

# Sequential Testing Problems for Bessel Processes

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Consider the motion of a Brownian particle that takes place either in a two-dimensional plane or in the three-dimensional space. Given that only the distance of the particle to the origin is being observed, the problem is to detect the true dimension as soon as possible and with minimal probabilities of the wrong terminal decisions. We solve this problem in the Bayesian formulation under any prior probability of the true dimension when the passage of time is penalised linearly.

## 1. Introduction

Imagine the motion of a Brownian particle that takes place either in a two-dimensional plane or in the three-dimensional space. Assuming that only the distance of the particle to the origin is being observed (Figure 1), the problem is to detect the true dimension as soon as possible and with minimal probabilities of the wrong terminal decisions. The purpose of the present paper is to derive the solution to this problem in the Bayesian formulation under any prior probability of the true dimension when the passage of time is penalised linearly.

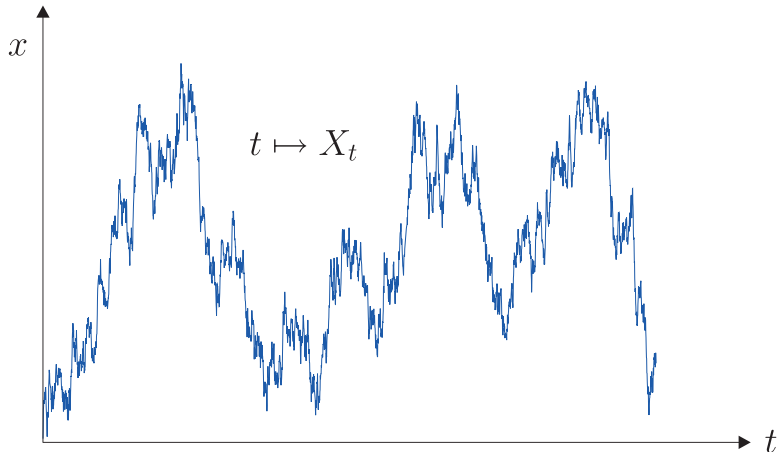
Denoting the distance of the Brownian particle to the origin by  $X$ , it is well known that  $X$  may be viewed as a Bessel process of dimension 2 or 3. We study the problem above by embedding it into the more general setting where a Bessel process  $X$  of dimension  $\delta_0 \geq 2$  or  $\delta_1 > \delta_0$  is being observed. In these cases 0 is known to be an entrance boundary point for  $X$  viewed as a diffusion process in  $[0, \infty)$  where  $X$  is also known to be recurrent when  $\delta_0 = 2$  and transient when  $\delta_1 > 2$ . Our methods are developed to treat these cases and we will leave other cases of  $\delta_0 \in [0, 2)$  with  $\delta_1 > \delta_0$  open for future development.

The loss to be minimised over sequential decision rules is expressed as the linear combination of the expected running time and the probabilities of the wrong terminal decisions with prior probabilities of the two dimensions given and fixed. This problem formulation of sequential testing dates back to [14] and has been extensively studied to date (see [3] and the references therein). The linear combination represents the Lagrangian and once the optimisation problem has been solved in this form it will also lead to the solution of the constrained problem where upper bounds are imposed on the probabilities of the wrong terminal decisions. Standard arguments show that the initial optimisation problem can be reduced to an optimal stopping problem for the posterior probability process  $\Pi$  of  $\delta_1$  given  $X$ . A canonical example is the

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**Figure 1.** Distance to the origin of a Brownian particle whose simulated motion that takes place either in a two-dimensional plane or in the three-dimensional space.

Brownian motion process with one or another *constant* drift (see [5] and [11]). This problem has also been solved in finite horizon (see [2] and the references therein). Books [12, Section 4.2] and [8, Section 21] contain expositions of these results and provide further details and references. In all these problems, however, the *signal-to-noise ratio* (defined as the difference between the two drifts divided by the diffusion coefficient) is constant. This is no longer the case in the sequential testing problem of the present paper and to our knowledge this is the first time that such a problem has been solved in the literature.

