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On the Russian Option: The Expected Waiting Time

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In the context of the Russian option of L. Shepp and A. N. Shiryaev, we present a new derivation of the solution for the underlying one-dimensional optimal stopping problem. Our method is not based on the smooth pasting guess and Itô formula, but only uses the strong Markov property. In addition, the exact formula is given for the expected waiting time of the optimal stopping strategy. Two different methods for this computation are presented. Both methods can be easily generalized to treat similar problems for general one-dimensional time-homogeneous diffusions.

1. Introduction

In this section we shall introduce the setting and recall basic facts on the Russian option as proposed by L. Shepp and A. N. Shiryaev in [5]. This will be complemented in Section 3 where the exact formula is found for the expected waiting time of the optimal stopping strategy.

For the Russian option [5] the fluctuations in the price of an asset are assumed to follow the Black-Scholes model (see [1] and [2]). Thus they are given by the geometric Brownian motion:

$$X_t = x \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right) \;,\;\; t \geq 0 \tag{1.1}$$

where $B = (B_t)_{t \geq 0}$ is standard Brownian motion on $(\Omega, \mathcal{F}, P)$ started at 0 under P , and $x > 0$ is given and fixed. The parameters $\mu \in \mathbf{R}$ (drift) and $\sigma > 0$ (volatility) are assumed to be known. The Russian option is a financial option of American type with the value (fair price):

$$V_*(x,s) = \sup_{\tau} E_{x,s}\left(e^{-r\tau} S_\tau\right) \tag{1.2}$$

where $S = (S_t)_{t \geq 0}$ is the maximum process for $X = (X_t)_{t \geq 0}$ started at $s \geq x > 0$ under P :

$$S_t = s \vee \max_{0 \leq u \leq t} X_u \tag{1.3}$$

while $r > 0$ (the interest rate) is given and fixed. The supremum in (1.2) is taken over all stopping times $\tau < \infty$ for X . It is crucial that no bound is imposed on τ . The exponential discounting $D = \left(e^{-rt}\right)_{t \geq 0}$ is applied, and it is shown in [5] that $V_*(x,s) < \infty$ if and only if $r > \mu$. In this case the explicit formula is given in [5] for the value of the Russian option:

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$$(1.4) \qquad V_*(x,s) = \frac{s}{\gamma_2 - \gamma_1}\left(\gamma_2\left(\frac{\alpha x}{s}\right)^{\gamma_1} - \gamma_1\left(\frac{\alpha x}{s}\right)^{\gamma_2}\right) \quad , \quad \text{if} \;\; \tfrac{s}{\alpha} \le x \le s$$

$$= s \quad , \quad \text{if} \;\; 0 < x \le \tfrac{s}{\alpha}$$

where the number $\alpha > 1$ is given by:

$$(1.5) \qquad \alpha = \left(\frac{1 - 1/\gamma_1}{1 - 1/\gamma_2}\right)^{1/(\gamma_2 - \gamma_1)}$$

with $\gamma_1 < 0$ and $\gamma_2 > 1$ being the two roots of the quadratic equation:

$$(1.6) \qquad \tfrac{\sigma^2}{2}\,\gamma^2 + \left(\mu - \tfrac{\sigma^2}{2}\right)\gamma - r = 0 \; .$$

The optimal stopping strategy (stopping time at which supremum in (1.2) is attained) is given by:

$$(1.7) \qquad \tau_* = \inf\,\Big\{\, t > 0 \;\Big|\; X_t \le S_t/\alpha \,\Big\} \; .$$

The Russian option (1.2) is characterized by “reduced regret” because the owner is paid the maximum asset price up to the time of exercise, and hence feels less remorse at not having exercised earlier. It should be mentioned that the results (1.4) and (1.7) in [5] were in essence derived (guessed) by applying the so-called principle of smooth fit (smooth pasting). For more information on this condition we shall refer the reader to [3] (see Remark on page 238). In [6] a different approach was taken where the key idea was to introduce a dual martingale measure which enables one to reduce the “two-dimensional” optimal stopping problem to a “one-dimensional” one. We begin next section by presenting the essential argument in this direction. This is followed by an elementary proof for the solution of the one-dimensional optimal stopping problem. While the proof in [6] is also based on the smooth pasting guess, which is then verified by application of Itô formula, the proof presented below is direct and is only based on the strong Markov property. In the third section we compute the expectation of the optimal stopping strategy. For the purpose of comparison this is done in two different ways. Both of these methods can be easily generalized to treat similar problems for general one-dimensional time-homogeneous diffusions.

