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On Wald-Type Optimal Stopping for Brownian Motion

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The solution is presented to all optimal stopping problems of the form:

$$\sup_{\tau} E\Big(G\big(|B_\tau|\big) - c\tau\Big)$$

where $B = (B_t)_{t\ge 0}$ is standard Brownian motion and the supremum is taken over all stopping times τ for B with finite expectation, while the map $G : \mathbf{R}_+ \to \mathbf{R}$ satisfies $G\big(|x|\big) \le c|x|^2 + d$ for some $d \in \mathbf{R}$ with $c > 0$ being given and fixed. The optimal stopping time is shown to be the hitting time by the reflecting Brownian motion $|B| = (|B_t|)_{t\ge 0}$ of the set of all (approximate) maximum points of the map $x \mapsto G(|x|) - cx^2$. The method of proof relies upon Wald's identity for Brownian motion and simple real analysis arguments. A simple proof of the Dubins-Jacka-Schwarz-Shepp-Shiryaev (square root of two) maximal inequality for randomly stopped Brownian motion is given as an application.

1. Introduction

The aim of this paper is to present the solution to a class of Wald's type optimal stopping problems for Brownian motion, and from this deduce some sharp inequalities which give bounds for the expectation of functionals of randomly stopped Brownian motion in terms of the expectation of the stopping time.

More precisely, let $B = (B_t)_{t\ge 0}$ be standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Then in this paper we find the solution to all optimal stopping problems of the following form. *Maximize the expectation:*

$$E\Big(G\big(|B_\tau|\big) - c\tau\Big) \tag{1.1}$$

over all stopping times τ *for* B *with finite expectation, where the measurable map* $G : \mathbf{R}_+ \to \mathbf{R}$ *satisfies* $G\big(|x|\big) \le c|x|^2 + d$ *for some* $d \in \mathbf{R}$ *with* $c > 0$ *being given and fixed.* It is shown that the optimal stopping time is the hitting time by the reflecting Brownian motion $|B| = (|B_t|)_{t\ge 0}$ of the set of all (approximate) maximum points of the map $x \mapsto G(|x|) - cx^2$.

The result just indicated will be presented in more detail in Section 2, as well as extended to all continuous local martingales by using the standard time change method. To conclude the introduction we should like to say that our main emphasis in this paper is on simplicity of our solution to the problem under consideration. Nevertheless, we will see in Section 3 that our

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method is flexible enough to provide a simple proof of the Dubins-Jacka-Schwarz-Shepp-Shiryaev inequality for Brownian motion which was firstly found in [2] (and independently in [4]), and then proved by an entirely different method in [3].

2. Wald's optimal stopping for Brownian motion

1. In this section we present the solution to the optimal stopping problem (1.1). For simplicity, we shall only consider the case where $G(|x|) = |x|^p$ for $0 < p \leq 2$, and it will be clear from our proof below that the case of general map G (satisfying the boundedness condition) could be treated by exactly the same method.

Thus, if $B = (B_t)_{t \geq 0}$ is standard Brownian motion, then the problem under consideration in this section is the following. *Maximize the expectation:*

$$E\Big(|B_\tau|^p - c\tau \Big) \tag{2.1}$$

over all stopping times τ *for* B *with finite expectation, where* $0 < p \leq 2$ *and* $c > 0$ *are given and fixed.*

First, it should be noted that in the case $p = 2$, we find by Wald's identity (see [5]) for Brownian motion ($E|B_\tau|^2 = E(\tau)$) that the expression in (2.1) equals $(1-c)E(\tau)$. Thus taking $\tau \equiv n$ or 0 for $n \geq 1$, depending on whether $0 < c < 1$ or $1 < c < \infty$, we see that the supremum equals $+\infty$ or 0 respectively. If $c = 1$, then the supremum equals 0 , and any stopping time τ for B is optimal. These facts solve the problem (2.1) in the case $p = 2$. The solution in the general case $0 < p < 2$ is formulated in the following theorem.

