, that $\mathcal{F}(a) \cap \mathcal{F}(b)$ is a closed disc. The following theorem shows that every arrangement of a and b in $\mathrm{RO}(\hat{\mathbb{R}}^3)$ such that $a \cdot b = 0$ and a, b and $a + b$ are all balls looks like this. The proof may be omitted without loss of understanding of subsequent material.

THEOREM 3.14. *Let $a, b \in \mathrm{RO}(\hat{\mathbb{R}}^3)$ be disjoint balls in $\hat{\mathbb{R}}^3$ such that $a + b$ is also a ball in $\hat{\mathbb{R}}^3$. Then $\mathcal{F}(a) \cap \mathcal{F}(b) \cap \mathcal{F}(a + b)$ is the locus of a Jordan curve, and $\mathcal{F}(a) \cap \mathcal{F}(b)$ is a closed disc.*

Proof. Let $c = -(a+b)$. By Lemma 3.6, both c and $a+c=-b$ are balls. Hence, we may treat b and c symmetrically. By Lemma 3.7, $\mathcal{F}(c) = \mathcal{F}(a+b)$. We denote the set $\mathcal{F}(a) \cap \mathcal{F}(b) \cap \mathcal{F}(c)$ by J.

As a first step, we show that $\mathcal{F}(a)$ is the disjoint union of the sets $\mathcal{F}(a) \cap (a+b)$, $\mathcal{F}(a) \cap (a+c)$ and J. By Lemma 3.13,

$$\begin{array}{rcl} \mathcal{F}(a) \cap (a+b) & = & \mathcal{F}(a) \setminus \mathcal{F}(c) \\ \mathcal{F}(a) \cap (a+c) & = & \mathcal{F}(a) \setminus \mathcal{F}(b). \end{array} \tag{1}$$

It follows that $(\mathcal{F}(a) \cap (a+b)) \cup (\mathcal{F}(a) \cap (a+c)) \cup J = \mathcal{F}(a)$. To prove disjointness note that equations (1) imply $(\mathcal{F}(a) \cap (a+b)) \cap J = (\mathcal{F}(a) \cap (a+c)) \cap J = \emptyset$; moreover, $(\mathcal{F}(a) \cap (a+b)) \cap (\mathcal{F}(a) \cap (a+c)) = \mathcal{F}(a) \cap ((a+b) \cdot (a+c)) = \mathcal{F}(a) \cap a = \emptyset$.

Let F be the frontier of the set $\mathcal{F}(a) \cap (a+b)$ in the space $\mathcal{F}(a)$. We show that $F = J$. Since $\mathcal{F}(a) \cap (a+b)$ and $\mathcal{F}(a) \cap (a+c)$ are disjoint open subsets of the space $\mathcal{F}(a)$, $p \in F$ implies that $p \notin \mathcal{F}(a) \cap (a+b)$ and also $p \notin \mathcal{F}(a) \cap (a+c)$. But then equations (1) yield $p \in J$. Conversely, suppose $p \in J$. By applying a homeomorphism if necessary, we may assume that $a+b$ is literally the set $B(O,1)$, and hence is convex. Now choose sequences $\{o_i\}$ from a and $\{q_i\}$ from b converging to p. For each i, the line segment joining o_i to q_i lies within $a+b$ and so certainly contains some point p_i on $\mathcal{F}(a) \cap (a+b)$. Since the sequence $\{p_i\}$ converges to p, we have $p \in F$. Thus, J is the frontier of $\mathcal{F}(a) \cap (a+b)$ in the space $\mathcal{F}(a)$. An exactly similar argument applies to $\mathcal{F}(a) \cap (a+c)$. We note in passing that this conclusion does not hold for an arbitrary partition a, b, c in $\mathrm{RO}(\hat{\mathbb{R}}^3)$.

Having identified the frontier of $\mathcal{F}(a) \cap (a+b)$ in the space $\mathcal{F}(a)$, we now show that $\mathcal{F}(a) \cap (a+b)$ is locally connected (in the space $\mathcal{F}(a)$) at every point p in that frontier. That is, we show that, given any $p \in J$ and $\epsilon > 0$, we can find a $\delta > 0$ such that any two points lying in $\mathcal{F}(a) \cap (a+b) \cap B(p,\delta)$ are connected in the set $\mathcal{F}(a) \cap (a+b) \cap B(p,\epsilon)$. Since local connectedness is a topological property we may continue to assume without loss of generality that $a+b = B(O,1)$. Let $p \in J$ and $\epsilon > 0$ be given. Since a and b are balls, and hence are locally connected at p, choose $\delta_a > 0$ and $\delta_b > 0$ such that any two points lying in $a \cap B(p,\delta_a)$ are connected in $a \cap B(p,\epsilon)$ and any two points lying in $b \cap B(p,\delta_b)$ are connected in $b \cap B(p,\epsilon)$. We show that $\delta := \min(\delta_a, \delta_b)$ has the properties stated above.

Let $p', p'' \in \mathcal{F}(a) \cap (a+b) \cap B(p,\delta)$. Because both p' and p'' are on the boundaries of both a and b, we can certainly find points p'_a, p''_a in a and p'_b, p''_b in b such that:

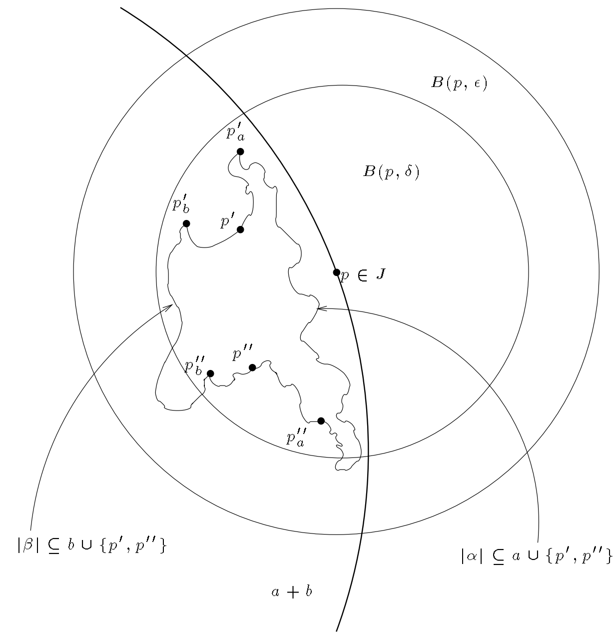

Figure 2. Local connectedness of $\mathcal{F}(a) \cap (a + b)$ in the space $\mathcal{F}(a)$ (Proof of Theorem 3.14).

p'_a is connected to p' by a Jordan arc lying in $(a \cup \{p'\}) \cap B(p, \delta)$
p'_b is connected to p' by a Jordan arc lying in $(b \cup \{p'\}) \cap B(p, \delta)$
p''_a is connected to p'' by a Jordan arc lying in $(a \cup \{p''\}) \cap B(p, \delta)$
p''_b is connected to p'' by a Jordan arc lying in $(b \cup \{p''\}) \cap B(p, \delta)$,

as shown in fig. 2. Since p'_a and p''_a are in $a \cap B(p, \delta)$, they are joined by a Jordan arc lying entirely in $a \cap B(p, \epsilon)$. Similarly, p'_b and p''_b are joined by a Jordan arc lying entirely in $b \cap B(p, \epsilon)$. Thus, we can find a Jordan arc α from p' to p'' lying entirely within $(a \cup \{p', p''\}) \cap B(p, \epsilon)$, and a Jordan arc β from p'' to p' lying entirely within $(b \cup \{p', p''\}) \cap B(p, \epsilon)$. Together, α and β form a Jordan curve $\gamma(s) : S^1 \to (a + b) \cap B(p, \epsilon)$ (where S^1 is the unit circle), since, by hypothesis, $p', p'' \in a + b$. Moreover, since we are assuming $a + b$ to be $B(O, 1)$, it is clear that $(a + b) \cap B(p, \epsilon)$ is convex, so that we can certainly shrink the locus of γ homotopically to a point within this set. That is, we can find a homotopy $F(s, t) : S^1 \times [0, 1] \to (a + b) \cap B(p, \epsilon)$ such that $F(s, 0) = \gamma(s)$ for all $s \in S^1$ and $\{F(s, 1) | s \in S^1\}$ is a singleton.