2. The Russian option

We begin this section by displaying the key idea behind the introduction of a dual martingale measure [6]. To do so we shall recall that $X = (X_t)_{t \ge 0}$ denotes geometric Brownian motion given as in (1.1), while the maximum process $S = (S_t)_{t \ge 0}$ is defined by (1.3), where $s \ge x > 0$ are given and fixed. Having (1.2) on mind we may write:

$$(2.1) \qquad E_{x,s}\Big(e^{-r\tau} S_\tau\Big) = E_{x,s}\left(e^{-r\tau}\,\frac{S_\tau}{X_\tau}\,X_\tau\right) = x\,E_{x,s}\left(e^{-(r-\mu)\tau}\,\frac{S_\tau}{X_\tau}\exp\left(\sigma B_\tau - \frac{\sigma^2}{2}\,\tau\right)\right) \; .$$

This indicates the change of measure:

(2.2) $$d\tilde{P} = \exp\left(\sigma B_t - \frac{\sigma^2}{2}\,t\right) dP\ .$$

Then by Girsanov's theorem [4] we know that $\tilde{B} = (\tilde{B}_t)_{t \ge 0} := (B_t - \sigma t)_{t \ge 0}$ is a Wiener process under $\tilde{P}$. Note that:

(2.3) $$X_t = x\,\exp\left(\sigma \tilde{B}_t + \left(\tilde{\mu} - \tfrac{\sigma^2}{2}\right) t\right)\ ,\ \ t \ge 0$$

with $\tilde{\mu} = \mu + \sigma^2$, and compare with (1.1). Set $\tilde{r} = r - \mu$, and consider the process:

(2.4) $$Z_t = \frac{S_t}{X_t}\ \ ,\ \ t \ge 0\ .$$

Then from (1.2) with (2.1)+(2.2) we easily find:

(2.5) $$V_*(x,s) = x \cdot \sup_\tau \tilde{E}_{s/x}\left(e^{-\tilde{r}\tau} Z_\tau\right)$$

where the supremum is taken over all stopping times $\tau < \infty$ for Z . Thus the problem (1.2) is reduced to the one-dimensional optimal stopping problem (2.5). A successful treatment of the latter is provided by the crucial fact that $Z = (Z_t)_{t \ge 0}$ is a diffusion process in $[1, \infty[$. In fact, by Itô's formula we easily find that the infinitesimal operator of Z in $]1, \infty[$ equals:

(2.6) $$\mathbf{L_Z} = \frac{\sigma^2}{2}\,z^2 \frac{\partial^2}{\partial z^2} - \mu z \frac{\partial}{\partial z}$$

while the boundary point 1 is a point of the instantaneous reflection.

In order to compute the payoff in (2.5), we shall consider the discounted optimal stopping problem for the diffusion Z described by the gain function $g(z) = z$ for $z \ge 1$ and discounting factor $\tilde{r} > 0$. The payoff for this problem is thus defined by:

(2.7) $$W_*(z) = \sup_\tau \tilde{E}_z\left(e^{-\tilde{r}\tau} Z_\tau\right)$$

for $z \ge 1$, where the supremum is taken over all stopping times $\tau < \infty$ for Z . Note that:

(2.8) $$V_*(x,s) = x\,W_*(s/x)$$

for $s \ge x > 0$.