Theorem 2.1 (Wald's optimal stopping for Brownian motion)
Let $B = (B_t)_{t \geq 0}$ *be standard Brownian motion, and let* $0 < p < 2$ *and* $c > 0$ *be given and fixed. Consider the optimal stopping problem:*

$$\sup_\tau E\Big(|B_\tau|^p - c\tau \Big) \tag{2.2}$$

where the supremum is taken over all stopping times τ *for* B *with finite expectation. Then the optimal stopping time (at which the supremum is attained) in (2.2) may be defined as follows:*

$$\tau^*_{p,c} = \inf \Big\{ t > 0 \ : \ |B_t| = \big(p/2c\big)^{1/(2-p)} \Big\} . \tag{2.3}$$

Moreover, for all stopping times τ *for* B *with finite expectation we have:*

$$E\Big(|B_\tau|^p - c\tau \Big) \leq \Big(\tfrac{2-p}{2}\Big) \Big(\tfrac{p}{2c}\Big)^{p/(2-p)} . \tag{2.4}$$

The upper bound in (2.4) is the best possible.

Proof. Given $0 < p < 2$ and $c > 0$, denote:

$$V_\tau(p,c) = E\Big(|B_\tau|^p - c\tau \Big) \tag{2.5}$$

whenever τ is a stopping time for B . Then by Wald's identity for Brownian motion we find out that the expression in (2.5) may be equivalently written in the following form:

$$
(2.6) \qquad V_\tau(p,c) = \int_{-\infty}^{\infty} \Big(|x|^p - cx^2 \Big)\, dP_{B_\tau}(x)
$$

whenever τ is a stopping time for B with $E(\tau) < \infty$. Our next step is to maximize the map $x \mapsto D(x) = |x|^p - cx^2$ over $\mathbf{R}$. For this, note that $D(-x) = D(x)$ for all $x \in \mathbf{R}$, and therefore it is enough to consider $D(x)$ for $x > 0$. We have $D'(x) = px^{p-1} - 2cx$ for $x > 0$, and hence we see that D attains its maximal value at the point $\pm(p/2c)^{1/(2-p)}$. Thus it is clear from (2.6) that the optimal stopping time in (2.2) might be defined by (2.3). This completes the first part of the proof.

Finally, inserting $\tau^* = \tau^*_{p,c}$ from (2.3) into (2.6), we easily find:

$$
V_{\tau^*}(p,c) = D\Big((p/2c)^{1/(2-p)} \Big) = \Big(\tfrac{2-p}{2} \Big) \Big(\tfrac{p}{2c} \Big)^{p/(2-p)} .
$$

This establishes (2.4) with the last statement of the theorem, and the proof is complete. □

Remark 2.2

The preceding proof shows that the solution to the problem (1.1) in the case of general map G (satisfying the boundedness condition) could be found by using exactly the same method: *The optimal stopping time is the hitting time by the reflecting Brownian motion* $|B| = (|B_t|)_{t \geq 0}$ *of the set of all (approximate) maximum points of the map* $x \mapsto D(x) = G(|x|) - cx^2$. (Here "approximate" stands to cover the case (in an obvious manner) when D does not attain its least upper bound on the real line.)

2. In the remaining part of this section we will explore some consequences of the inequality (2.4) in more detail. For this, let a stopping time τ for B with $E(\tau) < \infty$ and $0 < p < 2$ be given and fixed. Then from (2.4) we get:

$$
(2.7) \qquad E|B_\tau|^p \leq \inf_{c>0} \bigg(cE(\tau) + \Big(\tfrac{2-p}{2} \Big) \Big(\tfrac{p}{2c} \Big)^{p/(2-p)} \bigg) .
$$

It is elementary to compute that this infimum equals $(E\tau)^{p/2}$. In this way we obtain:

$$
(2.8) \qquad E|B_\tau|^p \leq \big(E\tau \big)^{p/2} \quad (0 < p \leq 2)
$$

with the constant 1 being the best possible in all of the inequalities. (Observe that this also follows by Wald's identity and Jensen's inequality in a straightforward way.)