Consider the set $X := \{F(s, t) | s \in S^1, t \in [0, 1]\} \cap \mathcal{F}(a)$. We show that p' and p'' are in the same component of X. Clearly X is closed in the space $\mathcal{F}(a)$. Newman [12] Chapter VI, Theorem 3.3, states that points in different components of a closed set Y in the closed plane $\hat{\mathbb{R}}^2$ are separated by some Jordan curve lying in the complement of Y. Since $\mathcal{F}(a)$ is homeomorphic to $\hat{\mathbb{R}}^2$, it follows that if p' and p'' are in different components of X, they are separated by some Jordan curve γ' in $\mathcal{F}(a) \setminus X$. Now, γ consists of two Jordan arcs—one in $a \cup \{p', p''\}$, and the other in $-a \cup \{p', p''\}$—where p' and p'' are separated on $\mathcal{F}(a)$ by γ'. Therefore γ and γ' are interlinked Jordan curves (i.e. form a Hopf link), whence, for some $t \in [0, 1]$, some $s \in S^1$, $F(s, t) \in |\gamma'|$, contradicting the fact that $|\gamma'| \subset \mathcal{F}(a) \setminus X$. Hence p' and p'' are connected by $X \subseteq \mathcal{F}(a) \cap (a + b) \cap B(p, \delta)$. This completes the proof of the claim that, in the space $\mathcal{F}(a)$, the set $\mathcal{F}(a) \cap (a+b)$ is locally connected at every point of its frontier. Exactly similar reasoning shows that the same holds for $\mathcal{F}(a) \cap (a + c)$. Certainly then, $\mathcal{F}(a) \cap (a + b)$ and $\mathcal{F}(a) \cap (a + c)$ are connected.

Newman [12], Chapter VI Theorem 14.4 states that, if a set D is locally connected at a point p in its frontier, then p is accessible from D. (That is: p is connected to every point in D by a Jordan arc lying in $D \cup \{p\}$.) Hence, in the space $\mathcal{F}(a)$, every point of the frontier of $\mathcal{F}(a) \cap (a+b)$ is accessible from $\mathcal{F}(a) \cap (a+b)$; and similarly for $\mathcal{F}(a) \cap (a+c)$. According to the converse of the Jordan curve theorem (Newman [12], Chapter V, Theorem 11.5), if a closed set F has two complementary domains in the closed plane, from both of which every point of F is accessible, then F is the locus of a Jordan curve. Since $\mathcal{F}(a)$ is homeomorphic to the closed plane, the result holds for this space too. Hence J is the locus of a Jordan curve as required. For the remainder of the theorem, by equations (1), $\mathcal{F}(a) \cap \mathcal{F}(b) = \mathcal{F}(a) \setminus (\mathcal{F}(a) \cap (a + c))$; but we have already shown that $\mathcal{F}(a) \cap (a + c)$ is one of the residual domains in $\mathcal{F}(a)$ of a Jordan curve. Hence, $\mathcal{F}(a) \cap \mathcal{F}(b)$ is a closed disc.

3.3. Partitioning closed space

The task of section 3.3 is to define certain polyhedral partitions of closed space called *cell partitions*. Cell partitions function, in effect, as triangulations of closed space.

DEFINITION 3.15. *A* cell q *is a quadruple* $\langle q(1), q(2), q(3), q(4)\rangle$ *of pairwise disjoint elements of* $\hat{R}$ *such that, for all nonempty* $J \subseteq \{1, 2, 3, 4\}$, *the element of* $\hat{R}$ *given by* $\sum_{j \in J} q(j)$ *is a ball in* $\hat{\mathbb{R}}^3$.

EXAMPLE 3.16. *Consider the regular open tetrahedron* t_0 *with vertices* $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. *Let* t_1, t_2, t_3, t_4 *be the four regular open tetrahedra (taken in some fixed order) having three vertices* *in common with* t_0 *and the point* $(1/4, 1/4, 1/4)$ *as the fourth vertex. Evidently, the quadruple* $q_0 = \langle t_1, t_2, t_3, t_4 \rangle$ *is a cell.*

The cell q_0 is in effect a tetrahedron whose faces (and hence, edges and vertices) are picked out by its internal division into components. The next result says that Example 3.16 is as general as we need, in the sense that cells are topologically indistinguishable from each other. Thus, a cell is topologically equivalent to the first Barrycentric subdivision of a polyhedral ball.

When we speak of a homeomorphism α mapping a cell q to a cell q', we of course mean that α maps each $q(j)$ $(1 \leq j \leq 4)$ setwise to $q'(j)$. To say that cells q and q' are similarly situated in $\hat{\mathbb{R}}^3$ is to say that some such $\alpha : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ exists.

THEOREM 3.17. *All cells are similarly situated in the space* $\hat{\mathbb{R}}^3$.

Proof. Let $\langle a, b, c, d \rangle$ be a cell. Since a, b, c, $a+b$, $b+c$, $a+c$ and $a+b+c$ are balls, by Theorem 3.14, the sets $\mathcal{F}(a) \cap \mathcal{F}(b)$, $\mathcal{F}(a) \cap \mathcal{F}(c)$, $\mathcal{F}(b) \cap \mathcal{F}(c)$ and $\mathcal{F}(a+b) \cap \mathcal{F}(c)$ are all closed discs. Letting $S = \mathcal{F}(a+b+c)$, it is then easy to show that the sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and $\mathcal{F}(c) \cap S$ must be arranged on S as shown in fig. 3a, up to similar situation. Moreover, letting $e = -(a+b+c+d)$, by Lemma 3.6, if B is a proper subset of $\{a, b, c, d\}$, $\sum(B \cup \{e\})$ is a ball. Thus, all of the sets $a+b+c$, d, e, $a+b+c+d$ and $a+b+c+e$ are balls. By Theorem 3.14 then, $\mathcal{F}(d) \cap S$ and $\mathcal{F}(e) \cap S$ are both closed discs, whose common frontier in the space S is the locus of some Jordan curve γ, say.

Consider how γ might be drawn on S. Since $a+d$ and $a+e$ are balls, by Theorem 3.14, $\mathcal{F}(a) \cap \mathcal{F}(d)$ and $\mathcal{F}(a) \cap \mathcal{F}(e)$ are closed discs. Similarly, $\mathcal{F}(b) \cap \mathcal{F}(d)$, $\mathcal{F}(b) \cap \mathcal{F}(e)$, $\mathcal{F}(c) \cap \mathcal{F}(d)$ and $\mathcal{F}(c) \cap \mathcal{F}(e)$ are closed discs. Hence γ divides each of the three sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and $\mathcal{F}(c) \cap S$ into two residual domains. Moreover, γ cannot pass through either of the points X or Y in fig. 3a; for otherwise, one of the sets $\mathcal{F}(a) \cap \mathcal{F}(d)$, $\mathcal{F}(b) \cap \mathcal{F}(d)$, $\mathcal{F}(c) \cap \mathcal{F}(d)$, $\mathcal{F}(a) \cap \mathcal{F}(e)$, $\mathcal{F}(b) \cap \mathcal{F}(e)$ or $\mathcal{F}(c) \cap \mathcal{F}(e)$ would have an isolated point, contradicting the fact that these regions are all closed discs. It is then easy to see that γ and the region $\mathcal{F}(d) \cap S$ it encloses must lie in S as shown in fig. 3b or fig. 3c, up to similar situation. But these two arrangements of a, b, c, d are obviously similarly situated.