A more general problem formulation for one-dimensional diffusion processes having one or another *non-constant* drift has been considered in the recent paper [3]. This reference serves as a starting point for the present paper and for future reference we will also make it explicit in the analysis below which arguments/results are applicable/valid in the general case as well. To recognise the Markovian structure in the optimal stopping problem referred to above one considers the *posterior probability process*  $\Pi$  of  $\delta_1$  given  $X$ , as well as the *posterior probability ratio process*  $\Phi$  of  $\delta_1$  given  $X$ , in addition to the observed process  $X$ . These considerations take place under the probability measure  $\mathbb{P}_\pi = \pi \mathbb{P}_1 + (1-\pi) \mathbb{P}_0$  where  $\pi$  is a prior probability of  $\delta_1$  being true and  $1-\pi$  is a prior probability of  $\delta_0$  being true. The process  $\Phi$  happens to coincide (up to the initial point) with the *likelihood ratio process*  $L$  of  $\mathbb{P}_1$  and  $\mathbb{P}_0$  given  $X$  that provides an explicit link to the observed process  $X$ . The two processes  $\Pi$  and  $\Phi$  are in one-to-one correspondence so that one of them is Markov if and only if the other is Markov. This is the case when the signal-to-noise ratio is constant. On the other hand, if the signal-to-noise ratio is not constant, then both  $\Pi$  and  $\Phi$  fail to be Markov processes. To remedy the situation, as noted in [3], one needs to account for  $X$  and then both  $(\Pi, X)$  and  $(\Phi, X)$  become Markov processes. This shows that if the signal-to-noise ratio is not constant, as in the sequential testing problem of the present paper, then the optimal stopping problem under consideration is inherently/fully two-dimensional and hence more difficult. Finding and fully characterising the solution to this problem is the main/principal result of the present paper.

The exposition of the material is organised as follows. In Section 2 we formulate the optimal stopping problem and recall the stochastic differential equations for  $\Pi$ ,  $\Phi$ ,  $L$  and  $X$  from [3, Section 2]. The stochastic differential equations for  $\Pi$ ,  $\Phi$  and  $X$  are expressed in terms of the

innovation process (standard Brownian motion) so that the stochastic differential equations for both  $(\Pi, X)$  and  $(\Phi, X)$  are fully coupled. This makes the analysis of the optimal stopping problem more complicated. In Section 3 we show that a measure change from  $\mathbb{P}_\pi$  to  $\mathbb{P}_0$  simplifies the matters in that the stochastic differential equations for both  $(\Pi, X)$  and  $(\Phi, X)$  become uncoupled in the second component. This is an important step that abandons the innovation process and makes the subsequent analysis possible. The resulting optimal stopping problem for  $(\Phi, X)$  is expressed in Bolza form and in Section 4 we disclose its Lagrange and Mayer formulations (see [8, Section 6] for the terminology). The Lagrange form is expressed in terms of the local time of  $\Phi$  that makes the problem more intuitive.

In Section 5 we make use of the fact that a Bessel process of dimension  $\delta \geq 2$  can be time changed into a geometric Brownian motion. This has a dramatic effect on the problem since the stochastic differential equations for the time-changed process  $(\hat{\Phi}, \hat{X})$  get completely decoupled and moreover reduce to two geometric Brownian motions driven by the same standard Brownian motion. To understand this phenomenon within a more general context, one may note that since the stochastic differential equations for the process  $(\Phi, X)$  are also driven by the same Brownian motion, we know that the resulting infinitesimal generator equation must be of parabolic type. Reducing this equation to its canonical form by means of a diffeomorphic transformation replaces the process  $(\Phi, X)$  by the process  $(U, \Phi)$  where  $U$  is a process of bounded variation. It turns out moreover that the process  $U$  coincides with the additive functional  $A$  which is used to time change  $(\Phi, X)$  to  $(\hat{\Phi}, \hat{X})$  through its inverse. Making use of the diffeomorphic transformation in the Bolza problem for  $(\hat{\Phi}, \hat{X})$ , or solving the stochastic differential equations for  $\hat{\Phi}$  and  $\hat{X}$  explicitly, we then show that this problem reduces to an optimal stopping problem for the time-space process  $(t+s, \hat{\Phi}_s)_{s \geq 0}$  where the initial time  $t$  is expressed in terms of the initial points of  $\Phi_0 = \pi/(1-\pi)$  and  $X_0 = x$  and as such can also be negative. The resulting optimal stopping problem for the time-space process is also expressed in Bolza form and in Section 6 we disclose its Lagrange and Mayer formulations.

Exploiting the equivalence of the optimal stopping problems for  $(\hat{\Phi}, \hat{X})$  and  $(t+s, \hat{\Phi}_s)_{s \geq 0}$  in Section 7 we prove the existence of the optimal stopping boundaries, describe their shape, and derive their asymptotic behaviour at zero and infinity. The proof of their monotonicity endorses by different/rigorous means the implication stated in [3, Lemma 2.1] that the optimal stopping boundaries are monotone if the signal-to-noise ratio is monotone. Making use of the established techniques for the treatment of time-space optimal stopping problems, in Section 8 we disclose the free-boundary problems which stand in one-to-one correspondence with the optimal stopping problems. Further, in Section 9 we show that the optimal stopping boundaries can be characterised as the unique solution to a coupled system of nonlinear Volterra integral equations. These equations can be used to find the optimal stopping boundaries numerically.