Next consider the case $2 < p < \infty$. Thus we shall look at $-V_\tau(p,c)$ instead of $V_\tau(p,c)$ in (2.5) and (2.6). By the same argument as for (2.6) we obtain:

$$
-V_\tau(p,c) = E\Big(c\tau - |B_\tau|^p \Big) = \int_{-\infty}^{\infty} \Big(cx^2 - |x|^p \Big)\, dP_{B_\tau}(x)
$$

where $2 < p < \infty$. The same calculation as in the proof of Theorem 2.1 shows that the map $x \mapsto -D(x) = cx^2 - |x|^p$ attains its maximal value over $\mathbf{R}$ at the point $\pm(p/2c)^{1/(2-p)}$. Thus as in the proof of Theorem 2.1 we find:

$$E\Big(c\tau - |B_\tau|^p\Big) \;\le\; \Big(\tfrac{p-2}{2}\Big)\Big(\tfrac{p}{2c}\Big)^{p/(2-p)} \;.$$

From this inequality we get:

$$\sup_{c>0}\bigg(cE(\tau) + \Big(\tfrac{2-p}{2}\Big)\Big(\tfrac{p}{2c}\Big)^{p/(2-p)}\bigg) \;\le\; E|B_\tau|^p \;.$$

The same calculation as for the proof of (2.8) shows that this supremum equals $(E\tau)^{p/2}$. Thus as above for (2.8) we obtain:

$$\big(E\tau\big)^{p/2} \le E|B_\tau|^p \quad (\, 2 \le p < \infty \,) \tag{2.9}$$

with the constant 1 being the best possible in all of the inequalities. (Observe that this also follows by Wald's identity and Jensen's inequality in a straightforward way.)

3. The previous calculations together with conclusions (2.8) and (2.9) indicate that the inequality (2.4)+(2.7) provide sharp estimates which are otherwise obtainable by a different method that relies upon convexity and Jensen's inequality (see Remark 2.4 below). This leads precisely to the main point of our observation: *The previous procedure can be repeated for any measurable map* G *satisfying the boundedness condition.* In this way we obtain a sharp estimate of the form:

$$E\Big(G\big(|B_\tau|\big)\Big) \;\le\; \gamma_{_G}(E\tau)$$

where $\gamma_{_G}$ is a map to be found (by maximizing and minimizing certain real valued functions of real variable). We formulate this more precisely in the next corollary.

Corollary 2.3

Let $B = (B_t)_{t\ge 0}$ *be standard Brownian motion, and let* $G : \mathbf{R} \to \mathbf{R}$ *be a measurable map. Then for any stopping time* τ *for* B *the inequality holds:*

$$E\Big(G\big(|B_\tau|\big)\Big) \;\le\; \inf_{c>0}\bigg(cE(\tau) + \sup_{x\in\mathbf{R}}\Big(G\big(|x|\big) - cx^2\Big)\bigg) \tag{2.10}$$

and is sharp whenever the right-hand side is finite. Similarly, if $H : \mathbf{R} \to \mathbf{R}$ *is a measurable map, then for any stopping time* τ *for* B *with finite expectation the inequality holds:*

$$\sup_{c>0}\bigg(cE(\tau) + \inf_{x\in\mathbf{R}}\Big(H\big(|x|\big) - cx^2\Big)\bigg) \le E\Big(H\big(|B_\tau|\big)\Big) \tag{2.11}$$

and is sharp whenever the left-hand side is finite.

Proof. It follows from the proof of Theorem 2.1 as indicated in Remark 2.2 and the lines above following it (or just straightforward by using Wald's identity). It should be noted that the boundedness condition on the maps G and H is contained in the non-triviality of the conclusions. □

Remark 2.4

As noted by a referee, if we set $H(x) = G(\sqrt{x})$ for $x \ge 0$, then we have:

$$\sup_{x\in\mathbf{R}}\Big(G\big(|x|\big) - cx^2\Big) \;=\; -\inf_{x\ge 0}\Big(cx - H(x)\Big) \;=\; -\tilde{H}(c)$$

where $\tilde{H}$ denotes the *concave conjugate* of H. Similarly, we have:

$$\inf_{c>0} \Big(cE(\tau) + \sup_{x \in \mathbf{R}} \Big(G(|x|) - cx^2 \Big) \Big) = \inf \Big(cE(\tau) - \tilde{H}(c) \Big) = \tilde{\tilde{H}}\big(E(\tau)\big) \ .$$

Thus (2.10) reads as follows:

$$E\Big(H\big(|B_\tau|^2\big) \Big) \ \leq \ \tilde{\tilde{H}}\Big(E(\tau) \Big) \ . \tag{2.12}$$

Moreover, since $\tilde{\tilde{H}}$ is the (smallest) concave function which dominates H, it is clear from a simple comparison that (2.12) also follows by Jensen's inequality. This provides an alternative way of looking at (2.10) and (2.11) and clarifies (2.7)+(2.8). (A similar remark might be directed to (2.11) with (2.9).) Note that (2.12) gets the following form:

$$E\Big(G\big(|B_\tau|\big) \Big) \ \leq \ G\Big(\sqrt{E(\tau)} \Big)$$

whenever $x \mapsto G(\sqrt{x})$ is *concave* on $\mathbf{R}_+$.

