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Controlling the Velocity of Brownian Motion by its Terminal Value

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Let $V = (V_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck velocity process solving

$$dV_t = -\beta V_t \; dt + \sigma \, dB_t$$

with $V_0 = 0$, where $\beta > 0$, $\sigma > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. Then the following inequality is satisfied:

$$E\Big(\max_{0 \leq t \leq \tau} |V_t| \Big) \leq s_*(\kappa; \beta, \sigma) + \frac{\kappa}{\beta} \, E\Big(e^{(\beta/\sigma^2) \, V_\tau^2} - 1 \Big)$$

for all bounded stopping times τ of V and all $\kappa > 0$, where $s_*(\kappa; \beta, \sigma) > 0$ is the unique zero point of the maximal solution of the equation

$$y' = \frac{\sigma^2}{2\kappa \displaystyle\int_y^x e^{(\beta/\sigma^2) z^2} \, dz}$$

staying strictly below the diagonal in $I\!R^2$. This inequality is sharp, and equality can be attained for each $\kappa > 0$. The following estimate is established:

$$s_*(\kappa; \beta, \sigma) \leq \frac{\sigma}{\sqrt{\beta}} \, \Psi^{-1} \bigg(\frac{\sqrt{\beta} \, \sigma}{2\kappa} \bigg)$$

where $\Psi(x) = \int_0^x e^{z^2} dz$. In particular, this yields the existence of a universal constant $C \geq \sqrt{2}$ such that

$$E\Big(\max_{0 \leq t \leq \tau} |V_t| \Big) \leq C \, \frac{\sigma}{\sqrt{\beta}} \sqrt{\log E\Big(e^{(\beta/\sigma^2) \, V_\tau^2} \Big)}$$

for all stopping times τ of V for which $(e^{(\beta/\sigma^2) \, V_{\tau \wedge t}^2})_{t \geq 0}$ is uniformly integrable. This inequality shows that the question of controlling the velocity of Brownian motion by its terminal value can be answered positively. Better versions of this inequality are also derived which go beyond the best value for C .

1. The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck *velocity process* $V = (V_t)_{t \geq 0}$ is a diffusion process which is a unique strong solution of the *Langevin* stochastic differential equation

(1.1) $$dV_t = -\beta V_t \; dt + \sigma \, dB_t$$

where $\beta > 0$, $\sigma > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion (see [7] and [1]). This

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process was initially introduced and later used by Ornstein and Uhlenbeck [13] as a model for the velocity of a Brownian particle in order to develop the theory of Brownian motion in accordance with Newtonian particle mechanics; the Ornstein-Uhlenbeck *position process*

$$X_t = x + \int_0^t V_r \, dr \tag{1.2}$$

is then another model for Brownian motion; if m is the mass of the Brownian particle, then in accordance with *Newton's law* $F = ma$, we may formally rewrite (1.1) as follows

$$m \frac{d^2 X_t}{dt^2} = -m\beta \, V_t + m\sigma \frac{dB_t}{dt} \tag{1.3}$$

where $-m\beta V_t$ represents a "frictional" force towards the origin (equilibrium state of velocity zero), and $m\sigma \, (dB_t/dt)$ represents a "fluctuating" white-noise force due to the media. The theory of Brownian motion $B_t \sim N(0,t)$, established earlier by Einstein and Smoluchowski, does not rely upon Newton's law, and does, moreover, make the concept of velocity meaningless (the trajectories of B are nowhere differentiable). For a Brownian motion under no influence of external force, it is known that both theories are in agreement with experiment, and to a large extent offer predictions which are numerically indistinguishable. For Brownian motion under influence of external force, however, the Einstein-Smoluchowski theory breaks down, while the Ornstein-Uhlenbeck theory remains successful (see [9] p.53-78).

