

The Relational Syllogistic Revisited

Ian Pratt-Hartmann

Abstract

The relational syllogistic is the extension of the language of Classical syllogisms in which predicates are allowed to feature transitive verbs with quantified objects. It is known that the relational syllogistic does not admit a finite set of syllogism-like rules whose associated (direct) derivation relation is sound and complete. We present a modest extension of this language which does.

1 Introduction

By the *Classical syllogistic*, we mean the language featuring the four sentence-forms of standard Aristotelian syllogisms: “Every p is a q ”, “Some p is a q ”, “No p is a q ” and “Some p is not a q ”, where p and q are substituted by common (count) nouns. By the *relational syllogistic*, we mean the extension of the Classical syllogistic in which predicates are additionally allowed to feature transitive verbs with quantified objects. For example, the following is a valid argument in this language.

Some artist admires no beekeeper
Every beekeeper admires some artist
Some artist is not a beekeeper

Indeed, consider any artist, a , who admires no beekeeper. If he is not a beekeeper himself, the conclusion is certainly true. On the other hand, if he is a beekeeper, the second premise guarantees the existence of an artist b , whom he admires, and who cannot therefore be a beekeeper (since otherwise a would not admire b), whence the conclusion is again true. Thus, the relational syllogistic allows us to formulate arguments that—subjectively at least—are more difficult than those we encounter in the Classical syllogistic.

For the Classical syllogistic, the familiar collection of syllogisms constitutes, together with one or two ancillary rules, a (sound and) complete inference system: if an argument in this language is valid, in the sense that there is no possible world making its premises true and its conclusion false, then that validity can be demonstrated using syllogistic inference steps alone. This was first shown, in a slightly weakened form, by Corcoran [1] and Smiley [3]. Specifically, these authors proved certain systems of syllogism-like rules to be (sound and) *refutation-complete*: if an argument is valid, then an absurdity can be derived from its premises together with the negation of its conclusion. Subsequently,

Pratt-Hartmann and Moss [2] obtained the following results: (i) there is a finite set of syllogism-like rules in the Classical syllogistic which is sound and complete (not just refutation-complete); (ii) there is a finite set of syllogism-like rules in the relational syllogistic which is sound and refutation complete; (iii) there is *no* finite set of syllogism-like rules in the relational syllogistic which is sound and complete. In other words, for the relational syllogistic, indirect reasoning is essential.

This paper presents a modest extension of the relational syllogistic, which additionally features sentences of the form: “If there are p ’s, there are q ’s.” The following is a valid argument in this language.

If there are artists, there are carpenters
 If there are beekeepers, there are dentists
 Every carpenter admires every electrician
 No carpenter admires any flautist
 Every dentist is an electrician
Every dentist is a flautist
 No artist is a beekeeper

Indeed, suppose some artist were a beekeeper. By the first two premises, some carpenter would exist, and so would some dentist. But the remaining four premises evidently rule out the possibility that both carpenters and dentists exist, since the latter would be electricians and flautists, and hence both admired and not admired by the former.

In the sequel, we provide a finite set of syllogism-like rules for this extended relational syllogistic, and show that the resulting proof system is sound and complete. That is: the impossibility of providing a such rule-set for the relational syllogistic can be overcome by a modest increase in expressive power. Thus, the existence of a proof-system defined by a finite set of syllogism-like rules does not represent a ‘boundary’ with respect to the expressiveness of fragments of natural language.

2 The languages \mathcal{S} , \mathcal{R} and \mathcal{RE}

2.1 Syntax and semantics

In this section, we define the formal languages: \mathcal{S} , \mathcal{R} and \mathcal{RE} , corresponding to the Classical syllogistic, the relational syllogistic and the extended relational syllogistic, respectively. We review known results concerning \mathcal{S} and \mathcal{R} and, in particular, give a sketch of the proof that \mathcal{R} does not admit a finite set of syllogistic rules defining a sound and complete proof system. The motivation for defining the language \mathcal{RE} as we have should at that point be obvious.

Let \mathbf{P} and \mathbf{R} be countably infinite sets. We call the elements of \mathbf{P} *unary atoms* and the elements of \mathbf{R} *binary atoms*; we use the (possibly decorated) letters o , p , q to range over unary atoms, and r to range over binary atoms. A *unary literal* is an expression of the form p or \bar{p} , and a *binary literal* is an

expression of the form r or \bar{r} ; we use ℓ, m to range over unary literals, and t to range over binary literals. The set of c -terms is defined by:

$$c := \ell \quad | \quad \forall(p, t) \quad | \quad \exists(p, t);$$

we use c, d to range over c -terms. The set of \mathcal{S} -formulas is defined by:

$$\varphi := \forall(p, \ell) \quad | \quad \exists(p, \ell);$$

the set of \mathcal{R} -formulas is defined by:

$$\varphi := \forall(p, c) \quad | \quad \exists(p, c);$$

and the set of \mathcal{RE} -formulas is defined by:

$$\varphi := \forall(p, c) \quad | \quad \exists(p, c) \quad | \quad \exists!(p, q).$$

We refer to formulas of the form $\exists(p, c)$ as *existential formulas*, and to formulas of the forms $\forall(p, c)$ and $\exists!(p, q)$ as *universal formulas*,

Intuitively, the elements of \mathbf{P} stand for common count-nouns—“artist”, “beekeeper”, “carpenter”, etc.—and those of \mathbf{R} for transitive verbs—“despises”, “en- vies”, etc. We then read the c -terms $\forall(p, r)$ and $\exists(p, r)$ as “thing which r ’s every p ” and “thing which r ’s some p ”, respectively; and we read the sentence forms $\forall(p, c)$ and $\exists(p, c)$ as “Every p is a c ” and “Some p is a c ”, respectively. Thus we have the following informal glosses for \mathcal{R} -formulas:

$\exists(\text{artist, beekeeper})$	Some artist is a beekeeper
$\forall(\text{artist, } \forall(\text{beekeeper, admire}))$	Every artist (is a thing which) admires every beekeeper.

A negative unary literal \bar{p} is then read as “is not a p ”, and a negative binary literal \bar{r} as “does not r ”, but with negation having narrow scope with respect to the quantification of the predicate. In idiomatic English:

$\exists(\text{artist, } \overline{\text{beekeeper}})$	Some artist is not a beekeeper
$\forall(\text{artist, } \forall(\text{beekeeper, } \overline{\text{admire}}))$	No artist admires <i>any</i> beekeeper.

Finally, the additional forms $\exists!(p, q)$ in \mathcal{RE} are read “If there are p ’s, then there are q ’s,” thus:

$\exists!(\text{artist, beekeeper})$	If there are artists, then there are beekeepers.
--------------------------------------	--------------------------------------------------

We use the symbol \mathcal{S} to denote the set of all \mathcal{S} -formulas, and similarly for \mathcal{R} and \mathcal{RE} . When the language in question (\mathcal{S} , \mathcal{R} or \mathcal{RE}) is clear from context, we speak simply of *formulas*, and we use φ, ψ to range over formulas of that language. Notice that none of these languages includes formulas such as $\forall(\bar{p}, q)$ or $\exists(p, \forall(\bar{q}, r))$. Thus, we cannot say “Every *non*-artist is a beekeeper” or “Some artist admires every *non*-beekeeper” etc. We comment on the significance of this restriction below.

The semantics of these languages is given in the standard way. A *structure* is a triple $\mathfrak{A} = \langle A, \{p^{\mathfrak{A}}\}_{p \in \mathbf{P}}, \{r^{\mathfrak{A}}\}_{r \in \mathbf{R}} \rangle$, where A is a non-empty set, $p^{\mathfrak{A}} \subseteq A$, for every $p \in \mathbf{P}$, and $r^{\mathfrak{A}} \subseteq A \times A$, for every $r \in \mathbf{R}$. The set A is called the *domain* of \mathfrak{A} . We extend the map $p \mapsto p^{\mathfrak{A}}$ to all unary and binary literals by setting, for any p, r :

$$\begin{aligned}\bar{p}^{\mathfrak{A}} &= A \setminus p^{\mathfrak{A}} \\ \bar{r}^{\mathfrak{A}} &= A \times A \setminus r^{\mathfrak{A}}\end{aligned}$$

and thence to all c-terms by setting, for any p, t :

$$\begin{aligned}\forall(p, t)^{\mathfrak{A}} &= \{a \in A \mid \text{for all } b \in p^{\mathfrak{A}}, \langle a, b \rangle \in t^{\mathfrak{A}}\} \\ \exists(p, t)^{\mathfrak{A}} &= \{a \in A \mid \text{there exists } b \in p^{\mathfrak{A}} \text{ such that } \langle a, b \rangle \in t^{\mathfrak{A}}\}.\end{aligned}$$

If $a \in c^{\mathfrak{A}}$, we say that a *satisfies* c (in \mathfrak{A}), and think of a as having the property denoted by c , according to the model \mathfrak{A} . If $\ell = \bar{p}$ is a negative unary literal, we write $\bar{\ell} = p$; similarly for binary literals. We write $\bar{\forall} = \exists$ and $\bar{\exists} = \forall$. For any c-term of the form $Q(p, t)$ with $Q \in \{\forall, \exists\}$, we write $\bar{c} = \bar{Q}(p, \bar{t})$. Thus, for any c-term c , $(\bar{c})^{\mathfrak{A}} = A \setminus c^{\mathfrak{A}}$.

Having defined the notion of satisfaction for c-terms relative to a structure, we define the truth-relation \models between structures and formulas as follows:

$$\begin{aligned}\mathfrak{A} \models \forall(p, c) &\text{ iff } p^{\mathfrak{A}} \subseteq c^{\mathfrak{A}} \\ \mathfrak{A} \models \exists(p, c) &\text{ iff } p^{\mathfrak{A}} \cap c^{\mathfrak{A}} \neq \emptyset \\ \mathfrak{A} \models \exists(p, q) &\text{ iff } p^{\mathfrak{A}} \neq \emptyset \text{ implies } q^{\mathfrak{A}} \neq \emptyset\end{aligned}$$

If $\mathfrak{A} \models \theta$, we say that θ is *true* in \mathfrak{A} . As usual, when Θ is a set of formulas, we write $\mathfrak{A} \models \Theta$ if $\mathfrak{A} \models \theta$ for all $\theta \in \Theta$; in this case, we say that \mathfrak{A} is a *model* of Θ , and that Θ is *satisfiable*. If, for every structure \mathfrak{A} , $\mathfrak{A} \models \Theta$ implies $\mathfrak{A} \models \theta$, then we write $\Theta \models \theta$, and we say the argument from *premises* Θ to *conclusion* θ is *valid*. A formula of the form $\exists(p, \bar{p})$ is evidently true in no model; indeed this is the only sentence form in \mathcal{RE} having this property. We consequently refer to any formula of this form as an *absurdity*, and use \perp to stand, indifferently, for any absurdity.

It is easy to see that the formulas $\exists(p, q)$ and $\exists(q, p)$ are true in exactly the same structures, as indeed are the formulas $\forall(p, \bar{q})$ and $\forall(q, \bar{p})$. For this reason, we henceforth regard these pairs as identical, silently transforming one into the other as required. Indeed, we allow ourselves to write *any* \mathcal{RE} -formula $\exists(p, c)$ as $\exists(c, p)$, and *any* \mathcal{RE} -formula $\forall(p, c)$ as $\forall(\bar{c}, \bar{p})$, again performing these transformations silently, as required. This (inessential) notational shortcut will allow inference rules to be stated more succinctly.

2.2 Proof-theory

Let \mathcal{L} be any of the languages \mathcal{S} , \mathcal{R} or \mathcal{RE} . A *sylogistic* rule (or simply: *rule*) in \mathcal{L} is a pair Θ/θ , where Θ is a finite set (possibly empty) of \mathcal{RE} -formulas, and θ

an \mathcal{RE} -formula. We call Θ the *antecedents* of the rule, and θ its *consequent*. We generally display rules with premises and conclusion separated by a horizontal line. Thus, the rules

$$\frac{\forall(q, o) \quad \exists(p, q)}{\exists(p, o)} \qquad \frac{\forall(q, \bar{o}) \quad \exists(p, q)}{\exists(p, \bar{o})}$$

correspond to the familiar syllogisms *Darii* and *Ferio*:

$$\frac{\text{Every } q \text{ is an } o \quad \text{Some } p \text{ is a } q}{\text{Some } p \text{ is an } o} \qquad \frac{\text{No } q \text{ is an } o \quad \text{Some } p \text{ is a } q}{\text{Some } p \text{ is not an } o}.$$

To save space, and using the convention that ℓ ranges over literals, we can merge these rules as follows:

$$\frac{\forall(q, \ell) \quad \exists(p, q)}{\exists(p, \ell)} \quad (\text{A4}').$$

We employ this convention heavily in the sequel, with ℓ ranging over unary literals, t over binary literals, c, d over c-terms, and ψ over formulas. We remark that the label (A4') is simply there to identify uses of this rule in derivations, and has no formal significance.

