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Uniform Convergence of Reversed Martingales

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A necessary and sufficient condition for the uniform convergence of a family of reversed martingales converging to a degenerated limiting process is given. The condition is expressed by means of regular convergence (in Hardy's sense) of corresponding means. It is shown that the given regular convergence is equivalent to Hoffmann-Jørgensen's eventually total boundedness in the mean which is necessary and sufficient for the uniform law of large numbers. Analogous results are carried out for families of reversed submartingales. By applying derived results several convergence statements are obtained which extend those from the uniform law of large numbers to the general reversed martingale case.

1. Introduction

Let $\{ \xi_n \mid n \geq 1 \}$ be a sequence of independent identically distributed random functions defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{A})$ and a common distribution law π , let T be a set, and let $f : S \times T \to \mathbf{R}$ be an arbitrary function. Suppose that the π-mean function associated to f :

$$M(t) = \int_S f(s,t) \; \pi(ds)$$

exists for all $t \in T$, then in [3], [5], [14] and [15] one can find a series of necessary and sufficient conditions for the following form of the law of large numbers:

$$\frac{1}{n} \sum_{j=1}^{n} f(\xi_j(\omega), t) \;\rightrightarrows\; M(t) \tag{1.1}$$

where the convergence is uniform over $t \in T$, for all $\omega \in \Omega$ outside some P-null set N from $\mathcal{F}$. Let us note if $f(\xi_j(\omega), \cdot)$ and $M(\cdot)$ belongs to $B(T)$ for all $j \geq 1$ and $\omega \in \Omega$, where $B(T)$ denotes the Banach space of all bounded real valued functions on T relative to the sup-norm, then (1.1) just states that the sequence of functions $\{ f(\xi_n, \cdot) \mid n \geq 1 \}$ from Ω into $B(T)$ satisfies the strong law of large numbers in the Banach space $B(T)$. A classical example which is covered by this setting is the well-known Glivenko-Cantelli theorem. In its case the function f is defined

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on $\mathbf{R} \times \mathbf{R}$ with values in $\mathbf{R}$ by $f(s,t) = 1_H(s,t)$, where $H = \{ (s,t) \in \mathbf{R} \times \mathbf{R} \mid s \leq t \}$, and it is very interesting and spectacular that the induced random functions $f(\xi_j, \cdot)$ for $j \geq 1$ are not necessarily measurable with respect to the Borel σ-algebra on $B(\mathbf{R})$, as well as not separably valued in $B(\mathbf{R})$, see [6]. Therefore the classical Banach space versions of the strong laws of large numbers based on measurability of random functions in question and separability of a Banach range space are too week to cover the first and the most natural example of the infinitely dimensional law of large numbers. And in [6] one can find infinitely dimensional versions of the strong law of large numbers which do not assume neither measurability nor separability, and which cover the Glivenko-Cantelli theorem, as well as many other uniform laws of large numbers for stochastic processes. It is well-known that the sequence:

$$X_n(t) \ = \ \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t)$$

forms a reversed martingale relative to the associated permutation invariant σ-algebras for every $t \in T$ for which $f(\cdot, t) \in L^1(\pi)$. Moreover, the same is true for sequences:

$$Y_n(t) \ = \ \frac{1}{n} \sum_{j=1}^{n} \Big[\ f(\xi_j, t) - M(t) \ \Big]^2$$

$$Z_n(t) \ = \ \frac{1}{n-1} \sum_{j=1}^{n} \Big[\ f(\xi_j, t) - \frac{1}{n} \sum_{k=1}^{n} f(\xi_k, t) \ \Big]^2$$

for every $t \in T$ for which $f(\cdot, t) \in L^2(\pi)$. Therefore having all these facts in mind, it is very natural to ask when does a family of *reversed* martingales (or submartingales) converge uniformly? In [7] one can find such conditions for the uniform convergence of families of *ordinary* martingales and submartingales, and it is indeed surprising that these conditions are very week compared to the necessary and sufficient conditions for the uniform law of large numbers. The main reason for this surprise lies in the well-known fact that reversed martingales and submartingales usually behave much more regularly than ordinary ones. And in this paper we will establish such conditions for the uniform convergence of reversed martingales and submartingales. We shall begin in the next section by introducing the basic terminology and notation and presenting some preliminary results needed in the rest of the paper. In particular, we shall summarize some elementary facts on regular convergence (in Hardy's sense) of double sequences. Then we shall obtain a series of necessary and sufficient conditions for the $a.s.$ and L^1-convergence of supremum of a countable family of reversed submartingales, provided that some mild limited condition on its means is satisfied, see Theorem 3.1, Corollary 3.2 and Remark 3.3. It is remarkable, which is pointed out by J. Hoffmann-Jørgensen in private communications, that these conditions actually describe regular convergence of a double sequence of given means, introduced by G. H. Hardy in [4] and independently rediscovered by F. Móricz in [9]. Moreover, one can easily deduce that *these conditions confirm a conjecture of J. Hoffmann-Jørgensen in [7] that considering uniform convergence, we have a case where ordinary martingales behave more regularly than reversed ones.* And using this result in the sequel we shall derive a necessary and sufficient condition for the uniform convergence of a family of reversed martingales (and submartingales) converging to a degenerated limiting process, which is indexed by a separable topological space, see Theorem 4.1 and Theorem 4.3 together with Theorem 4.7 and Theorem 4.11, respectively. Furthermore, it will be shown that the given condition for the uniform convergence of a family of reversed martingales is equivalent to the condition of eventually total boundedness in the mean discovered for the uniform law of large numbers by J. Hoffmann-Jørgensen in [5], see Theorem 4.11 and Remark 4.12. Applying derived results we shall obtain several convergence statements which extend some of those in [5] to the general reversed martingale case, see Corollary 4.2, Corollary 4.4 and Example 4.13. At the end in Example 4.14 we will see that the convergence estimates given in Theorem 4.1 and Theorem 4.3 can not be improved in general.

2. Preliminary facts

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a fixed probability space, and let $\{ \mathcal{F}_n \mid n \leq -1 \}$ be a given increasing sequence of sub-σ-algebras of $\mathcal{F}$ with the intersection $\mathcal{F}_{-\infty}$. If T is a non-empty set, then $\mathbf{R}^T$ denotes the set of all real valued functions defined on T , and $B(T)$ denotes the set of all *bounded* functions from $\mathbf{R}^T$. For $f \in \mathbf{R}^T$ and $A \subset T$ we put:

$$M_A(f) \;=\; \sup_{t \in A} f(t) \qquad \text{and} \qquad \| \, f \, \|_A = \sup_{t \in A} | \, f(t) \, | \; .$$

Then $\| \, f - g \, \|_T$ defines a metric on $\mathbf{R}^T$, not necessarily finite valued, but topologically equivalent to the bounded metric $\arctan \| \, f - g \, \|_T$. It is well-known that $B(T)$ is a Banach space, relative to the norm $\| \cdot \|_T$. A finite *cover* of T is a family $\gamma = \{ D_1, \ldots, D_n \}$ of non-empty subsets of T satisfying $T = \bigcup_{j=1}^n D_j$. The family of all finite covers of T will be denoted by $\Gamma(T)$. If (T, τ) is a topological space, then $\mathcal{C}(T) = \mathcal{C}(T, \tau)$ denotes the set of all real valued *τ-continuous* functions defined on T , $\mathcal{U}sc\,(T) = \mathcal{U}sc\,(T, \tau)$ denotes the set of all real valued *upper τ-semicontinuous* functions defined on T , and $\mathcal{L}sc\,(T) = \mathcal{L}sc\,(T, \tau)$ denotes the set of all real valued *lower τ-semicontinuous* functions defined on T . If (T, δ) is a pseudometric space, then $\mathcal{C}_u(T) = \mathcal{C}_u(T, \delta)$ denotes the set of all real valued *uniformly δ-continuous* functions defined on T . It is easy to check that $\mathcal{C}_u(T, \delta)$ is a $\| \cdot \|_T$-closed subspace of $\mathbf{R}^T$.

Let (T, τ) be a topological space, let $t \in T$ be a given point, and let $\mathcal{N}_t$ be the family of all open neighborhoods of the point t . Let us recall that for given $f \in \mathbf{R}^T$ and $G \in \mathcal{N}_t$, *the lower, upper and absolute oscillation* of f at t over G is respectively defined by:

$$W_G^+(t, f) \;=\; \sup_{s \in G} \, (f(t) - f(s))$$

$$W_G^-(t, f) \;=\; \sup_{s \in G} \, (f(s) - f(t))$$

$$W_G(t, f) \;=\; \max \{ \, W_G^+(t, f) \; , \; W_G^-(t, f) \, \} = \sup_{s \in G} \; | \, f(t) - f(s) \, | \; .$$

The lower, upper and absolute jump of f at t relative to τ is respectively defined by:

$$\partial^+(t, f) \;=\; \partial_\tau^+(t, f) \;=\; \inf_{G \in \mathcal{N}_t} W_G^+(t, f)$$

$$\partial^-(t,f) \;=\; \partial^-_\tau(t,f) \;=\; \inf_{G\in\mathcal{N}_t} W^-_G(t,f)$$

$$\partial(t,f) \;=\; \partial_\tau(t,f) \;=\; \max\,\{\ \partial^+_\tau(t,f)\ ,\ \partial^-_\tau(t,f)\ \} \;=\; \inf_{G\in\mathcal{N}_t} W_G(t,f)\ .$$

Clearly, we have:

(2.1) f is lower τ-semicontinuous at t, if and only if $\partial^+_\tau(t,f) = 0$

(2.2) f is upper τ-semicontinuous at t, if and only if $\partial^-_\tau(t,f) = 0$

(2.3) f is τ-continuous at t, if and only if $\partial_\tau(t,f) = 0$.