DEFINITION 3.18. *Let* t_0 *and* q_0 *be as defined in Example 3.16. Suppose* q *is any cell, and let* $\alpha : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ *be any homeomorphism mapping* q_0 *to* q. *We define a* face *of* q *to be the image, under* α, *of a face of the tetrahedron* t_0, *and we define the* edges *and* vertices *of* q *similarly. If* q' *is any other cell, we say that a face of* q *and a face of* q' *are*

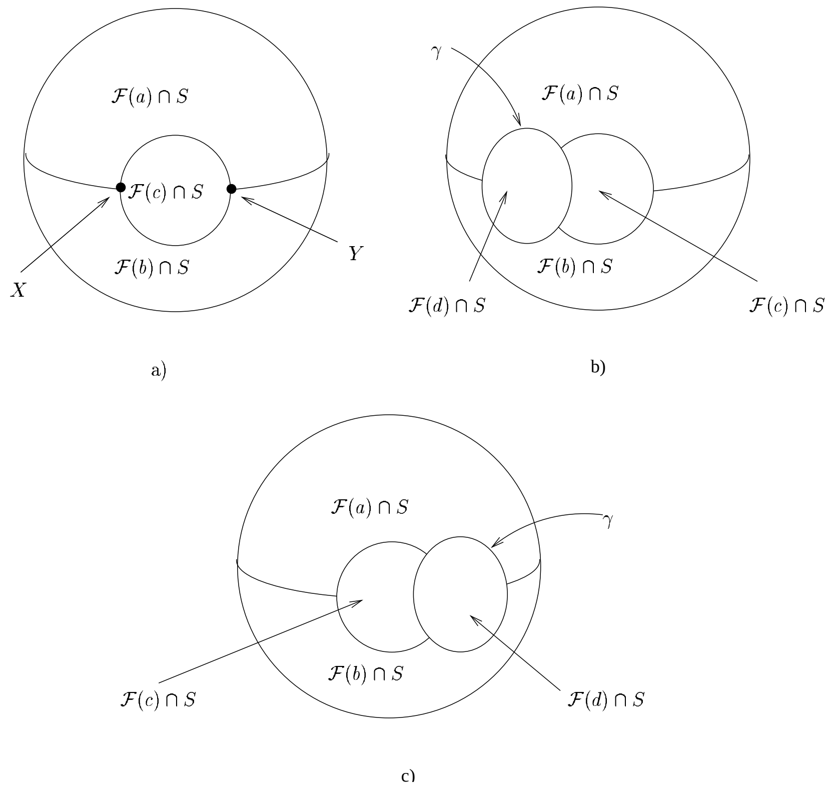

Figure 3. Possible arrangements of $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$, $\mathcal{F}(c) \cap S$ and $\mathcal{F}(d) \cap S$, where $S = \mathcal{F}(a + b + c)$ (Proof of Theorem 3.17).

corresponding *faces if they are images in this way of the same face of* t_0*; and similarly for edges and vertices.*

LEMMA 3.19. *The concepts introduced in Definition 3.18 are well-defined.*

Proof. Theorem 3.17 guarantees the existence of α, so we need only show that the definitions of faces, vertices and edges of q do not depend on the choice of α. But the faces, edges and vertices of t_0 can be characterized in terms of intersections of the sets $t_1^-, \ldots, t_4^-$, so that the faces, edges and vertices of q can be characterized in terms of the corresponding intersections of the sets $q(1)^-, \ldots, q(4)^-$.

NOTATION 3.20. *If* q *is a cell, denote by* K_q *the set of faces, edges and vertices of* q*. For readability, if* $K \subseteq K_q$ *we denote the set of points* $\bigcup K$ *by* $|K|$*. Likewise, we denote the set of points* $q(1)^- \cup \ldots \cup q(4)^-$ *by* $|q|$.

Thus, if t_0 and q_0 are as defined in Example 3.16, then $|K_{q_0}| = \mathcal{F}(t_0)$.

LEMMA 3.21. *Let q_0 be as defined in Example 3.16. Suppose $\alpha : |K_{q_0}| \to |K_{q_0}|$ is a homeomorphism fixing each element of K_{q_0} setwise. Then α can be extended to a homeomorphism $\alpha^+ : |q_0| \to |q_0|$ mapping q_0 to itself.*

Proof. Set $\alpha^+((1/4, 1/4, 1/4)) = (1/4, 1/4, 1/4)$, and then define α^+ on the rest of $|q_0|$ by linear interpolation. Obviously, α^+ fixes each $q_0(j)$ $(1 \leq j \leq 4)$ setwise.

LEMMA 3.22. *Let q and q' be cells. Suppose $\alpha : |K_q| \to |K_{q'}|$ is a homeomorphism mapping each element of K_q to the corresponding element of $K_{q'}$. Then α can be extended to a homeomorphism $\alpha^+ : |q| \to |q'|$ mapping q to q'.*

Proof. Denote the restriction of a homeomorphism γ to a set S by γ_S. Let q_0 be the cell defined in Example 3.16. By Theorem 3.17, let $\beta : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ be a homeomorphism taking q to q_0, and let $\beta' : \hat{\mathbb{R}}^3 \to \hat{\mathbb{R}}^3$ be a homeomorphism taking q' to q_0. Then the homeomorphism $\beta'_{|K_{q'}|} \circ \alpha \circ \beta^{-1}_{|K_{q_0}|}$ satisfies the conditions of Lemma 3.21, and hence can be extended to the whole of $|q_0|$ such that q_0 is mapped to itself. Let this extended homeomorphism be α^*. Then the homeomorphism $\beta'^{-1}_{|q_0|} \circ \alpha^* \circ \beta_{|q|}$ has the required properties.

LEMMA 3.23. *Let q and q' be cells, and let $K \subseteq K_q$. Suppose that α is a homeomorphism defined on $|K|$ mapping every $k \in K$ to the corresponding element of $K_{q'}$. Then α can be extended to a homeomorphism $\alpha^+ : |q| \to |q'|$ mapping q to q'. In particular, α^+ maps every $k \in K_q$ to the corresponding element in $K_{q'}$*

Proof. It is straightforward to extend α to the entire surface $|K_q|$ mapping every $k \in K_q$ to the corresponding element of $K_{q'}$. The result then follows by Lemma 3.22.

DEFINITION 3.24. *A cell partition is a tuple $\mathbf{q} = q_1, \ldots, q_N$ of cells satisfying the following conditions.*

1. *The $4N$ elements $q_i(j)$ $(1 \leq i \leq N,\ 1 \leq j \leq 4)$ form a partition in the Boolean algebra $\hat{R}$;*

2. *For each i, j $(1 \leq i \leq N,\ 1 \leq j \leq 4)$, there is exactly one pair i', j' with $i' \neq i$ such that $q_i(j) + q_{i'}(j')$ is connected.*

Thus, the faces, edges and vertices of the cells in a cell partition 'match up' exactly, just as in a triangulation.