We are invited to think of a set of rules X as specifying the allowed steps in a *derivation* involving a set of formulas, Θ , (the *premises*), and a formula, θ (the *conclusion*). Any derivation has the shape of a tree whose leaves are labelled with elements of Θ (repeats allowed), and whose root is labelled with θ . We write $\Theta \vdash_X \theta$ to indicate the existence of such a derivation. Thus, for example, if X contains the rule (A4'), the derivation

$$\frac{\forall(\text{beekeeper}, \overline{\text{dentist}}) \quad \frac{\forall(\text{artist}, \text{carpenter}) \quad \exists(\text{artist}, \text{beekeeper})}{\exists(\text{carpenter}, \text{beekeeper})} \quad (\text{A4}')}{\exists(\text{carpenter}, \overline{\text{dentist}})} \quad (\text{A4}')$$

shows that

$$\{\exists(\text{artist}, \text{beekeeper}), \forall(\text{artist}, \text{carpenter}), \forall(\text{beekeeper}, \overline{\text{dentist}})\} \vdash_X \exists(\text{carpenter}, \overline{\text{dentist}}).$$

It is important to realize in this regard that all proof systems considered here are *direct*, proceeding step-by-step from premises to conclusion. In particular, they incorporate no mechanism of *indirect* proof, in which a conclusion is derived by first supposing its negation and then deriving an absurdity. (We hasten to add that there is nothing *wrong* with indirect proofs: we are simply interested in the circumstances under which they are dispensable.)

The rule-sets X of most interest to us are those whose associated derivation relation, \vdash_X , constitutes necessary and sufficient conditions for entailment. Formally, we say that \vdash_X is *sound* if $\Theta \vdash_X \theta$ implies $\Theta \models \theta$, and *complete* if $\Theta \models \theta$ implies $\Theta \vdash_X \theta$. In this paper, we shall be concerned exclusively with *finite*

rule-sets X , since there trivially exist infinite (computable) rule-sets for \mathcal{S} , \mathcal{R} and \mathcal{RE} whose associated derivation relation is sound and complete. A set of formulas Θ is said to be *inconsistent* (with respect to a derivation relation \vdash) if there is a derivation of some absurdity from Θ . We say that \vdash_{X} is *refutation-complete* if any unsatisfiable set of formulas is inconsistent. It is easy to see that completeness implies refutation-completeness, but not, it transpires, vice versa. Indeed, the following results were shown in Pratt-Hartmann and Moss [2].

Theorem 2.1. *There exists a finite set X of syllogistic rules in \mathcal{S} such that \vdash_{X} is sound and complete.*

Theorem 2.2. *There exists a finite set X of syllogistic rules in \mathcal{R} such that \vdash_{X} is sound and refutation-complete.*

Theorem 2.3. *There exists no finite set X of syllogistic rules in \mathcal{R} such that \vdash_{X} is sound and complete.*

The sole result of this paper is

Theorem 2.4. *There exists a finite set X of syllogistic rules in \mathcal{RE} such that \vdash_{X} is sound and complete.*

We are now in a position to understand the motivation for the particular language, \mathcal{RE} , that we are concerned with in this paper. To do so, we need to consider the proof Theorem 2.3, which we sketch here informally; for full details, see Pratt-Hartmann and Moss [2], pp. 661 ff.

Sketch proof of Theorem 2.3. For n positive, let p_1, \dots, p_n be distinct unary atoms, r a binary atom, γ_n the \mathcal{R} -formula $\forall(p_1, \exists(p_n, r))$ and Γ_n the set of \mathcal{R} -formulas:

$$\begin{aligned} &\forall(p_i, \exists(p_{i+1}, r)) && (1 \leq i < n) \\ &\forall(p_1, \forall(p_n, r)). \end{aligned}$$

We claim that $\Gamma_n \models \gamma_n$. Indeed, suppose $\mathfrak{A} \models \Gamma_n$. If $p_1^{\mathfrak{A}} = \emptyset$, then trivially $\mathfrak{A} \models \gamma_n$; on the other hand, if $p_1^{\mathfrak{A}} \neq \emptyset$, a simple induction using the formulas $\forall(p_i, \exists(p_{i+1}, r))$ shows that $p_n^{\mathfrak{A}} \neq \emptyset$, whence, from $\forall(p_1, \forall(p_n, r))$, $\mathfrak{A} \models \gamma_n$. Call any formula of the form $\forall(p, p)$ *trivial*. For any i ($1 \leq i \leq n$), let $\Gamma_{n,i}$ be the result from removing from Γ_n the formula $\forall(p_i, \exists(p_{i+1}, r))$. It can be shown, by a detailed examination of cases, that $\Gamma_{n,i}$ has *no* non-trivial consequences expressible in the language \mathcal{R} that are not already elements of Γ_n . Now consider any finite set X of rules in \mathcal{R} , and let n be greater than the maximum number of premises in any of these rules. For any instance, Θ/θ , of one of these rules, if $\Theta \subseteq \Gamma_n$, then $\Theta \subseteq \Gamma_{n,i}$ for some i ($1 \leq i \leq n$). Hence, if the rules of X are valid, θ must be trivial or contained in Γ_n . In other words, the rules of X never yield non-trivial conclusions outside Γ_n , and in particular never yield γ_n . Therefore \vdash_{X} is not complete. \square

It is easy to see what is going wrong here: $\Gamma_{n,i}$ does indeed entail non-trivial conclusions of the form $\exists(p_1, p_j)$ for $j \leq i$; however, these formulas are not in \mathcal{R} , and so cannot feature in derivations in \mathcal{R} . But now suppose we work within the language \mathcal{RE} , where we can write rules

$$\frac{\forall(p, \exists(q, r))}{\exists(p, q)} \text{ (C4)} \quad \frac{\exists(o, p) \quad \exists(p, q)}{\exists(o, q)} \text{ (C2)} \quad \frac{\forall(p, \forall(q, t)) \quad \exists(p, q)}{\forall(p, \exists(q, t))} \text{ (D1)}.$$

We may derive $\exists(p_1, p_2)$, $\exists(p_1, p_3)$ and $\exists(p_1, p_4)$ as follows:

$$\frac{\frac{\frac{\forall(p_1, \exists(p_2, r))}{\exists(p_1, p_2)} \text{ (C4)} \quad \frac{\forall(p_2, \exists(p_3, r))}{\exists(p_2, p_3)} \text{ (C4)}}{\exists(p_1, p_3)} \text{ (C2)} \quad \frac{\forall(p_3, \exists(p_4, r))}{\exists(p_3, p_4)} \text{ (C4)}}{\exists(p_1, p_4)} \text{ (C2)}.$$

⋮

Continuing in this way, we can derive $\exists(p_1, p_i)$ for all i ($2 \leq i \leq n$), and hence derive γ_n , thus:

$$\frac{\begin{array}{c} \vdots \\ \exists(p_1, p_n) \end{array} \quad \forall(p, \forall(q, r))}{\forall(p, \exists(q, r))} \text{ (D1)}$$

The existence of such a derivation is no accident: the formulas $\exists(p_i, p_j)$ are precisely what we need to add to \mathcal{R} in order to secure the existence of a sound and complete set of rules.

We mentioned above that the languages \mathcal{S} and \mathcal{R} do not include the formulas $\forall(\bar{p}, q)$ or $\exists(\bar{p}, \bar{q})$, and indeed that \mathcal{R} does not feature the c-terms $\forall(\bar{p}, t)$ or $\exists(\bar{p}, t)$. That is, we cannot say in these languages “Every non-artist is a beekeeper”, or “Every artist admires some non-beekeeper”. Extensions of \mathcal{S} and \mathcal{R} featuring such ‘noun-level’ negation were investigated in detail in Pratt-Hartmann and Moss [2], where they are denoted by \mathcal{S}^\dagger and \mathcal{R}^\dagger , respectively. It was shown there that \mathcal{S}^\dagger has a finite system of syllogistic rules which is sound and complete; however, there can be no such system for \mathcal{R}^\dagger , even with un-restricted use of *reductio ad absurdum*. It was further shown that the satisfiability problem for \mathcal{R}^\dagger is EXPTIME-complete. It then follows from the fact that $\text{PTIME} \neq \text{EXPTIME}$ that no extension of \mathcal{R}^\dagger can have a sound and complete (or indeed sound and refutation-complete) system of syllogistic rules. Therefore, the languages \mathcal{S}^\dagger and \mathcal{R}^\dagger will not be further discussed in this paper.

2.3 A sound and complete rule-set for \mathcal{RE}

We now proceed to define a rule-set, RE, which we later show to be sound and complete for \mathcal{RE} , under application of the proof-theoretic machinery outlined above, thus proving Theorem 2.4. Since RE is very complicated, we divide the rules into groups, with brief remarks concerning their validity.

The first group consists of simple generalizations of familiar syllogistic rules:

$$\begin{array}{ccc} \frac{}{\forall(p, p)} \text{ (A1)} & \frac{\forall(p, q) \quad \forall(q, c)}{\forall(p, c)} \text{ (A2)} & \frac{\exists(p, q)}{\exists(p, p)} \text{ (A3)} \\ \\ \frac{\exists(p, c) \quad \forall(p, q)}{\exists(q, c)} \text{ (A4)} & \frac{\exists(p, c) \quad \forall(q, \bar{c})}{\exists(p, \bar{q})} \text{ (A5)} & \frac{\exists(p, q) \quad \forall(q, c)}{\exists(p, c)} \text{ (A6)} \\ \\ \frac{\exists(p, \bar{p})}{\psi} \text{ (A7)} & \frac{\forall(p, \bar{p})}{\forall(p, c)} \text{ (A8)} & \end{array}$$

Most of these rules should be self-explanatory: indeed, we have already met the rule (A4), though in a slightly restricted form (A4'), where c was limited to literals. The rule (A1) allows us to infer $\forall(p, p)$ from no premises. The rule (A7) is the rule of *ex falso quodlibet*, and allows us to infer anything from an absurdity. Do not confuse this rule with the strategy of *reductio ad absurdum*, which allows us to retract a premise and infer its negation having derived an absurdity: *reductio ad absurdum* cannot be understood—in the technical sense employed in this paper—as a syllogistic rule.

The second group consists of rules governing quantified predicates. In these rules, the subject-quantifier, Q , stands for either \forall or \exists .

$$\begin{array}{ccc} \frac{Q(o, \forall(p, t)) \quad \exists(p, q)}{Q(o, \exists(q, t))} \text{ (B1)} & \frac{Q(o, \forall(q, t)) \quad \forall(p, q)}{Q(o, \forall(p, t))} \text{ (B2)} \\ \\ \frac{Q(o, \exists(p, t)) \quad \forall(p, q)}{Q(o, \exists(q, t))} \text{ (B3)} & \frac{\forall(q, \bar{q})}{\forall(p, \forall(q, t))} \text{ (B4)} \\ \\ \frac{\exists(p, \exists(q, t))}{\exists(q, q)} \text{ (B5)}. & & \end{array}$$

These rules are almost trivially valid, and require no comment.

The third group consists of rules governing the quantifier $\exists!$

$$\begin{array}{ccc} \frac{}{\exists!(p, p)} \text{ (C1)} & \frac{\exists!(o, p) \quad \exists!(p, q)}{\exists!(o, q)} \text{ (C2)} & \frac{\forall(p, q)}{\exists!(p, q)} \text{ (C3)} \\ \\ \frac{\forall(p, \exists(q, r))}{\exists!(p, q)} \text{ (C4)} & \frac{\exists(q, q)}{\exists!(p, q)} \text{ (C5)} & \frac{\exists(p, p) \quad \exists!(p, q)}{\exists(q, q)} \text{ (C6)} \\ \\ \frac{\forall(q, \bar{q}) \quad \exists!(p, q)}{\forall(p, c)} \text{ (C7)}. & & \end{array}$$

Notice that $(B\exists)$ expresses the transitivity of the relation $\exists(p, q)$. The rule (C7) may be surprising at first sight. The premises assert that there are no qs , and that if there are ps , then there are qs ; it follows, of course, that there are no ps , and hence that every p is a c , for any c -term c .