Finally, for each $f \in \mathbf{R}^T$ we shall put:

$$\Delta^+(f) \;=\; \Delta^+_\tau(f) \;=\; \sup\,\{\ \partial^+_\tau(t,f)\ \mid\ t\in T\ \}$$

$$\Delta^-(f) \;=\; \Delta^-_\tau(f) \;=\; \sup\,\{\ \partial^-_\tau(t,f)\ \mid\ t\in T\ \}$$

$$\Delta(f) \;=\; \Delta_\tau(f) \;=\; \sup\,\{\ \partial_\tau(t,f)\ \mid\ t\in T\ \}\ .$$

Let us recall that a topological space T is called *hereditarily separable*, if every subspace of T is separable. It is well-known that every separable pseudometric space is hereditarily separable, but $\mathbf{R}^2$ with the product left (right) Sorgenfrey topology is an example of a separable normal space which is not hereditarily separable. In our considerations on hereditarily separable topologies the following statement will be useful:

(2.4) If τ_1 and τ_2 are hereditarily separable topologies on a set T such that τ_1 has a countable base, then there is a hereditarily separable topology ν on T finer than τ_1 and τ_2 .

Indeed, if we take a countable base $\mathcal{B}_1$ of the topology τ_1 which is closed under the formations of finite intersections, then it is easy to verify that the family:

$$\mathcal{B} = \{\ B_1 \cap G_2\ \mid\ B_1 \in \mathcal{B}_1\ ,\ G_2 \in \tau_2\ \}$$

is a base of a desired topology ν .

Let $\{\ X_n \mid -\infty < n \le -1\ \}$ be a sequence of integrable random variables defined on $(\Omega,\mathcal{F},P)$. Let us recall that the family $\{\ X_n\ ,\ \mathcal{F}_n \mid -\infty < n \le -1\ \}$ is called *a reversed martingale, submartingale* or *supermartingale*, if X_n is $\mathcal{F}_n$-measurable and $X_n = E\{\ X_{n+1} \mid \mathcal{F}_n\ \}$, $X_n \le E\{\ X_{n+1} \mid \mathcal{F}_n\ \}$ or $X_n \ge E\{\ X_{n+1} \mid \mathcal{F}_n\ \}$, for all $-\infty < n \le -2$, respectively. We shall refer the reader to [1] for basic properties and convergence statements of reversed martingales and submartingales which we are going to use mainly implicitly in the rest of this paper. Let us recall that an arbitrary function $Z : \Omega \to \mathbf{R}$ is called a real valued *random element* on $(\Omega,\mathcal{F},P)$. Let $\{\ Z_n \mid n \ge 1\ \}$ be a sequence of real valued random elements defined on $(\Omega, \mathcal{F}, P)$. In order to describe certain non-measurable random phenomenons which occur naturally in our next considerations, we shall introduce the following convergence notions:

(i) $Z_n \to 0 \ (a.s.)$, if there exists a P-null set $N \in \mathcal{F}$ such that $Z_n(\omega) \to 0$, $\forall \omega \in \Omega \backslash N$

(ii) $Z_n \to 0 \ (a.s.)^*$, if there exist random variables D_n defined on $(\Omega, \mathcal{F}, P)$, satisfying $D_n \to 0 \ (a.s.)$ and $|Z_n| \leq D_n$ for all $n \geq 1$

(iii) $Z_n \to 0 \ (P^*)$, if $P^*\{ \ |Z_n| \geq \varepsilon \ \} \to 0$, $\forall \varepsilon > 0$

(iv) $Z_n \to 0 \ (P_*)$, if $P_*\{ \ |Z_n| \geq \varepsilon \ \} \to 0$, $\forall \varepsilon > 0$

(v) $Z_n \to 0 \ (L^1)^*$, if $E^*|Z_n| \to 0$

(vi) $Z_n \to 0 \ (L^1)_*$, if $E_*|Z_n| \to 0$

where P^* and P_* denotes the outer and inner P-measure, and E^* and E_* denotes the upper and lower P-integral. Note if every function Z_n is measurable for $n \geq 1$, then the convergence notion in (i) and (ii) coincides with the notion of *P-almost surely convergence* (in next denoted by $(a.s.)$), the convergence notion in (iii) and (iv) coincides with the notion of *convergence in P-probability* (in next denoted by (P)), and the convergence notion in (v) and (vi) coincides with the notion of *convergence in $L^1(P)$* (in next denoted by (L^1)) of the given sequence of random variables $\{ \ Z_n \mid n \geq 1 \ \}$, respectively. For more informations on the convergence notions in (i)-(vi) we shall refer the reader to [11] and [12]. It is easy to establish that we have:

$$
\begin{array}{ccc}
(a.s.)^* & \Rightarrow & (a.s.) \\
\Downarrow & & \Downarrow \\
(P^*) & \Rightarrow & (P_*) \\
\Uparrow & & \Uparrow \\
(L^1)^* & \Rightarrow & (L^1)_*
\end{array}
\tag{2.5}
$$

and no other implication holds in general. Let $\{ \ Z_n \mid n \geq 1\}$ and $\{V_n \mid n \geq 1 \ \}$ be two sequences of random elements, and let (c) denote either of the following convergence notions: $(a.s.), (a.s.)^*, (P^*), (L^1)^*$, but neither (P_*) nor $(L^1)_*$, then it is easy to verify that we have:

(2.6) $Z_n \to 0 \ (c)$ and $V_n \to 0 \ (c) \ \Rightarrow \ Z_n + V_n \to 0 \ (c)$

(2.7) $Z_n \to 0 \ (P_*)$ and $V_n \to 0 \ (P^*) \ \Rightarrow \ Z_n + V_n \to 0 \ (P_*)$

(2.8) $Z_n \to 0 \ (L^1)_*$ and $V_n \to 0 \ (L^1)^* \ \Rightarrow \ Z_n + V_n \to 0 \ (L^1)_*$.

Let us recall that a double sequence of real numbers $\mathcal{A} = \{ \ a_{nk} \mid n, k \geq 1 \ \}$ is *convergent* (*in Pringsheim's sense*) to the limit A , if $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall n, k \geq p_\varepsilon$ we have $| \ A - a_{nk} \ | < \varepsilon$. In this case we shall write $A = \lim_{n,k \to \infty} a_{nk}$. And following G. H. Hardy [4] we say that a double sequence of real numbers $\mathcal{A} = \{ \ a_{nk} \mid n, k \geq 1 \ \}$ is *regularly convergent*

$($*in Hardy's sense*$)$ to the limit A , if all limits:

$$\lim_{n\to\infty} a_{nk} \ , \ \lim_{k\to\infty} a_{nk} \ , \ \lim_{n,k\to\infty} a_{nk}$$

exist, for all $n, k \geq 1$, and the last one is equal to A . In this case we necessarily have:

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{nk} = \lim_{k\to\infty} \lim_{n\to\infty} a_{nk} = \lim_{n,k\to\infty} a_{nk} = A \ .$$

In the results below we shall need the following lemma:

Lemma 2.1

Let $\mathcal{E} = \{ E_{nk} \mid n, k \geq 1 \}$ *be a double sequence of real numbers satisfying the following two conditions:*

(i) $E_{n1} \leq E_{n2} \leq E_{n3} \leq \dots$, *for all* $n \geq 1$

(ii) $E_{1k} \geq E_{2k} \geq E_{3k} \geq \dots$, *for all* $k \geq 1$.

Let $E_{n\infty} = \lim_{k\to\infty} E_{nk}$ *and* $E_{\infty k} = \lim_{n\to\infty} E_{nk}$ *for* $n, k \geq 1$. *Then the following five statements are equivalent:*

(1) $\mathcal{E}$ *is regularly convergent* $($*in Hardy's sense*$)$

(2) $\mathcal{E}$ *is convergent* $($*in Pringsheim's sense*$)$

(3) $-\infty < \lim_{k\to\infty} E_{\infty k} = \lim_{n\to\infty} E_{n\infty} < +\infty$

(4) $\forall \varepsilon > 0 \ , \ \exists p_\varepsilon \geq 1$ *such that* $\forall n, m, k, l \geq p_\varepsilon$ *we have* $\mid E_{nk} - E_{ml} \mid < \varepsilon$

(5) $\forall \varepsilon > 0 \ , \ \exists p_\varepsilon \geq 1$ *such that* $E_{p_\varepsilon \infty} - E_{\infty p_\varepsilon} < \varepsilon$.

Proof. The proof is straight forward and we shall leave the details to the reader.

□

3. Total boundedness in the mean

The essential conclusions in the proofs of the main results in [7] are provided by using Lemma V-2-9 in Neveu's book [10], which gives a mild sufficient condition for the $a.s.$-convergence of supremum of a countable family of ordinary submartingales $(\{ X_n^i \ , \ \mathcal{F}_n \mid n \geq 1 \} \ , \ i \in \mathbf{N})$. A close look into its proof shows that it has mainly due to the fact that the function:

$$(3.1) \qquad (n, k) \longmapsto E(\sup_{1\leq i\leq k} X_n^i)$$

is non-decreasing in each variable. This is no longer true in the general reversed submartingale case, where the function from (3.1) is still non-decreasing in k , but it is decreasing in n , and it is easy to see that Neveu's lemma can fail in this case in general. It turns out that this irregularity is the main technical detail which makes the uniform convergence of families of reversed submartingales much more harder to establish than for families of ordinary ones. Our next theorem gives an analogue of Neveu's lemma for reversed submartingales.

Theorem 3.1

Let $(\{ X_n^i \, , \, \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid i \in \mathbf{N})$ *be a countable family of reversed submartingales and let* $X_{-\infty}^i$ *denote the* $a.s.$ *limit of* X_n^i *, as* $n \to -\infty$ *, for all* $i \in \mathbf{N}$ *. If the following condition is satisfied:*

(i) $\forall \varepsilon > 0 \, , \, \exists p_\varepsilon \geq 1$ *such that* $\forall n, m, k, l \geq p_\varepsilon$ *we have:*

$$\mid E(\sup_{1 \leq i \leq k} X_{-n}^i) - E(\sup_{1 \leq j \leq l} X_{-m}^j) \mid < \varepsilon$$

then we have:

(ii) $\exists k_0 \geq 1$ *such that* $-\infty < \inf_{n \leq -1} E(\sup_{1 \leq i \leq k_0} X_n^i) \leq \inf_{n \leq -1} E(\sup_{i \in \mathbf{N}} X_n^i) < +\infty$

(iii) $\sup_{i \in \mathbf{N}} X_n^i \longrightarrow \sup_{i \in \mathbf{N}} X_{-\infty}^i \;\; (a.s.)$ *and* (L^1) *, as* $n \to -\infty$.