Remark 2.5

By using the standard time change method, we can generalize and extend the inequalities (2.10) and (2.11) to cover the case of all continuous local martingales. *Let $M = (M_t)_{t \geq 0}$ be a continuous local martingale with the quadratic variation process $[M] = ([M]_t)_{t \geq 0}$ such that $M_0 = 0$, and let G, H : $\mathbf{R}_+ \to \mathbf{R}$ be measurable functions. Then for any $t > 0$ for which $E([M]_t) < \infty$ the inequalities hold:*

$$E\Big(G\big(|M_t|\big) \Big) \ \leq \ \inf_{c>0} \Big(c\, E([M]_t) + \sup_{x \in \mathbf{R}} \Big(G\big(|x|\big) - cx^2 \Big) \Big) \tag{2.13}$$

$$\sup_{c>0} \Big(c\, E([M]_t) + \inf_{x \in \mathbf{R}} \Big(H\big(|x|\big) - cx^2 \Big) \Big) \leq E\Big(H\big(|M_t|\big) \Big) \tag{2.14}$$

and are sharp whenever the right-hand side in (2.13) and the left-hand side in (2.14) are finite. To prove the sharpness of (2.13) and (2.14) for every given and fixed $t > 0$, consider $M_t = B_{\alpha t + \tau_\beta}$ with $\alpha > 0$ and τ_β being the hitting time of some $\beta > 0$ by the reflecting Brownian motion $|B| = (|B_t|)_{t \geq 0}$. Letting $\alpha \to \infty$ and using (integrability) properties of τ_β (in the context of Corollary 2.3), by Burkholder-Gundy's inequalities (see [1]) and uniform integrability arguments we (eventually) finish with the inequalities (2.10) and (2.11) for optimal $\tau = \tau_\beta$, at least in the case when G allows the limiting procedures which are required (the case of general G could then follow by approximation). Thus the sharpness of (2.13)+(2.14) follows from the sharpness of (2.10)+(2.11).

3. Applications

In this section, as an application of our method and results obtained, we shall present a simple proof of the Dubins-Jacka-Schwarz-Shepp-Shiryaev (square root of two) maximal inequality for randomly stopped Brownian motion which was firstly found in [2] (and independently in [4]), and then proved by an entirely different method in [3]. The method of attack in the proof of (3.1) and (3.2) is based upon the trick of picking up the two martingales (3.3) and (3.5) with the properties desired. We shall begin by stating the two inequalities to be proved.

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion, and let τ be a stopping time for B with finite expectation. Then the following inequalities are sharp:

$$E\Big(\max_{0 \leq t \leq \tau} B_t \Big) \leq \sqrt{E\tau} \tag{3.1}$$

$$E\Big(\max_{0 \leq t \leq \tau} |B_t| \Big) \leq \sqrt{2}\, \sqrt{E\tau}\ . \tag{3.2}$$

1. We shall first deduce these inequalities by our method, and then show their sharpness by picking up the optimal stopping times (for which the equalities are attained). Our approach to the problem of establishing (3.1) is motivated by the fact that the process $(\max_{0 \leq s \leq t} B_s - B_t)_{t \geq 0}$ is equally distributed as the reflecting Brownian motion process $(|B_t|)_{t \geq 0}$ for which we have found optimal bound (2.4) (from where by (2.7) we get (2.8) with $p=1$), while $E(B_\tau) = 0$ whenever $E(\sqrt{\tau}) < \infty$. These observations clearly lead us to (3.1), at least for some stopping times. To extend this to all stopping times, we shall use a simple martingale argument.