The optimal stopping problem (2.7) is very pleasant, due to the simple structure lying behind. For the purpose of comparison we shall first recall how solution can be guessed by standard arguments based upon smooth pasting. Below, however, we present a completely different proof which does not uses this guess and only relies upon the strong Markov property. This proof is based on the idea that the optimal stopping time is to be the hitting time of a point (or closed interval) by Z , and our main aim in the proof is to reduce the supremum over all stopping times in (2.7) to the supremum over all hitting times. Such an approach is in accordance with the general exessive characterization of the payoff (see [7], Corollary 3, p.129), and often is proven useful in the treatment of the one-dimensional problems.

The smooth pasting guess: Assuming that an optimal stopping strategy in (2.7) exists, and that is the first hitting time of the closed set $[z_*, \infty[$ by the process Z, we are naturally led to consider the following one-dimensional Stephan problem (with moving boundary). Find $z_* \in [1, \infty[$ and $W_* : [1, z_*] \rightarrow \mathbf{R}$, such that:

$$\mathbf{L_Z} W_*(z) = \tilde{r}\, W_*(z) \quad (1 < z < z_*) \tag{2.9}$$

$$W_*(z_*) = z_* \tag{2.10}$$

$$\frac{\partial W_*}{\partial z}(1+) = 0 \quad (\text{normal reflection}) \tag{2.11}$$

$$\frac{\partial W_*}{\partial z}(z_*-) = 1 \quad (\text{smooth pasting}). \tag{2.12}$$

The general solution of the equation in (2.9) is given by the formula:

$$z \mapsto a z^{\gamma_1^*} + b z^{\gamma_2^*} \quad (a, b \in \mathbf{R}) \tag{2.13}$$

where $\gamma_1^* < 0$ and $\gamma_2^* > 1$ are the two roots of the quadratic equation:

$$\frac{\sigma^2}{2}\, \gamma^2 - \Big(\mu + \frac{\sigma^2}{2}\Big)\gamma - (r - \mu) = 0 \ . \tag{2.14}$$

We note that the conditions (2.10)-(2.12) determine $z_* \in]1, \infty[$ and $z \mapsto W_*(z)$ of the form in (2.13) uniquely. They are given by the formulas:

$$z_* = \left(\frac{1 - 1/\gamma_1^*}{1 - 1/\gamma_2^*} \right)^{1/(\gamma_2^* - \gamma_1^*)} \tag{2.15}$$

$$W_*(z) = \frac{z_*}{\gamma_2^* - \gamma_1^*} \left(\big(\gamma_2^* - 1\big) \Big(\frac{z}{z_*}\Big)^{\gamma_1^*} - \big(\gamma_1^* - 1\big) \Big(\frac{z}{z_*}\Big)^{\gamma_2^*} \right) \tag{2.16}$$

for $z \in [1, z_*]$. Hence by (2.8) we may guess the following result of L. Shepp and A. N. Shiryaev [5]. Our proof below does not follow this line, but is based on another idea as explained above.

Theorem 3.1

The value of the Russian option defined by (1.2) is given by the formula:

$$V_*(x, s) = \frac{z_*}{\gamma_2^* - \gamma_1^*}\, x \left(\big(\gamma_2^* - 1\big) \Big(\frac{s}{z_* \, x}\Big)^{\gamma_1^*} - \big(\gamma_1^* - 1\big) \Big(\frac{s}{z_* \, x}\Big)^{\gamma_2^*} \right) , \ \textit{if} \ \frac{s}{z_*} \leq x \leq s \tag{2.17}$$

$$= s \ , \ \textit{if} \ 0 < x \leq \frac{s}{z_*} \ .$$

The optimal stopping strategy in (1.2) is given by:

$$\tau_* = \inf \Big\{ t > 0 \ \Big| \ S_t / X_t \geq z_* \Big\} \ . \tag{2.18}$$

(Note that $\gamma_{1,2}^* = 1 - \gamma_{1,2}$, where $\gamma_{1,2}$ solve (1.6). Thus z_* in (2.15) equals α in (1.5).)