Conversely, if we have convergence in P*-probability in (iii) together with the following condition:*

(iv) $-\infty < \inf_{n \leq -1} E(\sup_{i \in \mathbf{N}} X_n^i) < +\infty$

then (i) holds.

Proof. First suppose that (i) holds, then letting $k \to \infty$ in (i) and using the monotone convergence theorem we may deduce:

$$\mid E(\sup_{i \in \mathbf{N}} X_{-p_\varepsilon}^i) - E(\sup_{1 \leq j \leq p_\varepsilon} X_{-p_\varepsilon}^j) \mid \leq \varepsilon$$

Hence we see that the last inequality in (ii) must be satisfied, and moreover one can easily verify that for every M subset of $\mathbf{N}$ the family:

(1) $$\{ \sup_{i \in M} X_n^i \, , \, \mathcal{F}_n \mid -\infty < n \leq p_\varepsilon \}$$

forms a reversed submartingale. Therefore by the reversed submartingale convergence theorem we may conclude:

(2) $$\sup_{i \in M} X_n^i \longrightarrow X_{-\infty}^M \;\; (a.s.)$$

(3) $$E(\sup_{i \in M} X_n^i) \longrightarrow E(X_{-\infty}^M)$$

(4) $$\sup_{i \in M} X_n^i \longrightarrow X_{-\infty}^M \ \ (L^1) \quad , \quad \text{if} \ \inf_{n \le -1} E(\sup_{i \in M} X_n^i) > -\infty$$

as $n \to -\infty$. Note if M is a finite subset of $\mathbf{N}$, then $X_{-\infty}^M = \sup_{i \in M} X_{-\infty}^i \ a.s.$. Therefore letting $n \to \infty$ in (i) and using (3) we may obtain:

$$| \, E(\sup_{1 \le i \le p_\varepsilon} X_{-\infty}^i) - E(\sup_{1 \le j \le p_\varepsilon} X_{-p_\varepsilon}^j) \, | \le \varepsilon \ .$$

Hence $\sup_{1 \le i \le p_\varepsilon} X_{-\infty}^i \in L^1(P)$, and thus by (3) we may conclude:

$$\inf_{n \le -1} E(\sup_{1 \le i \le p_\varepsilon} X_n^i) = E(\sup_{1 \le i \le p_\varepsilon} X_{-\infty}^i) > -\infty \ .$$

This completes the proof of (ii). Moreover $\inf_{n \le -1} E(\sup_{i \in \mathbf{N}} X_n^i) > -\infty$, and thus by (4) we have:

(5) $$\sup_{i \in \mathbf{N}} X_n^i \longrightarrow X_{-\infty}^{\mathbf{N}} \ \ (L^1)$$

as $n \to -\infty$. And in order to establish (iii) let us now note that by (2) and (5) it is enough to show that $X_{-\infty}^{\mathbf{N}} = \sup_{i \in \mathbf{N}} X_{-\infty}^i \ a.s.$. Since the inequality $X_{-\infty}^{\mathbf{N}} \ge \sup_{i \in \mathbf{N}} X_{-\infty}^i \ a.s.$ follows straight forward and $X_{-\infty}^{\mathbf{N}} \in L^1(P)$, for this last conclusion it is sufficient to show that:

(6) $$EX_{-\infty}^{\mathbf{N}} = E(\sup_{i \in \mathbf{N}} X_{-\infty}^i) \ .$$

And in order to deduce (6) let us note that the double sequence $\mathcal{E} = \{ \ E_{nk} \mid n,k \ge 1 \ \}$ defined by:

(7) $$E_{nk} = E(\sup_{1 \le i \le k} X_{-n}^i)$$

satisfies conditions (i) and (ii) in Lemma 2.1, and moreover condition (i) in Theorem 3.1 is exactly condition (4) in Lemma 2.1. Therefore by (3) in Lemma 2.1 and the monotone convergence theorem we may conclude:

$$EX_{-\infty}^{\mathbf{N}} = \lim_{n \to \infty} \lim_{k \to \infty} E(\sup_{1 \le i \le k} X_{-n}^i) = \lim_{k \to \infty} \lim_{n \to \infty} E(\sup_{1 \le i \le k} X_{-n}^i) = E(\sup_{i \in \mathbf{N}} X_{-\infty}^i) \ .$$

Thus (6) is established, and the proof of (iii) is complete.

Now suppose that (iv) holds and that we have convergence in P-probability in (iii). Then by (1) and the second inequality in (iv) the family $\{ \ \sup_{i \in \mathbf{N}} X_n^i \ , \ \mathcal{F}_n \mid -\infty < n \le n_1 \ \}$ is a reversed sub-martingale, for some $n_1 \le -1$, which by the first inequality in (iv) satisfies:

$$\inf_{n \le n_1} E(\sup_{i \in \mathbf{N}} X_n^i) > -\infty \ .$$

Therefore by (2) and (4) statement (iii) follows straight forward. Moreover, by (iii) and the monotone convergence theorem we may conclude:

$$-\infty < \lim_{n\to\infty} \lim_{k\to\infty} E(\sup_{1\le i\le k} X^i_{-n}) = \lim_{k\to\infty} \lim_{n\to\infty} E(\sup_{1\le i\le k} X^i_{-n}) < \infty \ .$$

Hence (i) follows directly by applying the implication (3) ⇒ (4) in Lemma 2.1 on the double sequence $\mathcal{E} = \{ E_{nk} \mid n,k \ge 1 \}$ defined by (7) above, and the proof is complete.

□

Corollary 3.2

Let $(\{ X^i_n \ , \ \mathcal{F}_n \mid -\infty < n \le -1 \} \mid i \in \mathbf{N})$ *be a countable family of reversed submartingales and let* $X^i_{-\infty}$ *denote the* $a.s.$ *limit of* X^i_n *, as* $n \to -\infty$ *, for all* $i \in \mathbf{N}$ *. Suppose that:*

$$-\infty < \inf_{n\le -1} E(\sup_{i\in\mathbf{N}} X^i_n) \ < +\infty$$

and let $E_{nk} = E(\sup_{1\le i\le k} X^i_{-n})$ *for all* $n,k \ge 1$ *. Then the family:*

$$\{ \sup_{i\in\mathbf{N}} X^i_n \ , \ \mathcal{F}_n \mid -\infty < n \le n_1 \}$$

is a reversed submartingale, for some $n_1 \le -1$ *, and* $\sup_{i\in\mathbf{N}} X^i_n$ *converges* P*-almost surely and in* $L^1(P)$ *, as* $n \to -\infty$ *. Moreover, we have:*

$$\sup_{i\in\mathbf{N}} X^i_n \longrightarrow \sup_{i\in\mathbf{N}} X^i_{-\infty} \quad (a.s.) \ \ \textit{and} \ \ (L^1)$$

as $n \to -\infty$ *, if and only if the double sequence* $\mathcal{E} = \{ E_{nk} \mid n,k \ge 1 \}$ *is regularly convergent* (*in Hardy's sense*).

□

Remark 3.3 Let us note that under the hypotheses in Corollary 3.2, the double sequence $\mathcal{E} = \{ E_{nk} \mid n,k \ge 1 \}$ satisfies conditions (i) and (ii) in Lemma 2.1, so it is regular convergent if and only if either of statements (2)-(5) in Lemma 2.1 is satisfied. Notice that we have:

$$E_{n\infty} = E(\sup_{i\in\mathbf{N}} X^i_{-n}) \quad \text{and} \quad E_{\infty k} = E(\sup_{1\le i\le k} X^i_{-\infty}).$$

In order to extend the result in Theorem 3.1 to not necessarily countable families of reversed submartingales, we shall introduce the following *definition*: Let T be a non-empty set, let

$$\{ X_n(t) \ , \ \mathcal{F}_n \ \mid \ -\infty < n \le -1 \}$$

be a reversed submartingale for $t \in T$, and let $X_{-\infty}(t)$ denote the $a.s.$-limit of $X_n(t)$, as $n \to -\infty$, for all $t \in T$. Let $D = \{ d_i \mid i \ge 1 \}$ be a countable subset of T , and let $D_n = \{ d_1, \dots, d_n \}$, for all $n \ge 1$. Then the family of reversed submartingales:

$$(\{ \ X_n(t) \ , \ \mathcal{F}_n \ | \ -\infty < n \leq -1 \ \} \ | \ t \in T \ )$$

is said to be *totally bounded in P-mean relative to* D , if either of the following nine equivalent conditions is satisfied, see Remark 3.3 and Lemma 2.1:

(i) The double sequence $\{ \ EM_{D_k}(X_{-n}) \ | \ n,k \geq 1 \ \}$ is regularly convergent (in Hardy's sense)

(ii) The double sequence $\{ \ EM_{D_k}(X_{-n}) \ | \ n,k \geq 1 \ \}$ is convergent (in Pringsheim's sense)

(iii) $-\infty < \lim\limits_{k\to\infty} EM_{D_k}(X_{-\infty}) = \lim\limits_{n\to\infty} EM_D(X_{-n}) < +\infty$,