## 2. Formulation of the problem

In this section we formulate the sequential testing problem under consideration and recall stochastic differential equations for the underlying stochastic processes (cf. [3, Section 2]). These traditional formulations will then be evaluated under a change of measure in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the Bessel process  $X$  of dimension  $\delta_0 \geq 2$  or  $\delta_1 > \delta_0$  with prior probabilities

$1-\pi$  and  $\pi$  respectively. The problem is to detect the true dimension as soon as possible and with minimal probabilities of the wrong terminal decisions. This problem belongs to the class of sequential testing problems as discussed in Section 1 above.

2. Standard arguments imply that the previous setting can be realised on a probability space  $(\Omega, \mathcal{F}, \mathbb{P}_\pi)$  with the probability measure  $\mathbb{P}_\pi$  decomposed as follows

$$(2.1) \quad \mathbb{P}_\pi = (1-\pi)\mathbb{P}_0 + \pi\mathbb{P}_1$$

for  $\pi \in [0, 1]$  where  $\mathbb{P}_i$  is the probability measure under which the observed Bessel process  $X$  has dimension  $\delta_i$  for  $i = 0, 1$ . This can be formally achieved by introducing an unobservable random variable  $\theta$  taking values 0 and 1 with probabilities  $1-\pi$  and  $\pi$  under  $\mathbb{P}_\pi$  and assuming that  $X$  after starting in  $[0, \infty)$  solves the stochastic differential equation

$$(2.2) \quad dX_t = [\mu_0(X_t) + \theta(\mu_1(X_t) - \mu_0(X_t))] dt + \sigma(X_t) dB_t$$

driven by a standard Brownian motion  $B$  that is independent from  $\theta$  under  $\mathbb{P}_\pi$  where we set

$$(2.3) \quad \mu_0(x) = \frac{\delta_0 - 1}{2x} \quad \& \quad \mu_1(x) = \frac{\delta_1 - 1}{2x} \quad \& \quad \sigma(x) = 1$$

for  $x > 0$  and  $\pi \in [0, 1]$ . We will often assume below that  $X$  starts at a strictly positive point and we will see below that this also yields solution when  $X$  starts at zero.

3. Being based upon the continued observation of  $X$ , the problem is to test sequentially the hypotheses  $H_0 : \theta = 0$  and  $H_1 : \theta = 1$  with a minimal loss. For this, we are given a sequential decision rule  $(\tau, d_\tau)$ , where  $\tau$  is a stopping time of  $X$  (i.e. a stopping time with respect to the natural filtration  $\mathcal{F}_t^X = \sigma(X_s | 0 \leq s \leq t)$  of  $X$  for  $t \geq 0$ ), and  $d_\tau$  is an  $\mathcal{F}_\tau^X$ -measurable random variable taking values 0 and 1. After stopping the observation of  $X$  at time  $\tau$ , the terminal decision function  $d_\tau$  takes value  $i$  if and only if the hypothesis  $H_i$  is to be accepted for  $i = 0, 1$ . With constants  $a > 0$  and  $b > 0$  given and fixed, the problem then becomes to compute the risk function

$$(2.4) \quad V(\pi) = \inf_{(\tau, d_\tau)} \mathbb{E}_\pi[\tau + aI(d_\tau = 0, \theta = 1) + bI(d_\tau = 1, \theta = 0)]$$

for  $\pi \in [0, 1]$  and find the optimal decision rule  $(\tau_*, d_{\tau_*}^*)$  at which the infimum in (2.4) is attained. Note that  $\mathbb{E}_\pi(\tau)$  in (2.4) is the expected waiting time until the terminal decision is made, and  $\mathbb{P}_\pi(d_\tau = 0, \theta = 1)$  and  $\mathbb{P}_\pi(d_\tau = 1, \theta = 0)$  in (2.4) are probabilities of the wrong terminal decisions respectively.