Proof: Consider the hitting time:

$$\tau_v = \inf \left\{ t > 0 \mid Z_t = v \right\} \tag{2.19}$$

where $1 \le z \le v < \infty$ and Z starts at z . Consider the function:

$$H(z) = H(z;v) = \tilde{E}_z\big(e^{-\tilde{r}\tau_v}\big) \tag{2.20}$$

By the general Markov process theory we know that $H(z)$ solves uniquely the system:

$$\mathbf{L_Z} H(z) = \tilde{r}\, H(z) \quad (\ 1 < z < v\ ) \tag{2.21}$$

$$H(v) = 1\ ;\ H'(1+) = 0\ . \tag{2.22}$$

This gives the explicit formula:

$$H(z) = \frac{\gamma_2^*}{\gamma_2^*\, v^{\gamma_1^*} - \gamma_1^*\, v^{\gamma_2^*}}\ z^{\gamma_1^*} \ -\ \frac{\gamma_1^*}{\gamma_2^*\, v^{\gamma_1^*} - \gamma_1^*\, v^{\gamma_2^*}}\ z^{\gamma_2^*}\ . \tag{2.23}$$

Hence we easily deduce that the function:

$$\psi(v) = \tilde{E}_1\Big(e^{-\tilde{r}\tau_v} Z_{\tau_v}\Big) = v\, \tilde{E}_1\Big(e^{-\tilde{r}\tau_v}\Big) = vH(1;v) \tag{2.24}$$

has a unique maximum point z_* on $[1,\infty[$ given by (2.15).

In addition introduce the following notation:

$$W(z;\tau) = \tilde{E}_z\Big(e^{-\tilde{r}\tau} Z_\tau\Big)\ . \tag{2.25}$$

Thus $W_*(z)$ in (2.7) is given by:

$$W_*(z) = \sup_\tau\ W(z;\tau)\ . \tag{2.26}$$

To show that $W_*(z) < \infty$ for $1 \le z < \infty$ we could truncate the underlying gain function $g(z) = z$ by setting $g_n(z) = g(z) \wedge n$, pass through the proof which follows with $g_n(z)$ instead of $g(z)$, and in the end let n tend to infinity. For simplicity, we shall work with $g(z)$.

Having z_* as the unique maximum point of $\psi(v)$ in (2.24), to solve the optimal stopping problem (2.26) we shall prove the following two facts:

$$W_*(z) = W_*(z_*)\, \tilde{E}_z\Big(e^{-\tilde{r}\tau_{z_*}}\Big) \quad (\ 1 \le z \le z_*\ ) \tag{2.27}$$

$$W_*(z_*) = z_*\ . \tag{2.28}$$

Before turning to their proofs, let us show how this might be used to end the proof.

So, assume that (2.27) and (2.28) are valid. Then by using strong Markov property, and the supermartingale property of the process $(e^{-\tilde{r}t} Z_t)_{t\ge 0}$ until Z hits 1 , we may write:

(2.29)
$$W(z;\tau) = \tilde{E}_z\Big(e^{-\tilde{r}\tau} Z_\tau \ ; \ \tau \le \tau_{z_*}\Big) + \tilde{E}_z\Big(e^{-\tilde{r}\tau} Z_\tau \ ; \ \tau > \tau_{z_*}\Big)$$
$$\le \tilde{E}_z\Big(e^{-\tilde{r}(\tau\wedge\tau_{z_*})} Z_{\tau\wedge\tau_{z_*}} \ ; \ \tau \le \tau_{z_*}\Big) + W_*(z_*)\, \tilde{E}_z\Big(e^{-\tilde{r}(\tau\wedge\tau_{z_*})} \ ; \ \tau > \tau_{z_*}\Big)$$
$$= \tilde{E}_z\Big(e^{-\tilde{r}(\tau\wedge\tau_{z_*})} Z_{\tau\wedge\tau_{z_*}}\Big) \le z$$

for any stopping time $\tau < \infty$ for Z and all $z \ge z_*$. This clearly shows that:

(2.30)
$$W_*(z) = z \quad (\ z_* \le z < \infty\ )$$

which taken together with (2.27)+(2.28) implies that the stopping time:

(2.31)
$$\tau_* = \inf\ \big\{\ t > 0 \mid Z_t \ge z_*\ \big\}$$

is the optimal stopping strategy in (2.26). Inserting $H(z;z_*)$ from (2.23) in (2.27), and using (2.28) and (2.30), we find the explicit formula for the function $W_*(z)$ in (2.26) on $[1,\infty[$ which is easily verified to satisfy (2.12), as well as (2.9)-(2.11). Thus, on $[1,z_*]$, it must be equal to the function given in (2.16). In other words, this proves our guess above and establishes (2.17) and (2.18). Thus the proof will be complete as soon as we prove (2.27) and (2.28).

Proof of (2.27): Let $1 \le z < z_*$ be given. Then by strong Markov property we find:

(2.32)
$$\tilde{E}_1\Big(e^{-\tilde{r}\tau_{z_*}} Z_{\tau_{z_*}}\Big) = \tilde{E}_z\Big(e^{-\tilde{r}\tau_{z_*}} Z_{\tau_{z_*}}\Big)\tilde{E}_1\Big(e^{-\tilde{r}\tau_z}\Big)$$
$$= z^{-1} W(z;\tau_{z_*})\, z\tilde{E}_1\Big(e^{-\tilde{r}\tau_z}\Big)\ .$$

This by definition of z_* clearly shows that $W(z;\tau_{z_*}) > z$ for $1 \le z < z_*$. Now, for any stopping time τ for Z given, define:

(2.33)
$$\tilde{\tau} = \begin{cases} \tau & ,\ \text{if}\ \ Z_\tau \ge z_* \\ \tau + \tau_{z_*} \circ \theta_\tau & ,\ \text{if}\ \ Z_\tau < z_*\ . \end{cases}$$

Then $\tilde{\tau}$ is a stopping time for Z, and by strong Markov property and the fact just deduced we get:

(2.34)
$$W(z;\tau) \le W(z,\tilde{\tau})$$

for all $z \ge 1$. Since $\tilde{\tau} \ge \tau_{z_*}$, hence we find:

(2.35)
$$W_*(z) = \sup_{\tau \ge \tau_{z_*}} W(z;\tau) \quad (\ 1 \le z \le z_*\ )$$

from where (2.27) follows by strong Markov property.

Proof of (2.28): Let $\tilde{W}_*(z)$ denote the unique function on $[1,\infty[$ which satisfies:

(2.36)
$$\mathbf{L_Z}\tilde{W}_*(z) = \tilde{r}\, \tilde{W}_*(z) \quad (\ 1 < z < \infty\ )$$

(2.37)
$$\tilde{W}_*(1) = W_*(1)\ ;\ \tilde{W}_*'(1+) = 0\ .$$

Then $\tilde{W}_*(z) = az^{\gamma_1^*} + bz^{\gamma_2^*}$ for some $a\,,\, b \in \mathbf{R}$ with $\gamma_1^* < 0$ and $\gamma_2^* > 1$ solving (2.14). Now, there could be the two possibilities:

(2.38) $$\tilde{W}_*(z) > z \quad \text{for all} \quad z \ge 1\ ;$$

(2.39) $$\exists\, \tilde{z} \ge 1 \quad \text{such that} \quad \tilde{W}_*(\tilde{z}) = \tilde{z}\ .$$

We shall first prove that (2.38) fails. For this, suppose that (2.38) holds. Since $\gamma_2^* > 1$, there exists $0 < r < 1$ such that:

(2.40) $$\frac{z}{\tilde{W}_*(z)} \le r$$

for all $z \ge 1$. By the martingale property of $(e^{-\tilde{r}t}\tilde{W}_*(Z_t))_{t\ge 0}$ under $\tilde{P}_1$, hence we easily get:

(2.41) $$\begin{aligned} W_*(1) &= \sup_\tau W(1;\tau) = \sup_\tau \tilde{E}_1\Big(e^{-\tilde{r}\tau} Z_\tau\Big) \\ &\le r\, \sup_\tau \tilde{E}_1\Big(e^{-\tilde{r}\tau}\tilde{W}_*(Z_\tau)\Big) \\ &= r\, \tilde{W}_*(1) = r\, W_*(1) \end{aligned}$$

which contradicts the fact that $W_*(1) < \infty$. Thus (2.38) fails, and (2.39) must be satisfied.