NOTATION 3.25. *If* $\mathbf{q} = q_1, \ldots, q_N$ *is a cell partition, we write* $\mathbf{q}(i,j)$ *to denote* $q_i(j)$ *for all* i, j *(*$1 \leq i \leq N$*,* $1 \leq j \leq 4$*). In addition, we write* $K_{\mathbf{q}}$ *for* $\cup_{1 \leq i \leq N} K_{q_i}$*.*

DEFINITION 3.26. *Let* $\mathbf{q} = q_1, \ldots, q_N$ *and* $\mathbf{q}' = q'_1, \ldots, q'_N$ *be cell partitions. We say that* $\mathbf{q}$ *and* $\mathbf{q}'$ *are* equivalent *if there is a function* $g : K_{\mathbf{q}} \rightarrow K_{\mathbf{q}'}$ *such that, for all* i *(*$1 \leq i \leq N$*),* g *maps every element in* K_{q_i} *to the corresponding element in* $K_{q'_i}$*.*

The main result of this section is:

LEMMA 3.27. *Equivalent cell partitions are similarly situated in* $\hat{\mathbb{R}}^3$*.*

Proof. Let $\mathbf{q}$ and $\mathbf{q}'$ and $g : K_{\mathbf{q}} \rightarrow K_{\mathbf{q}'}$ be as in Definition 3.26. Let α_0 be the empty homeomorphism. For $0 \leq j < N$, assume the homeomorphism α_j has been defined on the space $\bigcup_{1 \leq i \leq j} |q_i|$ such that, for all i ($1 \leq i \leq j$), α_j maps each cell q_i to the cell q'_i. Certainly then, α_j maps every k occurring in any of the K_{q_i} ($1 \leq i \leq j$) to the corresponding element in $K_{q'_i}$, which is by assumption just $g(k)$. Let K be the set of elements of $K_{q_{j+1}}$ occurring in any of the K_{q_i} ($1 \leq i \leq j$). Then the restriction of α_j to $|K|$ satisfies the conditions of Lemma 3.23, whence α_j can be extended to a homeomorphism α_{j+1} satisfying analogous conditions to α_j. Proceeding thus, we eventually obtain α_N mapping $\mathbf{q}$ to $\mathbf{q}'$. Since $\bigcup_{1 \leq i \leq N} |q_i| = \hat{\mathbb{R}}^3$, we have $\mathbf{q} \sim \mathbf{q}'$.

We finish with two technical results on cell partitions that will be useful later.

LEMMA 3.28. *Let* $\mathbf{q}$ *be an* N*-element cell partition, and let* $p, p' \in \hat{\mathbb{R}}^3$ *such that, for all* i, j *(*$1 \leq i \leq N$*,* $1 \leq j \leq 4$*),* $p \in \mathbf{q}(i,j)^-$ *if and only if* $p' \in \mathbf{q}(i,j)^-$*. Then there exists a homeomorphism of* $\hat{\mathbb{R}}^3$ *onto itself fixing every* $\mathbf{q}(i,j)$ *setwise, and mapping* p' *to* p*.*

Proof. Obvious.

LEMMA 3.29. *Let* $\mathbf{q}$ *and* $\mathbf{q}'$ *be equivalent* N*-element cell partitions, and let* $p \in \hat{\mathbb{R}}^3$ *such that, for all* i, j *(*$1 \leq i \leq N, 1 \leq j \leq 4$*),* $p \in \mathbf{q}(i,j)^-$ *if and only if* $p \in \mathbf{q}'(i,j)^-$*. Then there exists a homeomorphism of* $\hat{\mathbb{R}}^3$ *onto itself mapping each* $\mathbf{q}(i,j)$ *to* $\mathbf{q}'(i,j)$*, and fixing* p*.*

Proof. By Lemma 3.27, let α' be some homeomorphism of $\hat{\mathbb{R}}^3$ onto itself mapping $\mathbf{q}$ to $\mathbf{q}'$. Then we have, for all i, j ($1 \leq i \leq N, 1 \leq j \leq 4$) $\alpha'(p) \in \mathbf{q}'(i,j)^-$ if and only if $p \in \mathbf{q}'(i,j)^-$. By Lemma 3.28, let β be a homeomorphism of $\hat{\mathbb{R}}^3$ onto itself mapping $\mathbf{q}'$ to itself, and mapping $\alpha'(p)$ to p. Then $\alpha := \beta \circ \alpha'$ has the properties required by the lemma.

3.4. Partitioning open space

The task of section 3.4 is to modify the notion of a cell partition introduced in section 3.3 so that it applies to open space, $\mathbb{R}^3$. The main result is Lemma 3.38.

DEFINITION 3.30. *A polyhedron $a \in R$ is said to be an* A-ball *if the associated closed-space polyhedron $\hat{a} \in \hat{R}$ is a ball in $\hat{\mathbb{R}}^3$.*

EXAMPLE 3.31. *If a is a ball in the space $\mathbb{R}^3$, then a is an A-ball. If a is an A-ball, then so is $-a$. Furthermore, a half-space is an A-ball, as is a prism bounded at either one or both ends. However, a prism unbounded at both ends is not an A-ball.*

DEFINITION 3.32. *An* A-cell *is a quadruple $q = \langle q(1), q(2), q(3), q(4)\rangle \in R^4$ such that, for all nonempty $J \subseteq \{1, 2, 3, 4\}$, the polyhedron $\sum_{j \in J} q(j)$ is an A-ball.*

REMARK 3.33. *A quadruple $q \in R^4$ is an A-cell if and only if the associated quadruple $\hat{q} = \langle \hat{q}(1), \hat{q}(2), \hat{q}(3), \hat{q}(4)\rangle$ is a cell.*

DEFINITION 3.34. *An* A-cell partition *is a tuple $\mathbf{q} = q_1, \ldots, q_N$ of A-cells satisfying the following conditions.*

1. *The $4N$ elements $q_i(j)$ $(1 \leq i \leq N,\ 1 \leq j \leq 4)$ form a partition in the Boolean algebra R;*
2. *For each i, j $(1 \leq i \leq N,\ 1 \leq j \leq 4)$, there is exactly one pair i', j' with $i' \neq i$ such that $q_i(j) + q_{i'}(j')$ is connected.*

NOTATION 3.35. *Again, if $\mathbf{q} = q_1, \ldots, q_N$ is an A-cell partition, we write $\mathbf{q}(i, j)$ to denote $q_i(j)$.*

REMARK 3.36. *A tuple $\mathbf{q} = q_1, \ldots, q_N$ is an A-cell partition if and only if the associated tuple $\hat{\mathbf{q}} = \hat{q}_1, \ldots, \hat{q}_N$ is a cell partition.*

DEFINITION 3.37. *Let $\mathbf{q}$ and $\mathbf{q}'$ be A-cell partitions of length N. We say that $\mathbf{q}$ and $\mathbf{q}'$ are* equivalent *if*

1. *The associated cell partitions $\hat{\mathbf{q}}$ and $\widehat{\mathbf{q}'}$ are equivalent.*
2. *For all i, j $(1 \leq i \leq N,\ 1 \leq j \leq 4)$, $\mathbf{q}(i, j)$ is bounded if and only if $\mathbf{q}'(i, j)$ is bounded.*

LEMMA 3.38. *Equivalent A-cell partitions are similarly situated in the space* $\mathbb{R}^3$.

Proof. Let $\mathbf{q}$ and $\mathbf{q}'$ be equivalent A-cell partitions, each with N elements. Then $\hat{\mathbf{q}}$ and $\widehat{\mathbf{q}'}$ are equivalent cell partitions such that, for all i, j $(1 \leq i \leq N, 1 \leq j \leq 4)$, $\infty \in \hat{\mathbf{q}}(i,j)^-$ if and only if $\infty \in \widehat{\mathbf{q}'}(i,j)^-$. By Lemma 3.29, there exists a homeomorphism of $\hat{\mathbb{R}}^3$ onto itself mapping $\hat{\mathbf{q}}$ to $\widehat{\mathbf{q}'}$, and fixing ∞. Thus, $\alpha = \hat{\alpha} \setminus \{\langle \infty, \infty \rangle\}$ is a homeomorphism of $\mathbb{R}^3$ onto itself mapping $\mathbf{q}$ to $\mathbf{q}'$.

