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# Vitali Convergence Theorem for Upper Integrals

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It is shown that the Vitali convergence theorem remains valid for the $\mu$-upper integral. Using this result we prove completeness of the space $L^{<p>}(\mu)$ with respect to the $\|\cdot\|_p$-upper norm for $1 \leq p < \infty$, describe convergence of its elements in terms of the space $L^p(\mu)$ for $1 \leq p < \infty$, give a necessary and sufficient condition for a sequence from $L^{<p>}(\mu)$ to converge in the $\|\cdot\|_p$-upper norm to a function from $L^p(\mu)$ for $1 \leq p < \infty$, and deduce some convergence relations concerning the non-measurable objects under consideration.

## 1. Introduction

Let us recall that the fundamental *Vitali convergence theorem* states, see [2] (p.122,150): *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, let* $\{ f_n \mid n \geq 1 \}$ *be a sequence of functions in* $L^p(\mu)$ *for some* $1 \leq p < \infty$ *, and let* $f : X \to \mathbf{R}$ *be a given function. Then* $f$ *belongs to* $L^p(\mu)$ *and* $\| f_n - f \|_p$ *converges to zero as* $n \to \infty$ *, if and only if the following three conditions are satisfied:*

(1) $$f_n \to f \text{ in } \mu\text{-measure, as } n \to \infty$$

(2) $$\lim_{\mu(A) \to 0} \sup_{n \geq 1} \int_A | f_n |^p \, d\mu = 0$$

(3) $$\inf_{A \in \mathcal{A},\, \mu(A) < \infty} \sup_{n \geq 1} \int_{X \setminus A} | f_n |^p \, d\mu = 0 .$$

*Moreover, if* $f_n$ *converges* $\mu$*-a.a. to* $f$ *as* $n \to \infty$ *, and (2)+(3) is satisfied, then* $f$ *belongs to* $L^p(\mu)$ *, and* $\| f_n - f \|_p$ *converges to zero as* $n \to \infty$ . This theorem, along with the monotone and dominated convergence theorem and Fatou's lemma, is the principal result of the classical convergence theory for Lebesgue integrals, see [4], [6], [7] and [9]. Although different aspects of this theorem have been obtained, see for example [5] (p.223), the common framework of all these results is the measurability assumption on the functions in question. In contrast to all of these results, in this paper we show that the same theorem remains valid for functions which need not be measurable or satisfy any other restriction, provided that the ordinary integral is replaced by the upper integral. This is achieved by applying a method that uses the so-called envelopes of functions under consideration, see [3], and the classical Vitali convergence theorem which is becoming a particular case of the new result. As an application of this result we prove completeness of the spaces $L^{<p>}(\mu)$ introduced below with respect to the $\|\cdot\|_p$-upper norm for $1 \leq p < \infty$ ,

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and give a complete description of the $\|\cdot\|_p$-upper convergence in these spaces in terms of the classical spaces $L^p(\mu)$ with $1 \le p < \infty$ . We begin by recalling some preliminary facts.

## 2. Preliminary facts

Let $(X, \mathcal{A}, \mu)$ be a measure space, then $\mu^*$ denotes *the outer $\mu$-measure* and $\mu_*$ denotes *the inner $\mu$-measure*. If $C$ is an arbitrary subset of $X$ , then $C^*$ denotes *the $\mu$-hull* of $C$ , and $C_*$ denotes *the $\mu$-kernel* of $C$ , i.e. $C^*$ , $C_* \in \mathcal{A}$ and we have:

$$\mu(C^*) = \mu^*(C) \ , \ \mu(C_*) = \mu_*(C) \ .$$

It is easily verified that each subset $C$ of $X$ has the $\mu$-hull and the $\mu$-kernel , uniquely determined up to a $\mu$-nullset if $\mu^*(C) < \infty$ . In general, it is no restriction to assume that $C_* \subset C \subset C^*$ for all $C \subset X$ . Let $L^1(\mu)$ be the space of all $\mu$-integrable functions from $X$ into $\bar{\mathbf{R}}$ . Then *the upper* and *lower $\mu$-integral* of an arbitrary function $f$ from $X$ into $\bar{\mathbf{R}}$ are defined respectively as follows:.

$$\int^* f \, d\mu = \inf \{ \textstyle\int g \, d\mu \mid g \in L^1(\mu) \ , \ f \le g \}$$

$$\int_* f \, d\mu = \sup \{ \textstyle\int g \, d\mu \mid g \in L^1(\mu) \ , \ g \le f \} \ .$$

Let $\bar{\mathcal{M}}(\mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions from $X$ into $\bar{\mathbf{R}}$ . Then *the upper* and *lower $\mu$-envelope* of an arbitrary function $f$ from $X$ into $\bar{\mathbf{R}}$ are defined respectively as follows:

$$f^* = \mu\text{-ess inf} \{ g \in \bar{\mathcal{M}}(\mathcal{A}) \mid f \le g \}$$

$$f_* = \mu\text{-ess sup} \{ g \in \bar{\mathcal{M}}(\mathcal{A}) \mid g \le f \} \ .$$

It is well-known that if $\mu$ satisfies *Segal's localization principle*, then $f^*$ and $f_*$ exist for every function $f$ from $X$ into $\bar{\mathbf{R}}$ . Moreover, if $\mu$ is a $\sigma$-finite, or even $\Sigma$-finite measure, then $\mu$ satisfies Segal's localization principle, see [8]. The basic connection between the objects just introduced may be stated as follows. For this suppose that $\mu$ is a finitely founded measure (without infinite atoms) which satisfies Segal's localization principle. Then for an arbitrary function $f$ from $X$ into $\bar{\mathbf{R}}$ and for an arbitrary subset $C$ of $X$ the following relations are satisfied, see [8]:

(1) $$\int^* f \, d\mu = \begin{cases} \int f^* \, d\mu & , \ \text{if} \ f^* \in L(\mu) \\ +\infty & , \ \text{otherwise} \end{cases}$$