(iv) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall n,m,k,l \geq p_\varepsilon$ we have:

$$| \ EM_{D_k}(X_{-n}) - EM_{D_l}(X_{-m}) \ | < \varepsilon$$

(v) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall m \geq n \geq p_\varepsilon$ and $\forall k \geq l \geq p_\varepsilon$ we have:

$$EM_{D_k}(X_{-n}) - EM_{D_l}(X_{-m}) < \varepsilon$$

(vi) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall m \geq n \geq p_\varepsilon$ and $\forall l \geq k \geq p_\varepsilon$ we have:

$$| \ EM_{D_k}(X_{-n}) - EM_{D_l}(X_{-m}) \ | < \varepsilon$$

(vii) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall n,k \geq p_\varepsilon$ we have:

$$| \ EM_{D_k}(X_{-n}) - EM_{D_{p_\varepsilon}}(X_{-p_\varepsilon}) \ | < \varepsilon$$

(viii) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that $\forall n,k \geq p_\varepsilon$ we have:

$$EM_{D_k}(X_{-p_\varepsilon}) - EM_{D_{p_\varepsilon}}(X_{-n}) < \varepsilon$$

(ix) $\forall \varepsilon > 0$, $\exists p_\varepsilon \geq 1$ such that:

$$EM_D(X_{-p_\varepsilon}) - EM_{D_{p_\varepsilon}}(X_{-\infty}) < \varepsilon \ .$$

In this case the limit of the double sequence $\{ \ EM_{D_k}(X_{-n}) \ | \ n,k \geq 1 \ \}$ is equal to $EM_D(X_{-\infty})$, and we have:

$$\lim_{k\to\infty} \lim_{n\to\infty} EM_{D_k}(X_{-n}) = EM_D(X_{-\infty}) = \lim_{n\to\infty} \lim_{k\to\infty} EM_{D_k}(X_{-n}) \ .$$

Moreover then by Theorem 3.1 we have:

$$M_D(X_n) \longrightarrow M_D(X_{-\infty}) \quad (a.s.) \ \text{ and } \ (L^1)$$

as $n \to -\infty$.

4. The reversed martingale convergence theorem

In this section we shall extend the result in Theorem 3.1 to families of reversed martingales and submartingales indexed by a separable topological space. We first consider the reversed submartingale case in Theorem 4.1, and then we shall pass to its martingale version in Theorem 4.3. The results in Theorem 4.1 and Theorem 4.3 will be completed by forthcoming results in Theorem 4.7 and Theorem 4.11, respectively.

Theorem 4.1

Let $(\{ X_n(t) \ ,\ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed submartingales indexed by a separable topological space* T *, let* D *be a countable dense subset of* T *, and let* L *be a map from* T *into* $\mathbf{R}$ *. Suppose that:*

(i) $X_n(t) \to L(t)\ \ (a.s.)$ *, as* $n \to -\infty$ *, for all* $t \in T$

(ii) *The family of reversed submartingales*

$$(\{ (X_n(t) - L(t))^+ \ ,\ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$$

is totally bounded in P*-mean relative to* D

and either of the following two conditions is satisfied:

(iii) $L \in \mathcal{U}sc\ (T)$

(iv) T *is hereditarily separable and (ii) holds for each countable* D *subset of* T *.*

Then there exists a sequence of random variables $\{ V_n \mid n \leq -1 \}$ *satisfying:*

(v) $V_n \to 0\ \ (a.s.)$ *and* (L^1) *, as* $n \to -\infty$

(vi) $\| (X_n(\omega) - L)^+ \|_T \leq \Delta_n^+(\omega) + V_n(\omega)\ ,\ \ \forall \omega \in \Omega$

where $\Delta_n^+(\omega) = \Delta^+(X_n(\omega)) = \sup\{ \partial^+(t, X_n(\omega)) \mid t \in T \}$ *. In particular, if* $\Delta_n^+ \to 0\ \ (c)$ *, then* $\| (X_n - L)^+ \|_T \to 0\ \ (c)$ *, as* $n \to -\infty$ *, where* (c) *denotes either of the following convergence notions:* $(a.s.), (a.s.)^*, (P^*), (P_*), (L^1)^*, (L^1)_*$ *. Conversely, if (i) is satisfied and:*

$$\| (X_n - L)^+ \|_T \to 0\ \ (P_*)$$

as $n \to -\infty$ *, then (ii) holds for an arbitrary countable* D *subset of* T *satisfying* $\inf_{n \leq -1} E \| (X_n - L)^+ \|_D < \infty$ *.*

Proof. Let $f, g \in \mathbf{R}^T$ be arbitrary functions, let t be a point in T , and let $\mathcal{N}_t$ be the family of all open neighborhoods of t . Since D is dense in T , then for each $G, H \in \mathcal{N}_t$ there exists $d \in G \cap H \cap D$, and consequently we may deduce:

$$f(t) - g(t) = (f(t) - f(d)) + (g(d) - g(t)) + (f(d) - g(d)) \leq$$

$$\leq W_G^+(t, f) + W_H^-(t, g) + \| (f - g)^+ \|_D \ .$$

Taking infimums over all $G, H \in \mathcal{N}_t$ we obtain:

$$(f(t)-g(t))^+ \le \partial^+(t,f) + \partial^-(t,g) + \| (f-g)^+ \|_D$$

And taking supremum over all $t \in T$ we may conclude:

$$\| (f-g)^+ \|_T \le \Delta^+(f) + \Delta^-(g) + \| (f-g)^+ \|_D$$

for all $f,g \in \mathbf{R}^T$. In particular, by (2.2) we have:

$$\| (f-g)^+ \|_T \le \Delta^+(f) + \| (f-g)^+ \|_D$$

for all $f \in \mathbf{R}^T$ and all $g \in \mathcal{U}sc\ (T)$. Therefore (iii) implies:

$$\| (X_n(\omega) - L)^+ \|_T \le \Delta^+(X_n(\omega)) + \| (X_n(\omega) - L)^+ \|_D$$

for all $\omega \in \Omega$ and all $n \le -1$. Let us note that the family:

$$(\ \{\ (X_n(t) - L(t))^+\ ,\ \mathcal{F}_n \mid -\infty < n \le -1\ \}\ \mid\ t \in D\ )$$

is a countable family of reversed submartingales which by (i) satisfies:

$$(X_n(t) - L(t))^+ \longrightarrow 0\ \ (a.s.)$$

as $n \to -\infty$, for all $t \in D$. But then by (ii) and Theorem 3.1 we may conclude:

$$V_n := \| (X_n(t) - L(t))^+ \|_D \longrightarrow 0 \quad (a.s.)\ \text{ and }\ (L^1)$$

as $n \to -\infty$. Consequently, we see that (i),(ii) and (iii) imply (v) and (vi).

Next suppose that (i), (ii) and (iv) hold. Let τ be the given hereditarily separable topology on T . Note that:

$$\delta(t',t'') = \arctan \mid L(t') - L(t'') \mid$$

defines a separable pseudometric on T such that $L \in \mathcal{C}_u(T,\delta)$. Let τ_δ denote the topology generated by δ , then by (2.4) there exists a hereditarily separable topology ν on T which is finer than τ and τ_δ . Let D be a countable ν-dense subset of T . Now let us return to the beginning of the proof with the new separable topology ν on T and let us note that:

$$L \in \mathcal{C}_u(T,\delta) \subset \mathcal{C}(T,\nu) \subset \mathcal{U}sc\ (T,\nu)\ .$$

Thus (iii) holds if we replace τ by τ_δ . Since $\tau \subset \nu$, we have:

$$\partial^+_\nu(t,f) \le \partial^+_\tau(t,f)$$

for all $f \in \mathbf{R}^T$ and all $t \in T$. Therefore we may conclude:

$$\Delta^+_\nu(X_n(\omega)) \le \Delta^+_\tau(X_n(\omega))$$

for all $\omega \in \Omega$ and $n \ge 1$. Consequently, (v) and (vi) follow straight forward by the first part of the proof. Finally note that the convergence statement, stated after statement (vi) in the theorem, follows directly by (2.6)-(2.8).

Conversely, if $\| (X_n - L)^+ \|_T \to 0 \ \ (P_*)$, as $n \to -\infty$, then obviously:

$$\| (X_n - L)^+ \|_D \to 0 \ \ (P)$$

as $n \to -\infty$, for every countable D subset of T . But then (ii) follows by Theorem 3.1, and the proof is complete.

□

The next corollary is an easy consequence of the equivalence statement obtained in the previous theorem, and we shall leave its verification to the reader.

Corollary 4.2

Let $(\{ X_n(t) \ , \ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed submartingales indexed by a separable topological space* T *, let* D *be a countable dense subset of* T *, and let* L *be a map from* T *into* $\mathbf{R}$ *. Suppose that:*

(i) $X_n(t) \to L(t) \ \ (a.s.)$ *, as* $n \to -\infty$ *, for all* $t \in T$

(ii) $\| (X_n(\omega) - L)^+ \|_T \to 0 \ \ (P_*)$ *, as* $n \to -\infty$

and either of the following two conditions is satisfied:

(iii) $\inf_{n \leq -1} E \| (X_n - L)^+ \|_D < \infty$ *and* $L \in \mathcal{U}sc\,(T)$

(iv) $\inf_{n \leq -1} E \| (X_n - L)^+ \|_D < \infty$ *for each countable* D *subset of* T *, and* T *is hereditarily separable.*

If $\Delta^+(X_n) \to 0 \ \ (c)$ *, then* $\| (X_n - L)^+ \|_T \to 0 \ \ (c)$ *, as* $n \to -\infty$ *, where* (c) *denotes either of the following convergence notions:* $(a.s.), (a.s.)^*, (P^*), (P_*), (L^1)^*, (L^1)_*$ *. In particular, if* T *is hereditarily separable,* $X_n(\omega, \cdot) \in \mathcal{L}sc(T)$ *for* $a.a.\ \omega \in \Omega$ *and all* $n \leq -1$ *, and:*

$$\inf_{n \leq -1} E^* \| (X_n - L)^+ \|_T < \infty$$

then $\| (X_n - L)^+ \|_T \to 0 \ \ (P_*)$ *, implies* $\| (X_n - L)^+ \|_T \to 0 \ \ (a.s.)^*$ *and* $(L^1)^*$ *, as* $n \to -\infty$*.*

□

Theorem 4.3

Let $(\{ X_n(t) \ , \ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed martingales indexed by a separable topological space* T *, let* D *be a countable dense subset of* T *, and let* L *be a map from* T *into* $\mathbf{R}$ *. Suppose that:*

(i) $X_n(t) \to L(t) \ \ (a.s.)$ *, as* $n \to -\infty$ *, for all* $t \in T$

(ii) *The family of reversed submartingales*

$$(\{ \mid X_n(t) - L(t) \mid , \ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$$

is totally bounded in P-mean relative to D

and either of the following two conditions is satisfied:

(iii) $L \in \mathcal{C}(T)$

(iv) T *is hereditarily separable and (ii) holds for each countable* D *subset of* T .