4. To tackle the sequential testing problem (2.4) we consider the *posterior probability process*  $\Pi = (\Pi_t)_{t \geq 0}$  of  $H_1$  given  $X$  that is defined by

$$(2.5) \quad \Pi_t = \mathbb{P}_\pi(\theta = 1 | \mathcal{F}_t^X)$$

for  $t \geq 0$ . Noting that  $\mathbb{P}_\pi(d_\tau = 0, \theta = 1) = \mathbb{E}_\pi[(1-d_\tau)\Pi_\tau]$  and  $\mathbb{P}_\pi(d_\tau = 1, \theta = 0) = \mathbb{E}_\pi[d_\tau(1-\Pi_\tau)]$ , and defining  $\tilde{d}_\tau = I(a\Pi_\tau \geq b(1-\Pi_\tau))$  for any given  $(\tau, d_\tau)$ , it is easily seen that the problem (2.4) is equivalent to the optimal stopping problem

$$(2.6) \quad V(\pi) = \inf_{\tau} \mathbb{E}_\pi[\tau + M(\Pi_\tau)]$$

where the infimum is taken over all stopping times  $\tau$  of  $X$  and  $M(\pi) = a\pi \wedge b(1-\pi)$  for  $\pi \in [0, 1]$ . Letting  $\tau_*$  denote the optimal stopping time in (2.6), and setting  $c = b/(a+b)$ , these arguments also show that the optimal decision function in (2.4) is given by  $d_{\tau_*}^* = 0$  if  $\Pi_{\tau_*} < c$  and  $d_{\tau_*}^* = 1$  if  $\Pi_{\tau_*} \geq c$ . Thus to solve the initial problem (2.4) it is sufficient to solve the optimal stopping problem (2.6). If the *signal-to-noise ratio* defined by

$$(2.7) \quad \rho(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}$$

is constant for  $x > 0$ , then  $\Pi$  is known to be a one-dimensional Markov (diffusion) process so that the optimal stopping problem (2.6) can be tackled using established techniques both in infinite and finite horizon (see [8, Section 21]). Note that this is no longer the case in the setting of the present problem since from (2.3) we see that

$$(2.8) \quad \rho(x) = \frac{\gamma}{x} \neq \text{constant}$$

for  $x > 0$  where we set  $\gamma = (\delta_1 - \delta_0)/2$ .

5. To connect the process  $\Pi$  to the observed process  $X$  we consider the *likelihood ratio process*  $L = (L_t)_{t \geq 0}$  defined by

$$(2.9) \quad L_t = \frac{d\mathbb{P}_{1,t}}{d\mathbb{P}_{0,t}}$$

where  $\mathbb{P}_{0,t}$  and  $\mathbb{P}_{1,t}$  denote the restrictions of the probability measures  $\mathbb{P}_0$  and  $\mathbb{P}_1$  to  $\mathcal{F}_t^X$  for  $t \geq 0$ . By the Girsanov theorem one finds that

$$(2.10) \quad L_t = \exp \left( \int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)} dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)} ds \right)$$

for  $t \geq 0$ . A direct calculation based on (2.1) shows that the *posterior probability ratio process*  $\Phi = (\Phi_t)_{t \geq 0}$  of  $\theta$  given  $X$  that is defined by

$$(2.11) \quad \Phi_t = \frac{\Pi_t}{1 - \Pi_t}$$

can be expressed in terms of  $L$  (and hence  $X$  as well) as follows

$$(2.12) \quad \Phi_t = \Phi_0 L_t$$

for  $t \geq 0$  where  $\Phi_0 = \pi/(1-\pi)$ . Note that  $L_t$  in (2.10) is expressed in terms of a stochastic integral with respect to  $X$  and as such may not be an explicit functional of the observed sample path of  $X$  up to time  $t$ . We will see in Section 7 below that such an explicit functional can be determined and that this issue is closely related to the parabolic nature of the underlying partial differential equation.

6. To derive stochastic differential equations for the posterior processes  $\Pi$  and  $\Phi$  one may apply Itô's formula in (2.10) to find that

$$(2.13) \quad dL_t = \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)} L_t [dX_t - \mu_0(X_t) dt]$$

with  $L_0 = 1$ . Further applications of Itô's formula in (2.11) and (2.12) then show that

$$(2.14) \quad d\Pi_t = \rho(X_t)\Pi_t(1-\Pi_t)d\bar{B}_t$$

$$(2.15) \quad d\Phi_t = \rho^2(X_t)\frac{\Phi_t^2}{1+\Phi_t}dt + \rho(X_t)\Phi_t d\bar{B}_t$$

upon noting that  $X$  solves

$$(2.16) \quad dX_t = [\mu_0(X_t) + \Pi_t(\mu_1(X_t) - \mu_0(X_t))]dt + \sigma(X_t)d\bar{B}_t$$

where  $\bar{B} = (\bar{B}_t)_{t \geq 0}$  is the *innovation process* defined by

$$(2.17) \quad \bar{B}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left[ \frac{\mu_0(X_s)}{\sigma(X_s)} + \Pi_s \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right] ds$$

for  $t \geq 0$  from where we see by Lévy's characterisation theorem that  $\bar{B}$  is a standard Brownian motion with respect to  $(\mathcal{F}_t^X)_{t \geq 0}$  under  $\mathbb{P}_\pi$  for  $\pi \in [0, 1]$ .