(2) $$\int_* f \, d\mu = \begin{cases} \int f_* \, d\mu & , \ \text{if} \ f_* \in L(\mu) \\ -\infty & , \ \text{otherwise} \end{cases}$$

(3) $$(1_C)^* = 1_{C^*} \quad \text{and} \quad (1_C)_* = 1_{C_*}$$

(4) $$\mu^*(C) = \textstyle\int^* 1_C \, d\mu \quad \text{and} \quad \mu_*(C) = \textstyle\int_* 1_C \, d\mu \ .$$

Here $L(\mu)$ denotes the set of all $\mu$-measurable functions from $X$ into $\bar{\mathbf{R}}$ for which the $\mu$-integral exists (it may be $\pm\infty$). For the further properties of the objects introduced above we shall refer the reader to [1] and [8]. Most of these facts will be used implicitly throughout. Let us however recall here the following proposition on the stability of envelopes which plays an important role in the next considerations.

**Proposition 1.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu$ satisfies Segal's localization principle, and let $f : X \to \bar{\mathbf{R}}$ and $G : \bar{\mathbf{R}} \to \bar{\mathbf{R}}$ be given functions. If $G$ is increasing and left continuous, then we have:*

$$G(f)^* = G(f^*) \ .$$

*Similarly, if $G$ is increasing and right continuous, then we have:*

$$G(f)_* = G(f_*) \ .$$

**Proof.** A straight verification, see [8]. We shall leave the details to the reader.

□

In particular, for an arbitrary function $f$ from $X$ into $\bar{\mathbf{R}}$ and any $1 \leq p < \infty$ we have:

(5) $$\big(\, |\, f\,|^p\,\big)^* = \big(\, |\, f\,|^*\,\big)^p \quad \text{and} \quad \big(\, |\, f\,|^p\,\big)_* = \big(\, |\, f\,|_*\,\big)^p \ .$$

In addition, for an arbitrary subset $C$ of $X$ and an arbitrary function $f$ from $X$ into $\bar{\mathbf{R}}$ we define:

$$\int_C^* f \ d\mu = \int^* f \cdot 1_C \ d\mu \qquad \text{and} \qquad \int_{*C} f \ d\mu = \int_* f \cdot 1_C \ d\mu \ .$$

A very important fact in the considerations on the spaces $L^{<p>}(\mu)$ for $1 \leq p < \infty$ , which will be introduced in the next section, is contained in the next proposition. Roughly speaking, it says that dealing with functions from $L^{<p>}(\mu)$ for $1 \leq p < \infty$ there is no restriction to assume that the underlying measure $\mu$ satisfies Segal's localization principle.

**Proposition 2.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f$ be a real valued function on $X$ such that the set $\{f \neq 0\}$ has $\sigma$-finite $\mu$-measure, i.e. there are $F_i \in \mathcal{A}$ with $\mu(F_i) < \infty$ for $i \geq 1$ satisfying $\{f \neq 0\} = \bigcup_{i=1}^{\infty} F_i$ . Then there exist the upper and lower $\mu$-envelope $f^*$ and $f_*$ of $f$ . In particular, if $\int^* |\, f\,|^p \ d\mu < \infty$ for some $1 \leq p < \infty$ , then the set $\{\, f \neq 0\,\}$ has $\sigma$-finite $\mu$-measure, and there exist the upper and lower $\mu$-envelope $f^*$ and $f_*$ of $f$ .*

**Proof.** Put $F = \{f \neq 0\}$ and define $\nu = tr(\mu, F)$ , see [8] . Then $\nu$ is a $\sigma$-finite measure on $(X, \mathcal{A})$ , and hence $\nu$ satisfies Segal's localization principle. Let $f^*$ and $f_*$ be the upper and lower $\nu$-envelope of $f$ . Without loss of generality we can assume that $f^*(x) = g^*(x) = 0$ for all $x \in X \setminus F$ , and hence one can easily verify that $f^*$ and $f_*$ are the upper and lower $\mu$-envelope of $f$ . This fact completes the first part of the proposition. Further, if $f \in L^{<p>}(\mu)$ , then there exists a function $g \in L^1(\mu)$ such that $|f|^p \le g$ . Let $G = \{g > 0\}$ , then $G = \bigcup_{n=1}^{\infty}\{g > \frac{1}{n}\}$ and since $g \in L^1(\mu)$ , we have $\mu\{ g > \frac{1}{n} \} < \infty$ . This shows that $G$ has $\sigma$-finite $\mu$-measure. Since $\{f \neq 0\} \subset G$ , we see that the set $\{f \neq 0\}$ has $\sigma$-finite $\mu$-measure as well, and hence the proof follows by the first part of proposition. □

## 3. The space $L^{<p>}(\mu)$ for $1 \le p < \infty$

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \le p < \infty$ be a real number. Let us define:

$$L^{<p>}(\mu) = \{ f \in \mathbf{R}^X \mid \int^* |f|^p \, d\mu < \infty \}$$

Then evidently the following statements are equivalent:

(1) $f \in L^{<p>}(\mu)$

(2) $\exists\, g, h \in L^p(\mu)$ such that $g \le f \le h$ $\mu$-a.a.

(3) $\exists\, g \in L^p(\mu)$ such that $|f| \le g$ $\mu$-a.a.

Moreover, it is easily seen from proposition 2 in section 2 and (5) in section 2 that the statements (1)-(3) are equivalent to the following two statements as well:

(4) $f^*$ , $f_* \in L^p(\mu)$

(5) $|f|^* \in L^p(\mu)$ .