Then there exists a sequence of random variables $\{ V_n \mid n \leq -1 \}$ *satisfying:*

(v) $V_n \rightarrow 0 \ (a.s.)$ *and* (L^1) *, as* $n \rightarrow -\infty$

(vi) $\| X_n(\omega) - L \|_T \leq \Delta_n(\omega) + V_n(\omega) , \ \forall \omega \in \Omega$

where $\Delta_n(\omega) = \Delta(X_n(\omega)) = \sup\{ \partial(t, X_n(\omega)) \mid t \in T \}$ *. In particular, if* $\Delta_n \rightarrow 0 \ (c)$ *, then* $\| X_n - L \|_T \rightarrow 0 \ (c)$ *, as* $n \rightarrow -\infty$ *, where* (c) *denotes either of the following convergence notions:* $(a.s.), (a.s.)^*, (P^*), (P_*), (L^1)^*, (L^1)_*$ *. Conversely, if (i) is satisfied and:*

$$\| X_n - L \|_T \rightarrow 0 \ (P_*)$$

as $n \rightarrow -\infty$ *, then (ii) holds for an arbitrary countable* D *subset of* T *satisfying* $\inf_{n \leq -1} E \| X_n - L \|_D < \infty$ *.*

Proof. In the proof of Theorem 4.1 we have established the following inequality:

$$(f(t) - g(t))^+ \leq \partial^+(t, f) + \partial^-(t, g) + \| (f - g)^+ \|_D$$

for all $f, g \in \mathbf{R}^T$ and all $t \in T$. Hence one can easily deduce that we have:

$$\| f - g \|_T \leq \Delta(f) + \| f - g \|_D$$

for all $f \in \mathbf{R}^T$ and all $g \in \mathcal{C}(T)$. Therefore (iii) implies:

$$\| X_n(\omega) - L \|_T \leq \Delta(X_n(\omega)) + \| X_n - L \|_D$$

for all $\omega \in \Omega$ and all $n \leq -1$. Let us note that the family:

$$(\{ \mid X_n(t) - L(t) \mid , \ \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in D)$$

is a countable family of reversed submartingales which by (i) satisfies:

$$\mid X_n(t) - L(t) \mid \rightarrow 0 \ (a.s.)$$

as $n \rightarrow -\infty$, for all $t \in D$. Hence by (ii) and Theorem 3.1 we may conclude:

$$V_n := \| X_n - L \|_D \to 0 \quad (a.s.) \quad \text{and} \quad (L^1)$$

as $n \to -\infty$. Consequently, we see that (i), (ii) and (iii) imply (v) and (vi).

The rest of the proof is very similar to the corresponding last part in the proof of Theorem 4.1, and we shall leave the details to the reader.

□

Similarly to Corollary 4.2, the next corollary is an easy consequence of the equivalence statement obtained in the previous theorem, and we shall leave its verification to the reader.

Corollary 4.4

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed martingales indexed by a separable topological space* T *, let* D *be a countable dense subset of* T *, and let* L *be a map from* T *into* $\mathbf{R}$ *. Suppose that:*

(i) $X_n(t) \to L(t)$ $(a.s.)$ *, as* $n \to -\infty$ *, for all* $t \in T$

(ii) $\| X_n - L \|_T \to 0$ (P_*) *, as* $n \to -\infty$

and either of the following two conditions is satisfied:

(iii) $\inf_{n \leq -1} E \| X_n - L \|_D < \infty$ *and* $L \in \mathcal{C}(T)$

(iv) $\inf_{n \leq -1} E \| X_n - L \|_D < \infty$ *for each countable* D *subset of* T *, and* T *is hereditarily separable.*

If $\Delta(X_n) \to 0$ (c) *, then* $\| X_n - L \|_T \to 0$ (c) *, as* $n \to -\infty$ *, where* (c) *denotes either of the following convergence notions:* $(a.s.), (a.s.)^*, (P^*), (P_*), (L^1)^*, (L^1)_*$ *. In particular, if* T *is hereditarily separable,* $X_n(\omega, \cdot\,) \in \mathcal{C}(T)$ *for* $a.a.$ $\omega \in \Omega$ *and all* $n \leq -1$ *, and:*

$$\inf_{n \leq -1} E^* \| X_n - L \|_T < \infty$$

then $\| X_n - L \|_T \to 0$ (P_*) *, implies* $\| X_n - L \|_T \to 0$ $(a.s.)^*$ *and* $(L^1)^*$ *, as* $n \to -\infty$.

□

We shall continue our considerations by exploring condition (ii) in Theorem 4.1 and Theorem 4.3, in order to obtain an equivalent form which is more suitable for applications. The main results in this direction are established in Theorem 4.7 and Theorem 4.11 below, and they offer a convenient criterion for that condition. We shall use $X_n^{\pm}$ to denote $(X_n)^{\pm}$ for $n \geq 1$, respectively.

Proposition 4.5

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed submartingales indexed by a set* T *, let* D *be a countable subset of* T *, and suppose that the following two conditions are satisfied:*

(i) $X_n(t) \to 0 \;\; (a.s.)$ *, as* $n \to -\infty$, $\forall t \in T$

(ii) $\forall \varepsilon > 0$, $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$, $\exists d_1 \in D_1, \dots, d_{m_\varepsilon} \in D_{m_\varepsilon}$ *such that* $\forall N \le -1$, $\exists n_\varepsilon \le N$ *and* $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(1) $(X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j))^+ \le \Psi_j \;\; , \;\; \forall t \in D_j \;,\; \forall j = 1, \dots, m_\varepsilon$

(2) $P\{ \max\limits_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} < \varepsilon$.

Then the family of reversed submartingales:

$$(\{ X_n^+(t) \;,\; \mathcal{F}_n \;\mid -\infty < n \le -1 \} \mid t \in T)$$

is totally bounded in P*-mean relative to* D .

Proof. Let us first note that the family $\{ E\{ X_n^+(t) \mid \mathcal{F}_{-\infty} \} \;,\; \mathcal{F}_n \mid -\infty < n \le -1 \}$ forms a reversed submartingale, for every $t \in T$. Since obviously $\inf_{n \le -1} E[\, X_n^+(t) \,] > -\infty$, then $\{ X_n^+(t) \mid n \le -1 \}$ is uniformly integrable for every $t \in T$, and therefore we have:

$$E\{ X_n^+(t) \mid \mathcal{F}_{-\infty} \} \to 0 \quad (a.s.) \;\; \text{and} \;\; (L^1)$$

as $n \to -\infty$, for all $t \in T$. Thus for given $\varepsilon > 0$ and $t \in T$, there exists $n_{\varepsilon,t} \le -1$ such that:

(1) $$| \, E\{ X_n^+(t) \mid \mathcal{F}_{-\infty} \} \, | < \varepsilon$$

for all $n \le n_{\varepsilon,t}$. By (ii) we can find $\gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ and $d_1 \in D_1, \dots, d_{m_\varepsilon} \in D_{m_\varepsilon}$ such that $\forall N \le -1$, $\exists n_\varepsilon \le N$ and $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying (1) and (2) in (ii). In particular for $N = \min \{ n_{\varepsilon,d_1}, \dots, n_{\varepsilon,d_{m_\varepsilon}} \}$ there exists $n_\varepsilon \le N$ and $\Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying (1) and (2) in (ii). Hence we may conclude:

$$X_n^+(t) \le E\{ X_{n_\varepsilon}^+(t) \mid \mathcal{F}_n \} \le E\{ (X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j))^+ \mid \mathcal{F}_n \} +$$

$$+ E\{ X_{n_\varepsilon}^+(d_j) \mid \mathcal{F}_n \} \le E\{ \Psi_j \mid \mathcal{F}_n \} + E\{ X_{n_\varepsilon}^+(d_j) \mid \mathcal{F}_n \}$$

for all $n \le n_\varepsilon$ and all $t \in D_j$ with $j = 1, \dots m_\varepsilon$. Taking supremum over all $t \in D$ and using (1) we get:

$$\limsup_{n \to -\infty} \| X_n^+ \|_D \le \max_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} + \max_{1 \le j \le m_\varepsilon} E\{ X_{n_\varepsilon}^+(d_j) \mid \mathcal{F}_{-\infty} \} \le$$

$$\le \max_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} + \varepsilon \; .$$

Hence by (2) we easily obtain $P\{ \limsup_{n \to -\infty} \| X_n^+ \|_D > 2\varepsilon \} < \varepsilon$, for all $\varepsilon > 0$. Thus we may conclude $\limsup_{n \to -\infty} \| X_n^+ \|_D = 0 \;\; a.s.$. By (1) in (ii) one can easily deduce $E \| X_{n_\varepsilon}^+ \|_D < \infty$, and therefore the family $\{ \| X_n^+ \|_D \;,\; \mathcal{F}_n \mid -\infty < n \le n_\varepsilon \}$ forms a non-negative reversed submartingale which converges P-almost surely to zero, as $n \to -\infty$. But then $E \| X_n^+ \|_D \to 0$, as $n \to -\infty$, and by (5) in Lemma 2.1 and Remark 3.3 we may conclude that the family of reversed submartingales $(\{ X_n^+(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ is totally bounded in P-mean relative to D . This fact completes the proof.