7. From (2.14) and (2.15) it is evident that  $\Pi$  and  $\Phi$  cannot be Markov processes unless the signal-to-noise ratio  $\rho$  defined in (2.7) is constant. If  $\rho$  is not constant such as in (2.8) above then one needs to look at (2.14)+(2.16) and (2.15)+(2.16) as two systems of stochastic differential equations for the pairs of processes  $(\Pi, X)$  and  $(\Phi, X)$  respectively. It is well known (see e.g. [10, pp 158–163]) that when these systems have a unique weak solution then  $(\Pi, X)$  and  $(\Phi, X)$  are (time-homogeneous) strong Markov processes under  $\mathbb{P}_\pi$  for  $\pi \in [0, 1]$ . Recalling known sufficient conditions for the existence and uniqueness of weak solutions (see e.g. [10, pp 166–173]) we see that this is the case whenever  $x \mapsto \mu_0(x)$ ,  $x \mapsto \mu_1(x)$  and  $x \mapsto \sigma(x)$  are continuous with  $\mu_0(x) \neq \mu_1(x)$  and  $\sigma(x) > 0$  for all  $x$  in the state space of  $X$  (possibly excluding entrance boundary points). Note that these conclusions are not confined to the setting of Bessel processes but hold generally in the sequential testing problems for diffusion processes  $X$  solving (2.2) when the drift equals either  $\mu_0$  or  $\mu_1$  depending on the outcome of the unobservable random variable  $\theta$ .

8. The preceding considerations show that the optimal stopping problem (2.6) is inherently/fully two-dimensional with the pairs of processes  $(\Pi, X)$  and  $(\Phi, X)$  solving (2.14)+(2.16) and (2.15)+(2.16) being strong Markov when (2.8) holds. This fact makes the subsequent analysis of these problems more challenging than when the signal-to-noise ratio  $\rho$  defined in (2.7) is constant. The analysis of (2.6) performed in [3] is based on the stochastic differential equations (2.14)+(2.16) and (2.15)+(2.16) under the probability measure  $\mathbb{P}_\pi$  for  $\pi \in [0, 1]$ . In this case one sees that these two systems of stochastic differential equations are fully coupled (as both  $\Pi$  and  $X$  as well as  $\Phi$  and  $X$  enter both (2.14) and (2.16) as well as (2.15) and (2.16) respectively). This makes the analysis of (2.6) more involved. In the next section we will see that a change of measure argument simplifies the setting and decouples the systems (2.14)+(2.16) and (2.15)+(2.16) in the second equation so that the analysis of (2.6) becomes easier and more penetrating. This change of measure argument is not confined to the Bessel process setting and holds in general. Moreover, another major difficulty encountered in [3] is that both  $\Pi$  and  $X$  as well as  $\Phi$  and  $X$  enter the diffusion coefficient in (2.14) and (2.15) respectively. This makes the use of comparison theorems for the systems of stochastic

differential equations (2.14)+(2.16) and (2.15)+(2.16) more challenging. We will see in Section 5 below that time change arguments remove the dependence of the diffusion coefficient on the process  $X$  in both systems and in fact completely decouple the two equations in both systems. This change of time argument is confined to the Bessel process setting and it will enable us in Section 7 to reduce the optimal stopping problem (2.6) to a solvable form.

### 3. Measure change

In this section we show that changing the measure  $\mathbf{P}_\pi$  for  $\pi \in [0, 1]$  to  $\mathbf{P}_0$  in the optimal stopping problems (2.6) above provides crucial simplifications of the setting which makes the subsequent analysis possible. The change of measure arguments are presented in the proof of Lemma 1. Recalling that the systems of stochastic differential equations (2.14)+(2.16) and (2.15)+(2.16) are equivalent our focus in the sequel will be on the system (2.15)+(2.16) for the pair of processes  $(\Phi, X)$  after showing that this system takes a simpler form under the new measure  $\mathbf{P}_0$ . This is then followed by a reformulation of the optimal stopping problem (2.6) in terms of  $(\Phi, X)$  under the new measure  $\mathbf{P}_0$  in Proposition 2 below.