By (3) we easily see that $L^{<p>}(\mu)$ for $1 \le p < \infty$ is a real vector space. Moreover, by subadditivity of the $\mu$-upper integral we find that:

(6) $$\| f \|_p^* = \Big\{ \int^* |f|^p \, d\mu \Big\}^{\frac{1}{p}}$$

with $f \in L^{<p>}(\mu)$ defines a pseudonorm on $L^{<p>}(\mu)$ for $1 \le p < \infty$ , see [8]. The same symbol $L^{<p>}(\mu)$ will denote the set of all equivalence classes $[f]$ of functions $f$ in $L^{<p>}(\mu)$ for $1 \le p < \infty$ , where the equivalence relation is defined as follows:

$$f \sim g \;\Leftrightarrow\; \| f - g \|_p^* = 0 \ .$$

By definition of the $\mu$-upper integral one can easily verify that we have:

$$f \sim g \;\Leftrightarrow\; \mu^*\{ |f - g| > 0 \} = 0 \;\Leftrightarrow\; f = g \;\; \mu\text{-a.a.}$$

Thus $L^{<p>}(\mu)$ for $1 \le p < \infty$ is a normed linear space with the norm being defined as follows:

$$\| [f] \|_p^* = \| f \|_p^* = \Big\{ \int^* |f|^p \, d\mu \Big\}^{\frac{1}{p}}$$

for all $[f] \in L^{<p>}(\mu)$ . As usual we shall identify classes with functions which belong to them.

The norm $\|\cdot\|_p^*$ will be called *the* $\|\cdot\|_p$*-upper norm* of $L^{<p>}(\mu)$ for all $1 \leq p < \infty$ . Let us note that from proposition 2.2 and (2.5) we get:

$$\text{(7)} \qquad \| f \|_p^* = \Big\{ \int^* | f |^p \, d\mu \Big\}^{\frac{1}{p}} = \Big\{ \int (| f |^*)^p d\mu \Big\}^{\frac{1}{p}}$$

A natural question arises immediately. Is the normed space $\big(L^{<p>}(\mu) \, , \, \| \cdot \|_p^* \big)$ for $1 \leq p < \infty$ complete? To give the answer to this question we shall extend the fundamental Vitali convergence theorem to the form which involve arbitrary functions and uses the upper integral instead of the ordinary one. For this we shall first in the next section introduce definitions and deduce statements that will be of use later on.

## 4. Vitali convergence theorem for upper integrals

Let $(X, \mathcal{A}, \mu)$ be a measure space , let $\{ f_n \mid n \geq 1 \}$ be a sequence of functions from $X$ into $\mathbf{R}$ , and let $f : X \to \mathbf{R}$ be a given function. Then we say that the sequence $\{ f_n \mid n \geq 1 \}$ :

(a) *converges* $(a.a.)(\mu)$ to $f$ , and write $f_n \overset{(a.a.)}{\longrightarrow} f$ , if there is $N \in \mathcal{A}$ with $\mu(N) = 0$ , such that $| f_n(x) - f(x) | \to 0$ for all $X \setminus N$ as $n \to \infty$

(b) *converges* $(a.a.)^*(\mu)$ resp. $(a.a.)_*(\mu)$ to $f$ , and write $f_n \overset{(a.a.)^*}{\longrightarrow} f$ resp. $f_n \overset{(a.a.)_*}{\longrightarrow} f$ , if $| f_n - f |^*$ resp. $| f_n - f |_*$ converges $(a.a.)(\mu)$ to zero

(c) *converges* $\mu$*-uniformly* to $f$ , and write $f_n \overset{u}{\longrightarrow} f$ , if $\forall \varepsilon > 0$ , $\exists N_\varepsilon \in \mathcal{A}$ with $\mu(N_\varepsilon) < \varepsilon$ , such that $f_n$ converges uniformly to $f$ on $X \setminus N_\varepsilon$ as $n \to \infty$

(d) *converges* $(m^p)^*$ resp. $(m^p)_*$ to $f$ , and write $f_n \overset{(m^p)^*}{\longrightarrow} f$ resp. $f_n \overset{(m^p)_*}{\longrightarrow} f$ , if $\int^* | f_n - f |^p \, d\mu$ resp. $\int_* | f_n - f |^p \, d\mu$ converges to zero as $n \to \infty$ , where $1 \leq p < \infty$

(e) *converges* $(\mu^*)$ resp. $(\mu_*)$ to $f$ , and write $f_n \overset{(\mu^*)}{\longrightarrow} f$ resp. $f_n \overset{(\mu_*)}{\longrightarrow} f$ , if $\mu^* \{ | f_n - f | > \varepsilon \} \to 0$ resp. $\mu_* \{ | f_n - f | > \varepsilon \} \to 0$ as $n \to \infty$ .

The concept of the Cauchy sequence is supposed to be defined naturally. One could verify that the following convergence implications are satisfied, and no other holds in general:

(1) $(a.a.)^* \Rightarrow (a.a.) \Rightarrow (a.a.)_*$

(2) $(a.a.)^* \Rightarrow (\mu^*)$ , if $\mu$ is finite

(3) $(a.a.)_* \Rightarrow (\mu_*)$ , if $\mu$ is finite

(4) $(a.a.) \Rightarrow (\mu)$ , if $\mu$ is finite and all of the functions are measurable

(5) $(\mu^*) \Leftrightarrow (\mu) \Leftrightarrow (\mu_*)$ , if all of the functions are measurable

(6) $(m^p)^* \Rightarrow (\mu^*)$

(7) $(m^p)_* \Rightarrow (\mu_*)$

(8) $(m^p)^* \Leftrightarrow (m^p) \Leftrightarrow (m^p)_*$ , if all of the functions are measurable

(9) $(m^p) \Rightarrow (\mu)$ , if all of the functions are measurable

(10) $(u) \Rightarrow (\mu^*)$

(11) $(u) \Rightarrow (a.a)$

The next lemma concerns the converse of (10). It will be of use in the proof of theorem 3 below.