1. The solution of (1.1) started at $v \in I\!R$ is explicitly given by

$$V_t = e^{-\beta t} v + \sigma e^{-\beta t} \int_0^t e^{\beta r} dB_r \tag{1.4}$$

which is easily verified by Itô formula (see also [12] p.361). This shows that the Ornstein-Uhlenbeck velocity process $V = (V_t)_{t \geq 0}$ is a *Gaussian* process satisfying

$$V_t \sim N\Big(e^{-\beta t} v \, , \, \tfrac{\sigma^2}{2\beta}\big(1 - e^{-2\beta t}\big)\Big) \tag{1.5}$$

for all t. Thus, no matter what the initial value v is, the equilibrium state of the process at infinity is distributed according to a single Gaussian law; that is:

$$V_t \rightarrow N\big(0, \tfrac{\sigma^2}{2\beta}\big) \tag{1.6}$$

as $t \rightarrow \infty$. Moreover, if $v \sim N(0, \frac{\sigma^2}{2\beta})$ in (1.4) is taken independently from B, then $V = (V_t)_{t \geq 0}$ is a *stationary* Gaussian process (sometimes called a *coloured noise*). In fact, these processes are the only stationary Gaussian Markov processes (see [12] p.81).

2. By (1.4) we find that the Ornstein-Uhlenbeck position process (1.2) is explicitly given by

$$X_t = x + \frac{1}{\beta}\Big(v - V_t + \sigma B_t\Big) \; . \tag{1.7}$$

It is a Gaussian process with mean $E(X_t) = x + \beta^{-1}(1 - e^{-\beta t})\, v$ and variance $D(X_t) = (\sigma^2/\beta^2)$ $t^2 + (\sigma^2/2\beta^3)(-3 + 4e^{-\beta t} - e^{-2\beta t})$. Thus, if $\beta \rightarrow \infty$ and $\sigma \rightarrow \infty$ so that $\hat{\sigma} := \sigma/\beta$ remains constant, then we have the following equilibrium relation

$$(1.8) \qquad \big(X_t\big)_{t\ge 0} \overset{\sim}{\longrightarrow} \Big(x+\hat{\sigma} B_t\Big)_{t\ge 0}$$

no matter what the initial velocity v is. This indicates that the Einstein-Smoluchowski (idealized) theory is a good approximation of the Ornstein-Uhlenbeck (Newtonian) theory for a Brownian particle under no influence of external force.

3. The Ornstein-Uhlenbeck velocity process (1.4) (started at $v=0$ for simplicity) can be obtained as a time-space change of standard Brownian motion:

$$(1.9) \qquad V_t = \frac{\sigma}{\sqrt{2\beta}}\, e^{-\beta t}\, B(e^{2\beta t}-1)\ .$$

If $V = (V_t)_{t\ge 0}$ is a stationary Ornstein-Uhlenbeck velocity process started at $v \sim N(0, \frac{\sigma^2}{2\beta})$ independently from B , then the following representation is valid:

$$(1.10) \qquad V_t = \frac{\sigma}{\sqrt{2\beta}}\, e^{-\beta t}\, B(e^{2\beta t})\ .$$

These facts are well-known and can easily be verified by means of standard time-change techniques; they appear naturally in some *nonlinear optimal stopping* problems (see [10]).

4. Similarly to the fact that Brownian motion can be approximated by means of a discrete random walk, the Ornstein-Uhlenbeck velocity process can be approximated by an *Ehrenfest urn* model for diffusion of particles through a porous membrane (see [6] p.171-172).

5. The Ornstein-Uhlenbeck velocity process has found a large number of applications in modelling various other random phenomena. Perhaps the most notable lately are some models for *stochastic volatility* (see [8] p.154-155).

2. Description of the problem

Given an Ornstein-Uhlenbeck velocity process started at zero, our main objective in this note is to study the question on how to control the maximal value of this process taken up to a random instant of time by means of its terminal value.

Our aim is motivated by the fact that the Ornstein-Uhlenbeck velocity process is neither a martingale, nor a positive submartingale, for which such bounds are successfully established and play a significant role in probability theory (see [12] p.50-52), but nevertheless, due to its special relation to standard Brownian motion (Section 1), we feel that such a question is of interest and can be answered positively. Moreover, in this note we deal only with the Ornstein-Uhlenbeck *velocity* process, and yet this is another interesting aspect of the question above, not present in the martingale framework of standard Brownian motion B where the concept of velocity is meaningless. Thus, it is interesting to see to which extent the control of the maximal position of Brownian motion B by means of the terminal value carries over to a similar control of the maximal velocity; our first result below ((3.3)+(3.4)) shows that we have to pass from a polynomial in the former case (see [3] and [4]) to the exponential function of the terminal value in the latter; this dichotomy is then balanced in (3.12) by another appearance of the logarithm function (see (3.19)+(3.20)).