1. In the sequel we let  $\mathbf{P}_{\pi, \tau}$  denote the restriction of the measure  $\mathbf{P}_\pi$  to  $\mathcal{F}_\tau^X$  for  $\pi \in [0, 1]$  where  $\tau$  is a stopping time of  $X$ .

**Lemma 1.** *The following identity holds*

$$(3.1) \quad \frac{d\mathbf{P}_{\pi, \tau}}{d\mathbf{P}_{0, \tau}} = \frac{1 - \pi}{1 - \Pi_\tau}$$

for all stopping times  $\tau$  of  $X$  and all  $\pi \in [0, 1]$ .

**Proof.** A standard rule for the Radon-Nikodym derivatives based on (2.1) gives

$$(3.2) \quad \Pi_\tau = \mathbf{P}_\pi(\theta=1 | \mathcal{F}_\tau^X) = (1 - \pi) \mathbf{P}_0(\theta=1 | \mathcal{F}_\tau^X) \frac{d\mathbf{P}_{0, \tau}}{d\mathbf{P}_{\pi, \tau}} + \pi \mathbf{P}_1(\theta=1 | \mathcal{F}_\tau^X) \frac{d\mathbf{P}_{1, \tau}}{d\mathbf{P}_{\pi, \tau}} = \pi \frac{d\mathbf{P}_{1, \tau}}{d\mathbf{P}_{\pi, \tau}}$$

for any  $\tau$  and  $\pi$  as above given and fixed since  $\mathbf{P}_0(\theta=1) = 0$  and  $\mathbf{P}_1(\theta=1) = 1$ . Using the identity (2.1) again this shows that

$$(3.3) \quad \frac{d\mathbf{P}_{\pi, \tau}}{d\mathbf{P}_{1, \tau}} = \pi + (1 - \pi) \frac{d\mathbf{P}_{0, \tau}}{d\mathbf{P}_{1, \tau}} = \frac{\pi}{\Pi_\tau}$$

from where by (2.9) we see that

$$(3.4) \quad L_\tau = \frac{d\mathbf{P}_{1, \tau}}{d\mathbf{P}_{0, \tau}} = \frac{1 - \pi}{\pi} \frac{\Pi_\tau}{1 - \Pi_\tau}$$

as stated in (2.11) and (2.12). From (3.3) and (3.4) we find that

$$(3.5) \quad \frac{d\mathbf{P}_{\pi, \tau}}{d\mathbf{P}_{0, \tau}} = \frac{d\mathbf{P}_{\pi, \tau}}{d\mathbf{P}_{1, \tau}} \frac{d\mathbf{P}_{1, \tau}}{d\mathbf{P}_{0, \tau}} = \frac{\pi}{\Pi_\tau} \frac{1 - \pi}{\pi} \frac{\Pi_\tau}{1 - \Pi_\tau} = \frac{1 - \pi}{1 - \Pi_\tau}$$

as claimed in (3.1) and the proof is complete.  $\square$

2. From (2.12) and (2.13) we see that the stochastic differential equations (2.15) and (2.16) for  $(\Phi, X)$  under the measure  $\mathbb{P}_0$  simplify to read as follows

$$(3.6) \quad d\Phi_t = \rho(X_t)\Phi_t dB_t$$

$$(3.7) \quad dX_t = \mu_0(X_t) dt + \sigma(X_t) dB_t$$

where (3.7) follows from (2.2) upon recalling that  $\theta$  equals 0 under  $\mathbb{P}_0$ . Recall that  $\rho$  in (3.6) is given by (2.7) above, and  $\mu_0$  and  $\sigma$  in (3.7) are given in (2.3) above. The stochastic differential equations (3.6)+(3.7) also hold in general under  $\mathbb{P}_0$  whenever  $\mu_0 \neq \mu_1$  and  $\sigma > 0$  in (2.12) are continuous and for the reasons stated at the end of Section 2 in this case we know that  $(\Phi, X)$  is a strong Markov process under  $\mathbb{P}_0$ . Note also from (2.10) and (2.12) with (3.7) that under  $\mathbb{P}_0$  we have

$$(3.8) \quad \Phi_t = \Phi_0 \exp \left( \int_0^t \rho(X_s) dB_s - \frac{1}{2} \int_0^t \rho^2(X_s) ds \right)$$

for  $t \geq 0$ . The stochastic differential equation (2.14) for the process  $\Pi$  takes a slightly more complicated form under the measure  $\mathbb{P}_0$  and given that this equation is equivalent to (3.6) due to (2.11) we will not state it explicitly. Thus our focus in the sequel will be on the system (3.6)+(3.7) for the pair of processes  $(\Phi, X)$  under the measure  $\mathbb{P}_0$ .