Proof of (3.1): Set $S_t = \max_{0 \leq s \leq t} B_s$ for $t \geq 0$. Since $(B_t^2 - t)_{t \geq 0}$ is a martingale, and $(S_t - B_t)_{t \geq 0}$ is equally distributed as $(|B_t|)_{t \geq 0}$, we see that:

$$Z_t = c\Big(\big(S_t - B_t \big)^2 - t \Big) + 1/4c \tag{3.3}$$

is a martingale (with respect to the natural filtration which is known to be the same as the natural filtration of B). Using $E(B_\tau) = 0$, by Doob's optional sampling theorem and the elementary inequality $x - ct \leq c(x^2 - t) + 1/4c$, we find:

$$E\big(S_\tau - c\tau\big) = E\big(S_\tau - B_\tau - c\tau\big) \leq E\big(Z_\tau\big) = E(Z_0) = 1/4c$$

for any bounded stopping time τ . Hence we get:

$$E\big(S_\tau\big) \leq \inf_{c>0} \Big(cE(\tau) + 1/4c \Big) = \sqrt{E\tau}$$

for any bounded stopping time τ . Passing to the limit, we obtain (3.1) for all stopping times with finite expectation. This completes the proof of (3.1).

2. Next we extend (3.1) to any continuous local martingale $M = (M_t)_{t \geq 0}$ with $M_0 = 0$. For this, note that by the time change and (3.1) we obtain:

$$E\Big(\max_{0 \leq s \leq t} M_s \Big) = E\Big(\max_{0 \leq s \leq t} B_{[M]_s} \Big) = E\Big(\max_{0 \leq s \leq [M]_t} B_s \Big) \leq \sqrt{E\big([M]_t\big)} \tag{3.4}$$

for all $t > 0$.

3. In the next step we will apply (3.4) to the continuous martingale M defined by:

$$M_t = E\Big(|B_\tau| - E\big(|B_\tau|\big) \ \big|\ \mathcal{F}_{t \wedge \tau} \Big) \tag{3.5}$$

for $t \geq 0$. In this way we get:

$$E\Big(\max_{0\le t<\infty} E\big(|B_\tau|-E(|B_\tau|)\mid \mathcal{F}_{t\wedge\tau}\big)\Big) \le \sqrt{E\big(|B_\tau|-E(|B_\tau|)\big)^2}\ . \tag{3.6}$$

We now pass to the proof of the 'square root of two' inequality.

Proof of (3.2): Since $\sqrt{A-x^2}+x \le \sqrt{2A}$ for $0<x<\sqrt{A}$, by (3.6) we find:

$$E\Big(\max_{0\le t\le\tau}|B_t|\Big) = E\Big(\max_{0\le t<\infty}|B_{t\wedge\tau}|\Big) \le E\Big(\max_{0\le t<\infty} E\big(|B_\tau|\mid \mathcal{F}_{t\wedge\tau}\big)\Big) =$$

$$= E\Big(\max_{0\le t<\infty} E\big(|B_\tau|-E(|B_\tau|)\mid \mathcal{F}_{t\wedge\tau}\big)\Big) + E(|B_\tau|)$$

$$\le \sqrt{E\big(|B_\tau|-E(|B_\tau|)\big)^2} + E(|B_\tau|)$$

$$= \sqrt{E(\tau)-\big(E(|B_\tau|)\big)^2} + E(|B_\tau|) \le \sqrt{2E(\tau)}\ .$$

This establishes (3.2) and completes the first part of the proof.

4. To prove the sharpness of (3.1) one may take the stopping time:

$$\tau_1^* = \inf\ \{\, t>0 : |B_t| = a \,\}$$

for any $a>0$. Then the equality in (3.1) is attained. It follows by Wald's identity. Note that for any $a>0$ the stopping time τ_1^* could be equivalently (in distribution) defined by:

$$\tau_1^* = \inf\ \Big\{\, t>0\ :\ \max_{0\le s\le t} B_s - B_t \ge a \,\Big\}\ .$$

5. To prove the sharpness of (3.2) one may take the stopping time:

$$\tau_2^* = \inf\ \Big\{\, t>0\ :\ \max_{0\le s\le t} |B_s| - |B_t| \ge a \,\Big\}$$

for any $a>0$. Then it is easily verified that $E\big(\max_{0\le t\le\tau_2^*}|B_t|\big) = 2a$ and $E(\tau_2^*) = 2a^2$ (see [3]). Thus the equality in (3.2) is attained, and the proof of the sharpness is complete.

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