The fourth group concerns interactions between formulas with quantified predicates and formulas involving the quantifier \exists

$$\frac{\forall(p, \forall(q, t)) \quad \exists(p, q)}{\forall(p, \exists(q, t))} \quad (D1)$$

$$\frac{\exists(q, q) \quad \exists(p, o) \quad \forall(o, q) \quad \forall(o, \forall(p, r))}{\exists(q, \forall(p, r))} \quad (D2)$$

$$\frac{\exists(q, q) \quad \exists(p, o) \quad \forall(o, \bar{p}) \quad \forall(o, q)}{\exists(q, \bar{p})} \quad (D3)$$

$$\frac{\forall(p, \forall(q', r)) \quad \forall(o, q) \quad \forall(o, q') \quad \exists(p, o)}{\forall(p, \exists(q, r))} \quad (D4)$$

$$\frac{\forall(o, \forall(o', t')) \quad \forall(o, \forall(o', \bar{t}')) \quad \exists(p, o) \quad \exists(q, o')}{\forall(p, \forall(q, t))} \quad (D5)$$

$$\frac{\forall(o, \forall(o', t')) \quad \forall(o, \forall(o', \bar{t}')) \quad \exists(p, o) \quad \exists(q, o')}{\forall(p, \bar{q})} \quad (D6).$$

Consider (D1). The premises assert that every p is related by t to every q , and that, if ps exist then q 's exist. If no ps exist, the conclusion is vacuously true; otherwise, we may choose some q to which every p is related, and the conclusion is certainly true.

The fifth group allows records slightly less obvious inferences involving formulas with quantified predicates. Rule $(\exists\forall)$ has the form of the argument displayed on the first page of this article, and we have already demonstrated its validity; the other rules in this group may be treated similarly.

$$\frac{\forall(o, \forall(p, t)) \quad \exists(o, \forall(q, \bar{t}))}{\forall(p, \bar{q})} \quad (E1) \qquad \frac{\forall(q, \exists(p, t)) \quad \exists(p, \forall(q, \bar{t}))}{\exists(p, \bar{q})} \quad (E2)$$

$$\frac{\exists(p, \exists(p, t)) \quad \forall(q, \forall(q, \bar{t}))}{\exists(p, \bar{q})} \quad (E3).$$

The sixth group allows us to conclude, from premises with quantified predi-

cates, a conclusion of the form $\exists(p, \bar{q})$:

$$\frac{\exists(p, \forall(o, t)) \quad \forall(q, \forall(q, \bar{t})) \quad \forall(o, p) \quad \exists(q, o)}{\exists(p, \bar{q})} \text{ (F1)}$$

$$\frac{\exists(p, \forall(q, t)) \quad \forall(q, \forall(o, \bar{t})) \quad \forall(o, p) \quad \exists(q, o)}{\exists(p, \bar{q})} \text{ (F2)}$$

$$\frac{\forall(o', \exists(p, t)) \quad \forall(o', \forall(q, \bar{t})) \quad \exists(q, o') \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F3)}$$

$$\frac{\forall(o, \forall(o', t)) \quad \forall(o, \forall(q, \bar{t})) \quad \exists(q, o) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(p, p)}{\forall(p, \bar{q})} \text{ (F4)}$$

$$\frac{\forall(o', \exists(p, t)) \quad \forall(q, \forall(q, \bar{t})) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F5)}$$

$$\frac{\forall(o', \forall(q, t)) \quad \forall(q, \exists(p, \bar{t})) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F6)}$$

$$\frac{\forall(o', \forall(o, t)) \quad \forall(q, \forall(o, \bar{t})) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(q, o) \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F7)}$$

$$\frac{\forall(o', \forall(o, t)) \quad \forall(q, \forall(q, \bar{t})) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(q, o) \quad \forall(o, p) \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F8)}$$

$$\frac{\forall(o', \forall(q, t)) \quad \forall(q, \forall(o, \bar{t})) \quad \exists(q, o') \quad \forall(o', p) \quad \exists(q, o) \quad \forall(o, p) \quad \exists(p, p)}{\exists(p, \bar{q})} \text{ (F9)}.$$

We demonstrate the validity of the most complicated of these rules, (F9). From the last premise, $\exists(p, p)$, let a be some p . If a is not a q , the conclusion is true. Otherwise, from the premises $\exists(q, o')$ and $\exists(q, o)$, let b, b' satisfy o, o' , respectively. From $\forall(o, p)$, and $\forall(o', p)$, b and b' are both ps . If either of these is not a q , the conclusion is true. Otherwise, from $\forall(q, \forall(o, \bar{t}))$, and $\forall(o', \forall(q, t))$, b' both is and is not related to b by t —contradiction. The validity of rules (F1)–(F8) may be demonstrated by similar arguments.

The seventh group allows us to conclude, from premises with quantified predicates, a conclusion of the form $\forall(p, \bar{q})$.

For any $o_1, o_2, o_3, o_4 \in \{p, q\}$:

$$\frac{\forall(o_1, \forall(p', t)) \quad \forall(o_2, \forall(q', \bar{t})) \quad \forall(o_3, p') \quad \forall(o_4, q')}{\forall(p, \bar{q})} \text{ (G1)}$$

$$\frac{\forall(o_1, \forall(q', t)) \quad \forall(o_2, \forall(q', \bar{t})) \quad \exists(o_3, o') \quad \forall(o', p') \quad \forall(o', q')}{\forall(p, \bar{q})} \text{ (G2)}$$

$$\frac{\exists(o_1, o) \quad \forall(o, \forall(p', t)) \quad \forall(o, \forall(q', \bar{t})) \quad \forall(o_2, p') \quad \forall(o_3, q')}{\forall(p, \bar{q})} \text{ (G3)}$$

$$\frac{\exists(p, o) \quad \exists(q, o') \quad \forall(o, \forall(p', t)) \quad \forall(o, \forall(q', \bar{t})) \quad \forall(o', p') \quad \forall(o', q')}{\forall(p, \bar{q})} \text{ (G4)}.$$

To demonstrate the validity of (G1), suppose, to the contrary, that some p is a q . Choose any such element, a . Then, no matter how o_1, \dots, o_4 are chosen from p and q , the premises guarantee that a is a p' and a q' , and hence both is and is not related to itself by t —contradiction. The validity of rules (F2)–(F4) may be demonstrated by similar arguments.

We make no claim regarding the non-redundancy of the rule-set RE.

3 The proof

This section is devoted entirely to the proof of Theorem 2.4. Soundness of \vdash_{RE} is straightforward, since, as may readily be verified, every rule in RE is valid. It therefore remains to show that, if Φ is a set of \mathcal{RE} -formulas, and ψ is an \mathcal{RE} -formula such that $\Phi \models \psi$, then $\Phi \vdash_{\text{RE}} \psi$. Since RE is the only rule-set we shall be concerned with in the sequel, we henceforth write \vdash in place of \vdash_{RE} . In addition, we continue to take the letters o, p, q to range over unary atoms, ℓ, m over unary literals, r over binary atoms, t over binary literals, c, d over c-terms, and φ, ψ over formulas, without further comment.

3.1 Two transitive closures

Let Π be a set of universal formulas. Define the binary relation $\xrightarrow{\Pi}$ to be the smallest reflexive, transitive relation on unary atoms satisfying

$$p \xrightarrow{\Pi} q \quad \text{if } \Pi \text{ contains } \forall(p, q), \forall(p, \exists(q, t)) \text{ or } \exists(p, q).$$

The key lemma regarding $\xrightarrow{\Pi}$ is:

Lemma 3.1. *If $p \xrightarrow{\Pi} q$, then $\Phi \vdash \exists(p, q)$.*

Proof. Immediate given the rules

$$\frac{}{\exists(p, p)} \text{ (C1)} \qquad \frac{\exists(o, p) \quad \exists(p, q)}{\exists(o, q)} \text{ (C2)}$$

$$\frac{\forall(p, q)}{\exists(p, q)} \text{ (C3)} \qquad \frac{\forall(p, \exists(q, r))}{\exists(p, q)} \text{ (C4)}.$$

□

Let Π be a set of universal formulas. Define the binary relation $\xRightarrow{\Pi}$ to be the smallest reflexive, transitive relation on c-terms satisfying

$$\begin{array}{ll} p \xRightarrow{\Pi} c & \text{if } \forall(p, c) \in \Pi \\ \forall(q, t) \xRightarrow{\Pi} \forall(p, t) & \text{if } \forall(p, q) \in \Pi \\ \exists(p, t) \xRightarrow{\Pi} \exists(q, t) & \text{if } \forall(p, q) \in \Pi. \end{array}$$

The key lemma regarding $\xRightarrow{\Pi}$ is:

Lemma 3.2. *If $p \xRightarrow{\Pi} d$, then $\Pi \vdash \forall(p, d)$.*

Proof. Immediate given the rules

$$\frac{}{\forall(p, p)} \text{ (A1)} \qquad \frac{\forall(p, q) \quad \forall(q, c)}{\forall(p, c)} \text{ (A2)}$$

$$\frac{\forall(o, \exists(p, t)) \quad \forall(p, q)}{\forall(o, \exists(q, t))} \text{ (B3)} \qquad \frac{\forall(o, \forall(q, t)) \quad \forall(p, q)}{\forall(o, \forall(p, t))} \text{ (B2)}.$$

□

Lemma 3.3. *If $p \xRightarrow{\Pi} q$, then $\forall(q, t) \xRightarrow{\Pi} \forall(p, t)$ and $\exists(p, t) \xRightarrow{\Pi} \exists(q, t)$.*

Proof. Obvious. □

3.2 Some technical machinery

Let s be any set of c-terms. We think of s as a description of an individual: that individual satisfies all the c-terms in s . For any set Σ of existential formulas, define

$$U_{\Sigma} = \{\{p, c\} \mid \exists(p, c) \in \Sigma\}$$

We think of U_{Σ} as a collection of (descriptions of) individuals that obviously exist on the assumption that the formulas in Σ are true. In addition, for any

set Σ of existential formulas any set Π of universal formulas, define

$$\begin{aligned} O_{\Sigma, \Pi} &= \left\{ o \mid p \xrightarrow{\Pi} o \text{ for some } p \in \bigcup U_{\Sigma} \right\} \\ &\quad \bigcup \left\{ o \mid p \xrightarrow{\Pi} o \text{ for some } \exists(p, t) \in \bigcup U_{\Sigma} \right\} \\ V_{\Sigma, \Pi} &= U_{\Sigma} \cup \{ \{o\} \mid o \in O_{\Sigma, \Pi} \}. \end{aligned}$$

Think of $O_{\Sigma, \Pi}$ as the set of unary predicates that must be instantiated on the assumption that the formulas in $\Sigma \cup \Pi$ are true; and think of $V_{\Sigma, \Pi}$ as a set of (descriptions of) individuals that exist on the assumption that the formulas in $\Sigma \cup \Pi$ are true.

For any set s of c-terms and any set Π of universal formulas, define

$$s^{\Pi} = \left\{ c \mid p \xrightarrow{\Pi} c \text{ for some } p \in s \right\}.$$

We think of s^{Π} as an elaboration of the description s : any individual described by s is in fact described by s^{Π} , provided the formulas in Π are all true. We remark in passing that it is useful to imagine s^{Π} as the union of a sequence s_0, s_1, \dots of sets of c-terms, defined inductively as follows

$$\begin{aligned} s_0 &= s \\ s_{3k+1} &= \{ c \mid p \in s_{3k} \text{ and } \forall(p, c) \in \Pi \} \\ s_{3k+2} &= \{ \forall(p, t) \mid \forall(q, t) \in s_{3k+1} \text{ and } \forall(p, q) \in \Pi \} \\ s_{3(k+1)} &= \{ \exists(q, t) \mid \exists(p, t) \in s_{3k+2} \text{ and } \forall(p, q) \in \Pi \}. \end{aligned}$$

We refer to the sequence s_0, s_1, \dots as the *staged construction* of s^{Π} .

Finally, define

$$W_{\Sigma, \Pi} = \{ s^{\Pi} \mid s \in V_{\Sigma, \Pi} \}.$$

We think of $W_{\Sigma, \Pi}$ as a collection of elaborated descriptions of the individuals that must exist on the assumption that the formulas in $\Sigma \cup \Pi$ are true. Sometimes, things can go wrong with $W_{\Sigma, \Pi}$. Suppose $w \in W_{\Sigma, \Pi}$ and, for some c-term c , both c and \bar{c} are elements of w : in that case, the description w cannot be satisfied, and we say that $W_{\Sigma, \Pi}$ contains a *local defect* at w . Or suppose that $u, v \in W_{\Sigma, \Pi}$ and, for some unary atoms p, q and binary atom r , u contains both $\forall(p, r)$ and $\forall(q, \bar{r})$, while v contains both p and q : in that case, the descriptions u and v cannot be simultaneously satisfied, and we say that $W_{\Sigma, \Pi}$ contains a *global defect* involving u and v .