Finally, since $(e^{-\tilde{r}t}\tilde{W}_*(Z_t))_{t\ge 0}$ is a martingale under $\tilde{P}_1$, we have:

(2.42) $$\tilde{W}_*(1) = \tilde{E}_1\Big(e^{-\tilde{r}\tau}\tilde{W}_*(Z_\tau)\Big)$$

whenever $\tau \le \tau_{\tilde{z}}$ is a stopping time for Z . Therefore by (2.37) and (2.39) we find:

(2.43) $$\begin{aligned} \tilde{z}\, \tilde{E}_1\Big(e^{-\tilde{r}\tau_{\tilde{z}}}\Big) &= \tilde{E}_1\Big(e^{-\tilde{r}\tau_{\tilde{z}}}\, \tilde{W}_*(Z_{\tau_{\tilde{z}}})\Big) = \tilde{W}_*(1) \\ &= W_*(1) \ge \tilde{E}_1\Big(e^{-\tilde{r}\tau_{z_*}}\, Z_{\tau_{z_*}}\Big) = z_*\, \tilde{E}_1\Big(e^{-\tilde{r}\tau_{z_*}}\Big)\ . \end{aligned}$$

This by definition of z_* implies $\tilde{z} = z_*$, and hence (2.28) follows by using (2.27) . The proof of the theorem is complete. □

3. The expected waiting time

In the setting for the Russian option in Section 1, consider the expected waiting time of the optimal stopping strategy:

(3.1) $$m(x,s) = E_{x,s}\big(\tau_*\big)$$

with τ_* given in (1.7) or (2.18). Our main goal in this section is to find an explicit formula for $m(x,s)$. For the purpose of comparison, this will be done in two different ways.

The first approach is two-dimensional, and a "missing condition" is compensated by the observation that $m(x,s)$ must be constant on the diagonal. This determines $m(x,s)$ uniquely.

(Similarly to the present case, the case $\alpha = 1$ in [3] contains an extra information, since it is possible to compute the value $m(s_*, s_*)$ from its very definition, or just argue that $m(x,s)$ is constant on the diagonal.) The second approach is one-dimensional and relies upon the ideas used in the proof of Theorem 3.1. Both approaches are simple and can be easily generalized to treat similar problems for general one-dimensional time-homogeneous diffusions.

Theorem 4.1

The expected waiting time of the optimal stopping strategy (1.7) for the Russian option (1.2) is given by the formula:

$$(3.2)\qquad m(x,s) = \frac{2}{\sigma^2\Delta^2}\left(\frac{1}{s^\Delta}\left(\left(\frac{s}{\alpha}\right)^\Delta - x^\Delta\right) - \left(\log\left(\frac{s}{\alpha}\right)^\Delta - \log x^\Delta\right)\right) \quad \textit{if} \quad \Delta := 1 - \tfrac{2\mu}{\sigma^2} \neq 0$$

$$= \frac{1}{\sigma^2}\left(\log s\left(\log x^2 - \log\left(\frac{s}{\alpha}\right)^2\right) - \left(\log^2 x - \log^2\left(\frac{s}{\alpha}\right)\right)\right) \quad \textit{if} \quad 1 - \tfrac{2\mu}{\sigma^2} = 0$$

whenever $\frac{s}{\alpha} \leq x \leq s$. (For $0 < x \leq \frac{s}{\alpha}$ we have $m(x,s) = 0$.)