4. Defining relations in $\mathcal{L}$

In this section, we establish directly that certain relations are definable by means of $\mathcal{L}$-formulas.

LEMMA 4.1. *The formula* $\phi(x_1, x_2) := \forall y (C(x_1, y) \rightarrow C(x_2, y))$ *is satisfied by the pair* a_1, a_2 *in* R *if and only if* $a_1 \leq a_2$.

Proof. Straightforward.

Since the part-of relation $\leq$ is $\mathcal{L}$-definable over R, we may henceforth use the symbol $\leq$ as a shorthand in $\mathcal{L}$-formulas. It follows in addition that the Boolean functions and constants $+$, $\cdot$, $-$, 1 and 0 are also $\mathcal{L}$-definable over R; so again, we may henceforth use these symbols in formulas without further ado.

The following lemma shows that the language $\mathcal{L}$ allows us to talk about points in $\mathbb{R}^3$. The idea is that a point p is represented by a pair of regions $a_1, a_2 \in R$ such that $a_1^- \cap a_2^- = \{p\}$.

LEMMA 4.2. *There exist formulas* $\phi(x_1, x_2)$, $\phi'(x_1, x_2)$ *and* $\phi''(x_1, x_2)$ *satisfying the following conditions for all* $a_1, a_2, a_3 \in R$.

1. $R \models \phi[a_1, a_2]$ *if and only if* $a_1^- \cap a_2^-$ *is a singleton.*
2. *If* $a_1^- \cap a_2^- = \{p\}$ *for some point* $p \in \mathbb{R}^3$, *then* $R \models \phi'[a_1, a_2, a_3]$ *if and only if* $p \in a_3$
3. *If* $a_1^- \cap a_2^- = \{p\}$ *for some point* $p \in \mathbb{R}^3$, *then* $R \models \phi''[a_1, a_2, a_3]$ *if and only if* $p \in a_3^-$

Proof. Let:

$$\begin{aligned}
\phi(x_1,x_2) &:= C(x_1, x_2) \wedge \\
&\quad \forall y_1 \forall y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge C(y_1, x_2) \wedge C(y_2, x_1) \rightarrow C(y_1, y_2)) \\
\phi'(x_1, x_2, x_3) &:= \exists y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \wedge \neg C(y_1, -x_3)) \\
\phi''(x_1, x_2, x_3) &:= \forall y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \rightarrow C(y_1, x_3)).
\end{aligned}$$

The previous lemma gives us the right to include expressions such as, for example, $x_1 \cap x_2^- \neq \emptyset$ or $\mathcal{F}(x_1) \cap \mathcal{F}(x_2) \subseteq \mathcal{F}(x_1) \cap \mathcal{F}(x_2)$ etc. in formulas, with the obvious interpretation over R.

LEMMA 4.3. *Let $c(x)$ abbreviate the formula*

$$\neg \exists y_1 \exists y_2 (x \cap y_1^- \neq \emptyset \wedge x \cap y_2^- \neq \emptyset \wedge x \subseteq y_1^- \cup y_2^- \wedge x \cap y_1^- \cap y_2^- = \emptyset).$$

Then, for all $a \in R$, $R \models c[a]$ if and only if a is connected.

Proof. The if-direction follows instantly from the definition of connectedness. For the only-if direction, we need to show that a non-connected element of $a \in R$ can be partitioned into two disjoint, nonempty sets which are the closures in a of *elements of* R. But this is simple given that a is evidently the sum in R of finitely many connected polyhedra.

Of course, an exactly similar technique can be used to express the connectedness of sets topologically dependent on elements of R, such as, for instance $\mathcal{F}(a_1)$, $\mathcal{F}(a_1) \cap \mathcal{F}(a_2)$, etc. Thus, we may allow ourselves to include expressions such as $c(x_1)$, $c(\mathcal{F}(x_1) \cap \mathcal{F}(x_2))$, etc. in formulas to denote the connectedness of the sets in question.

LEMMA 4.4. *There exists an $\mathcal{L}$-formula $\kappa(x)$ such that, for all $a \in R$, $R \models \kappa[a]$ if and only if K_5 is embeddable in $\mathcal{F}(a)$.*

Proof. The graph K_5 is evidently embeddable in $\mathcal{F}(a)$ if and only if there exist polyhedra v_i $(1 \leq i \leq 5)$ and $e_{i,j}$ $(1 \leq i < j \leq 5)$, all disjoint from a and from each other, satisfying the following conditions:

1. For all i $(1 \leq i \leq 5)$, $v_i^- \cap a^-$ is a singleton
2. For all i, j $(1 \leq i < j \leq 5)$, $e_{i,j}^- \cap a^-$ is connected
3. For all i, j, i', j' $(1 \leq i < j \leq 5,\ 1 \leq i' < j' \leq 5)$, $\{i, j\} \cap \{i', j'\} = \emptyset$ implies $e_{i,j}^- \cap e_{i',j'}^- \cap a^- = \emptyset$, and $\{i, j\} \cap \{i', j'\} = \{k\}$ implies $e_{i,j}^- \cap e_{i',j'}^- \cap a^- = v_k^- \cap a^-$.

But these conditions are expressible in $\mathcal{L}$ over R by the foregoing lemmas.

We note in passing that, in the above proof, the polyhedra $e_{i,j}$ are not themselves required to be connected—only the sets $e_{i,j}^- \cap a^- = \mathcal{F}(e_{i,j}) \cap \mathcal{F}(a)$.

LEMMA 4.5. *Let $\gamma(x)$ be the formula*

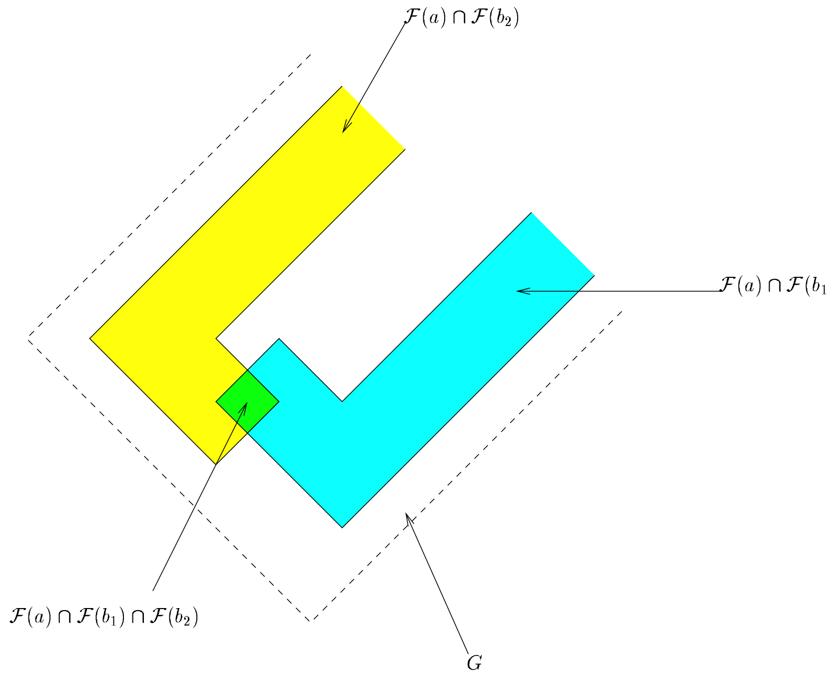

Figure 4. Arrangement of $\mathcal{F}(a) \cap \mathcal{F}(b_1)$ and $\mathcal{F}(a) \cap \mathcal{F}(b_2)$ on G (Proof of Lemma 4.5).

$$\begin{aligned}\exists y_1 \exists y_2 (y_1 \cdot x = 0 \wedge y_2 \cdot x = 0 \wedge c(\mathcal{F}(x) \cap \mathcal{F}(y_1) \cap \mathcal{F}(y_2)) \wedge \\ c(\mathcal{F}(x) \setminus \mathcal{F}(y_1)) \wedge c(\mathcal{F}(x) \setminus \mathcal{F}(y_2)) \wedge \\ \neg c(\mathcal{F}(x) \setminus (\mathcal{F}(y_1) \cup \mathcal{F}(y_2)))).\end{aligned}$$

For all $a \in R$, the following hold.