□

Proposition 4.6

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed submartingales indexed by a set* T *, let* D *be a countable subset of* T *, and suppose that the following condition is satisfied:*

(i) $X_n(t) \to 0 \;\; (a.s.)$, as $n \to -\infty$, $\forall t \in T$.

If the family of reversed submartingales:

$$(\{ X_n^+(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$$

is totally bounded in P-mean relative to D , then we have:

(ii) $\forall \varepsilon > 0$, $\forall \gamma = \{D_1, \dots, D_m\} \in \Gamma(D)$, $\forall d_1 \in D_1, \dots, d_m \in D_m$ and $\forall N \leq -1$, $\exists n_\varepsilon \leq N$ and $\exists \Psi_1, \dots, \Psi_m \in L^1(P)$ satisfying:

(1) $(X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j))^+ \leq \Psi_j$, $\forall t \in D_j$, $\forall j = 1, \dots, m$

(2) $E(\max_{1 \leq j \leq m} \Psi_j) < \varepsilon$.

Proof. Let us first note that by (5) in Lemma 2.1 and Remark 3.3 the family of reversed submartingales $(\{ X_n^+(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ is totally bounded in P-mean relative to D , if and only if $E \parallel X_n^+ \parallel_D \to 0$, as $n \to -\infty$. Let $\varepsilon > 0$, $\gamma = \{ D_1, \dots, D_m \} \in \Gamma(D)$ and $d_1 \in D_1, \dots, d_m \in D_m$ be given, then we have:

$$(X_n(t) - X_n(d_j))^+ \leq X_n^+(t) + X_n^-(d_j)$$

for all $n \leq -1$, all $t \in D_j$ and all $j = 1, \dots, m$. Hence we find:

$$\sup_{t \in D_j} (X_n(t) - X_n(d_j))^+ \leq \parallel X_n^+ \parallel_{D_j} + X_n^-(d_j)$$

for all $n \leq -1$ and all $j = 1, \dots, m$. Since we have $\inf_{n \leq -1} E[X_n^-(t)] > -\infty$, then $\{ X_n^-(t) \mid n \leq -1 \}$ is uniformly integrable for every $t \in T$, and therefore we may deduce that $\{ \max_{1 \leq j \leq m} X_n^-(d_j) \mid n \leq -1 \}$ is uniformly integrable. Using this fact we may conclude that:

$$E(\max_{1 \leq j \leq m} X_n^-(d_j)) \to 0$$

as $n \to -\infty$. Thus for given $N \leq -1$, there exists $n_\varepsilon \leq N$ satisfying:

$$E \parallel X_{n_\varepsilon}^+ \parallel_D + E(\max_{1 \leq j \leq m} X_{n_\varepsilon}^-(d_j)) < \varepsilon$$

and the proof follows easily by putting:

$$\Psi_j = \| X^+_{n_\varepsilon} \|_{D_j} + X^-_{n_\varepsilon}(d_j)$$

for all $j = 1, \dots, m$.

$\square$

Theorem 4.7

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed submartingales indexed by a set* T *, let* D *be a countable subset of* T *, and suppose that the following condition is satisfied:*

(i) $X_n(t) \to 0 \ (a.s.)$ *, as* $n \to -\infty$, $\forall t \in T$.

Then the following three statements are equivalent:

(1) $\forall \varepsilon > 0$, $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$, $\exists d_1 \in D_1, \dots, d_{m_\varepsilon} \in D_{m_\varepsilon}$ *such that* $\forall N \leq -1$, $\exists n_\varepsilon \leq N$ *and* $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(i) $(X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j))^+ \leq \Psi_j \ , \ \forall t \in D_j \ , \ \forall j = 1, \dots, m_\varepsilon$

(ii) $P\{ \max_{1 \leq j \leq m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} < \varepsilon$

(2) $\forall \varepsilon > 0$, $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$, $\exists d_1 \in D_1, \dots, d_{m_\varepsilon} \in D_{m_\varepsilon}$ *such that* $\forall N \leq -1$, $\exists n_\varepsilon \leq N$ *and* $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(i) $(X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j))^+ \leq \Psi_j \ , \ \forall t \in D_j \ , \ \forall j = 1, \dots, m_\varepsilon$

(ii) $E(\max_{1 \leq j \leq m_\varepsilon} \Psi_j) < \varepsilon$

(3) *The family of reversed submartingales:*

$$(\{ X^+_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$$

is totally bounded in P-mean relative to D .

Proof. The implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ follow by Proposition 4.5 and Proposition 4.6, respectively. In order to prove the implication $(2) \Rightarrow (1)$ take $\varepsilon > 0$, then $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ and $\exists d_1 \in D_1, \dots, d_{m_\varepsilon} \in D_{m_\varepsilon}$, such that $\forall N \leq -1$, $\exists n_\varepsilon \leq N$ and $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying (i) in (2) and:

$$E(\max_{1 \leq j \leq m_\varepsilon} \Psi_j) < \varepsilon^2 \ .$$

Hence by Markov's inequality we may conclude:

$$P\{ \max_{1 \leq j \leq m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} \leq P\{ E\{ \max_{1 \leq j \leq m_\varepsilon} \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} \leq \frac{1}{\varepsilon} E(\max_{1 \leq j \leq m_\varepsilon} \Psi_j) < \varepsilon \ .$$

Thus (i) and (ii) in (1) are fulfilled, and the proof is complete.

□

Remark 4.8 Under the hypotheses in Theorem 4.7, suppose that the σ-algebra $\mathcal{F}_{-\infty}$ is degenerated, i.e. $P(F) \in \{0,1\}$, $\forall F \in \mathcal{F}_{-\infty}$. Then $E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} = E(\Psi_j)$, for all $j = 1, \ldots, m_\varepsilon$, and condition (ii) in (1) is equivalent to the following condition:

(1) $$\max_{1 \le j \le m_\varepsilon} E(\Psi_j) \le \varepsilon$$

provided that $\varepsilon \le 1$. Also note if $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \le -1 \} \mid t \in T)$ is a family of reversed submartingales indexed by a set T , and L is a map from T into $\mathbf{R}$ satisfying $X_n(t) \to L(t)$ $(a.s.)$, as $n \to -\infty$, $\forall t \in T$, and if we put $Y_n(t) = X_n(t) - L(t)$, for all $n \le -1$ and all $t \in T$, then $(\{ Y_n(t) , \mathcal{F}_n \mid -\infty < n \le -1 \} \mid t \in T)$ becomes a family of reversed submartingales satisfying hypotheses in Theorem 4.7. Hence we see that Theorem 4.7 offers criterions for condition (ii) in Theorem 4.1.

Now we shall pass to the martingale versions of the preceding three results. We could see in Theorem 4.11 and Remark 4.12 below, that our condition of total boundedness in the mean is actually equivalent to Hoffmann-Jørgensen's condition of eventually total boundedness in the mean, which is necessary and sufficient for the uniform law of large numbers, see [5].

Proposition 4.9

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \le -1 \} \mid t \in T)$ *be a family of reversed martingales indexed by a set* T *, let* D *be a countable subset of* T *, let* L *be a map from* T *into* $\mathbf{R}$ *, and suppose that the following two conditions are satisfied:*

(i) $X_n(t) \to L(t)$ $(a.s.)$ *, as* $n \to -\infty$, $\forall t \in T$

(ii) $\forall \varepsilon > 0$, $\exists n_\varepsilon \le -1$, $\exists \gamma = \{ D_1, \ldots, D_{m_\varepsilon} \} \in \Gamma(D)$ *and* $\exists \Psi_1, \ldots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(1) $$| X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') | \le \Psi_j \ , \ \forall t', t'' \in D_j \ , \ \forall j = 1, \ldots, m_\varepsilon$$

(2) $$P\{ \max_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} < \varepsilon \ .$$

Then the family of reversed submartingales:

$$(\{ \mid X_n(t) - L(t) \mid , \mathcal{F}_n \mid -\infty < n \le -1 \} \mid t \in T)$$

is totally bounded in P-mean relative to D .

Proof. For given $\varepsilon > 0$, there exists $n_\varepsilon \le -1$, $\gamma = \{ D_1, \ldots, D_{m_\varepsilon} \} \in \Gamma(D)$ and $\Psi_1, \ldots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying (1) and (2) in (i). For each $j = 1, \ldots, m_\varepsilon$ choose a point d_j in

D_j , then for given $t \in D_j$ we have:

$$| X_{n_\varepsilon}(t) | \leq | X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j) | + | X_{n_\varepsilon}(d_j) | \leq$$
$$\leq \Psi_j + | X_{n_\varepsilon}(d_j) | \leq \sum_{j=1}^{m_\varepsilon} \big(\Psi_j + | X_{n_\varepsilon}(d_j) | \big) .$$

Since the map on the right side above belongs to $L^1(P)$, taking supremum over all $t \in D$, we get $\| X_{n_\varepsilon} \|_D \in L^1(P)$. Therefore the family $\{ \| X_n \|_D , \mathcal{F}_n \mid -\infty < n \leq n_\varepsilon \}$ forms a reversed submartingale, and in particular we have:

$$M := \lim_{n \to -\infty} E \| X_n \|_D < \infty .$$

And since $| L(t) | = \lim_{n \to -\infty} E | X_n(t) | \leq \lim_{n \to -\infty} E \| X_n \|_D = M$ for every $t \in D$, then $\| L \|_D < \infty$, or equivalently $L \in B(D)$. Therefore it is no restriction to assume that the cover $\gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ satisfying (1) and (2) in (ii), also satisfies the following condition:

$$| L(t') - L(t'') | < \varepsilon$$

for all $t', t'' \in D_j$ and all $j = 1, \dots, m_\varepsilon$. Hence for given $t \in D_j$ with $1 \leq j \leq m_\varepsilon$, and for every $n \leq n_\varepsilon$ we may deduce:

$$| X_n(t) - L(t) | \leq | X_n(t) - X_n(d_j) | + | X_n(d_j) - L(d_j) | + | L(d_j) - L(t) | \leq$$
$$\leq E\{ |X_{n_\varepsilon}(t) - X_{n_\varepsilon}(d_j)| \mid \mathcal{F}_n \} + | X_n(d_j) - L(d_j) | + \varepsilon \ \leq E\{ \Psi_j \mid \mathcal{F}_n \} +$$
$$+ | X_n(d_j) - L(d_j) | + \varepsilon \ \leq \max_{1 \leq j \leq m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_n \} + \max_{1 \leq j \leq m_\varepsilon} | X_n(d_j) - L(d_j) | + \varepsilon .$$