Our approach is based on the fact that the Ornstein-Uhlenbeck velocity process satisfies the strong Markov property. It would be interesting to see how this compares to any other approach which would be based more substantially on the Gaussian property too (see also [5]).

1. To formulate the above problem more precisely, let us assume we are given an Ornstein-Uhlenbeck velocity process $V = (V_t)_{t \geq 0}$ satisfying (1.1) and starting at some $v \in I\!R$, where $\beta > 0$, $\sigma > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Then

$$|V| = \big(|V_t|\big)_{t \geq 0} \tag{2.1}$$

is a non-negative diffusion process having the same infinitesimal characteristics in $(0, \infty)$ as the diffusion V, and 0 as an instantaneously reflecting boundary point. Due to the symmetry of the process V, it is no restriction to assume that $v \geq 0$.

Associate with $|V|$ its maximum process

$$S_t = \max_{0 \leq r \leq t} |V_t| \tag{2.2}$$

and note that $S_0 = |V_0| = v$ under P. Setting $S_t := S_t \vee s$ for $s \geq v$, the process S may start at any $s \geq v$, and to indicate that under P the process $(|V_t|, S_t)$ starts at (v, s), we write $P_{v,s}$ instead of P; thus $P_{v,s}\big(|V_0| = v, S_0 = s\big) = 1$ for any given $s \geq v \geq 0$.

2. Given a continuous map $v \mapsto c(v) > 0$, consider the optimal stopping problem with payoff

$$\Delta(v, s) = \sup_\tau E_{v,s}\Big(S_\tau - \int_0^\tau c\big(|V_t|\big)\, dt \Big) \tag{2.3}$$

where the supremum is taken over all stopping times τ of V for which the integral has finite expectation. It follows from the general result of [11] that this problem has a solution (the payoff is finite and there exists an optimal stopping time), if and only if the following differential equation

$$g'(s) = \frac{\sigma^2\, L'\big(g(s)\big)}{2\, c\big(g(s)\big)\, \big(L(s) - L\big(g(s)\big)\big)} \quad (s \in I\!R) \tag{2.4}$$

admits a maximal solution which stays strictly below the diagonal in $I\!R^2$ (the *maximality principle*). In this equation we denote by L the scale function of $|V|$; it is explicitly given by

$$L(v) = \int_0^v e^{(\beta/\sigma^2) z^2}\, dz \tag{2.5}$$

for all $v \geq 0$. If $s \mapsto g_*(s)$ denote the maximal solution of (2.4), then the stopping time

$$\tau_* = \inf \big\{\, t > 0 : |V_t| \leq g_*(S_t) \,\big\} \tag{2.6}$$

is optimal for the problem (2.3), and the payoff is explicitly given by

$$\begin{aligned} \Delta(v, s) &= s_* + \int_0^v \big(L(v) - L(z)\big) c(z)\; m(dz) \quad , \ \text{if} \ \ 0 \leq v \leq s \leq s_* \\ &= s + \int_{g_*(s)}^v \big(L(v) - L(z)\big) c(z)\; m(dz) \quad , \ \text{if} \ \ g_*(s) \leq v \leq s \ \text{ and } \ s \geq s_* \\ &= s \quad , \ \text{if} \ \ 0 \leq v \leq g_*(s) \ \text{ and } \ s \geq s_* \end{aligned} \tag{2.7}$$

where s_* is the zero point of $s \mapsto g_*(s)$. In (2.7) we denote by $m(dz)$ the *speed measure* of $|V|$; it is explicitly given by

$$m(dz) \;=\; \frac{2\,dz}{\sigma^2 L'(z)}\;. \tag{2.8}$$

3. In view of our problem formulated above, the crucial question becomes how to choose the cost function $v \mapsto c(v)$ in the optimal stopping problem (2.3). The following considerations suggest how this can be done.