**Lemma 1.**

*Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $\{ f_n \mid n \geq 1 \}$ *be a sequence of arbitrary functions from* $X$ *into* $\mathbf{R}$ *. Suppose that:*

$$\mu^*\{ \, |f_n - f_m| > \varepsilon \, \} \to 0$$

*as* $n, m \to \infty$ *for all* $\varepsilon > 0$ *. Then there exist a subsequence* $\{ f_{n_k} \mid k \geq 1 \}$ *of* $\{ f_n \mid n \geq 1 \}$ *and a function* $f$ *from* $X$ *into* $\mathbf{R}$*, such that* $\{ f_{n_k} \mid k \geq 1 \}$ *converges* $\mu$*-uniformly to* $f$ *.*

**Proof**. By induction, for each $\varepsilon_k = 2^{-k}$ there exists $n_k > n_{k-1}$ , such that we have:

$$\mu^*\{ \, |f_n - f_m| > \varepsilon_k \, \} < \varepsilon_k$$

for all $n, m \geq n_k$ , where $k \geq 1$ and $n_0 = 1$ . Let $N_k = \{ \, |f_{n_k} - f_{n_{k+1}}| > \varepsilon_k \, \}^*$ be the $\mu$-hull of $\{ \, |f_{n_k} - f_{n_{k+1}}| > \varepsilon_k \}$ , then $\mu(N_k) < 2^{-k}$ and $|f_{n_k}(x) - f_{n_{k+1}}(x)| \leq 2^{-k}$ for all $x \in X \setminus N_k$ and all $k \geq 1$ . Put $M_k = \bigcup_{i=k}^{\infty} N_i$ , then $\mu(M_k) < 2^{-k+1}$ , and for each $x \in X \setminus M_k$ we have:

(1) $$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{m=k}^{\infty} |f_{n_m}(x) - f_{n_{m+1}}(x)| < 2^{-k+1}$$

for all $j > i \geq k \geq 1$ . Since $\{ X \setminus M_k \mid k \geq 1 \}$ is an increasing sequence of subsets of $X$ , then from (1) we see that $\{ f_{n_k}(x) \mid k \geq 1 \}$ is a Cauchy sequence for each $x \in \bigcup_{k=1}^{\infty} X \setminus M_k$ , and thus it converges to some $f(x)$ . Moreover, the sequence $\{ f_{n_k} \mid k \geq 1 \}$ converges uniformly to $f$ on each of the sets $X \setminus M_k$ for $k \geq 1$ . Putting $f(x) = c$ for all $x \in \bigcap_{k=1}^{\infty} M_k$ , where $c$ is a real number, we see that $\{ f_{n_k} \mid k \geq 1 \}$ converges $\mu$-uniformly to $f$ , and the proof is complete. □

**Corollary 2.**

*Let* $(X, \mathcal{A}, \mu)$ *be a measure space, and let* $\{ f_n \mid n \geq 1 \}$ *be a sequence of functions from* $X$ *into* $\mathbf{R}$ *. Then we have:*

(1) *If* $f_n \xrightarrow{(\mu^*)} f$ *, for some function* $f$ *from* $X$ *into* $\mathbf{R}$*, then there exists a subsequence* $\{ f_{n_k} \mid k \geq 1 \}$ *of* $\{ f_n \mid n \geq 1 \}$ *such that* $\{ f_{n_k} \mid k \geq 1 \}$ *converges* $\mu$*-uniformly to* $f$ *.*

(2) *If* $\mu^*\{ |f_n - f_m| > \varepsilon \} \to 0$ *as* $n, m \to \infty$ *for all* $\varepsilon > 0$ *, then there exists a function* $f$ *from* $X$ *into* $\mathbf{R}$ *such that* $f_n \xrightarrow{(\mu^*)} f$ *.*

**Proof**. (1): Since $\mu^*\{ |f_n - f_m| > \varepsilon \} \leq \mu^*\{ |f_n - f| > \varepsilon/2 \} + \mu^*\{ |f_m - f| > \varepsilon/2 \}$ for all $n, m \geq 1$ and all $\varepsilon > 0$ , we see that $\mu^*\{ |f_n - f_m| > \varepsilon \} \to 0$ as $n, m \to \infty$ for all $\varepsilon > 0$ , and the proof of (1) follows straightforward from lemma 1.

(2): By lemma 1 and (10) above there exists a subsequence $\{ f_{n_k} \mid k \geq 1 \}$ of $\{ f_n \mid n \geq 1 \}$ and a function $f$ from $X$ into $\mathbf{R}$ , such that $\{ f_{n_k} \mid k \geq 1 \}$ converges $(\mu^*)$ to $f$ . Since

$\mu^*\{\,|\,f_n-f\,|>\varepsilon\} \le \mu^*\{\,|\,f_n-f_{n_k}|>\varepsilon/2\}+\mu^*\{\,|\,f_{n_k}-f\,|>\varepsilon/2\}$ for all $n,k \ge 1$ and all $\varepsilon > 0$, we see that $\{\,f_n \mid n \ge 1\}$ converges $(\mu^*)$ to $f$, and the proof of (2) is complete.

□

The main result of the section may be now stated as follows.

**Theorem 3. (Generalized Vitali convergence theorem)**

*Let $(X,\mathcal{A},\mu)$ be a measure space, let $\{f_n \mid n \ge 1\}$ be a sequence of functions from $L^{<p>}(\mu)$ for some $1 \le p < \infty$, and let $f : X \to \mathbf{R}$ be a given function. Then $f$ belongs to $L^{<p>}(\mu)$ and $\|\,f_n - f\,\|_p^*$ converges to zero as $n \to \infty$, if and only if the following three conditions are satisfied:*

(1) $$f_n \overset{(\mu^*)}{\longrightarrow} f$$

(2) $$\lim_{\mu(A)\to 0}\ \sup_{n\ge 1} \int_A^* |\,f_n\,|^p\ d\mu = 0$$

(3) $$\inf_{A\in\mathcal{A},\,\mu(A)<\infty}\ \sup_{n\ge 1} \int_{X\setminus A}^* |\,f_n\,|^p\ d\mu = 0\ .$$

**Proof**. We begin with the next lemma which is also of interest in itself.

**Lemma 4.**

*Let $(X,\mathcal{A},\mu)$ be a measure space, let $f,g$ be arbitrary functions from $X$ into $\mathbf{R}$, and let $A \in \mathcal{A}$ be a measurable set. Then we have:*