1. *If $\mathcal{F}(a)$ is connected and unbounded, then $R \models \gamma[a]$.*

2. *If $\mathcal{F}(a)$ is a sphere in $\mathbb{R}^3$, then $R \not\models \gamma[a]$.*

Proof. Suppose $\mathcal{F}(a)$ is connected and unbounded. By the definition of R, a is a Boolean combination of finitely many half-spaces, corresponding to a finite set of planes, say, $\pi_1, \ldots, \pi_m$; it is then easy to see that $\mathcal{F}(a) \subseteq \pi_1 \cup \cdots \cup \pi_m$. Since $\mathcal{F}(a)$ is unbounded, we can draw in $\mathcal{F}(a)$ a rectangular figure G, unbounded on one side (dotted lines in figure 4), such that G intersects only one of the π_i. Let $b_1, b_2 \in R$ be laminas, infinitely extended in one direction, and placed on G (on the outside of a) so that $\mathcal{F}(a) \cap \mathcal{F}(b_1)$ and $\mathcal{F}(a) \cap \mathcal{F}(b_2)$ are arranged as shown. Since $\mathcal{F}(a)$ is connected, $\mathcal{F}(a) \setminus \mathcal{F}(b_1)$ and $\mathcal{F}(a) \setminus \mathcal{F}(b_2)$ are also connected; and since G lies on just one of the π_i, $\mathcal{F}(a) \setminus (\mathcal{F}(b_1) \cup \mathcal{F}(b_2))$ is not connected. Thus $R \models \gamma[a]$.

Newman [12], Chapter V, Theorem 9.2 states that, if the common part of two closed sets F_1 and F_2 in the closed plane is connected, then two points which are connected in the complement of F_1 and in the complement of F_2 are connected in the complement of $F_1 \cup F_2$. Since a sphere in $\mathbb{R}^3$ is homeomorphic to the closed plane, the theorem applies to this space as well. It immediately follows that, if $\mathcal{F}(a)$ is a sphere, then $R \not\models \gamma[a]$.

NOTATION 4.6. *In the sequel, we take $cc(x)$ to abbreviate the formula*

$$x \neq 0 \wedge x \neq 1 \wedge c(x) \wedge c(-x).$$

Recall from Definition 3.8 that polyhedra satisfying this formula are said to be cc.

LEMMA 4.7. *For all $a \in R$, $R \models cc[a] \wedge \neg\kappa[a] \wedge \neg\gamma[a]$ if and only if $\mathcal{F}(a)$ is a sphere.*

Proof. The if-direction is immediate given Lemma 4.5. The other direction follows by Corollary 3.12 and Lemmas 3.9, 4.4 and 4.5.

LEMMA 4.8. *Let $a \in R$ such that $R \models cc[a] \wedge \neg\kappa[a] \wedge \gamma[a]$. Then a is unbounded.*

Proof. Suppose a is bounded. Since a is cc and K_5 cannot be embedded in a, Corollary 3.12 implies that $\mathcal{F}(a)$ is a sphere. But then Lemma 4.5 implies that a does not satisfy $\gamma(x)$.

COROLLARY 4.9. *There exists a formula $\beta(x)$ such that, for all $a \in R$, $R \models \beta[a]$ if and only if a is bounded.*

Proof. Let

$$\begin{aligned}\beta(x) := \exists y \exists z (x \leq y \wedge y \cdot z = 0 \wedge \\ cc(y) \wedge \neg\kappa(y) \wedge \neg\gamma(y) \wedge \\ cc(z) \wedge \neg\kappa(z) \wedge \gamma(z)).\end{aligned}$$

If a is bounded, let $b \in R$ be a ball in $\mathbb{R}^3$ such that $a \leq b$; and let $c \in R$ be an A-ball such that $\mathcal{F}(c)$ is unbounded and $b \cdot c = 0$. Thus, b and c are suitable witnesses for y and z in $\beta(x)$, so that $R \models \beta[a]$.

Conversely, suppose that $R \models \beta[a]$. Let b and c be witnesses for y and z. By Lemma 4.7, $\mathcal{F}(b)$ is a sphere, whence either b or $-b$ is a ball in $\mathbb{R}^3$ (but not both). By Lemma 4.8, c is unbounded, and so intersects the complement of every ball in $\mathbb{R}^3$, whence b is a ball in $\mathbb{R}^3$, so that a is bounded.

For all $a \in R$, $\infty \in \hat{a}$ if and only if $-a \in R$ is bounded, and $\infty \in \hat{a}^-$ if and only if a is unbounded (where the closure operator $^-$ refers to the topology on $\hat{\mathbb{R}}^3$). Since $\beta(x)$ expresses the property of boundedness over R, we may write expressions such as $\infty \in \hat{x}$, $\infty \in \mathcal{F}(\hat{x})$, etc. in $\mathcal{L}$-formulas with the obvious interpretation over R. By repeating the reasoning of the preceding lemmas, we may further write expressions such as $c(\hat{x}_1)$, $c(\mathcal{F}(\hat{x}_1) \cap \mathcal{F}(\hat{x}_2))$, etc. in formulas, again, with the obvious interpretation. It follows:

LEMMA 4.10. *There exists a formula $\hat{\kappa}(x)$ satisfied by $a \in R$ if and only if K_5 is embeddable in $\mathcal{F}(\hat{a})$.*

COROLLARY 4.11. *There exists a formula satisfied by $a \in R$ if and only if a is an A-ball in $\mathbb{R}^3$.*

Proof. By Theorem 3.11 and Lemma 4.10, $cc(x) \wedge \neg\hat{\kappa}(x)$ has the required property.

Putting together all the expressiveness results in this section, we have

COROLLARY 4.12. *For all $N > 0$, there exists a formula $\mu_N(\bar{z})$ such that, for any $4N$-tuple $\bar{c}$, $R \models \mu_N(\bar{c})$ if and only if $\bar{c}$ is an N-element A-cell partition.*

5. Model-Theoretic Analysis

Section 5 gathers together the material of sections 3 and 4 to prove the main results.

LEMMA 5.1. *Let $\bar{c}$ be a $4N$-tuple of polyhedra forming an N-element A-cell partition in R. Then we can find a topologically complete formula $\gamma(\bar{z})$ such that $R \models \gamma[\bar{c}]$.*

Proof. Let $\bar{z} = z_1, \ldots, z_{4N}$. By Corollary 4.12, let $\mu_N(\bar{z})$ be satisfied in R by exactly those $4N$-tuples which form N-element A-cell partitions. Now consider the collection of all formulas satisfied by $\bar{c}$ of the forms:

$$\star(\hat{z}_1^-) \cap \ldots \cap \star(\hat{z}_{4N}^-) = \emptyset, \qquad \star(\hat{z}_1^-) \cap \ldots \cap \star(\hat{z}_{4N}^-) \neq \emptyset,$$

where $\star Z$ is either Z or $\hat{\mathbb{R}}^3 \setminus Z$ for any set-denoting expression Z. Again, we may regard such formulas as belonging to $\mathcal{L}$ by the definability results proved in section 4. Finally, consider all the formulas satisfied by $\bar{c}$ of the forms $\beta(z_i)$ and $\neg\beta(z_i)$ (where β is as given in Corollary 4.9) with $1 \leq i \leq 4N$. The conjunction of all the above formulas is satisfied only by $4N$-tuples forming N-element A-cell partitions equivalent to $\bar{c}$. The result then follows from Lemma 3.38.