Proof 1 (Guess): In order to determine a differential equation for $m(x,s)$, we should note that the process (X,S) being inside the region $s/\alpha < x < s$ (at the vertical level s) moves only horizontally (only the first coordinate changes). Thus its infinitesimal operator in this region coincides with the infinitesimal operator of the process X :

$$(3.3)\qquad \mathbf{L_X} = \frac{\sigma^2}{2}\, x^2 \frac{\partial^2}{\partial x^2} + \mu\, x\, \frac{\partial}{\partial x} \ .$$

Since the optimal stopping strategy τ_* from (1.7) might be viewed as the exit time of the "diffusion" (X,S) from an open set, by the general Markov process theory we know that $m(x,s)$ in the region $s/\alpha < x < s$ should satisfy the equation:

$$(3.4)\qquad \mathbf{L_X}\, m(x,s) = -1 \ .$$

In order to determine a solution from (3.4) uniquely, we should find some boundary conditions. One condition is obvious. If (X,S) starts from the stopping boundary, we must stop instantly:

$$(3.5)\qquad m(x,s)\Big|_{x=s/\alpha} = 0 \ .$$

The next condition is less obvious. This is the condition of normal reflection at the diagonal:

$$(3.6)\qquad \frac{\partial}{\partial s} m(x,s)\Big|_{x=s} = 0 \ .$$

Since this condition involves a derivative over s , the given conditions are not sufficient to determine a solution from (3.4) uniquely. To do this we shall need an additional information, which relies upon the observation:

$$m(x,s)\Big|_{x=s} = const. \tag{3.7}$$

This is clear by strong Markov property and the fact that the process $Z = S/X$ defined in (2.4) takes value 1 any time it hits diagonal. (As soon as we are at the diagonal, the game starts from the very beginning.)

In order to solve the equation (3.4) under conditions (3.5)-(3.7), we shall note that (3.4) with (3.3) is an equation of Cauchy type:

$$\frac{\sigma^2}{2} x^2 y'' + \mu\, x\, y' \,=\, -1 \ . \tag{3.8}$$

Substitution $x = e^t$ leads to the explicit solution. We may distinguish two cases.

Case 1: $\Delta := 1 - 2\mu/\sigma^2 \neq 0$. Then the general solution of (3.4) is given by:

$$m(x,s) \,=\, A(s)\, x^{\Delta} + B(s) + \frac{2}{\sigma^2 - 2\mu} \log x \ . \tag{3.9}$$

The conditions (3.5) and (3.6) get the following form respectively:

$$A(s) \left(\frac{s}{\alpha} \right)^{\Delta} + B(s) + \frac{2}{\sigma^2 - 2\mu} \log \left(\frac{s}{\alpha} \right) = 0 \tag{3.10}$$

$$s^{\Delta} A'(s) + B'(s) \,=\, 0 \ . \tag{3.11}$$

Solving (3.10) and (3.11) in $A(s)$ and $B(s)$, and inserting this into (3.9) gives:

$$m(x,s) = \left(\frac{C}{s^{\Delta/(1-\alpha^{\Delta})}} - \frac{2}{\sigma^2 \Delta^2} \frac{1}{s^{\Delta}} \right) \left(x^{\Delta} - \left(\frac{s}{\alpha} \right)^{\Delta} \right) + \frac{2}{\sigma^2 \Delta} \left(\log x - \log \left(\frac{s}{\alpha} \right) \right) \tag{3.12}$$

where $C \in \mathbf{R}$ is an undetermined constant. To determine C we can use (3.7), and this gives $C = 0$. Taking this into account in (3.12), we obtain (3.2) for $\Delta \neq 0$.

Case 2: $1 - 2\mu/\sigma^2 = 0$. Then the general solution of (3.4) is given by:

$$m(x,s) \,=\, A(s)\, \log x + B(s) - \frac{1}{\sigma^2} \log^2 x \ . \tag{3.13}$$