(4) $$(f \cdot 1_A)^* = f^* \cdot 1_A$$

(5) $$|\,f^*| \le |\,f\,|^*$$

(6) $$|\,f^* - g^*| \le |\,f - g\,|^*$$

(7) $$|\,f_* - g_*| \le |\,f - g\,|^*$$

*provided that all relations are well-defined.*

**Proof of lemma 4**: (4): Since $f \le f^*$ then $f\cdot 1_A \le f^*\cdot 1_A$, and thus $(f\cdot 1_A)^* \le f^*\cdot 1_A$ by definition of the $\mu$-upper envelope. Conversely, let $\{\,f_n \mid n \ge 1\}$ and $\{\,g_n \mid n \ge 1\}$ be sequences of $\mathcal{A}$-measurable functions from $X$ into $\bar{\mathbf{R}}$ such that $f_n \ge f$ and $g_n \ge f\cdot 1_A$ with $f_n \downarrow f^*$ and $g_n \downarrow (f\cdot 1_A)^*$. Then $h_n = (g_n \wedge f_n)\cdot 1_A + f_n \cdot 1_{A^c}$ is $\mathcal{A}$-measurable, and $h_n \ge f$ with $h_n \downarrow \{(f\cdot 1_A)^* \wedge f^*\} + f^*\cdot 1_{A^c}$. Hence by definition of the upper $\mu$-envelope we find $f^* = (f\cdot 1_A)^* \wedge f^* + f^*\cdot 1_{A^c}$. Therefore $f^*\cdot 1_A = \{(f\cdot 1_A)^* \wedge f^*\}\cdot 1_A \le (f\cdot 1_A)^*$, and the proof of (4) is complete.

(5): Since $f \le |\,f\,| \le |\,f\,|^*$, then $f^* \le |\,f\,|^*$. Since $-f \le |\,f\,| \le |\,f\,|^*$, then $-f^* = (-f)_* \le |\,f\,|^*$. Thus $f^* \ge -\,|\,f\,|^*$, and the proof of (5) is complete.

(6): Since $f-g \le |\,f-g\,| \le |\,f-g\,|^*$, then $f \le |\,f-g\,|^* + g \le |\,f-g\,|^* + g^*$. Therefore $f^* \le |\,f-g\,|^* + g^*$. This shows that $f^* - g^* \le |\,f-g\,|^*$, and by symmetry $f^* - g^* \ge -\,|\,f-g\,|^*$. This completes the proof of (6).

(7): Since $f \le |\,f-g\,|^* + g$, then $f_* \le |\,f-g\,|^* + g$. Therefore $f_* \le |\,f-g\,|^* + g_*$, which easily completes the proof of (7). $\square$

To continue the main proof, suppose first that $\| f_n - f \|_p^* \to 0$ as $n \to \infty$ . By Markov′s inequality for upper integrals we have, see [8]:

$$\mu^*\{ \mid f_n - f \mid > \varepsilon \} \le \frac{1}{\varepsilon^p} \int^* \mid f_n - f \mid^p \, d\mu$$

for all $n \ge 1$ and all $\varepsilon > 0$ . Hence (1) follows. Furthermore, by (3.7) we have:

$$\| f_n - f \|_p^* = \Big\{ \int ( \mid f_n - f \mid^* )^p \, d\mu \Big\}^{\frac{1}{p}}$$

for all $n \ge 1$ . It shows that $\mid f_n - f \mid^* \xrightarrow{(m^p)} 0$ . By (6) in lemma 4 we find:

(8) $$\big| \mid f_n \mid^* - \mid f \mid^* \big| \le \big| \mid f_n \mid - \mid f \mid \big|^* \le \mid f_n - f \mid^*$$

for all $n \ge 1$ . Hence $\mid f_n \mid^* \xrightarrow{(m^p)} \mid f \mid^*$ . By (3.7) and (4) in lemma 4 we have:

$$\int_A^* \mid f_n \mid^p \, d\mu \; = \int_A ( \mid f_n \mid^* )^p \, d\mu$$

for all $A \in \mathcal{A}$ . Therefore (2) and (3) follow by the classical Vitali convergence theorem.

Conversely, suppose that conditions (1)-(3) are valid. Then by (1), and (1) in corollary 2, there exists a subsequence $\{ f_{n_k} \mid k \ge 1 \}$ of $\{ f_n \mid n \ge 1 \}$ such that $f_{n_k}$ converges $(a.a)(\mu)$ to $f$ . Given $\varepsilon > 0$ , by (3) there exists $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) < \infty$ such that:

$$\sup_{n \ge 1} \int^* \mid f_n \mid^p \cdot 1_{X \setminus A_\varepsilon} \, d\mu \; < \varepsilon \; .$$

Hence by Fatou′s lemma for upper integrals we find, see [8]:

$$\int^* \mid f \mid \cdot 1_{X \setminus A_\varepsilon} \, d\mu \le \liminf_{k \to \infty} \int^* \mid f_{n_k} \mid^p \cdot 1_{X \setminus A_\varepsilon} \, d\mu \; < \infty \; .$$

By proposition 2.2 this implies that $\{ f \ne 0 \} \cap (X \setminus A_\varepsilon)$ has $\sigma$-finite $\mu$-measure. Thus $\{ f \ne 0 \}$ has $\sigma$-finite $\mu$-measure, and again by proposition 2.2 we see that there exist the upper and lower $\mu$-envelope of $f$ , as well as of $\mid f \mid$ and $\mid f - f_n \mid$ for all $n \ge 1$ . In addition, we have $\mu^*\{ \mid f_n - f \mid > \varepsilon \} = \mu\{ \mid f_n - f \mid^* > \varepsilon \}$ for all $n \ge 1$ and all $\varepsilon > 0$ , see [8]. Hence by (1) we get:

(9) $$\mid f_n - f \mid^* \xrightarrow{(\mu)} 0$$

as $n \to \infty$ . Therefore by (8) we obtain $\mid f_n \mid^* \xrightarrow{(\mu)} \mid f \mid^*$ , and thus by the classical Vitali convergence theorem we have $\mid f \mid^* \in L^p(\mu)$ , or in other words $f \in L^{<p>}(\mu)$ . Further, given $A \in \mathcal{A}$ , by (3.7) and (4) in lemma 4 we have:

(10) $$\Big\{ \int_A ( \mid f_n - f \mid^* )^p \, d\mu \Big\}^{\frac{1}{p}} \le \Big\{ \int_A ( \mid f_n \mid^* )^p \, d\mu \Big\}^{\frac{1}{p}} + \Big\{ \int_A ( \mid f \mid^* )^p \, d\mu \Big\}^{\frac{1}{p}}$$

Since $f \in L^{<p>}(\mu)$ , hence by (2) we easily find that:

$$\text{(11)} \qquad \lim_{\mu(A)\to 0} \sup_{n\geq 1} \int_A (\mid f_n - f \mid^*)^p \; d\mu = 0 \; .$$

Furthermore, let $\varepsilon > 0$ be given, then by (3) there exists $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) < \infty$ such that:

$$\text{(12)} \qquad \sup_{n\geq 1} \int_{X\setminus A_\varepsilon}^* \mid f_n \mid^p \, d\mu \; < \frac{\varepsilon}{2^p} \; .$$

Since $f \in L^{<p>}(\mu)$ , we can find $B_\varepsilon \in \mathcal{A}$ with $\mu(B_\varepsilon) < \infty$ such that:

$$\text{(13)} \qquad \int_{X\setminus B_\varepsilon}^* \mid f \mid^p \; d\mu < \frac{\varepsilon}{2^p} \; .$$

But then $C_\varepsilon = A_\varepsilon \cup B_\varepsilon \in \mathcal{A}$ with $\mu(C_\varepsilon) < \infty$ , and by (10), (12) and (13) we have

$$\Big\{ \int_{X\setminus C_\varepsilon} (\mid f_n - f \mid^*)^p \; d\mu \Big\}^{\frac{1}{p}} \leq \Big\{ \int_{X\setminus C_\varepsilon} (\mid f_n \mid^*)^p \; d\mu \Big\}^{\frac{1}{p}} + \Big\{ \int_{X\setminus C_\varepsilon} (\mid f \mid^*)^p \; d\mu \Big\}^{\frac{1}{p}} \leq \varepsilon^{\frac{1}{p}}$$

Hence we may conclude:

$$\text{(14)} \qquad \int_{X\setminus C_\varepsilon} (\mid f_n - f \mid^*)^p \, d\mu < \varepsilon \; .$$

From (9), (11) and (14) we see that the sequence $\{ \mid f_n - f \mid^* \mid \; n \geq 1 \}$ satisfies the conditions of the classical Vitali convergence theorem. Therefore $\mid f_n - f \mid^* \to 0$ converges $(m^p)$ to zero, or in other words $\| f_n - f \|_p^* \to 0$ as $n \to \infty$ , and the proof of theorem 3 is complete.

□

**Corollary 5.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{ f_n \mid n \geq 1 \}$ be a sequence of functions from $L^{<p>}(\mu)$ for some $1 \leq p < \infty$ which converges $(a.a.)^*(\mu)$ to a function $f : X \to \mathbf{R}$ . Then $f$ belongs to $L^{<p>}(\mu)$ and $\| f_n - f \|_p^*$ converges to zero as $n \to \infty$ , if and only if the following two conditions are satisfied:*

$$\text{(1)} \qquad \lim_{\mu(A)\to 0} \sup_{n\geq 1} \int_A^* \mid f_n \mid^p \; d\mu = 0$$

$$\text{(2)} \qquad \inf_{A\in\mathcal{A}, \mu(A)<\infty} \sup_{n\geq 1} \int_{X\setminus A}^* \mid f_n \mid^p \; d\mu = 0 \; .$$

**Proof**. If $\| f_n - f \|_p^* \to 0$ for $n \to \infty$ , then (1) and (2) follow by theorem 3. Conversely, if (1) and (2) hold, then a close look at the proof of theorem 3 shows that its relations (11) and (14) are satisfied. Since by our assumption $\mid f_n - f \mid^*$ converges $(a.a.)(\mu)$ to zero, the proof follows straightforward by applying the almost everywhere version of the classical Vitali convergence theorem stated in the beginning of section 1, see [2] (p.150).

□

**Remark 1.** Consider the measure space $(X, \mathcal{A}, \mu) = ([0,1], \mathcal{B}([0,1]), \lambda)$ , and take a sequence $A_n \subset [0,1]$ such that $\mu^*(A_n) = 1$ for $n \geq 1$ and $A_n \downarrow \emptyset$ as $n \to \infty$ . Then $f_n = 1_{A_n} \to 0$ everywhere as $n \to \infty$ , and (1) and (2) in corollary 5 are satisfied. However $\int^* \mid f_n \mid \; d\mu = \mu^*(A_n) = 1$ does not converge to zero as $n \to \infty$ . This example shows that the $(a.a.)^*(\mu)$-convergence in corollary 5 can not in general be replaced by the $(a.a.)(\mu)$-convergence.

Moreover, by (10) above we also see that Egoroff's theorem [2] (p.149) can not be established for upper integrals, for the converse implication in (11) does not hold although $\mu$ is finite.

**Remark 2.** It is easily verified that conditions (2) and (3) in theorem 3, and conditions (1) and (2) in corollary 5 respectively, may be replaced by the following single condition:

$$\lim_{m\to\infty} \sup_{n\geq 1} \int_{A_m}^{*} |\, f_n \,|^p \, d\mu \,=\, 0$$

with $\{\, A_m \mid m \geq 1 \}$ being an arbitrary decreasing sequence of sets in $\mathcal{A}$ with empty intersection.