From (1.1) we read that the infinitesimal generator of V is given by

$$\mathbb{L}_V \;=\; -\beta\, v\frac{\partial}{\partial v} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial v^2}\;. \tag{2.9}$$

If $F(x) = G(x^2)$ is sufficiently smooth, then by Itô formula we find:

$$F(V_t) = F(v) + \int_0^t \big(\mathbb{L}_V F\big)(V_r)\; dr + \int_0^t \sigma F'(V_r)\; dB_r\;. \tag{2.10}$$

By applying the optional sampling theorem (see [12] p.65) to the continuous local martingale $(M_t)_{t\geq 0} \;=\; (\int_0^t F'(V_r)\,dB_r)_{t\geq 0}$ in (2.10), we see that

$$E_{v,s}\bigg(\int_0^\tau \big(\mathbb{L}_V F\big)(V_r)\; dr\bigg) \;=\; E_{v,s}\big(F(V_\tau)\big) - F(v) \tag{2.11}$$

for all stopping times τ of V for which the following condition is fulfilled:

$$E_{v,s}\bigg(\int_0^\tau \Big(F'(V_r)\Big)^2 dr\bigg)^{1/2} \;<\; \infty\;. \tag{2.12}$$

Yet another sufficient condition for (2.11) is obtained similarly by passing to a localization sequence of stopping times for $(M_t)_{t\geq 0}$; this shows that (2.11) is satisfied, whenever $(F(V_{\tau\wedge t}))_{t\geq 0}$ is uniformly integrable, and $v \mapsto (\mathbb{L}_V F)(v)$ is non-negative for instance.

Motivated in part by (2.4), define the cost function in (2.3) by $c(v) = \kappa\, L'(v) = \kappa\, e^{(\beta/\sigma^2)v^2}$, where $\kappa > 0$ is arbitrary and not specified, and note that $F(v) = (\kappa/\beta)(e^{(\beta/\sigma^2)v^2} - 1)$ satisfies $\mathbb{L}_V F \;=\; c$. Thus (2.11) shows that

$$E\bigg(\int_0^\tau e^{(\beta/\sigma^2)V_r^2} dr\bigg) \;=\; \frac{1}{\beta}\, E\Big(e^{(\beta/\sigma^2)V_\tau^2} - 1\Big) \tag{2.13}$$

for all stopping times τ of V for which either of the conditions specified above holds; in particular, note that (2.13) holds for all bounded stopping times τ of V . Inserting $c(v) = \kappa\, e^{(\beta/\sigma^2)v^2}$ into (2.4), we see that this equation takes the following form:

$$g'(s) \;=\; \frac{\sigma^2}{2\kappa \displaystyle\int_{g(s)}^s e^{(\beta/\sigma^2)z^2}\, dz}\;. \tag{2.14}$$

It is a nonlinear equation, and our problem in essence is reduced to detecting the maximal solution of this equation staying strictly below the diagonal in $I\!R^2$. This is the key argument in the proof of the main result which we state in the next section.

3. The result and proof

The main result of this note is contained in the following theorem. Observe that the simple estimate (3.2) stated below is considerably improved later on in (3.11), but yet we want to point it out for the useful role it plays in the derivation of (3.3), as well as to highlight technicalities of the final improvement (3.12). The first part of the theorem which we state now should be read together with the second part which is stated below; see also Corollary 3.2.

Theorem 3.1

(I): Let $V = (V_t)_{t \ge 0}$ be the Ornstein-Uhlenbeck velocity process starting at zero and satisfying (1.1). Then the following inequality is satisfied:

$$E\Big(\max_{0 \le t \le \tau} |V_t| \Big) \le s_*(\kappa;\beta,\sigma) + \kappa\, E\Big(\int_0^\tau e^{(\beta/\sigma^2)V_r^2}\, dr \Big) \tag{3.1}$$

for all stopping times τ of V and all $\kappa > 0$, where $s_*(\kappa;\beta,\sigma) > 0$ is the unique zero point of the maximal solution $s \mapsto g_*(s)$ of the equation (2.14) staying strictly below the diagonal in $I\!R^2$. This inequality is sharp, and equality can be attained for each $\kappa > 0$ at $\tau_*(\kappa)$ from (2.6).

The following estimate is valid (see (3.11) below):

$$s_*(\kappa;\beta,\sigma) \le \frac{\sigma^2}{2\kappa} \tag{3.2}$$

for all $\beta > 0$, where equality is attained if $\beta \downarrow 0$. In particular, this yields (see (3.12) below)

$$E\Big(\max_{0 \le t \le \tau} |V_t| \Big) \le \sigma \sqrt{2\, E\Big(\int_0^\tau e^{(\beta/\sigma^2)V_r^2}\, dr \Big)} \tag{3.3}$$

for all stopping times τ of V . The constant $\sigma\sqrt{2}$ is best possible in this inequality.