Given a set $W_{\Sigma, \Pi}$, let $A = W_{\Sigma, \Pi} \times \{0, 1\}$, and define structures \mathfrak{A} and $\bar{\mathfrak{A}}$

over A by setting, for all $p \in \mathbf{P}$ and $r \in \mathbf{R}$:

$$\begin{aligned}
p^{\mathfrak{A}} &= p^{\bar{\mathfrak{A}}} = \{(u, i) \in U \mid p \in u, i \in \{0, 1\}\} \\
r^{\mathfrak{A}} &= \left\{ \langle (u, i), (\{p\}^{\Pi}, 1) \rangle \in U \times U \mid \exists (p, r) \in u \right\} \cup \\
&\quad \left\{ \langle (u, i), (v, j) \rangle \in U \times U \mid \text{there exists } p \in v \text{ s.t. } \forall (p, r) \in u \right\} \\
r^{\bar{\mathfrak{A}}} &= A \times A \setminus \\
&\quad \left(\left\{ \langle (u, i), (\{p\}^{\Pi}, 1) \rangle \in U \times U \mid \exists (p, \bar{r}) \in u \right\} \cup \right. \\
&\quad \left. \left\{ \langle (u, i), (v, j) \rangle \in U \times U \mid \text{there exists } p \in v \text{ s.t. } \forall (p, \bar{r}) \in u \right\} \right).
\end{aligned}$$

Write $\mathfrak{A}_{\Sigma, \Pi} = \mathfrak{A}$ and $\bar{\mathfrak{A}}_{\Sigma, \Pi} = \bar{\mathfrak{A}}$ when we need to make the parameters Σ and Π explicit.

A word of motivation is in order here. For any description $w \in W_{\Sigma, \Pi}$, the domain A contains a pair of objects: $(w, 0)$ and $(w, 1)$. It helps to think of these objects as individuals which ‘want’ to satisfy the description w ; the construction of \mathfrak{A} and $\bar{\mathfrak{A}}$ ensures that, in felicitous circumstances, these desires are satisfied. The doubling of individuals is motivated by the need to provide witnesses for certain c-terms: in \mathfrak{A} , the individual $(\{q\}^{\Pi}, 1)$ serves as witnesses for c-terms of the form $\exists(q, r)$, while the individual $(\{q\}^{\Pi}, 0)$ serves as a witness for c-terms of the form $\exists(q, \bar{r})$; in $\bar{\mathfrak{A}}$, these roles are reversed. Both structures interpret unary atoms in the natural way: (w, i) is taken to satisfy p just in case $p \in w$. For binary atoms, \mathfrak{A} and $\bar{\mathfrak{A}}$ employ opposite strategies, with \mathfrak{A} , roughly speaking, making extensions as small as possible, and $\bar{\mathfrak{A}}$ making them as large as possible. In particular, for elements $a = (u, i)$ and $b = (v, j)$, the structure \mathfrak{A} takes a to stand in relation r to b if and only if, for some unary atom q , either $\exists(q, r) \in u$ and $b = (\{q\}^{\Pi}, 1)$ or $\forall(q, r) \in v$ and $q \in v$. By contrast, the structure $\bar{\mathfrak{A}}$ takes a always to stand in relation r to b unless, for some unary atom q , either $\exists(q, \bar{r}) \in u$ and $b = (\{q\}^{\Pi}, 1)$ or $\forall(q, \bar{r}) \in v$ and $q \in v$. Equivalently, $\bar{\mathfrak{A}}$ takes a to stand in relation r to b if and only if, for every unary atom q , $\exists(q, \bar{r}) \in u$ implies $b \neq (\{q\}^{\Pi}, 1)$ and $\forall(q, \bar{r}) \in v$ implies $q \notin v$.

Lemma 3.4. *Let Σ be any set of existential formulas and Π any set of universal formulas. Let $c \in u \in W_{\Sigma, \Pi}$ and $i \in \{0, 1\}$. If $W_{\Sigma, \Pi}$ contains no defects, then $(u, 0)$ and $(u, 1)$ satisfy c in both $\mathfrak{A}_{\Sigma, \Pi}$ and $\bar{\mathfrak{A}}_{\Sigma, \Pi}$.*

Proof. Write $\mathfrak{A} = \mathfrak{A}_{\Sigma, \Pi}$ and $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_{\Sigma, \Pi}$. We show that $a \in c^{\mathfrak{A}}$. If c is of any of the forms p , $\exists(p, r)$, $\forall(p, r)$, this is immediate by the construction of \mathfrak{A} . If $c = \bar{p}$, then $p \notin u$, since W contains no local defects, whence $a \in c^{\mathfrak{A}}$. If $c = \exists(p, \bar{r})$, then, $c \in u \in W_{\Sigma, \Pi}$ implies $\{p\}^{\Pi} \in W_{\Sigma, \Pi}$, and hence $b = (\{p\}^{\Pi}, 0) \in p^{\mathfrak{A}}$. To establish $a \in c^{\mathfrak{A}}$, it suffices to show that $\langle a, b \rangle \notin r^{\mathfrak{A}}$. By definition of \mathfrak{A} , $\langle a, b \rangle \in r^{\mathfrak{A}}$ only if there exists $q \in \{p\}^{\Pi}$ such that $\forall(q, r) \in u$. But then, by Lemma 3.3 and the fact that u is closed under $\xrightarrow{\Pi}$, we have $\forall(p, r) \in u$, contrary to the hypothesis that $W_{\Sigma, \Pi}$ contains no local defect. If $c = \forall(p, \bar{r})$, suppose $b = (v, j) \in p^{\mathfrak{A}}$, so that $p \in v$. To establish $a \in c^{\mathfrak{A}}$, it suffices to show that

$\langle a, b \rangle \notin r^{\mathfrak{A}}$. By definition of \mathfrak{A} , $\langle a, b \rangle \in r^{\mathfrak{A}}$ only if, for some q , either (i) $v = \{q\}^{\Pi}$, $j = 1$ and $\exists(q, r) \in u$, or (ii) $q \in v$ and $\forall(q, r) \in u$. In case (i), by Lemma 3.3 and the fact that u is closed under $\xrightarrow{\Pi}$, we have $\exists(p, r) \in u$, contrary to the hypothesis that $W_{\Sigma, \Pi}$ contains no local defect. Case (ii) is instantly contrary to the hypothesis that $W_{\Sigma, \Pi}$ contains no global defect.

To show that $a \in c^{\mathfrak{A}}$, we proceed in exactly the same way, except that references to r and \bar{r} are exchanged. \square

Lemma 3.5. *Let Σ be any set of existential formulas and Π any set of universal formulas. If $W_{\Sigma, \Pi}$ contains no defects, then both $\mathfrak{A}_{\Sigma, \Pi}$ and $\bar{\mathfrak{A}}_{\Sigma, \Pi}$ are models of $\Sigma \cup \Pi$.*

Proof. Write \mathfrak{A} ambiguously for $\mathfrak{A}_{\Sigma, \Pi}$ or $\bar{\mathfrak{A}}_{\Sigma, \Pi}$, and A for the domain of \mathfrak{A} . Consider first any $\varphi = \exists(p, c) \in \Sigma$. Then $a = (\{p, c\}^{\Pi}, 1) \in A$. By Lemma 3.4, $a \in p^{\mathfrak{A}} \cap c^{\mathfrak{A}}$; therefore, $\mathfrak{A} \models \varphi$. Consider next any $\varphi = \forall(p, c) \in \Pi$, and suppose $a = (u, i) \in p^{\mathfrak{A}}$. By definition of \mathfrak{A} , $p \in u$. Taking account of the fact that u is closed under the relation $\xrightarrow{\Pi}$, we have $c \in u$, whence, by Lemma 3.4, $a \in c^{\mathfrak{A}}$; therefore, $\mathfrak{A} \models \varphi$. Consider finally any $\varphi = \exists!(p, q) \in \Pi$, and suppose $a = (u, i) \in p^{\mathfrak{A}}$. By definition of \mathfrak{A} , $p \in u$, whence $q \in O_{\Sigma, \Pi}$, whence $b = (\{q\}^{\Pi}, 1) \in A$. By definition of \mathfrak{A} , $b \in q^{\mathfrak{A}}$; therefore, $\mathfrak{A} \models \varphi$. \square

3.3 The completeness proof

Fix a set of formulas Φ and a formula ψ . Our task is to demonstrate that, if $\Phi \models \psi$, then $\Phi \vdash \psi$. We consider the various forms of ψ separately. Write $\Phi = \Lambda \cup \Gamma$, where Λ is the set of universal formulas in Φ , and Γ , the set of existential formulas in Φ . (Recall that formulas of the form $\exists!(p, q)$ count as universal.)

Lemma 3.6. *If $p \in w \in W_{\Lambda, \Gamma}$, then $\Phi \vdash \exists(p, p)$.*

Proof. We first prove the result for $p \in s \in V_{\Lambda, \Gamma}$. If $s = \{p, c\} \in U_{\Lambda}$, the result is immediate from the definition of U_{Λ} and the rule (A3). So we may assume $p = c$ and either there exists $\{p', c'\} \in U_{\Lambda}$ such that $p' \xrightarrow{\Pi} p$, or there exists $\{p', \exists(q', r')\} \in U_{\Lambda}$ such that $q' \xrightarrow{\Pi} p$. Noting Lemma 3.1, we have either of the derivations

$$\frac{\frac{\exists(p', c')}{\exists(p', p')} \text{ (A3)} \quad \frac{\vdots}{\exists!(p', p)} \text{ (C6)}}{\exists(p, p)} \quad \frac{\frac{\exists(p', \exists(q', r'))}{\exists(q', q')} \text{ (B5)} \quad \frac{\vdots}{\exists!(q', p)} \text{ (C6)}}{\exists(p, p)}$$

Finally, suppose $p' \xrightarrow{\Pi} p$ for some $p' \in s \in V_{\Lambda, \Gamma}$. By Lemma 3.2, $\Phi \vdash \forall(p', p)$,

whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p', p')} \quad \frac{\vdots}{\forall(p', p)}}{\exists(p, p')} \text{ (A4)} \quad \frac{\vdots}{\forall(p', p)} \text{ (A6)}}{\exists(p, p)} \text{ (A6)}.$$

□

Let us say that c-terms c and d are *sisters* if they are identical, or if there exist $Q \in \{\forall, \exists\}$, unary literals p and q , and binary literal t , such that $c = Q(p, t)$ and $d = Q(q, t)$. Notice in particular that no pair of c-terms $\forall(p, r)$ and $\forall(q, \bar{r})$ can be sisters.

Lemma 3.7. *Let u be an element of $W_{\Lambda, \Gamma}$.*

- (i) *If $c, d \in u$ with c and d not sisters, then there exist $o \in u$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, c)$ and $\Phi \vdash \bar{Q}(o, d)$.*
- (ii) *If $q, d \in u$, then $\Phi \vdash \exists(q, d)$.*

Proof. Any element of $W_{\Lambda, \Gamma}$ has the form s^Γ , with $s = \{p, c\} \in V_{\Lambda, \Gamma}$; recall that we earlier described the staged construction of s^Γ . Suppose c is introduced at stage i and d at stage j : we proceed by induction on $i + j$.

If $i = j = 0$ then, since c and d are not identical, $\{c, d\} = \{p, q\} \in U_\Lambda$, whence $\Phi \vdash \exists(p, c)$, by definition of U_Λ . Moreover, from rule (A1), we have $\Phi \vdash \forall(p, p)$. Putting $o = p$ thus secures the lemma. Suppose, then $i + j > 0$, and assume without loss of generality that c is introduced in stage s_m with $m > 0$. If $m = 3k + 1$, let $p \in s_{3k}$ such that $\forall(p, c) \in \Gamma$. If $p = d$, then, by Lemma 3.6, $\Phi \vdash \exists(d, d)$, so that putting $o = p$ again secures the lemma. On the other hand, if $p \neq d$, then, since p is not the sister anything but itself, we have by inductive hypothesis o and Q such that $\Phi \vdash Q(o, p)$ and $\Phi \vdash \bar{Q}(o, d)$. But then we have either of the derivations

$$\frac{\frac{\frac{\vdots}{\forall(o, p)} \quad \frac{\vdots}{\forall(p, c)}}{\forall(o, c)} \text{ (A2)} \quad \frac{\frac{\frac{\vdots}{\exists(o, p)} \quad \frac{\vdots}{\forall(p, c)}}{\exists(o, c)} \text{ (A6)}}{\exists(o, c)} \text{ (A6)}.$$

If $m = 3k + 2$, let $c = \forall(p, t)$ and suppose $\forall(q, t) \in s_{3k+1}$ with $\forall(p, q) \in \Gamma$. If c is not d 's sister, then neither is $\forall(q, t)$, so, by inductive hypothesis, let o and Q be such that $\Phi \vdash Q(o, p)$ and $\Phi \vdash \bar{Q}(o, d)$. But then we have the derivation

$$\frac{\frac{\frac{\vdots}{Q(o, \forall(q, t))} \quad \frac{\vdots}{\forall(p, q)}}{Q(o, \forall(p, t))} \text{ (B2)}}{Q(o, \forall(p, t))} \text{ (B2)}.$$

If $m = 3(k + 1)$, we proceed similarly, but using rule $(\exists\forall)$ instead of $(\forall\forall)$.