According to the preceding results the next definitions seem to be convenient. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mathcal{F}$ be a subset of $L^{<p>}(\mu)$ for some $1 \leq p < \infty$ . Then $\mathcal{F}$ is called:

(a) *$p$-uniformly absolutely $\mu^*$-continuous*, if the condition is satisfied:

$$\lim_{\mu(A)\to 0} \sup_{f\in\mathcal{F}} \int_{A}^{*} |\, f \,|^p \, d\mu \,=\, 0$$

(b) *$p$-uniformly finitely $\mu^*$-founded*, if the condition is satisfied:

$$\inf_{A\in\mathcal{A}, \mu(A)<\infty} \sup_{f\in\mathcal{F}} \int_{X\setminus A}^{*} |\, f \,|^p \, d\mu \,=\, 0$$

(c) *$p$-uniformly $\mu^*$-integrable*, if $\mu$ is finite and the condition is satisfied:

$$\lim_{c\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>c\}}^{*} |\, f \,|^p \, d\mu \,=\, 0 \;.$$

In particular, if $\mathcal{F} = \{\, f_n \mid n \geq 1 \,\}$ is a sequence of functions from $L^{<p>}(\mu)$ for some $1 \leq p < \infty$ , and if $f$ is an arbitrary function from $X$ into $\mathbf{R}$ , then by theorem 3 the following two statements are equivalent:

(12) $f \in L^{<p>}(\mu)$ and $\|\, f_n - f \,\|_p^* \to 0$ as $n \to \infty$

(13) $\mathcal{F}$ is $p$-uniformly absolutely $\mu^*$-continuous and $p$-uniformly finitely $\mu^*$-founded

**Proposition 6.**

*If $\mu$ is a finite measure, then a family $\mathcal{F}$ in $L^{<p>}(\mu)$ for some $1 \leq p < \infty$ is $p$-uniformly $\mu^*$-integrable, if and only if $\mathcal{F}$ is $p$-uniformly absolutely $\mu^*$-continuous and $\|\cdot\|_p^*$-bounded as follows* $\sup_{f\in\mathcal{F}} \|\, f \,\|_p^* \,<\, \infty$ .

**Proof**. First suppose that $\mathcal{F}$ is $p$-uniformly $\mu^*$-integrable. Then by subadditivity of the $\mu$-upper integral we easily obtain:

$$\begin{aligned}\int^{*} |\, f \,|^p \, d\mu &\leq \int_{\{|f|>c\}}^{*} |\, f \,|^p \, d\mu + \int_{\{|f|\leq c\}}^{*} |\, f \,|^p \, d\mu \\ &\leq \int_{\{|f|>c\}}^{*} |\, f \,|^p \, d\mu \,+\, c^p \cdot \mu(X) \,\leq\, \varepsilon \,+\, c^p \cdot \mu(X)\end{aligned}$$

for large enough $c \in \mathbf{R}$ . This implies $\sup\limits_{f\in\mathcal{F}} \| f \|_p^* < \infty$ . Similarly for any $A \in \mathcal{A}$ we get:

$$\int_A^* | f |^p \, d\mu \le \int_{A\cap\{|f|>c\}}^* | f |^p \, d\mu + \int_{A\cap\{|f|\le c\}}^* | f |^p \, d\mu$$
$$\le \int_{\{|f|>c\}}^* | f |^p \, d\mu + c^p \cdot \mu(A) .$$

Hence we easily find that $\mathcal{F}$ is $p$-uniformly absolutely $\mu^*$-continuous. Conversely, suppose that $\mathcal{F}$ is $p$-uniformly absolutely $\mu^*$-continuous. Then by (4) in lemma 4 we have:

$$\int_{\{|f|>c\}}^* | f |^p \, d\mu \le \int (| f |^*)^p \cdot 1_{\{|f|^*>c\}} \, d\mu .$$

By Markov′s inequality for upper integrals we have, see [8]:

$$\mu^*\{ |f| > c^{\frac{1}{p}} \} \le \frac{1}{c} \int^* | f |^p \, d\mu .$$

Hence we get:

$$\sup_{f\in\mathcal{F}} \mu\{ |f|^* > c^{\frac{1}{p}} \} \le \frac{1}{c} \sup_{f\in\mathcal{F}} \int^* | f |^p \, d\mu .$$

Therefore we have:

$$\lim_{c\to\infty} \sup_{f\in\mathcal{F}} \mu\{ | f |^* > c \} = 0 .$$

From this equality and the hypothesis it follows easily that $\mathcal{F}$ is $p$-uniformly $\mu^*$-integrable. This fact completes the proof. □

## 5. Completeness of the space $L^{<p>}(\mu)$ for $1 \le p < \infty$

This section presents some fundamental facts on the space $L^{<p>}(\mu)$ for $1 \le p < \infty$ . We begin by showing that these spaces are Banach spaces.

**Theorem 1.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space, then $\big( L^{<p>}(\mu), \| \cdot \|_p^* \big)$ is a complete normed space for all $1 \le p < \infty$ .*

**Proof.** Let $\{ f_n \mid n \ge 1 \}$ be a Cauchy sequence in $L^{<p>}(\mu)$ . Then by the generalized Vitali convergence theorem 4.3 we have:

(1) $$\mu^*\{ | f_n - f_m | > \varepsilon\} \to 0$$

as $n, m \to \infty$ for all $\varepsilon > 0$ . Using facts from lemma 4.4 one can easily complete corresponding computations and verify that (2) and (3) in theorem 4.3 are satisfied. Moreover, by (2) in corollary 4.2 and (1) just deduced we see that (1) in theorem 4.3 is also satisfied with some function $f$ from $X$ into $\mathbf{R}$ . Therefore by theorem 4.3 we have that $\| f_n - f \|_p^*$ converges to zero. Already this fact completes the proof. Moreover, by (1) just deduced, lemma 4.1 and (4.11) we find that there exists a subsequence $\{ f_{n_k} \mid k \ge 1 \}$ of $\{ f_n \mid n \ge 1 \}$ and a function $f : X \to \mathbf{R}$ such that