(II) : The following identity holds in either (3.1) or (3.3):

$$E\Big(\int_0^\tau e^{(\beta/\sigma^2)V_r^2}\, dr \Big) = \frac{1}{\beta}\, E\Big(e^{(\beta/\sigma^2)V_\tau^2} - 1 \Big) \tag{3.4}$$

for all stopping times τ of V satisfying either

$$E\Big(\int_0^\tau V_r^2\, e^{2(\beta/\sigma^2)V_r^2}\, dr \Big)^{1/2} < \infty \tag{3.5}$$

or that the process $(e^{(\beta/\sigma^2)V_{\tau\wedge t}^2})_{t \ge 0}$ is uniformly integrable, the latter condition being more general. In particular, the condition (3.5) is satisfied, and thus (3.4) holds as well, whenever the stopping time τ is bounded.

Proof. (I): Motivated by our considerations in the previous section, consider the optimal stopping problem (2.3) with the cost function $c(v) = \kappa\, e^{(\beta/\sigma^2)v^2}$ where $\kappa > 0$. Then we know that this problem has a solution, if and only if the differential equation (2.14) admits a maximal solution which stays strictly below the diagonal in $I\!R^2$. By using a simple comparison argument we will now prove that such a solution exists.

1. Consider the following equation:

$$h'(s) = \frac{\sigma^2}{2\kappa\,\big(s-h(s)\big)} \quad (s \in I\!R)\ . \tag{3.6}$$

Observe that $h_*(s) = s - \sigma^2/2\kappa$ solves this equation, and moreover it is easily verified that $s \mapsto h_*(s)$ is the maximal solution of (3.6) which stays strictly below the diagonal in $I\!R^2$. To do so, pass to the inverse function $t \mapsto h^{-1}(t)$ in (3.6), and note that this equation becomes linear, and thus admits the general solution in closed form which is found easily.

2. A simple comparison of (2.14) and (3.6) shows that

$$g'(s) \le h'(s) \tag{3.7}$$

for all $s \in I\!R$. Thus, if we consider (2.14) as the initial value problem with $g(\sigma^2/2\kappa) = 0$, and use Picard's theorem step-by-step to the right from $\sigma^2/2\kappa$, we obtain a solution $s \mapsto g_1(s)$ of (2.14) satisfying $g_1(\sigma^2/2\kappa) = 0$ and staying strictly below $s \mapsto h_*(s)$ on $[\sigma^2/2\kappa, \infty)$; the solution $s \mapsto g_1(s)$ cannot hit $s \mapsto h_*(s)$ at any $s > \sigma^2/2\kappa$, since this would violate (3.7); observe also that $g_1'(\sigma^2/2\kappa) \le 1$. Thus, in particular, the solution $s \mapsto g_1(s)$ stays strictly below the diagonal on $[\sigma^2/2\kappa, \infty)$.

3. We now claim that this solution extends to $(-\infty, \sigma^2/2\kappa)$ so that there it also stays below the diagonal. This is again seen by applying Picard's theorem step-by-step to the left from $\sigma^2/2\kappa$. By doing so we obtain a solution $s \mapsto g_1(s)$ on $(-\infty, \sigma^2/2\kappa)$ which stays between the diagonal and $s \mapsto h_*(s)$; it cannot hit the diagonal, since this would violate the fact of (2.14) that $g_1'(s+) = +\infty$ if $g_1(s+) = s$, neither it can hit $s \mapsto h_*(s)$, since this would violate (3.7). (From (2.14) it is also clear that $g_1(s)$ approaches the diagonal very rapidly as $s \to -\infty$.)