Taking supremum over all $t \in D$ we may conclude:

$$\limsup_{n \to -\infty} \| X_n - L \|_D \leq \limsup_{n \to -\infty} \max_{1 \leq j \leq m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_n \} +$$
$$+ \limsup_{n \to -\infty} \max_{1 \leq j \leq m_\varepsilon} | X_n(d_j) - L(d_j) | + \varepsilon \ = \max_{1 \leq j \leq m_\varepsilon} \limsup_{n \to -\infty} E\{ \Psi_j \mid \mathcal{F}_n \} +$$
$$+ \max_{1 \leq j \leq m_\varepsilon} \limsup_{n \to -\infty} | X_n(d_j) - L(d_j) | + \varepsilon \ = \max_{1 \leq j \leq m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} + \varepsilon .$$

Hence by (2) in (ii) we easily obtain:

$$P\{ \limsup_{n \to -\infty} \| X_n - L \|_D > 2\varepsilon \} < \varepsilon$$

for all $\varepsilon > 0$. Since $L \in B(D)$, thus the family $\{ \| X_n - L \|_D , \mathcal{F}_n \mid -\infty < n \leq n_\varepsilon \}$ forms a non-negative reversed submartingale which converges P-almost surely to zero, as $n \to -\infty$. But then $E \| X_n - L \|_D \to 0$ as $n \to -\infty$, and by (5) in Lemma 2.1 and Remark 3.3 we may conclude that the family of reversed submartingales $(\{ | X_n(t) - L(t) | , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ is totally bounded in P-mean relative to D . This fact completes the proof. □

Proposition 4.10

Let $(\{ X_n(t) , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ *be a family of reversed martingales indexed by a set* T *, let* D *be a countable subset of* T *, let* L *be a map from* T *into* $\mathbf{R}$ *, and suppose that the following two conditions are satisfied:*

(i) $X_n(t) \to L(t)$ $(a.s.)$ *, as* $n \to -\infty$, $\forall t \in T$

(ii) $\inf_{n \leq -1} E \parallel X_n \parallel_D < \infty$.

If the family of reversed submartingales:

$$(\{ \mid X_n(t) - L(t) \mid , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$$

is totally bounded in P-mean relative to D *, then we have:*

(iii) $\forall \varepsilon > 0$, $\exists n_\varepsilon \leq -1$, $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ *and* $\exists \Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(1) $\mid X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') \mid \leq \Psi_j$, $\forall t', t'' \in D_j$, $\forall j = 1, \dots, m_\varepsilon$

(2) $E(\max_{1 \leq j \leq m_\varepsilon} \Psi_j) < \varepsilon$.

Proof. Let us first note that by (5) in Lemma 2.1 and Remark 3.3 the family of reversed submartingales $(\{ \mid X_n(t) - L(t) \mid , \mathcal{F}_n \mid -\infty < n \leq -1 \} \mid t \in T)$ is totally bounded in P-mean, if and only if the following condition is satisfied:

(1) $$\lim_{n \to -\infty} E \parallel X_n - L \parallel_D = 0$$

Therefore under our assumptions the family $\{ \parallel X_n - L \parallel_D , \mathcal{F} \mid -\infty < n \leq n_0 \}$ forms a reversed submartingale, for some $n_0 \leq -1$. Since we obviously have:

$$\parallel L \parallel_D \leq E \parallel X_n - L \parallel_D + E \parallel X_n \parallel_D$$

for all $n \leq -1$, then by (ii) we get $\parallel L \parallel_D < \infty$, or equivalently $L \in B(D)$. Now for a given cover $\gamma = \{ D_1, \dots, D_m \} \in \Gamma(D)$ let us define:

$$\Psi_{C,n} = \max_{1 \leq j \leq m} \sup_{t', t'' \in D_j} \mid X_n(t') - X_n(t'') \mid$$

for all $n \leq -1$. Then we have:

$$\Psi_{C,n} \leq \max_{1 \leq j \leq m} \sup_{t', t'' \in D_j} (\mid X_n(t') \mid + \mid X_n(t'') \mid) \leq 2 \parallel X_n \parallel_D$$

for all $n \leq -1$. Since by (ii) the family $\{ \parallel X_n \parallel_D , \mathcal{F}_n \mid -\infty < n \leq n_0 \}$ forms a reversed submartingale obviously satisfying $\inf_{n \leq n_0} E \parallel X_n \parallel_D > -\infty$ for $n_0 \leq -1$, then $\{ \parallel X_n \parallel_D \mid n \leq n_0 \}$ is uniformly integrable, and therefore the family of random variables:

$$\{ \ \Psi_{C,n} \ \mid \ C \in \Gamma(D) \ , \ n \leq n_0 \ \}$$

is uniformly integrable. Thus for given $\varepsilon > 0$, there is $0 < \delta < \varepsilon/2$ satisfying:

(2) For every $F \in \mathcal{F}$ satisfying $P(F) < \delta$ we have:

$$\int_F \Psi_{C,n} \ dP \ < \ \varepsilon/2$$

for all $C \in \Gamma(D)$ and all $n \leq n_0$.

And for such $\delta > 0$, using the fact that L is bounded on D , we can find a finite cover $\gamma = \{ \ D_1, \dots, D_{m_\varepsilon} \ \} \in \Gamma(D)$ such that:

$$\mid L(t') - L(t'') \mid \ < \ \delta/2$$

for all $t', t'' \in D_j$ and all $j = 1, \dots, m_\varepsilon$. Let us note that:

$$\mid X_n(t') - X_n(t'') \mid \ \leq \ \mid X_n(t') - L(t') \mid + \mid L(t') - L(t'') \mid + \mid X_n(t'') - L(t'') \mid$$

for all $t', t'' \in D_j$ and all $j = 1, \dots, m_\varepsilon$, so taking supremum over all $t', t'' \in D_j$ and maximum over all $j = 1, \dots, m_\varepsilon$ we may deduce:

$$\Psi_{C,n} = \max_{1 \leq j \leq m_\varepsilon} \ \sup_{t', t'' \in D_j} \ \mid X_n(t') - X_n(t'') \mid \ \leq \ 2 \parallel X_n - L \parallel_D + \ \delta/2$$

Hence by (1) we have:

$$P\{ \ \Psi_{C,n} > \delta \ \} \leq P\{ \ \parallel X_n - L \parallel_D > \delta/4 \ \} < \delta$$

for all $n \leq n_\varepsilon$ with some $n_\varepsilon \leq -1$. Therefore by (2) we may conclude:

$$E(\Psi_{C,n}) = E(\Psi_{C,n} \cdot 1_{\{ \ \Psi_{C,n} \leq \delta \ \}}) + E(\Psi_{C,n} \cdot 1_{\{ \ \Psi_{C,n} > \delta \ \}}) \leq \delta + \varepsilon/2 < \varepsilon$$

for all $n \leq n_\varepsilon$. The proof now follows easily by putting:

$$\Psi_j = \sup_{t', t'' \in D_j} \ \mid X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') \mid$$

for all $j = 1, \dots, m_\varepsilon$.

□

Theorem 4.11

Let $(\{ \ X_n(t) \ , \ \mathcal{F}_n \mid -\infty < n \leq -1 \ \} \ \mid \ t \in T \ )$ *be a family of reversed martingales indexed by a set* T *, let* D *be a countable subset of* T *, let* L *be a map from* T *into* $\mathbf{R}$ *, and suppose that the following two conditions are satisfied:*

(i) $X_n(t) \rightarrow L(t) \ \ (a.s.)$ *, as* $n \rightarrow -\infty$, $\forall t \in T$

(ii) $\inf_{n \leq -1} E \parallel X_n \parallel_D < \infty$.

Then the following three statements are equivalent:

(1) $\forall\varepsilon > 0$, $\exists n_\varepsilon \le -1$, $\exists\gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ *and* $\exists\Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(i) $| X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') | \le \Psi_j$, $\forall t', t'' \in D_j$, $\forall j = 1, \dots, m_\varepsilon$

(ii) $P\{ \max_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} < \varepsilon$

(2) $\forall\varepsilon > 0$, $\exists n_\varepsilon \le -1$, $\exists\gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ *and* $\exists\Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ *satisfying:*

(i) $| X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') | \le \Psi_j$, $\forall t', t'' \in D_j$, $\forall j = 1, \dots, m_\varepsilon$

(ii) $E(\max_{1 \le j \le m_\varepsilon} \Psi_j) < \varepsilon$

(3) *The family of reversed submartingales:*

$$(\{| X_n(t) - L(t) |, \mathcal{F}_n \mid -\infty < n \le -1\} \mid t \in T)$$

is totally bounded in P-mean relative to D .

Proof. The implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ follow by proposition 4.9 and proposition 4.10, respectively. In order to prove the implication $(2) \Rightarrow (1)$ take $\varepsilon > 0$, then there exists $n_\varepsilon \le -1$, $\gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(D)$ and $\Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying (i) in (2), and:

$$E(\max_{1 \le j \le m_\varepsilon} \Psi_j) < \varepsilon^2 .$$

Hence by Markov's inequality we may conclude:

$$P\{ \max_{1 \le j \le m_\varepsilon} E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} \le P\{ E\{ \max_{1 \le j \le m_\varepsilon} \Psi_j \mid \mathcal{F}_{-\infty} \} > \varepsilon \} \le \frac{1}{\varepsilon} E(\max_{1 \le j \le m_\varepsilon} \Psi_j) < \varepsilon .$$

Thus (i) and (ii) in (1) are fulfilled, and the proof is complete.