The second statement of the lemma follows from the first by a single application of either (A4) or (A6). □

Lemma 3.8. *If $W_{\Lambda, \Gamma}$ contains a local defect, then Φ is inconsistent.*

Proof. Let $u \in W_{\Lambda, \Gamma}$ contain c, \bar{c} . By Lemma 3.7 (i), there exists $o \in u$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, c)$ and $\Phi \vdash \bar{Q}(o, \bar{c})$. Thus, we have the derivation

$$\frac{\begin{array}{c} \vdots \\ Q(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \bar{Q}(o, \bar{c}) \end{array}}{\exists(o, \bar{o})} \quad (\text{A5}).$$

□

Lemma 3.9. *If $W_{\Lambda, \Gamma}$ contains a global defect, then Φ is inconsistent.*

Proof. Suppose $\forall(p, r), \forall(q, \bar{r}) \in u \in W_{\Lambda, \Gamma}$ and $p, q \in v \in W_{\Lambda, \Gamma}$. By Lemma 3.7 (i), there exists $o \in u$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, \forall(p, r))$ and $\Phi \vdash \bar{Q}(o, \forall(q, \bar{r}))$. And by Lemma 3.7 (ii), $\Phi \vdash \exists(p, q)$. Thus, we have the derivation

$$\frac{\frac{\begin{array}{c} \vdots \\ Q(o, \forall(p, r)) \end{array} \quad \begin{array}{c} \vdots \\ \exists(p, q) \end{array}}{Q(o, \exists(q, r))} \quad (\text{B1}) \quad \begin{array}{c} \vdots \\ \bar{Q}(o, \forall(q, \bar{r})) \end{array}}{\exists(o, \bar{o})} \quad (\text{A5}).$$

□

Lemma 3.10. *Suppose $\Phi \models \psi$, where ψ is of either of the forms $\exists(p, q)$ or $\exists(p, \exists(q, t))$. Then $\Phi \vdash \psi$.*

Proof. If $W_{\Lambda, \Gamma}$ contains any local or global defects, then Φ is inconsistent, whence $\Phi \vdash \psi$ by rule (A7). Otherwise, let $\mathfrak{A} = \mathfrak{A}_{\Lambda, \Gamma}$, so that by Lemma 3.5, $\mathfrak{A} \models \Phi$, whence $\mathfrak{A} \models \psi$. If $\psi = \exists(p, q)$, then, by definition of \mathfrak{A} , there exists $u \in W_{\Lambda, \Gamma}$ such that $p, q \in u$. By Lemma 3.7 (ii), $\Phi \vdash \exists(p, q)$. If $\psi = \exists(p, \exists(q, t))$, let us assume first of all that t is positive, and write $t = r$. Then there exists $a = (u, i)$ and $b = (v, j)$ in $W_{\Lambda, \Gamma}$ such that $a \in p^{\mathfrak{A}}$, $b \in q^{\mathfrak{A}}$ and $\langle a, b \rangle \in r^{\mathfrak{A}}$. By definition of \mathfrak{A} , $q \in v$. Furthermore, either (a) $\exists(o, r) \in u$ and $v = \{o\}^*$, or (b) $\forall(o, r) \in u$ and $o \in v$. In case (a), $\Phi \vdash \forall(o, q)$ by Lemma 3.2, and $\Phi \vdash \exists(p, \exists(o, r))$ by Lemma 3.7 (ii), whence we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(p, \exists(o, r)) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, q) \end{array}}{\exists(p, \exists(q, r))} \quad (\text{B3}).$$

In case (b), by Lemma 3.7 (ii), $\Phi \vdash \exists(o, q)$, and $\Phi \vdash \exists(p, \forall(o, r))$, whence we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(p, \forall(o, r)) \end{array} \quad \begin{array}{c} \vdots \\ \exists(o, q) \end{array}}{\exists(p, \exists(q, r))} \quad (\text{B1}).$$

Finally, if $t = \bar{r}$ is negative, we proceed in exactly the same way, but use $\bar{\mathfrak{A}}$ in place of \mathfrak{A} . □

We next deal with conclusions of the forms $\forall(p, q)$, $\forall(p, \exists(q, t))$, $\exists(q, \forall(p, t))$, $\exists(p, q)$, $\forall(p, \bar{q})$, and $\forall(p, \forall(q, t))$. We note in connection with the machinery introduced in Sec. 3.2 that, if Σ and Σ' are sets of existential formulas and Γ a set of universal formulas, then $W_{\Sigma \cup \Sigma', \Gamma} = W_{\Sigma, \Gamma} \cup W_{\Sigma', \Gamma}$.

Lemma 3.11. *If $W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contains a local defect, then $\Phi \vdash \forall(p, \bar{q})$.*

Proof. Observe that the statement of the lemma would be unaffected by exchanging p and q . Suppose $u \in W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contain both c and \bar{c} . If $u \in W_{\Lambda, \Gamma}$, then the result is secured by Lemma 3.8 and rule (A7). So we may assume $u \in W_{\{\exists(p, q)\}, \Gamma}$. By Lemmas 3.1 and 3.2, and exchanging p and q if necessary, either (a) $\Gamma \vdash \forall(p, c)$ and $\Gamma \vdash \forall(q, \bar{c})$, or (b) there exists o such that $\Gamma \vdash \exists(p, o)$, $\Gamma \vdash \forall(o, c)$ and $\Gamma \vdash \forall(o, \bar{c})$. Hence, we have either of the derivations

$$\frac{\begin{array}{c} \vdots \\ \forall(p, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, \bar{c}) \end{array}}{\forall(p, \bar{q})} \text{ (A2)} \quad \frac{\begin{array}{c} \vdots \\ \forall(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, \bar{c}) \end{array}}{\forall(o, \bar{o})} \text{ (A2)} \quad \frac{\begin{array}{c} \vdots \\ \exists(p, o) \end{array}}{\forall(p, \bar{q})} \text{ (C7)}.$$

□

Lemma 3.12. *If $W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contains a global defect, then $\Phi \vdash \forall(p, \bar{q})$.*

Proof. Observe that the statement of the lemma would be unaffected by exchanging p and q . Suppose $u \in W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contains both $\forall(p', t)$ and $\forall(q', \bar{t})$, and $v \in W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contains both p' and q' . If $u, v \in W_{\Lambda, \Gamma}$, the result is secured by Lemma 3.12. So we may assume that either u or v is in $W_{\{\exists(p, q)\}, \Gamma}$. We have three cases.

(i) Suppose first that $u \in W_{\{\exists(p, q)\}, \Gamma}$ and $v \in W_{\Lambda, \Gamma}$. By Lemmas 3.1 and 3.2, and exchanging p and q if necessary, either (a) $\Gamma \vdash \forall(p, \forall(p', t))$ and $\Gamma \vdash \forall(q, \forall(q', \bar{t}))$, or (b) there exists o such that $\Gamma \vdash \exists(p, o)$, $\Gamma \vdash \forall(o, \forall(p', t))$ and $\Gamma \vdash \forall(o, \forall(q', \bar{t}))$ (this incorporates the possibility that $o = p$). Furthermore, by Lemma 3.7 (ii), $\Phi \vdash \exists(p', q')$. In case (a), we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(p, \forall(p', t)) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \forall(q, \forall(q', \bar{t})) \end{array} \quad \begin{array}{c} \vdots \\ \exists(p', q') \end{array}}{\forall(q, \exists(p', \bar{t}))} \text{ (B1)}}{\forall(p, \bar{q})} \text{ (A2)}.$$

In case (b), we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(o, \forall(p', t)) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \forall(o, \forall(q', \bar{t})) \end{array} \quad \begin{array}{c} \vdots \\ \exists(p', q') \end{array}}{\forall(o, \exists(p', \bar{t}))} \text{ (B1)}}{\forall(o, \bar{o})} \text{ (A2)} \quad \frac{\begin{array}{c} \vdots \\ \exists(p, o) \end{array}}{\forall(p, \bar{q})} \text{ (C7)}.$$

(ii) Suppose next that $u \in W_{\Lambda, \Gamma}$ and $v \in W_{\{\exists(p, q)\}, \Gamma}$. By Lemma 3.7 (i), there exist o, Q such that $\Phi \vdash Q(o, \forall(p', t))$ and $\Phi \vdash \bar{Q}(o, \forall(q', \bar{t}))$. By Lemmas 3.1 and 3.2, and exchanging p and q if necessary, either (a) $\Gamma \vdash \forall(p, p')$ and $\Gamma \vdash \forall(q, q')$, or (b) there exists o' such that $\Gamma \vdash \exists(p, o')$, $\Gamma \vdash \forall(o', p')$ and $\Gamma \vdash \forall(o', q')$ (this incorporates the possibility that $o' = p$). In case (a), we have the derivation

$$\frac{\frac{\frac{\frac{\vdots}{Q(o, \forall(p', t))} \quad \frac{\vdots}{\bar{Q}(o, \forall(q', \bar{t}))}}{\forall(p', \bar{q}')} \text{ (E1)} \quad \frac{\vdots}{\forall(p, p')} \text{ (B1)}}{\forall(p, \bar{q}')} \text{ (A2)}}{\forall(p, \bar{q})} \text{ (A2)}.$$

In case (b), we have the derivation

$$\frac{\frac{\frac{\frac{\vdots}{Q(o, \forall(p', t))} \quad \frac{\vdots}{\bar{Q}(o, \forall(q', \bar{t}))}}{\forall(p, \bar{q}')} \text{ (E1)} \quad \frac{\vdots}{\forall(o', p')} \text{ (A2)}}{\forall(o', \bar{q}')} \text{ (A2)} \quad \frac{\vdots}{\forall(o', q')} \text{ (A2)}}{\forall(o', o')} \text{ (A2)} \quad \frac{\vdots}{\exists(p, o')} \text{ (C7)}}{\forall(p, \bar{q})} \text{ (C7)}.$$

(iii) Suppose, finally, that $u, v \in W_{\{\exists(p, q)\}, \Gamma}$. Here we have four sub-cases to consider. (a) If $u = v = \{\exists(p, q)\}^\Gamma$, then by Lemma 3.2, there exist $o_1, o_2, o_3, o_4 \in \{p, q\}$ such that $\Gamma \vdash \forall(o_1, \forall(p', t))$, $\Gamma \vdash \forall(o_2, \forall(q', \bar{t}))$, $\Gamma \vdash \forall(o_3, p')$, and $\Gamma \vdash \forall(o_4, q')$. Thus we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o_1, \forall(p', t))} \quad \frac{\vdots}{\forall(o_2, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o_3, p')} \quad \frac{\vdots}{\forall(o_4, q')}}{\forall(p, \bar{q})} \text{ (G1)}.$$

(b) If $u = \{p, q\}^\Gamma$ and $v = \{o'\}^\Gamma$, where $o' \in O_{\{\exists(p, q)\}, \Gamma}$, then, by Lemmas 3.1 and 3.2, there exist $o_1, o_2, o_3 \in \{p, q\}$ such that $\Gamma \vdash \forall(o_1, \forall(p', t))$, $\Gamma \vdash \forall(o_2, \forall(q', \bar{t}))$, $\Gamma \vdash \exists(o_3, o')$, $\Gamma \vdash \forall(o', p')$ and $\Gamma \vdash \forall(o', q')$. Thus we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o_1, \forall(p', t))} \quad \frac{\vdots}{\forall(o_2, \forall(q', \bar{t}))} \quad \frac{\vdots}{\exists(o_3, o')} \quad \frac{\vdots}{\forall(o', p')} \quad \frac{\vdots}{\forall(o', q')}}{\forall(p, \bar{q})} \text{ (G2)}.$$