4. In this way we have proved that the equation (2.14) admits a global solution $s \mapsto g_1(s)$ which stays strictly below the diagonal in $I\!R^2$. Passing then to the equivalent integral formulation of (2.14), and using the monotone convergence theorem, we may conclude that there exists a maximal solution $s \mapsto g_*(s)$ of (2.14) which stays strictly below the diagonal in $I\!R^2$. The existence of the maximal solution implies that the problem (2.3) has a solution; from (2.7) with $v = s = 0$ we find that (3.1) holds. Observe that $s \mapsto g_*(s)$ depends on κ , β and σ ; therefore we write $s_*(\kappa;\beta,\sigma)$ to denote its unique zero point. Finally, since $g_1(\sigma^2/2\kappa) = 0$, by the maximality property of $s \mapsto g_*(s)$ we see that (3.2) holds. Inserting this into (3.1), and minimising the right-hand side over all $\kappa > 0$, we obtain (3.3) by easy calculations. Observe also that equality in (3.2) is attained if we let $\beta \downarrow 0$.

(II): These facts follow from our considerations about (2.11)-(2.13) in the previous section. The proof is complete. □

Remarks: 1. Variational ideas applied in the theorem above are generally well-known. In a study of similar maximal inequalities for Bessel processes they were used very effectively in [2]. Similar ideas are also used in [3] and [4].

2. By (1.5) we can compute that

$$E\Big(e^{(\beta/\sigma^2)V_t^2}\Big) = e^{\beta t} \tag{3.8}$$

for all t. Inserting this via (3.4) into the right-hand side of (3.3), we obtain the following estimate:

$$E\Big(\max_{0\le r\le t}|V_r|\Big) \le \sigma\sqrt{\frac{2}{\beta}\Big(e^{\beta t}-1\Big)} \tag{3.9}$$

for all t. Letting $\beta\downarrow 0$, and using that $V_t\to\sigma B_t$, we see that (3.9) reduces to the inequality $E(S_t)\le\sqrt{2t}$ where $S_t=\max_{0\le r\le t}|B_r|$; recall that $E(|B_t|)=\sqrt{(2/\pi)\,t}$.

3. Similarly, if we let $\beta\downarrow 0$ in (3.3), we obtain the following inequality:

$$E\Big(\max_{0\le r\le \tau}|B_r|\Big) \le \sqrt{2E(\tau)} \tag{3.10}$$

for all stopping times τ of B. This inequality is known to be sharp (see [2]).

4. While for standard Brownian motion B we have $B_t\sim N(0,t)$ and therefore the variance of B_t equals t and tends to infinity as $t\to\infty$, for the Ornstein-Uhlenbeck velocity process V satisfying (1.1) we have (1.5) and thus the variance of V_t remains bounded by the constant $\sigma^2/2\beta$. Thus it is not surprising that to control the maximum of the process we have to pass from a polynomial in the former case (recall results of [3] and [4]) to the exponential function of the terminal value in the latter (see also Remark 8 below). It may also be observed that for $V_\infty\sim N(0,\sigma^2/2\beta)$ we have $E(e^{\lambda V_\infty^2})<\infty$ if and only if $\lambda<\beta/\sigma^2$; this is in agreement with the apparent fact that the left-hand side of (3.1) or (3.3) tends to infinity as τ does so.

5. In view of (1.9) and the law of iterated logarithm for Brownian motion, we see that the estimate (3.9) is much too crude; we expect the left-hand side in (3.9) to increase at least as slow as $\sqrt{\log(t)}$ as $t\to\infty$. The reason for this lack of precision comes from the estimate (3.2). There we expect $s_*(\kappa;\beta,\sigma)$ to increase to infinity at least as slow as $\log(1/\kappa)$ or even $\sqrt{\log(1/\kappa)}$ when $\kappa\downarrow 0$. For this reason we continue by obtaining a sharper estimate of $s_*(\kappa;\beta,\sigma)$ which is aimed to match the law-of-iterated-logarithm prediction.

Theorem 3.1 (continued)

The following refinement of (3.2) is valid (see (3.23) below):

$$s_*(\kappa;\beta,\sigma) \le \frac{\sigma}{\sqrt{\beta}}\,\Psi^{-1}\Big(\frac{\sqrt{\beta}\,\sigma}{2\kappa}\Big) \tag{3.11}$$

for all $\kappa>0$, $\beta>0$ and $\sigma>0$, where $\Psi(x)=\int_0^x e^{z^2}\,dz$. This estimate is sharp.