□

Remark 4.12 Under the hypotheses in Theorem 4.11, suppose that the σ-algebra $\mathcal{F}_{-\infty}$ is degenerated, i.e. $P(F) \in \{0, 1\}$, $\forall F \in \mathcal{F}_{-\infty}$. Then $E\{ \Psi_j \mid \mathcal{F}_{-\infty} \} = E(\Psi_j)$, for all $j = 1, \dots, m_\varepsilon$, and condition (ii) in (1) is equivalent to the following condition:

(1) $$\max_{1 \le j \le m_\varepsilon} E(\Psi_j) \le \varepsilon$$

provided that $\varepsilon \le 1$.

We turn to applications of the preceding results in the next example. In this paper we only present those that concern the uniform law of large numbers in a straightforward way. For further applications in statistics we shall refer the reader to [13].

Example 4.13

Let $(S, \mathcal{A}, \pi)$ be a probability space, let $(\Omega, \mathcal{F}, P) = (S^{\mathbf{N}}, \mathcal{A}^{\mathbf{N}}, \pi^{\mathbf{N}})$ be its countable product, and let $\xi_n(\omega) = \omega_n$ for $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ and $n \geq 1$. Then $\{ \xi_n \mid n \geq 1 \}$ is a sequence of independent identically distributed random functions defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{A})$ and with a common distribution law π. Let T be a set, and let $f : S \times T \to \mathbf{R}$ be an arbitrary function. Then one can find in [5] that the following five statements are equivalent:

(1) The map f belongs to the space $LLN_0(B(T), \pi)$, or equivalently there exists $m \in B(T)$ such that:

$$\| \, m - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \, \|_T \longrightarrow 0 \quad (a.s.) \ \text{ and } \ (P^*) \ , \text{ as } \ n \to \infty$$

(2) $\int^* \| \, f(\xi_1) \, \|_T \ dP < \infty$ and there exists $m \in B(T)$ such that:

$$\| \ m - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \ \|_T \longrightarrow 0 \quad (P^*) \ , \text{ as } \ n \to \infty$$

(3) There exists $m \in B(T)$ such that:

$$\| \ m - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \ \|_T \longrightarrow 0 \quad (a.s.)^* \ , \text{ as } \ n \to \infty$$

(4) There exists $m \in B(T)$ such that:

$$\| \ m - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \ \|_T \longrightarrow 0 \quad (L^1)^* \ , \text{ as } \ n \to \infty$$

(5) The map f is *eventually totally bounded in π-mean*, or equivalently the following three conditions are satisfied:

(i) The map $s \mapsto f(s,t)$ is π-measurable, $\forall t \in T$

(ii) $\int^* \| \, f(s) \, \|_T \ \pi(ds) \ < \infty$

(iii) For each $\varepsilon > 0$, there exists $\gamma = \{ \, D_1, \dots, D_{m_\varepsilon} \, \} \in \Gamma(T)$ such that:

$$\inf_{n \geq 1} E^*(\sup_{t', t'' \in D_j} \ | \, \frac{1}{n} \sum_{i=1}^{n} f(t', \xi_i) - \frac{1}{n} \sum_{i=1}^{n} f(t'', \xi_i) \, | \,) < \varepsilon \quad , \quad \forall j = 1, \dots, m_\varepsilon \ .$$

In this case we necessarily have:

(6) $f(\,\cdot\,, t) \in L^1(\pi)$, for all $t \in T$

(7) If $M(t) = \int_S f(s,t) \ \pi(ds)$ is the π-mean function associated to f, then $M \in B(T)$ and:

$$\| \ M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \ \|_T \longrightarrow 0 \quad (c) \ , \text{ as } \ n \to \infty$$

where (c) denotes either of the following convergence notions: $(a.s.), (a.s.)^*, (P^*), (P_*), (L^1)^*$, $(L^1)_*$. Also to prove the implication $(1) \Rightarrow (2)$ the following statement is established in [5]:

(8) If $\| \; m - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) \; \|_T \longrightarrow 0 \;\; (a.s.)$, as $n \to \infty$, for some $m \in B(T)$, then $\int^* \| \, f(\xi_1) \, \|_T \;\, dP < \infty$.

Let us put $X_{-n}(t) = (1/n) \sum_{j=1}^n f(\xi_j, t)$ for all $n \geq 1$ and all $t \in T$, and suppose that (6) holds. Then $(\{ \; X_n(t) \;,\; \mathcal{F}_n \mid -\infty < n \leq -1 \; \} \mid t \in T \;)$ is a family of reversed martingales indexed by T , where $\mathcal{F}_n$ is the permutation invariant σ-algebra of order $-n$ based on the random function $\xi = (\xi_1, \xi_2, \, \dots \,)$ for $n \leq -1$, see [1] . Consequently we can apply Theorem 4.1 and Theorem 4.3 together with Theorem 4.7 and Theorem 4.11, respectively, as well as their consequences in Corollary 4.2 and Corollary 4.4. Furthermore, let us note that for the converse statement in Theorem 4.3 it is enough to require $L = M \in B(T)$ in order that:

$$\| \, X_{-n} - L \, \|_T = \| \, M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \, \|_T \longrightarrow 0 \;\; (a.s.) \; , \text{ as } \; n \to \infty$$

implies the validity of (ii) in Theorem 4.3 for each countable D subset of T . Indeed, in this case we have:

$$\| \, X_{-n} - L \, \|_D = \| \, M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \, \|_D \longrightarrow 0 \;\; (P) \;\; , \text{ as } \; n \to \infty$$

and by (8) we find:

$$\inf_{n \geq 1} E \, \| \, X_{-n} - L \, \|_D = \inf_{n \geq 1} E \, \| \, M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \, \|_D \leq$$

$$\leq \; \| \, M \, \|_T + \int^* \| \, f(\xi_1) \, \|_T \;\, dP < \infty \; .$$

Hence the above statement follows by Theorem 3.1. Also note that our Corollary 4.4 extends the implications among (1)-(5) to the general reversed martingale case. Moreover, by (8) and Corollary 4.4 we may easily conclude, if $L = M \in \mathcal{C}(T) \cap B(T)$, or T is hereditarily separable and $L \in B(T)$, and if the following two conditions are satisfied:

$$\| \, X_{-n} - L \, \|_T = \| \, M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \, \|_T \longrightarrow 0 \;\; (a.s.) \;\; , \text{ as } \; n \to \infty$$

$$\Delta(X_{-n}) \to 0 \;\; (c) \; , \text{ as } \; n \to \infty \; ,$$

then we have:

$$\| \, X_{-n} - L \, \|_T = \| \, M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \, \|_T \longrightarrow 0 \;\; (c)$$

as $n \to \infty$, where (c) denotes either of the following convergence notions: $(a.s.)^*, (P^*)$, $(L^1)^*, (L^1)_*$. In particular, if T is hereditarily separable, $L \in B(T)$, $X_{-n}(\omega, \cdot) \in \mathcal{C}(T)$ for $a.s.\ \omega \in \Omega$ and all $n \geq 1$, and:

$$\| X_{-n} - L \|_T = \| M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \|_T \longrightarrow 0 \ \ (a.s.) \ \text{, as } n \to \infty \text{ ,}$$

then we have:

$$\| X_{-n} - L \|_T = \| M - \frac{1}{n} \sum_{j=1}^{n} f(\xi_j, t) \|_T \longrightarrow 0 \quad (a.s.)^* \ \text{ and } \ (L^1)^*$$

as $n \to \infty$. Finally, let us note that condition (iii) in (5) can be reformulated in the following way:

(9) $\forall \varepsilon > 0$, $\exists n_\varepsilon \leq -1$, $\exists \gamma = \{ D_1, \dots, D_{m_\varepsilon} \} \in \Gamma(T)$ and $\Psi_1, \dots, \Psi_{m_\varepsilon} \in L^1(P)$ satisfying:

(i) $| X_{n_\varepsilon}(t') - X_{n_\varepsilon}(t'') | \leq \Psi_j$, $\forall t', t'' \in D_j$, $\forall j = 1, \dots, m_\varepsilon$

(ii) $\max_{1 \leq j \leq m_\varepsilon} E(\Psi_j) < \varepsilon$.

Moreover, by the Hewitt-Savage 0-1 law the σ-algebra $\mathcal{F}_{-\infty} = \bigcap_{n=1}^{\infty} \mathcal{F}_{-n}$ is degenerated, see [1]. By using Remark 4.12 hence we could in fact observe that Theorem 4.11 examines the eventually total boundedness in the mean condition for general families of reversed martingales, as well as Theorem 4.7 and Remark 4.8 its reflection to families of reversed submartingales.

Example 4.14

Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$, let $T = [0,1]$ with Euclidean topology, and let $\{ Z_n \ , \ n \geq 1 \}$ be an arbitrary sequence of non-negative random elements defined on Ω . Let us define $X_{-n}(t, \omega) = Z_n(\omega) \cdot 1_t(\omega)$ and $X(t, \omega) = 0$, for all $\omega \in \Omega$, all $t \in T$ and all $n \geq 1$. Then the family:

$$\{ X_n(t) \ , \ \mathcal{F}_n \ \mid \ -\infty < n \leq -1 \}$$

forms a reversed martingale, for all $t \in T$, where $\{ \mathcal{F}_n \ , \ n \leq -1 \}$ may be an arbitrary increasing sequence of σ-algebras on Ω included in $\mathcal{F}$. Clearly, we have:

$$\| X_{-n}(\omega) - X(\omega) \|_T = \| (X_{-n}(\omega) - X(\omega))^+ \|_T =$$

$$= \Delta(X_{-n}(\omega)) \ = \ \Delta^+(X_{-n}(\omega)) = Z_n(\omega)$$

for all $\omega \in \Omega$ and all $n \geq 1$. Hence we may conclude that inequality (vi) given in Theorem 4.1 and Theorem 4.3 can not be improved in general.

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