3. We now show that the optimal stopping problem (2.6) admits a transparent reformulation under the measure  $\mathbb{P}_0$  in terms of the process  $\Phi$  solving (3.6) with (3.7). Recall that  $\Phi$  starts at  $\Phi_0 = \pi/(1-\pi)$  and this dependence on the initial point will be indicated by a superscript to  $\Phi$  when needed.

**Proposition 2.** *The value function  $V$  from (2.6) satisfies the identity*

$$(3.9) \quad V(\pi) = (1-\pi) \hat{V}(\pi)$$

where the value function  $\hat{V}$  is given by

$$(3.10) \quad \hat{V}(\pi) = \inf_{\tau} \mathbb{E}_0 \left[ \int_0^{\tau} (1 + \Phi_t^{\pi/(1-\pi)}) dt + \hat{M}(\Phi_{\tau}^{\pi/(1-\pi)}) \right]$$

for  $\pi \in [0, 1)$  with  $\hat{M}(\varphi) = a\varphi \wedge b$  for  $\varphi \in [0, \infty)$  and the infimum in (3.10) is taken over all stopping times  $\tau$  of  $X$ .

**Proof.** With  $\pi \in [0, 1)$  given and fixed, and dropping the superscript from  $\Phi$  in the sequel for simplicity, by the monotone and dominated convergence theorems it is enough to show that

$$(3.11) \quad \mathbb{E}_{\pi}[\tau + M(\Pi_{\tau})] = (1-\pi) \mathbb{E}_0 \left[ \int_0^{\tau} (1 + \Phi_t) dt + \hat{M}(\Phi_{\tau}) \right]$$

for all bounded stopping times  $\tau$  of  $X$ . For this, suppose that such a stopping time  $\tau$  is given and fixed, and note by (3.1) that

$$(3.12) \quad \mathbb{E}_{\pi}[\tau + M(\Pi_{\tau})] = (1-\pi) \mathbb{E}_0 \left[ \frac{\tau}{1-\Pi_{\tau}} + \frac{M(\Pi_{\tau})}{1-\Pi_{\tau}} \right] = (1-\pi) \mathbb{E}_0 [\tau(1 + \Phi_{\tau}) + \hat{M}(\Phi_{\tau})]$$



where in the final equality we use (2.11) above. Integration by parts then gives

$$(3.13) \quad t\Phi_t = \int_0^t \Phi_s ds + \int_0^t s d\Phi_s$$

where the final term defines a continuous local martingale in view of (3.6) above. Making use of a localising sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (3.13) that

$$(3.14) \quad \mathbb{E}_0(\tau\Phi_\tau) = \mathbb{E}_0\left(\int_0^\tau \Phi_t dt\right).$$

Inserting this back into (3.12) we obtain (3.11) as claimed and the proof is complete.  $\square$

4. Note that the identities (3.9) and (3.10) are not confined to the setting of Bessel processes but hold generally in the sequential testing problems for diffusion processes  $X$  solving (2.2) when the drift equals either  $\mu_0$  or  $\mu_1$  depending on the outcome of the unobservable random variable  $\theta$ . If  $\rho$  is not constant such as in (2.8) above then to tackle the resulting optimal stopping problem (3.10) for the strong Markov process  $(\Phi, X)$  solving (3.6)+(3.7) we will enable  $(\Phi, X)$  to start at any point  $(\varphi, x)$  in  $[0, \infty) \times [0, \infty)$  under the probability measure  $\mathbf{P}_{\varphi, x}^0$  (where we move 0 from the subscript to a superscript for notational reasons) so that the optimal stopping problem (3.10) extends as follows

$$(3.15) \quad \hat{V}(\varphi, x) = \inf_{\tau} \mathbb{E}_{\varphi, x}^0 \left[ \int_0^\tau (1 + \Phi_t) dt + \hat{M}(\Phi_\tau) \right]$$

for  $(\varphi, x) \in [0, \infty) \times [0, \infty)$  with  $\mathbf{P}_{\varphi, x}^0((\Phi_0, X_0) = (\varphi, x)) = 1$  where the infimum in (3.15) is taken over all stopping times  $\tau$  of  $(\Phi, X)$ . In this way we have reduced the initial sequential testing problem (2.4) to the optimal stopping problem (3.15) for the strong Markov process  $(\Phi, X)$  solving the system of stochastic differential equations

$$(3.16) \quad d\Phi_t = \gamma \frac{\Phi_t}{X_t} dB_t$$

$$(3.17) \quad dX_t = \frac{\delta_0 - 1}{2X_t} dt + dB_t$$

under the measure  $\mathbf{P}_{\varphi, x}^0$  with  $(\varphi, x) \in [0, \infty) \times [0, \infty)$ . Note that this optimal stopping problem is inherently/fully two-dimensional.