The conditions (3.5) and (3.6) get the following form respectively:

$$A(s) \log \left(\frac{s}{\alpha} \right) + B(s) - \frac{1}{\sigma^2} \log^2 \left(\frac{s}{\alpha} \right) = 0 \tag{3.14}$$

$$\big(\log s \big)\, A'(s) + B'(s) \,=\, 0 \ . \tag{3.15}$$

Solving (3.14) and (3.15) in $A(s)$ and $B(s)$, and inserting this into (3.13) gives:

$$m(x,s) = \left(C\, s^{1/\log \alpha} + \frac{2}{\sigma^2} \log s \right) \left(\log x - \log \left(\frac{s}{\alpha} \right) \right) - \frac{1}{\sigma^2} \left(\log^2 x - \log^2 \left(\frac{s}{\alpha} \right) \right) \tag{3.16}$$

where $C \in \mathbf{R}$ is an undetermined constant. To determine C we can use (3.7), and this again gives $C = 0$. Taking this into account in (3.16), we obtain (3.2) for $1-2\mu/\sigma^2 = 0$. This completes the first proof. $\square$

Proof 2: Suppose X starts at $x \in \,]s/\alpha, s[$, and consider the first exit time from this interval:

$$\tau = \inf \left\{ t > 0 \mid X_t \notin \,]s/\alpha, s[\right\} . \tag{3.17}$$

Then clearly $\tau_* \geq \tau$, and we may write:

$$\tau_* = \begin{cases} \tau & , \text{ if } X_\tau = s/\alpha \\ \tau + \tau_* \circ \theta_\tau & , \text{ if } X_\tau = s \ . \end{cases} \tag{3.18}$$

Hence by strong Markov property we get:

$$m(x,s) = E_{x,s}(\tau_*) = E_{x,s}(\tau) + P_{x,s}\big(X_\tau = s\big)\, m(s,s) \ . \tag{3.19}$$

Thus to compute $m(x,s)$ we need to find $G(x) = E_{x,s}(\tau)$, $H(x) = P_{x,s}\big(X_\tau = s\big)$ and $m(s,s)$. The first two functions are easily determined by the general Markov process theory. They solve the following systems respectively:

$$\mathbf{L_X} G(x) = -1 \quad (\ s/\alpha < x < s \ ) \tag{3.20}$$

$$G(s/\alpha) = G(s) = 0 \ ; \tag{3.21}$$

$$\mathbf{L_X} H(x) = 0 \quad (\ s/\alpha < x < s \ ) \tag{3.22}$$

$$H(s/\alpha) = 1 - H(s) = 0 \tag{3.23}$$

where $\mathbf{L_X}$ is the infinitesimal operator of X given in (3.3).

In order to determine $m(s,s)$ we may use Itô's formula to verify that $Z_t = S_t/X_t$, $t \geq 0$ under $P_{x,s}$ is a diffusion in $[1,\infty[$ started at $z = s/x$ having in $]1,\infty[$ the infinitesimal operator:

$$\mathbf{L_Z} = \frac{\sigma^2}{2}\, z^2 \frac{\partial^2}{\partial z^2} + (\sigma^2 - \mu)\, z\, \frac{\partial}{\partial z} \tag{3.24}$$

while the boundary point 1 is a point of the instantaneous reflection. It should be observed that $\tau_* = \inf \{ \ t > 0 \mid Z_t = \alpha \ \}$, and $Z = (Z_t)_{t \geq 0}$ under $P_{s,s}$ starts at 1 . Thus $m(s,s) = E_{s,s}(\tau_*) = M(1)$, where $M(z) = E_z(\tau_*)$ and Z under P_z starts at $z \geq 1$. By the general Markov process theory the function $M(z)$ solves uniquely the system:

$$\mathbf{L_Z} M(z) = -1 \quad (\ 1 < z < \alpha \ ) \tag{3.25}$$

$$M(\alpha) = 0 \ ; \ M'(1+) = 0 \ . \tag{3.26}$$

The equations (3.20), (3.22) and (3.25) are of Cauchy type, so the general solutions are easily found explicitly. The two conditions following each of these equations determine $G(x)$, $H(x)$ and $M(z)$ uniquely. Matching these formulas into (3.19) with $m(s,s) = M(1)$, we obtain the explicit formula for $m(x,s)$ given in (3.2). This ends the proof of the theorem.

□

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