$f_{n_k}$ converges $(a.a.)(\mu)$ to $f$ as $k \to \infty$ . Since $\{ f_n \mid n \ge 1 \}$ is a Cauchy sequence in $L^{<p>}(\mu)$ , then for given $\varepsilon > 0$ there exists $n_0 \ge 1$ such that we have:

$$\int^* \mid f_n - f_m \mid^p d\mu < \varepsilon \ .$$

for all $n, m \ge n_0$ . Hence by Fatou′s lemma for upper integrals we get, see [8]:

$$\int^* \mid f_n - f \mid^p d\mu \ \le \ \liminf_{k\to\infty} \int^* \mid f_n - f_{n_k} \mid d\mu \ \le \ \varepsilon$$

for all $n \ge n_0$ . This shows that $\| f_n - f \|_p^* \to 0$ for $n \to \infty$ , and the proof is complete. □

**Proposition 2.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space, then the statements are satisfied:*

(1) *If* $f_n \stackrel{(m^p)^*}{\longrightarrow} f$ , *then* $f_n^* \stackrel{(m^p)}{\longrightarrow} f^*$ *and* $(f_n)_* \stackrel{(m^p)}{\longrightarrow} f_*$ .

(2) *If* $f_n \stackrel{(\mu^*)}{\longrightarrow} f$ , *then* $f_n^* \stackrel{(\mu)}{\longrightarrow} f^*$ *and* $(f_n)_* \stackrel{(\mu)}{\longrightarrow} f_*$ .

(3) *If* $f_n \stackrel{(a.a.)^*}{\longrightarrow} f$ , *then* $f_n^* \stackrel{(a.a.)}{\longrightarrow} f^*$ *and* $(f_n)_* \stackrel{(a.a.)}{\longrightarrow} f_*$ .

**Proof**. (1): Both statements follow straightforward by (3.7) and (6)+(7) in lemma 4.4.

(2)+(3): Since $\mu^*\{ \mid f_n - f \mid > \varepsilon \} = \mu\{ \mid f_n - f \mid^* > \varepsilon \}$ for all $n \ge 1$ and all $\varepsilon > 0$ , all statements follow straightforward by (6)+(7) in lemma 4.4. □

**Corollary 3.**

*Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{ f_n \mid n \ge 1 \}$ be a sequence from $L^{<p>}(\mu)$ for some $1 \le p < \infty$ . Then $\{ f_n \mid n \ge 1 \}$ converges in $L^{<p>}(\mu)$ to some $f \in L^p(\mu)$ , if and only if the following two conditions are satisfied:*

(1) $$\lim_{n,m\to\infty} \int^* \mid f_n - f_m \mid^p d\mu \ = \ 0$$

(2) $$\lim_{n\to\infty} \int^* \mid f_n \mid^p d\mu \ = \ \lim_{m\to\infty} \int_* \mid f_m \mid^p d\mu \ .$$

**Proof.** If $f_n \stackrel{(m^p)^*}{\longrightarrow} f$ , then $\{ f_n \mid n \ge 1 \}$ is a Cauchy sequence in $L^{<p>}(\mu)$ , so (1) holds. Since $f \in L^p(\mu)$ , we have $\mid f \mid^* = \mid f \mid_* = \mid f \mid$ . Hence by (1) in proposition 2 and (2.5) we get:

$$\int^* \mid f_n \mid^p d\mu \ = \ \int (\, \mid f_n \mid^* )^p \ d\mu \ \ \longrightarrow \ \ \int \mid f \mid^p d\mu$$

$$\int_* \mid f_n \mid^p d\mu \ = \ \int (\, \mid f_n \mid_* )^p \ d\mu \ \ \longrightarrow \ \ \int \mid f \mid^p d\mu$$

as $n \to \infty$ . This establishes (2). Conversely if (1) holds, then by theorem 1 there exists $g \in L^{<p>}(\mu)$ such that $f_n$ converges $(m^p)^*$ to $g$ as $n \to \infty$ . Hence by (1)+(2) in proposition

2 and (2) we get:

$$\int^* \mid g \mid^p d\mu = \lim_{n\to\infty} \int^* \mid f_n \mid^p d\mu = \lim_{n\to\infty} \int_* \mid f_n \mid^p d\mu = \int_* \mid g \mid^p d\mu \ .$$

Hence we see that $g \in L^p(\mu)$ , and the proof is complete. □

**Remark 1**. Note that under the hypotheses of corollary 3 its condition (1) is equivalent to the conditions (1)+(2)+(3) in theorem 4.3. See also (1)+(2) in corollary 4.5, and recall the single condition from remark 4.2, as well as the condition of $p$-uniformly $\mu^*$-integrability of $\mathcal{F} = \{ f_n \mid n \geq 1 \}$ stated in section 4 when $\mu$ is finite.

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## REFERENCES


[1] ANDERSEN, N. T. (1985). The calculus of non-measurable functions and sets. *Various Publ. Series* Vol. 36.

[2] DUNFORD, N. *and* SCHWARTZ, J. T. (1958). *Linear Operators*, Vol 1. Interscience Publ. Inc.

[3] EAMES, W. *and* MAY, L. E. (1967). Measurable cover functions. *Canadian Math. Bull.* 10 (519-523).

[4] FATOU, P. J. L. (1906). Séries trigonométriques et séries de Taylor. *Acta Math.* 30 (335-340).

[5] HAHN, H. *and* ROSENTHAL, A. (1948). *Set Functions*. The University of New Mexico Press.

[6] LEBESGUE, H. (1910). Sur l'intégration des fonctions discontinues. *Ann. Scient. Ecole Normale Sup.* (Ser 3) 27 (364-450).

[7] LEVI, B. (1906). Sopra l'integrazione delle serie. *Rend. Istituto Lombardo di Sci. e Lett.* (Ser 2) 39 (775-780).

[8] PESKIR, G. (1991). Perfect measures and maps. *Math. Inst. Aarhus, Preprint Ser.* No. 26, (34 pp).

[9] VITALI, G. (1907). Sull'integrazione per serie. *Rend. del Circolo Mat. di Palermo* 23, (137-155).


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