(c) If $v = \{p, q\}^\Gamma$ and $u = \{o\}^\Gamma$, where $o' \in O_{\{\exists(p, q)\}, \Gamma}$, then, by Lemmas 3.1 and 3.2, there exist $o_1, o_2, o_3 \in \{p, q\}$ such that $\Gamma \vdash \forall(o_1, o)$, $\Gamma \vdash \forall(o, \forall(p', t))$, $\Gamma \vdash \forall(o, \forall(q', \bar{t}))$, $\Gamma \vdash \forall(o_2, p')$ and $\Gamma \vdash \forall(o_3, q')$. Thus we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(o_1, o)} \quad \frac{\vdots}{\forall(o, \forall(p', t))} \quad \frac{\vdots}{\forall(o, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o_2, p')} \quad \frac{\vdots}{\forall(o_3, q')}}{\forall(p, \bar{q})} \text{ (G3)}.$$

(d) If $u = \{o\}^\Gamma$ and $v = \{o'\}^\Gamma$, where $o, o' \in O_{\{\exists(p,q)\}, \Gamma}$, then by Lemma 3.1, we have $\Gamma \vdash \forall(o, \forall(p', r))$, $\Gamma \vdash \forall(o, \forall(q', \bar{r}))$, $\Gamma \vdash \forall(o', p')$, and $\Gamma \vdash \forall(o', q')$; and by Lemma 3.1, exchanging p and q if necessary, we have $\Gamma \vdash \exists(p, o)$, and either $\Gamma \vdash \exists(p, o')$ or $\Gamma \vdash \exists(q, o')$. Thus we have either the derivation

$$\frac{\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \exists(p, o) & \exists(q, o') & \forall(o, \forall(p', t)) & \forall(o, \forall(q', \bar{t})) & \forall(o', p') & \forall(o', q') \end{array}}{\forall(p, \bar{q})} \quad (\text{G4})$$

or the derivation

$$\frac{\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \exists(p, o) & \exists(p, o') & \forall(o, \forall(p', t)) & \forall(o, \forall(q', \bar{t})) & \forall(o', p') & \forall(o', q') \end{array}}{\frac{\forall(p, \bar{p})}{\forall(p, \bar{q})}} \quad (\text{G4}). \quad (\text{A8}).$$

□

Lemma 3.13. *Suppose $\Phi \models \psi$, where ψ is of any of the formulas $\forall(p, q)$, $\forall(p, \exists(q, t))$, $\exists(q, \forall(p, t))$ or $\exists(p, q)$. Then $\Phi \vdash \psi$.*

Proof. Consider the set $W = W_{\Lambda \cup \{\exists(p,p)\}, \Gamma}$. If this set contains a defect, then $\Phi \vdash \forall(p, \bar{p})$ by Lemmas 3.11 and 3.12. Further, if $\psi = \exists(q, \forall(p, t))$, then since $\Phi \models \exists(q, q)$, Lemma 3.10 guarantees $\Phi \vdash \exists(q, q)$. Hence we have one of the derivations:

$$\frac{\begin{array}{c} \vdots \\ \forall(p, \bar{p}) \end{array}}{\forall(p, q)} \quad (\text{A8}) \quad \frac{\begin{array}{c} \vdots \\ \forall(p, \bar{p}) \end{array}}{\forall(p, \exists(q, t))} \quad (\text{A8}) \quad \frac{\begin{array}{c} \vdots \\ \exists(q, q) \end{array} \quad \frac{\forall(p, \bar{p})}{\forall(q, \forall(p, t))} \quad (\text{B4})}{\exists(q, \forall(p, t))} \quad (\text{A6}) \quad \frac{\begin{array}{c} \vdots \\ \forall(p, \bar{p}) \end{array}}{\forall(p, q)} \quad (\text{A8})}{\exists(p, q)} \quad (\text{C3}),$$

as required. Therefore, we may assume that W contains no defect. Let $\mathfrak{A} = \mathfrak{A}_{\Lambda \cup \{\exists(p,p)\}, \Gamma}$ and $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_{\Lambda \cup \{\exists(p,p)\}, \Gamma}$. By Lemma 3.4, both \mathfrak{A} and $\bar{\mathfrak{A}}$ are models of Φ , and hence of ψ . We consider the various forms of ψ in turn.

$\psi = \forall(p, q)$: By construction of \mathfrak{A} , $a = (\{p\}^\Gamma, 1) \in p^{\mathfrak{A}}$. Since $\mathfrak{A} \models \psi$, $a \in q^{\mathfrak{A}}$, whence $q \in \{p\}^\Gamma$, whence $\Phi \vdash \forall(p, q)$, by Lemma 3.2.

$\psi = \forall(p, \exists(q, t))$: We assume first that t is positive, and write $t = r$. By construction of \mathfrak{A} , $a = (\{p\}^\Gamma, 1) \in p^{\mathfrak{A}}$. Since $\mathfrak{A} \models \psi$, there exists $b \in q^{\mathfrak{A}}$ such that $\langle a, b \rangle \in r^{\mathfrak{A}}$. Thus, there exists q' such that either (i) $b = (\{q'\}^\Gamma, 1)$ with $\exists(q', r) \in \{p\}^\Gamma$ and $q \in \{q'\}^\Gamma$, or (ii) $b = (v, j)$ with $\forall(q', r) \in \{p\}^\Gamma$ and $q, q' \in v$. In case (i), by Lemma 3.2, $\Phi \vdash \forall(p, \exists(q', r))$ and $\Phi \vdash \forall(q', q)$, whence we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(p, \exists(q', r)) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q', q) \end{array}}{\forall(p, \exists(q, r))} \quad (\text{B3}).$$

In case (ii), either (a) $v \in W_{\Lambda, \Gamma}$, in which case $\Phi \vdash \exists(q', q)$, by Lemma 3.7 (ii), or, (b) $v = \{o\}^\Gamma$, where $o \in O_{\{\exists(p, p)\}, \Gamma}$, in which case, by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(p, o)$, $\Phi \vdash \forall(o, q)$, and $\Phi \vdash \forall(o, q')$. Hence, we have either of the derivations

$$\frac{\frac{\vdots}{\forall(p, \forall(q', r))} \quad \frac{\vdots}{\exists(q, q')}}{\forall(p, \exists(q, r))} \text{ (B3)} \quad \frac{\frac{\vdots}{\forall(p, \forall(q', r))} \quad \frac{\vdots}{\forall(o, q)} \quad \frac{\vdots}{\forall(o, q')} \quad \frac{\vdots}{\exists(p, o)}}{\forall(p, \exists(q, r))} \text{ (D4)}.$$

If $t = \bar{r}$ is negative, then we proceed as before, but replacing \mathfrak{A} by $\bar{\mathfrak{A}}$.

$\psi = \exists(q, \forall(p, t))$: Assume first that $t = r$ is positive, and recall our earlier observation that $\Phi \vdash \exists(q, q)$. By construction, $b = (\{p\}^\Gamma, 0) \in p^\mathfrak{A}$. Since $\mathfrak{A} \models \psi$, let $a = (u, i)$ be such that $a \in q^\mathfrak{A}$ and $\langle a, b \rangle \in r^\mathfrak{A}$. Thus there exists p' such that $\forall(p', r) \in u$ and $p' \in \{p\}^\Gamma$. Now, either: (a) $u \in W_{\Lambda, \Gamma}$; or (b) $u = \{o\}^\Gamma$, where $\{o\} \in V_{\{\exists(p, p)\}, \Gamma}$. In case (a), $\Phi \vdash \exists(q, \forall(p', r))$ by Lemma 3.7 (ii), and $\Phi \vdash \forall(p, p')$ by Lemma 3.2, whence we have the derivation

$$\frac{\frac{\vdots}{\exists(q, \forall(p', r))} \quad \frac{\vdots}{\forall(p, p')}}{\exists(q, \forall(p, r))} \text{ (B2)}.$$

In case (b), by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(p, o)$, $\Phi \vdash \forall(o, q)$, $\Phi \vdash \forall(o, \forall(p', r))$ and $\Phi \vdash \forall(p, p')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(q, q)} \quad \frac{\vdots}{\exists(p, o)} \quad \frac{\frac{\frac{\vdots}{\forall(o, \forall(p', r))} \quad \frac{\vdots}{\forall(p, p')}}{\forall(o, \forall(p, r))} \text{ (B2)} \quad \frac{\vdots}{\forall(o, q)}}{\exists(q, \forall(p, r))} \text{ (D2)}.$$

If $t = \bar{r}$ is negative, then we proceed as before, but replacing \mathfrak{A} by $\bar{\mathfrak{A}}$.

$\psi = \exists(p, q)$: By construction, $a = (\{p_0\}^\Gamma, 1) \in p^\mathfrak{A}$. Since $\mathfrak{A} \models \psi$, let $b = (v, j)$ be such that $b \in q^\mathfrak{A}$. Thus, $q \in v$. Either (a) $u \in W_{\Lambda, \Gamma}$ or (b) $u = \{o\}^\Gamma$ where $o \in O_{\{\exists(p, p)\}, \Gamma}$. In case (a), $\Phi \vdash \exists(q, q)$, by Lemma 3.7 (ii), and we have the derivation

$$\frac{\frac{\vdots}{\exists(q, q)}}{\exists(p, q)} \text{ (C5)}.$$

In case (b), $\Phi \vdash \exists(p, q)$ by Lemma 3.1, and we are done. \square

Lemma 3.14. *Suppose $\Phi \models \forall(p, \bar{q})$. Then $\Phi \vdash \forall(p, \bar{q})$.*

Proof. Let $\mathfrak{A} = \mathfrak{A}_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$. By construction, $\mathfrak{A} \not\models \psi$, and hence $\mathfrak{A} \not\models \Phi$. By Lemma 3.4, $W_{\Lambda \cup \{\exists(p, q)\}, \Gamma}$ contains a defect. By Lemmas 3.11 and 3.12, $\Phi \vdash \forall(p, \bar{q})$. \square

Lemma 3.15. *If $W_{\Lambda \cup \{\exists(p,p), \exists(q,q)\}, \Gamma}$ contains a defect, then, for any binary literal t , $\Phi \models \forall(p, \forall(q, t))$.*

Proof. If $W_{\Lambda \cup \{\exists(p,p)\}, \Gamma}$ contains a defect, then, by Lemmas 3.11 and 3.12 (setting $q = p$), $\Phi \vdash \forall(p, \bar{p})$. Similarly, if $W_{\Lambda \cup \{\exists(q,q)\}, \Gamma}$ contains a defect, then $\Phi \vdash \forall(q, \bar{q})$. Hence we have either of the derivations

$$\frac{\begin{array}{c} \vdots \\ \forall(p, \bar{p}) \end{array}}{\forall(p, \forall(q, t))} \text{ (A8)} \qquad \frac{\begin{array}{c} \vdots \\ \forall(q, \bar{q}) \end{array}}{\forall(p, \forall(q, t))} \text{ (B4)}.$$

The only other possibility is a global defect involving one element of $W_{\{\exists(p,p)\}, \Gamma}$ and another of $W_{\{\exists(q,q)\}, \Gamma}$. Suppose, then, $\forall(p', t'), \forall(q', \bar{t}') \in u \in W_{\{\exists(p,p)\}, \Gamma}$, and $p', q' \in v \in W_{\{\exists(q,q)\}, \Gamma}$. By Lemmas 3.1 and 3.2, there exist o, o' such that $\Phi \vdash \exists(p, o)$, $\Phi \vdash \forall(o, \forall(p', t'))$, $\Phi \vdash \forall(o, \forall(q', \bar{t}'))$, $\Phi \vdash \exists(q, o')$, $\Phi \vdash \forall(o', p')$ and $\Phi \vdash \forall(o', q')$. Thus, we have the derivation

$$\frac{\frac{\begin{array}{c} \vdots \\ \forall(o, \forall(p', t')) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \forall(o', p') \end{array}}{\forall(o, \forall(o', t'))} \text{ (B2)}}{\forall(o, \forall(o', t'))} \quad \frac{\frac{\begin{array}{c} \vdots \\ \forall(o, \forall(q', \bar{t}')) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \forall(o', q') \end{array}}{\forall(o, \forall(o', \bar{t}'))} \text{ (B2)}}{\forall(o, \forall(o', \bar{t}'))} \quad \frac{\begin{array}{c} \vdots \\ \exists(p, o) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \exists(q, o') \end{array}}{\forall(p, \forall(q, t))} \text{ (D5)}}{\forall(p, \forall(q, t))}$$