Consequently, the following refinement of (3.3) is valid (see (3.19) and (3.20) below):

$$E\Big(\max_{0\le t\le\tau}|V_t|\Big) \le \frac{\sigma}{\sqrt{\beta}}\left(\Big(\Psi^{-1}\circ\rho^{-1}\Big)\big(\beta E(I_\tau)\big)+\frac{1}{2}\,\frac{\beta E(I_\tau)}{\rho^{-1}\big(\beta E(I_\tau)\big)}\right) \tag{3.12}$$

for all stopping times τ of V, where $\rho(x)=2x^2\exp\big(-(\Psi^{-1}(x))^2\big)$ and $I_\tau=\int_0^\tau e^{(\beta/\sigma^2)V_r^2}\,dr$. This inequality is sharp.

Proof. We shall establish (3.11) by applying the following trick which will enable us to extend the simple argument used in the proof above when considering the initial value problem (2.14) with $g(\sigma^2/2\kappa) = 0$. It was observed above that the solution $s \mapsto g_1(s)$ satisfies $g_1'(\sigma^2/2\kappa) \leq 1$.

On the other hand, if we compute the second derivative of any function $s \mapsto g(s)$ satisfying (2.14), we find that $g''(s) \leq 0$ if and only if

$$L'(s) - L'\big(g(s)\big) g'(s) \geq 0 \tag{3.13}$$

where $L'(s) = e^{(\beta/\sigma^2)s^2}$. In particular, we see that $g'(s) \leq 1$ will imply that $g''(s) \leq 0$ whenever s satisfies $g(s) < s$. This fact combined with our observation above that $g_1'(\sigma^2/2\kappa) \leq 1$ motivates us to consider the initial value problem (2.14) with $g(s_*) = 0$, and find the smallest positive $s_* \leq \sigma^2/2\kappa$ satisfying this property with $g'(s_*) \leq 1$. In other words, we shall be looking for the smallest positive $s_* \leq \sigma^2/2\kappa$ satisfying

$$\frac{\sigma^2}{2\kappa} \leq \int_0^{s_*} e^{(\beta/\sigma^2)z^2}\, dz \ . \tag{3.14}$$

The solution $s \mapsto g_2(s)$ obtained from (2.14) will then satisfy $g_2(s_*) = 0$, and clearly this solution will stay below the straight line $s \mapsto s - s_*$ for all $s > s_*$; otherwise $g_2''(s) \leq 0$ for some $s > s_*$ would be violated. In particular, the solution $s \mapsto g_2(s)$ stays below the diagonal in $I\!R^2$, and thus we may conclude that $s_*(\kappa; \beta, \sigma) \leq s_*$. This now can be quantitatively expressed by (3.11) if one makes use of (3.14).

Inserting (3.11) into (3.1), and minimising the right-hand side over all $\kappa > 0$, we obtain (3.12). This verification is somewhat lengthy, but quite straightforward. The proof is complete. □

6. To obtain a better understanding of (3.12), the following estimates show useful:

$$\frac{e^{x^2} - 1}{2x} \leq \int_0^x e^{z^2}\, dz \leq \frac{e^{x^2}}{x} \quad \big(\forall x > 0\big) \ . \tag{3.15}$$

By using these estimates it is possible to verify through elementary calculations that

$$\lim_{x \to \infty} \frac{\big(\Psi^{-1} \circ \rho^{-1}\big)(x)}{\sqrt{\log(x)}} = 1 \tag{3.16}$$

$$\frac{x}{\rho^{-1}(x)} \leq 2 \quad \big(\forall x > 0\big) \tag{3.17}$$

$$\lim_{x \to 0} \frac{x}{\rho^{-1}(x)} = \lim_{x \to \infty} \frac{x}{\rho^{-1}(x)} = 0 \ . \tag{3.18}$$

Observe also that the map $x \mapsto \rho(x)$ is strictly increasing on $[0, \infty)$.