LEMMA 5.2. *If $\bar{a}$ is any tuple of polyhedra, then there exists a $4N$-tuple $\bar{c}$ of polyhedra, for some N, such that $\bar{c}$ forms an N-element A-cell partition, and each element of $\bar{a}$ can be expressed as the sum of zero or more elements of $\bar{c}$.*

Proof. Given $a_1, \ldots, a_n \in R$, consider the corresponding elements $\hat{a}_1, \ldots, \hat{a}_n \in \hat{R}$. Certainly, we can find a triangulation of $\hat{\mathbb{R}}^3$, with $\hat{b}_1, \ldots, \hat{b}_N \in \hat{R}$ as the 3-simplices, in such a way that each $\hat{a}_i$ is the sum of zero or more of the $\hat{b}_j$. Now divide each simplex $\hat{b}_j$ into 4 smaller simplices $\hat{c}_{j,1}, \ldots, \hat{c}_{j,4}$ forming a cell with the faces, edges and vertices of the triangulation in the obvious way. We thus obtain a $4N$-tuple $\hat{c}_{1,1}, \ldots, \hat{c}_{N,4}$ forming a cell-partition, such that each $\hat{a}_i$ is the sum of zero or more of the $\hat{c}_j$. Now let $\bar{c} = c_{1,1}, \ldots, c_{N,4}$ be the associated A-cell partition.

We are now in a position to prove our first main result.

Proof. [of Theorem 2.8] Let $\bar{a} \in R^n$. By Lemma 5.2, let $\bar{c}$ be an A-cell partition whose elements sum to the elements of $\bar{a}$. By Lemma 5.1, let $\gamma(\bar{z})$ be a topologically complete formula satisfied by $\bar{c}$. Then $\bar{a}$ satisfies a formula $\phi(x_1, \ldots, x_n)$ of the form

$$\exists \bar{z} \left(\gamma(\bar{z}) \wedge \bigwedge_{1 \leq i \leq n} x_i = \sum Z_i \right),$$

where, for all i $(1 \leq i \leq n)$, Z_i is some subset of the variables in $\bar{z}$. The topological completeness of this formula follows easily from that of γ.

Now to the analysis of the alternative models of $\mathrm{Th}(R)$. We briefly recapitulate some standard definitions and results in model theory.

DEFINITION 5.3. *Let T be a theory over some fixed signature. A formula $\phi(\bar{x})$ is* complete *in T if, for all formulas $\theta(\bar{x})$, exactly one of $T \models \phi \rightarrow \theta$ and $T \models \phi \rightarrow \neg\theta$ holds. A structure $\mathfrak{A}$ is* atomic *if every n-tuple in A satisfies a complete formula in* $\mathrm{Th}(\mathfrak{A})$.

Atomicity is important because of Theorems 5.5 and 5.6 (Chang and Keisler [2] sec. 2.3.3):

DEFINITION 5.4. *A structure $\mathfrak{A}$ is said to be* prime *if, for any structure $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies that $\mathfrak{A}$ can be elementarily embedded in $\mathfrak{B}$.*

THEOREM 5.5. *A structure is countable and atomic if and only if it is prime.*

If a theory has a prime model, then it is unique, in the following sense.

THEOREM 5.6. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are countable atomic models and* $\mathfrak{A} \equiv \mathfrak{B}$, *then* $\mathfrak{A} \simeq \mathfrak{B}$.

Thus, prime models are regarded as the simplest models of their theories. The task before us is to establish that the structure R is atomic, and to identify Q as its prime model.

The following results concerning the collections R and Q are easily established:

LEMMA 5.7. *Let* $a_1, \ldots, a_n, b_1, \ldots, b_n, a \in R$ *such that* $a_1, \ldots, a_n \sim b_1, \ldots, b_n$. *Then there exists* $b \in R$ *such that* $a_1, \ldots, a_n, a \sim b_1, \ldots, b_n, b$.

LEMMA 5.8. *Let* $a_1, \ldots, a_n \in Q$ *and let* $b \in R$. *Then there exists* $a \in Q$ *such that* $a_1, \ldots, a_n, a \sim a_1, \ldots, a_n, b$.

Lemmas 5.7 and 5.8 imply the following two useful results:

LEMMA 5.9. *Let* $\bar{a}, \bar{b} \in R^n$ *such that* $\bar{a} \sim \bar{b}$ *in* $\mathbb{R}^3$. *Then* $\bar{a}$ *and* $\bar{b}$ *satisfy the same formulas.*

Proof. Suppose $R \models \phi[\bar{a}]$. We show by structural induction on ϕ that $R \models \phi[\bar{b}]$. The only nontrivial case is where $\phi(\bar{x})$ has the form $\exists y \phi(\bar{x}, y)$. Since $R \models \phi[\bar{a}]$, choose $a \in R$ such that $R \models \psi[\bar{a}, a]$; by Lemma 5.7, choose $b \in R$ such that $\bar{a}, a \sim \bar{b}, b$. By inductive hypothesis, $R \models \psi[\bar{b}, b]$; hence $R \models \phi[\bar{b}]$, as required.

LEMMA 5.10. *The structure* Q *is an elementary submodel of* R.

Proof. According to the Tarski-Vaught lemma (Hodges [9] p. 55), if $\mathfrak{A} \subseteq \mathfrak{B}$ and, for any n-tuple $\bar{a}$ of A and any formula $\phi(\bar{x})$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{B} \models \phi[\bar{b}]$, there exists $a \in A$ such that $\mathfrak{B} \models \psi[\bar{a}, a]$, then $\mathfrak{A} \preceq \mathfrak{B}$.

By construction, Q is a substructure of R. Let $\bar{a}$ be an n-tuple of Q, and let $\phi(\bar{x})$ be any formula of the form $\exists y \psi(\bar{x}, y)$ such that $R \models \phi[\bar{a}]$. Then there exists $b \in R$ such that $R \models \psi[\bar{a}, b]$. By Lemma 5.8, there exists $a \in Q$ such that $\bar{a}, a \sim \bar{a}, b$. By Lemma 5.9, $R \models \psi[\bar{a}, a]$.

THEOREM 5.11. *The structure* R *is atomic.*

Proof. By Lemma 5.9, every topologically complete formula is complete in $\mathrm{Th}(R)$. The result then follows by Theorem 2.8.

Finally, we have our second main result.

Proof. [of Theorem 2.10] By Lemma 5.10 and Theorem 5.11, Q is an atomic model of $\mathrm{Th}(R)$. The result follows by Theorem 5.5.

Finally, we observe that the order of derivation between Theorems 2.8 and 2.10 can actually be reversed! To prove Theorem 2.10 without first proving Theorem 2.8, recall that Corollary 4.12 ensures the existence of a formula $\mu_N(\bar{z})$ satisfied by a $4N$-tuple $\bar{c}$ if and only if $\bar{c}$ forms an N-element A-cell partition. But it is an immediate consequence of Lemma 3.38 that, for a fixed $N > 0$, there are only finitely many N-element A-cell partitions up to similar situation. By Lemma 5.9 then, every A-cell partition satisfies a complete formula, whence, by Lemma 5.2, every tuple from R satisfies a complete formula. To derive Theorem 2.8 from Theorem 2.10 requires that we show that every complete formula is topologically complete. In fact, this follows with relative ease, as described in using Pratt and Schoop [16] Theorem 5.13. The strategy is to show how automorphisms of R (considered as a structure) induce homeomorphisms from the original space $\mathbb{R}^3$ onto itself. (This argument depends on the fact, established in Corollary 4.9, that the property of boundedness is $\mathcal{L}$-definable.) The interested reader can find the details in the source just cited. In some ways, this reversed order of derivation is more elegant and general, in that it can be used to obtain information about alternative models of somewhat less expressive languages than $\mathcal{L}$. However, for most purposes, the order of derivation adopted here is probably more straightforward.