□

Lemma 3.16. *Suppose $\Phi \models \forall(p, \forall(q, t))$. Then $\Phi \vdash \forall(p, \forall(q, t))$.*

Proof. Assume for the moment t is positive and write $t = r$. Let $\mathfrak{A} = \mathfrak{A}_{\Lambda \cup \{\exists(p,p), \exists(q,q)\}, \Gamma}$. If $W_{\Lambda \cup \{\exists(p,p), \exists(q,q)\}, \Gamma}$ contains a defect, then the conclusion follows by Lemma 3.15. Otherwise, by Lemma 3.4, $\mathfrak{A} \models \Phi$, whence $\mathfrak{A} \models \psi$, and, by construction, the domain of \mathfrak{A} contains $a = (\{p\}^\Gamma, 1)$ and $b = (\{q\}^\Gamma, 0)$. Since $a \in p^{\mathfrak{A}}$ and $b \in q^{\mathfrak{A}}$, we have $\langle a, b \rangle \in r^{\mathfrak{A}}$, whence $\forall(q', r) \in \{p\}^\Gamma$ for some $q' \in \{q\}^\Gamma$. By Lemma 3.2, $\Gamma \vdash \forall(p, \forall(q', r))$ and $\Gamma \vdash \forall(q, q')$, whence we have the derivation:

$$\frac{\begin{array}{c} \vdots \\ \forall(p, \forall(q', r)) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \forall(q, q') \end{array}}{\forall(p, \forall(q, r))} \text{ (B2)}.$$

If t is negative, we write $t = \bar{r}$ and proceed as above, but replacing \mathfrak{A} by $\bar{\mathfrak{A}} = \bar{\mathfrak{A}}_{\Lambda \cup \{\exists(p,p), \exists(q,q)\}, \Gamma}$. □

We turn finally to conclusions of the form $\exists(p, \bar{q})$. Fixing the unary atoms p and q , consider the set $W_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$, together with the structures $\mathfrak{A}_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$ and $\bar{\mathfrak{A}}_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$, as defined in Sec. 3.2. By way of preparing the ground, let us examine $W_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$ and its relation to $W_{\Lambda, \Gamma}$. Evidently, if $p \notin \bigcup W_{\Lambda, \Gamma}$, then $W_{\Lambda, \Gamma \cup \{\forall(p,q)\}} = W_{\Lambda, \Gamma}$. So let us assume $p \in \bigcup W_{\Lambda, \Gamma}$, and consider $w \in W_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$. Let us write $w = s^{(\Gamma \cup \{\forall(p,q)\})}$, where $s \in V_{\Lambda, \Gamma \cup \{\forall(p,q)\}}$. We may distinguish two cases: either $s \in V_{\Lambda, \Gamma}$ or $V_{\{\exists(q,q)\}, \Gamma}$. But note that, in either

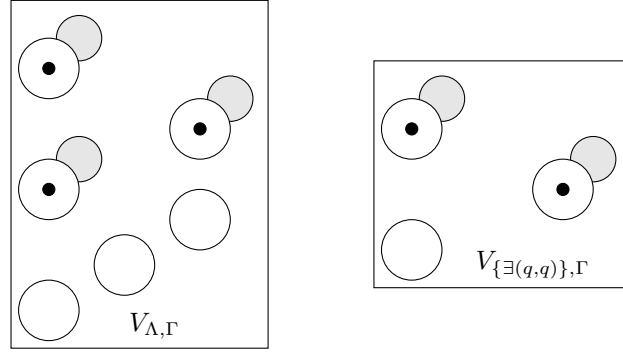


Figure 1: Schematic illustration of $W_{\Lambda, \Gamma \cup \{\forall(p, q)\}}$ when $p \in \bigcup W_{\Lambda, \Gamma}$: white circles illustrate the elements s^Γ , where s is in either $V_{\Lambda, \Gamma}$ or $V_{\{\exists(q, q)\}, \Gamma}$, as indicated by the labels in the rectangles; black dots indicate the presence of the unary atom p in these sets; grey blobs indicate the sets of added elements $\{q\}^\Gamma$.

case, if $p \notin s^\Gamma$, then $s^{\Gamma \cup \{\forall(p, q)\}} = s^\Gamma$; otherwise $s^{\Gamma \cup \{\forall(p, q)\}} = s^\Gamma \cup \{q\}^\Gamma$. Fig. 1 illustrates the four possible kinds of elements of $W_{\Lambda, \Gamma \cup \{\forall(p, q)\}}$ that result. Indeed, if c, d are c -terms realized in the same element w of $W_{\Lambda, \Gamma \cup \{\forall(p, q)\}}$, one of the following six cases obtains. To aid readability, we write $[q]^\Gamma$ for the set of unary atoms $\{o \mid q \xrightarrow{\Pi} o\} = O_{\{\exists(q, q)\}, \Gamma}$.

- (i) there exists $w' \in W_{\Lambda, \Gamma}$ such that $c, d \in w'$;
- (ii) there exists $w' \in W_{\Lambda, \Gamma}$ such that $c, p \in w'$, and $d \in \{q\}^\Gamma$;
- (iii) there exists $w' \in W_{\Lambda, \Gamma}$ such that $d, p \in w'$, and $c \in \{q\}^\Gamma$;
- (iv) $p \in \bigcup W_{\Lambda, \Gamma}$, and there exists $o \in [q]^\Gamma$ such that $c, d \in \{o\}^\Gamma$;
- (v) $p \in \bigcup W_{\Lambda, \Gamma}$, and there exists $o \in [q]^\Gamma$ such that $c, p \in \{o\}^\Gamma$ and $d \in \{q\}^\Gamma$;
- (vi) $p \in \bigcup W_{\Lambda, \Gamma}$, and there exists $o \in [q]^\Gamma$ such that $d, p \in \{o\}^\Gamma$ and $c \in \{q\}^\Gamma$.

This six-fold division of cases will be used extensively in the next two lemmas.

Lemma 3.17. *If $W_{\Lambda, \Gamma \cup \{\forall(p, q)\}}$ contains a local defect, then $\Phi \vdash \exists(p, \bar{q})$.*

Proof. We have six cases to consider. (i) If $c, \bar{c} \in w \in W_{\Lambda, \Gamma}$, Φ is inconsistent, by Lemma 3.8, and the result follows by rule (A7). (ii) If $c, p \in w \in W_{\Lambda, \Gamma}$, and $\bar{c} \in \{q\}^\Gamma$, then by Lemma 3.7 (ii), $\Phi \vdash \exists(p, c)$ and by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, \bar{c})$; hence we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(p, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, \bar{c}) \end{array}}{\exists(p, \bar{q})} \text{ (A5)}.$$

(iii) If $\bar{c}, p \in w \in W_{\Lambda, \Gamma}$, and $c \in \{q\}^\Gamma$, we replace c by \bar{c} and proceed as in case (ii). (iv) If $p \in \bigcup W_{\Lambda, \Gamma}$, $o \in [q]^\Gamma$ and $c, d \in \{o\}^\Gamma$, then, by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$, and by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, c)$ and $\Gamma \vdash \forall(o, \bar{c})$; hence we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o, c)} \quad \frac{\vdots}{\forall(o, \bar{c})}}{\forall(o, \bar{o})} \quad \frac{\vdots}{\exists(q, o)} \quad \frac{\vdots}{\exists(p, p)}}{\frac{\forall(q, \bar{p})}{\exists(p, \bar{q})}} \quad \text{(A2)} \quad \text{(C7)} \quad \text{(A5)}.$$

(v) If $p \in \bigcup W_{\Lambda, \Gamma}$, $o \in [q]^\Gamma$, $c, p \in \{o\}^\Gamma$ and $\bar{c} \in \{q\}^\Gamma$, then, by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$, and by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, c)$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, \bar{c})$; hence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p, p)} \quad \frac{\vdots}{\exists(q, o)} \quad \frac{\vdots}{\forall(o, p)} \quad \frac{\frac{\forall(o, c) \quad \forall(q, \bar{c})}{\forall(o, \bar{q})}}{\exists(p, \bar{q})} \quad \text{(A2)}}{\exists(p, \bar{q})} \quad \text{(D3)}$$

(vi) If $p \in \bigcup W_{\Lambda, \Gamma}$, $o \in [q]^\Gamma$, $\bar{c}, p \in \{o\}^\Gamma$ and $c \in \{q\}^\Gamma$, we replace c by \bar{c} and proceed as in case (v). \square

Lemma 3.18. *If $W_{\Lambda, \Gamma \cup \{\forall(p, q)\}}$ contains a global defect, then $\Phi \vdash \forall(p, \bar{q})$.*

Proof. Let u, v be elements of $W_{\Lambda, \Delta}$ and t a binary literal such that $\forall(p', t), \forall(q', \bar{t}) \in u$, and $p', q' \in v$. We apply the six-fold distinction noted above to both u and v , yielding thirty-six cases in total.

(i) Suppose $\forall(p', t), \forall(q', \bar{t}) \in u' \in W_{\Lambda, \Gamma}$. By Lemma 3.7 (i), there exist o, Q such that $\Phi \vdash Q(o, \forall(p', t))$ and $\Phi \vdash \bar{Q}(o, \forall(q', \bar{t}))$. Hence we have the derivation

$$\frac{\frac{\vdots}{Q(o, \forall(p', t))} \quad \frac{\vdots}{\bar{Q}(o, \forall(q', \bar{t}))}}{\forall(p', \bar{q}')} \quad \text{(E1)}$$

Recall that $\forall(p', \bar{q}')$ is symmetric in p' and q' . We consider the cases: (a) $p', q', \in v' \in W_{\Lambda, \Gamma}$; (b) $p', p \in v' \in W_{\Lambda, \Gamma}$, and $q' \in \{q\}^\Gamma$; (c) $q', p \in v' \in W_{\Lambda, \Gamma}$, and $p' \in \{q\}^\Gamma$; (d) $p', q' \in \{o\}^\Gamma$, where $o \in [q]^\Gamma$, and $p \in \bigcup W_{\Lambda, \Gamma}$; (e) $p', p \in \{o\}^\Gamma$, where $o \in [q]^\Gamma$, and $p \in \bigcup W_{\Lambda, \Gamma}$; (f) $q', p \in \{o\}^\Gamma \in [q]^\Gamma$, $p' \in \{q\}^\Gamma$, and $p \in \bigcup W_{\Lambda, \Gamma}$. If (a), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', q')$, and we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(p', \bar{q}')} \quad \frac{\vdots}{\exists(p', q')}}{\exists(p', \bar{p}')} \quad \text{(A5)}}{\exists(p, \bar{q})} \quad \text{(A7)}$$

If (b), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', p)$, and by Lemma 3.2, $\Phi \vdash \forall(q, q')$. Hence, we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(p', \bar{q}')} \quad \frac{\vdots}{\exists(p, p')}}{\exists(p, \bar{q}')} \quad (A6) \quad \frac{\vdots}{\forall(\bar{q}', \bar{q})} \quad (A5)}{\exists(q, \bar{q})}$$

Case (c) is identical to (b) by transposing p' and q' . If (d), by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, p')$, and $\Gamma \vdash \forall(o, q')$, and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. Hence, we have the derivation:

$$\frac{\frac{\frac{\frac{\vdots}{\forall(p', \bar{q}')} \quad \frac{\vdots}{\forall(o, p')}}{\forall(o, \bar{q}')} \quad (A2) \quad \frac{\vdots}{\forall(o, q')} \quad (A2)}{\forall(o, \bar{o})} \quad \frac{\frac{\vdots}{\exists(q, o)} \quad (C7) \quad \frac{\vdots}{\exists(p, p)} \quad (A6)}{\forall(p, \bar{q})} \quad (A6)}{\exists(p, \bar{q})}$$

If (e), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, p')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, q')$, and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. Hence, we have the derivation:

$$\frac{\frac{\frac{\frac{\vdots}{\forall(p', \bar{q}')} \quad \frac{\vdots}{\forall(o, p')}}{\forall(o, \bar{q}')} \quad (A2) \quad \frac{\vdots}{\forall(q, q')} \quad (A2)}{\forall(o, \bar{q})} \quad \frac{\vdots}{\forall(o, p)} \quad (D3)}{\exists(p, \bar{q})} \quad (D3)$$

Case (f) is identical to (e) by transposing p' and q' .