Corollary 3.2

Let $V = (V_t)_{t \geq 0}$ *be the Ornstein-Uhlenbeck velocity process starting at zero and satisfying (1.1). Then there exists a universal constant* $C > 0$ *such that (see Remark 9 below)*

$$E\bigg(\max_{0 \leq t \leq \tau} |V_t|\bigg) \leq C\, \frac{\sigma}{\sqrt{\beta}} \sqrt{\log\Big(1 + \beta E(I_\tau)\Big)} \tag{3.19}$$

for all stopping times τ *of* V *, where* $I_\tau = \int_0^\tau e^{(\beta/\sigma^2)V_r^2}\, dr$ *. In particular, we have*

$$E\Big(\max_{0\le t\le\tau}|V_t|\Big) \le C\,\frac{\sigma}{\sqrt{\beta}}\sqrt{\log E\Big(e^{(\beta/\sigma^2)V_\tau^2}\Big)} \tag{3.20}$$

for all stopping times τ *of* V *satisfying either (3.5) or that the process* $(e^{(\beta/\sigma^2)V_{\tau\wedge t}^2})_{t\ge 0}$ *is uniformly integrable.*

Proof. From (3.12) with (3.16) and (3.17) we have

$$E\Big(\max_{0\le t\le\tau}|V_t|\Big) \le (1+\varepsilon)\,\frac{\sigma}{\sqrt{\beta}}\Big(\sqrt{\log\Big(\beta E(I_\tau)\Big)}+1\Big) \tag{3.21}$$

whenever $E(I_\tau) \ge M_\varepsilon$ (large enough). Since $1+\sqrt{\log(\beta x)} \le A\sqrt{\log(1+\beta x)}$ for $\beta x \ge M_A$ (large enough) where $A > 1$, from (3.21) we get

$$E\Big(\max_{0\le t\le\tau}|V_t|\Big) \le A\,(1+\varepsilon)\,\frac{\sigma}{\sqrt{\beta}}\Big(\sqrt{\log\Big(1+\beta E(I_\tau)\Big)}\;\Big) \tag{3.22}$$

whenever $E(I_\tau) \ge M_\varepsilon \vee M_A$. Finally, since $\sqrt{\beta x} \le B\sqrt{\log(1+\beta x)}$ for $0 \le \beta x \le M_\varepsilon \vee M_A$ for some $B > 0$ large enough, combining (3.3) and (3.22) we obtain (3.19) with $C = A(1+\varepsilon)\vee B\sqrt{2}$. The inequality (3.20) follows from (3.19) and Part II of Theorem 3.1. The proof is complete. □

7. Observe in (3.11) that $\Psi^{-1}(x) \le x$ for all $x \ge 0$; thus the right-hand side in (3.11) is always smaller than the right-hand side in (3.2); therefore the right-hand side in (3.12) is always smaller than the right-hand side in (3.3). In fact, it is possible to verify by (3.15) that

$$\Psi^{-1}(x) \le (1+\varepsilon)\sqrt{\log(x)} \tag{3.23}$$

for all $x \ge x_\varepsilon$ (large enough); thus the improvement (3.12) upon (3.3) is substantial for large $E(I_\tau)$; this is expressed in a more readable way through the appearance of the logarithm function in (3.19); note also from the proof above that (3.19) has been obtained by merging (3.12) and (3.3) together; observe from the facts just pointed out, however, that no matter what value for C in (3.19) is found the best possible in the proof above, the inequality (3.12) will always give at least as good bound, although not as elegant in appearance.

8. Note by Jensen's inequality that $\log E(e^{(\beta/\sigma^2)V_\tau^2}) \ge (\beta/\sigma^2)\, E(V_\tau^2)$; since the variance of V_t remains bounded over all t , and the left-hand side in (3.20) tends to infinity when τ does so, we see that $\log E(e^{(\beta/\sigma^2)V_\tau^2})$ in (3.20) cannot generally be replaced by $(\beta/\sigma^2)\, E(V_\tau^2)$; observe, however, that the inequality (3.20) may be formally viewed as such an estimate if we neglect the expectation sign; for instance, if V_τ is identically constant, then (3.20) says that the maximum of $|V_t|$ over $0 \le t \le \tau$ is controlled by the terminal value $|V_\tau|$ whenever the uniform integrability condition is fulfilled; in view of the specific form of the drift term in (1.1) which always directs the process to the origin, it does not come quite with surprise.

9. By using a different method it is possible to prove that the second expectation sign in (3.19) can be pulled out in front of the square-root and logarithm sign (see [5]); in view of Jensen's inequal-

ity this bound is better; moreover, it is possible to prove that this inequality is two-sided; although it is difficult to compute these bounds explicitly, the knowledge of their existence helps to grasp the real nature of the error when passing to the terminal-value bound in (3.12) or (3.20); observe, however, that no such maneuver of the expectation sign can be applied directly to (3.12) or (3.20).

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