6. Discussion and open problems

At this point, we have a fairly clear understanding of the theory of polyhedra in the first-order language $\mathcal{L}$ based on Whitehead's contact-relation. In particular, we have two closely related results concerning, on the one hand, the expressive power of this language, and on the other, the possibilities for interpreting it within some alternative, region-based model of space. First: the formulas of $\mathcal{L}$ suffice to characterize any tuple of polyhedra up to the relation of similar situation. Second: any region-based model of space must *either* falsify some statement true in the familiar polyhedral interpretation *or* must contain a copy of the rational polyhedra as an elementary submodel.

It is natural to ask whether the results obtained here for R can be generalized to other, more liberal domains of quantification. Of particular interest is the collection S of regular open *semialgebraic* subsets of $\mathbb{R}^3$ (van den Dries [19], p. 168). Again, S determines an $\mathcal{L}$-structure by interpreting the binary predicate C in the usual way; it is then easy to show that S and R are elementarily equivalent and that Theorems 2.8 and 2.10 apply with R replaced by S. (See Pratt and Lemon [14] sec. 6 for a more detailed discussion in the two-dimensional case.) Since generalization to the semialgebraic case adds nothing new, we have kept to the polyhedral case in this paper for perspicuity.

More problematic is the liberalization of quantification to domains containing non-tame regions, for example, the whole of $\mathrm{RO}(\mathbb{R}^3)$. Restricting quantification to regular open (or closed) sets has now become standard in treatments of mereotopology (Randell *et al.* [17], p. 166, Borgo *at al.* [1], p. 221, Gotts *et al.* [8], sec. 3.). Motivations vary, but, roughly speaking, the idea seems to be that regular open subsets of $\mathbb{R}^3$ do a good job of modelling the regions occupied by physical objects, because they help to suppress awkward and meaningless questions as to whether such regions include their boundary points. So it is natural to wonder about the mereotopology that results if we interpret $\mathcal{L}$ in the usual way over the whole of $\mathrm{RO}(\mathbb{R}^3)$.

However, the choice of $\mathrm{RO}(\mathbb{R}^3)$ as our domain of quantification is less attractive than might at first appear. Certainly, $\mathrm{RO}(\mathbb{R}^3)$ contains subsets of $\mathbb{R}^3$ which could not possibly usefully model the regions occupied by physical objects. For example, the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 | 0 < x_1 < 1, -1 - x_1 < x_2 < 1/\sin x_1, 0 < x_3 < 1\}$$

is seen to be regular open, but has an infinitely wiggly boundary that cannot, it seems, be physically realized. This is precisely the sort of pathological behaviour which cannot arise in R (or S), and whose elimination has partly motivated the development of mereotopology. Moreover, the bad behaviour of such regions makes the structure $\mathrm{RO}(\mathbb{R}^3)$ much harder to understand than R. It is easily shown that $\mathrm{RO}(\mathbb{R}^3)$ is not elementarily equivalent (as an $\mathcal{L}$-structure) to R; but little else is known about it.

Liberalizing our domain of quantification still further, we could even interpret $\mathcal{L}$ over the entire power set $\mathbb{P}(\mathbb{R}^3)$. Again, it can be shown that this structure is elementarily equivalent to neither R nor $\mathrm{RO}(\mathbb{R}^3)$. However, it also transpires that $\mathcal{L}$ has virtually no useful expressive power when interpreted over $\mathbb{P}(\mathbb{R}^3)$. For example, the subset-relation, which we showed in Lemma 4.1 to be defined over R by a very simple formula, is not definable over $\mathbb{P}(\mathbb{R}^3)$ by *any* formula of $\mathcal{L}$. Since $\mathcal{L}$ is so inexpressive over this domain of interpretation, we have not investigated its mereotopology in any detail.

Where do these results leave the program of developing region-based models of physical space? The answer may depend on our original motivation. Insofar as that motivation was metaphysical—prompted by a conviction of the ontological primacy of regions—then the results achieved here are surely negative in character. Yes, space can be modelled as a structure of primitive regions related to each other by a primitive binary relation of contact; however, assuming the search for alternative spatial ontologies to be constrained by elementary equivalence to the familiar polyhedral model, nothing results but a copy of the familiar rational polyhedra together (possibly) with a collection of additional regions which are to all intents and purposes redundant. To be sure, one may simply reject the necessity of elementary equivalence to R. But this raises the problematic issue of deciding how an alternative spatial ontology *is* then to be constrained.

On the other hand, insofar as our motivation was simply to develop an elementary theory for topological reasoning about well-behaved regions in three-dimensional space, our results are more positive in character. For they show that recognizing only the regular open polyhedra as primitive spatial entities, and only *contact* as a primitive spatial relation, leads to an expressive language and a nontrivial elementary theory about which useful results can be derived. Quite what value such languages may have in real applications—for example in problems of spatial reasoning in Artificial Intelligence—is unclear at present. On the one hand, the difficulty of obtaining accurate coordinate information about the locations and shapes of objects in a real environment suggests the usefulness, in principle, of qualitative characterizations. On the other hand, it is difficult to think of practical spatial reasoning problems where purely topological reasoning is of much use. The fairest assessment is probably that the utility of such languages is still undetermined.

Form a technical point of view, three salient open questions remain. The first concerns the extension of the results obtained above to higher dimensions. The answer to this question hinges on the existence of a formula such as $\mu_N(\bar{z})$ of Corollary 4.12, stating that $\bar{z}$ forms an N-element partition of a sort which can be realized in only finitely many ways up to similar situation. (Almost all of the material in sections 3 and 4 was in effect devoted to this task.) It is unclear whether the philosophical significance of the mereotopology of higher-dimensional spaces is sufficient to justify the effort involved; as far as the author is aware, this topic has never seriously been addressed.

The second question concerns the model theory of domains of quantification less well-behaved than R, but still included in $\mathbb{P}(\mathbb{R}^3)$—most obviously, the collection of regions $\mathrm{RO}(\mathbb{R}^3)$. The analogue of Theorem 2.8 for $\mathrm{RO}(\mathbb{R}^3)$ fails, by a simple counting argument. Moreover, the fact that Theorem 2.10 can be used to derive Theorem 2.8, as outlined briefly at the end of section 5, further suggests that Theorem 2.10 fails for $\mathrm{RO}(\mathbb{R}^3)$. However, this latter derivation requires that the property of boundedness be definable over $\mathrm{RO}(\mathbb{R}^3)$, an issue which has not been settled. This question of the behaviour of $\mathcal{L}$ over non-tame domains of quantification is philosophically interesting, because its answer will determine to what extent the mereotopology of space depends on which subsets of $\mathbb{R}^3$ count as regions.

The third question concerns the prospects for characterizing the set of $\mathcal{L}$-sentences $\mathrm{Th}(R)$ syntactically. It can certainly be shown that this set is not recursively enumerable. (The corresponding result for the regular open polygons in the open plane was established by Dornheim [6]; similar techniques can be used in the three-dimensional case.) Hence, $\mathrm{Th}(R)$ has no recursive axiomatization. However, Pratt and Schoop [15] were able to characterize the theories of closely related structures in the plane using an infinitary proof-rule to encode the assumption of finite decomposability into cells. It seems certain that a similar approach would work for the present case; however, it should be admitted that the details are likely to be complicated, and the payoffs of a syntactical characterization of $\mathrm{Th}(R)$ perhaps somewhat limited.

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