(ii) Suppose $\forall(p', t), p \in u' \in W_{\Lambda, \Gamma}$, and $\forall(q, \bar{t}) \in \{q_0\}^\Gamma$. By Lemma 3.7 (ii), $\Phi \vdash \exists(p, \forall(p', t))$, and by Lemma 3.2, $\Phi \vdash \forall(q, \forall(q', \bar{t}))$. We consider the cases (a)–(f) regarding v , exactly as in Case (i). If (a), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', q')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\exists(p', q')}}{\exists(p, \exists(q', t))} \quad (B1) \quad \frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad (A5)}{\exists(p, \bar{q})}$$

If (b), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', p)$, and by Lemma 3.2, $\Phi \vdash \forall(q, q')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\exists(p, p')}}{\exists(p, \exists(p, t))} \quad (B3) \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(q, \forall(q, \bar{t}))} \quad (B2)}{\exists(p, \bar{q})} \quad (E3)$$

If (c), by Lemma 3.7 (ii), $\Phi \vdash \exists(p, q')$, and by Lemma 3.2, $\Phi \vdash \forall(q, p')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\forall(q, p')}}{\exists(p, \forall(q, t))} \text{ (B2)} \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\exists(p, q')}}{\forall(q, \exists(p, \bar{t}))} \text{ (B1)}}{\exists(p, \bar{q})} \text{ (E2)}.$$

If (d), by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, o)$, $\Phi \vdash \forall(o, p')$, and $\Phi \vdash \forall(o, q')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\forall(o, p')}}{\exists(p, \forall(o, t))} \text{ (B2)} \quad \frac{\frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o, q')}}{\forall(q, \forall(o, \bar{t}))} \text{ (B2)} \quad \frac{\vdots}{\exists(q, o)}}{\forall(q, \exists(o, \bar{t}))} \text{ (A5)}}{\exists(p, \bar{q})} \text{ (D1)}.$$

If (e), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, p')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, q')$. Hence, we have the derivation:

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\forall(o, p')}}{\exists(p, \forall(o, t))} \text{ (B2)} \quad \frac{\frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(q, \forall(q, \bar{t}))} \text{ (B2)} \quad \frac{\vdots}{\forall(o, p)} \quad \frac{\vdots}{\exists(q, o)}}{\exists(p, \bar{q})} \text{ (F1)}.$$

If (f), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, q')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, p')$. Hence, we have

$$\frac{\frac{\frac{\vdots}{\exists(p, \forall(p', t))} \quad \frac{\vdots}{\forall(q, p')}}{\exists(p, \forall(q, t))} \text{ (B2)} \quad \frac{\frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o, q')}}{\forall(q, \forall(o, \bar{t}))} \text{ (B2)} \quad \frac{\vdots}{\forall(o, p)} \quad \frac{\vdots}{\exists(q, o)}}{\exists(p, \bar{q})} \text{ (F2)}.$$

(iii) Suppose $\forall(q', t), p \in u' \in W_{\Lambda, \Gamma}$, and $\forall(p', \bar{t}) \in \{q\}^\Gamma$. We argue exactly as for case (ii), with p and q' exchanged, and t replaced by \bar{t} .

(iv) Suppose $\forall(p', t), \forall(q', \bar{t}) \in \{o'\}^\Gamma$, where $o' \in [q]^\Gamma$, and $p \in \bigcup W_{\Lambda, \Gamma}$. By Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, o)$, $\Phi \vdash \forall(o', \forall(p', t))$, and $\Phi \vdash \forall(o', \forall(q', \bar{t}))$; and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. We consider the cases (a)–(f) regarding v exactly as for (i). If (a), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', q')$, whence we have the

derivation

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\exists(p', q')}}{\forall(o', \exists(q', t))} \text{ (B1)} \quad \frac{\frac{\vdots}{\forall(o', \forall(q', \bar{t}))} \quad \frac{\vdots}{\exists(q, o')}}{\forall(o', \bar{o}')} \text{ (A2)} \quad \frac{\frac{\vdots}{\exists(q, \bar{p})} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, \bar{q})} \text{ (C2)}}{\frac{\vdots}{\forall(q, \bar{p})} \quad \frac{\vdots}{\exists(p, \bar{q})}} \text{ (A6)}$$

If (b), by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p')$, and by Lemma 3.2, $\Phi \vdash \forall(q, q')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\exists(p, p')}}{\forall(o', \exists(p, t))} \text{ (B1)} \quad \frac{\frac{\frac{\vdots}{\forall(o', \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(o', \forall(q, \bar{t}))} \text{ (B2)} \quad \frac{\frac{\vdots}{\exists(q, o')} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, q)} \text{ (F3)}}{\exists(p, q)}$$

Case (c) is identical to (b) by transposing p' and q' and replacing t by \bar{t} . If (d), by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, o)$, $\Phi \vdash \forall(o, p')$ and $\Phi \vdash \forall(o, q')$, whence we have the derivations

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\forall(o, p')}}{\forall(o', \forall(o, t))} \text{ (B2)} \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o, q')}}{\forall(o', \forall(o, \bar{t}))} \text{ (B2)},$$

and hence the derivation

$$\frac{\frac{\frac{\frac{\vdots}{\forall(o', \forall(o, t))} \quad \frac{\vdots}{\forall(o', \forall(o, \bar{t}))} \quad \frac{\vdots}{\exists(q, o)} \quad \frac{\vdots}{\exists(q, o')}}{\forall(q, \bar{p})} \text{ (D6)} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, \bar{q})} \text{ (A2)}.$$

If (e), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, p')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, q')$, and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. Hence, we have the derivations:

$$\frac{\frac{\frac{\vdots}{\forall(o, \forall(p', t))} \quad \frac{\vdots}{\forall(o', p')}}{\forall(o, \forall(o', t))} \text{ (B2)} \quad \frac{\frac{\frac{\vdots}{\forall(o, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(o, \forall(q, \bar{t}))} \text{ (B2)},$$

and hence the derivation

$$\frac{\frac{\frac{\frac{\vdots}{\forall(o, \forall(o', t))} \quad \frac{\vdots}{\forall(o, \forall(q, \bar{t}))} \quad \frac{\vdots}{\exists(q, o)} \quad \frac{\vdots}{\exists(q, o')}}{\forall(p, \bar{q})} \quad \frac{\vdots}{\exists(p, p)}}{\forall(p, \bar{q})} \text{ (F4)}$$

Case (f) is identical to (e) by transposing p' and q' and replacing t by \bar{t} .

(v) Suppose $\forall(p', t), p \in \{o'\}^\Gamma$, where $o' \in [q]^\Gamma$, $\forall(q', t) \in \{q\}^\Gamma$ and $p \in \bigcup W_{\Lambda, \Gamma}$. By Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, o')$, $\Phi \vdash \forall(o', \forall(p', t))$, $\Phi \vdash \forall(o', p)$, $\forall(q, \forall(q', \bar{t}))$; and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. We consider the cases (a)–(f) regarding v exactly as for (i). If (a), by Lemma 3.7 (ii), $\Phi \vdash \exists(p', q')$, whence we have the derivation

$$\frac{\frac{\frac{\frac{\vdots}{\forall(o', p)}{\forall(o', \forall(p', t))} \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\exists(p', q')}}{\forall(q, \exists(p', \bar{t}))} \quad (A2)}{\forall(o', \bar{q})} \quad (A2)}{\forall(o', p)} \quad \frac{\vdots}{\exists(q, o')} \quad \frac{\vdots}{\exists(p, p)} \quad (D3)}{\exists(p, \bar{q})} \quad (D3)$$

If (b), by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p')$, and by Lemma 3.2, $\Phi \vdash \forall(q, q')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\exists(p, p')}}{\forall(o', \exists(p, t))} \quad (B1) \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(q, \forall(q, \bar{t}))} \quad (B2) \quad \frac{\vdots}{\exists(q, o')} \quad \frac{\vdots}{\forall(o', p)} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, \bar{q})} \quad (F5)$$

If (c), by Lemma 3.7 (ii), $\Phi \vdash \exists(p, q')$, and by Lemma 3.2, $\Phi \vdash \forall(q, p')$, whence we have the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\forall(q, p')}}{\forall(o', \forall(q, t))} \quad (B2) \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\exists(p, q')}}{\forall(q, \exists(p, \bar{t}))} \quad (B1) \quad \frac{\vdots}{\exists(q, o')} \quad \frac{\vdots}{\forall(o', p)} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, \bar{q})} \quad (F6)$$

If (d), by Lemmas 3.1 and 3.2, $\Phi \vdash \exists(q, o)$, $\Phi \vdash \forall(o, p')$ and $\Phi \vdash \forall(o, q')$, whence we have the derivations

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\forall(o, p')}}{\forall(o', \forall(o, t))} \quad (B2) \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(o, q')}}{\forall(q, \forall(o, \bar{t}))} \quad (B2)$$

and hence the derivation

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(o, t))} \quad \frac{\vdots}{\forall(q, \forall(o, \bar{t}))} \quad \frac{\vdots}{\exists(q, o')} \quad \frac{\vdots}{\forall(o', p)} \quad \frac{\vdots}{\exists(q, o)} \quad \frac{\vdots}{\exists(p, p)}}{\exists(p, \bar{q})} \quad (F7)$$

If (e), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, p')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, q')$, and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. Hence, we have the derivations:

$$\frac{\frac{\frac{\vdots}{\forall(o', \forall(p', t))} \quad \frac{\vdots}{\forall(o, p')}}{\forall(o', \forall(o, t))} \quad (B2) \quad \frac{\frac{\frac{\vdots}{\forall(q, \forall(q', \bar{t}))} \quad \frac{\vdots}{\forall(q, q')}}{\forall(q, \forall(q, \bar{t}))} \quad (B2),$$

and hence the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(o', \forall(o, t)) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, \forall(q, \bar{t})) \end{array} \quad \begin{array}{c} \vdots \\ \exists(q, o') \end{array} \quad \begin{array}{c} \vdots \\ \forall(o', p) \end{array} \quad \begin{array}{c} \vdots \\ \exists(q, o) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, p) \end{array} \quad \begin{array}{c} \vdots \\ \exists(p, p) \end{array}}{\exists(p, \bar{q})} \quad (\text{F8})$$

If (f), then by Lemmas 3.1 and 3.2, $\Gamma \vdash \exists(q, o)$, $\Gamma \vdash \forall(o, q')$, $\Gamma \vdash \forall(o, p)$ and $\Gamma \vdash \forall(q, p')$, and by Lemma 3.7 (ii), $\Phi \vdash \exists(p, p)$. Hence, we have the derivations:

$$\frac{\begin{array}{c} \vdots \\ \forall(o', \forall(p', t)) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, p') \end{array}}{\forall(o', \forall(q, t))} \quad (\text{B2}) \quad \frac{\begin{array}{c} \vdots \\ \forall(q, \forall(q', \bar{t})) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, q') \end{array}}{\forall(q, \forall(o, \bar{t}))} \quad (\text{B2}),$$

and hence the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(o', \forall(q, t)) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, \forall(o, \bar{t})) \end{array} \quad \begin{array}{c} \vdots \\ \exists(q, o') \end{array} \quad \begin{array}{c} \vdots \\ \forall(o', p) \end{array} \quad \begin{array}{c} \vdots \\ \exists(q, o) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, p) \end{array} \quad \begin{array}{c} \vdots \\ \exists(p, p) \end{array}}{\exists(p, \bar{q})} \quad (\text{F9})$$

(vi) Suppose $\forall(q', t), p \in \{o'\}^\Gamma$, where $o' \in [q]^\Gamma$, $\forall(p', t) \in \{q\}^\Gamma$ and $p \in \bigcup W_{\Lambda, \Gamma}$. We argue exactly as for case (v), with p and q' exchanged, and t replaced by \bar{t} . \square

Lemma 3.19. *Suppose $\Phi \models \exists(p, \bar{q})$. Then $\Phi \vdash \exists(p, \bar{q})$.*

Proof. Let $\mathfrak{A} = \mathfrak{A}_{\Lambda, \Delta}$. By construction, $\mathfrak{A} \models \forall(p, q)$, and hence $\mathfrak{A} \not\models \exists(p, \bar{q})$. By Lemma 3.4, W contains a defect. By Lemmas 3.17 and 3.18, $\Phi \vdash \exists(p, \bar{q})$. \square

Proof of Theorem 2.4. Lemmas 3.10, 3.13, 3.14 and 3.19. \square

References

- [1] John Corcoran. Completeness of an ancient logic. *Journal of Symbolic Logic*, 37(4):696–702, 1972.
- [2] Ian Pratt-Hartmann and Lawrence S. Moss. Logics for the relational syllogistic. *Review of Symbolic Logic*, 2(4):647–683, 2009.
- [3] T.J. Smiley. What is a syllogism? *Journal of Philosophical Logic*, 2:135–154, 1973.