## 4. Lagrange and Mayer formulations

The optimal stopping problem (3.15) is Bolza formulated. In this section we derive its Lagrange and Mayer reformulations which are helpful in the subsequent analysis of the problem.

1. We first consider the Lagrange reformulation of the optimal stopping problem (3.15).

**Proposition 3.** *The value function  $\hat{V}$  from (3.15) can be expressed as*

$$(4.1) \quad \hat{V}(\varphi, x) = \inf_{\tau} \mathbb{E}_{\varphi, x}^0 \left[ \int_0^\tau (1 + \Phi_t) dt - \frac{a}{2} \ell_\tau^{b/a}(\Phi) \right] + \hat{M}(\varphi)$$

for  $(\varphi, x) \in [0, \infty) \times [0, \infty)$  where  $\ell_\tau^{b/a}(\Phi)$  is the local time of  $\Phi$  at  $b/a$  and  $\tau$  given by

$$(4.2) \quad \ell_\tau^{b/a}(\Phi) = \mathbf{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\tau I\left(\frac{b}{a} - \varepsilon \leq \Phi_t \leq \frac{b}{a} + \varepsilon\right) d\langle \Phi, \Phi \rangle_t$$

and the infimum in (4.1) is taken over all stopping times  $\tau$  of  $(\Phi, X)$ .

**Proof.** Note that  $\varphi \mapsto \hat{M}(\varphi) = a\varphi \wedge b$  is a concave function on  $[0, \infty)$  with  $\hat{M}'(d\varphi) = -a\delta_{b/a}(d\varphi)$  where  $\delta_{b/a}$  is the Dirac measure at  $b/a$ . By the Itô-Tanaka formula we therefore find using (3.16) that

$$(4.3) \quad \begin{aligned} \hat{M}(\Phi_t) &= \hat{M}(\Phi_0) + \int_0^t \hat{M}'_\pm(\Phi_s) d\Phi_s + \frac{1}{2} \int_{-\infty}^\infty \ell_t^\psi(\Phi) \hat{M}'(d\psi) \\ &= \hat{M}(\Phi_0) + \int_0^t \hat{M}'_\pm(\Phi_s) \gamma \frac{\Phi_s}{X_s} dB_s - \frac{a}{2} \ell_t^{b/a}(\Phi) \end{aligned}$$

for  $t \geq 0$  where the second term on the right-hand side defines a continuous local martingale. Making use of a localisation sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (4.3) that

$$(4.4) \quad \mathbf{E}_{\varphi, x}^0[\hat{M}(\Phi_\tau)] = \hat{M}(\varphi) - \frac{a}{2} \mathbf{E}_{\varphi, x}^0[\ell_\tau^{b/a}(\Phi)]$$

for  $(\varphi, x) \in [0, \infty) \times [0, \infty)$  and all stopping times  $\tau$  of  $(\Phi, X)$ . Inserting (4.4) into (3.15) we obtain (4.1) as claimed and the proof is complete.  $\square$

The Lagrange reformulation (4.1) of the optimal stopping problem (3.15) reveals the underlying rationale for continuing vs stopping in a clearer manner. Indeed, recalling that the local time process  $t \mapsto \ell_t^{b/a}(\Phi)$  strictly increases only when  $\Phi_t$  is at  $b/a$ , and that  $\ell_t^{b/a}(\Phi) \sim \sqrt{t}$  is strictly larger than  $\int_0^t (1 + \Phi_s) ds \sim t$  for small  $t$ , we see from (4.1) that it should never be optimal to stop at  $\varphi = b/a$  and the incentive for stopping should increase the further away  $\Phi_t$  gets from  $b/a$ . We will see in Section 7 below that these informal conjectures can be formalised and this will give a new proof of the known fact in the wider diffusion setting that the set  $\{(\varphi, x) \in [0, \infty) \times [0, \infty) \mid \varphi = b/a\}$  is contained in the continuation set of the sequential testing problem (2.4).

Note that the Lagrange reformulation (4.1) of the optimal stopping problem (3.15) is not confined to the setting of Bessel processes but holds generally in the sequential testing problems for diffusion processes  $X$  solving (2.2) when the drift equals either  $\mu_0$  or  $\mu_1$  depending on the outcome of the unobservable random variable  $\theta$ .

2. We next consider the Mayer reformulation of the optimal stopping problem (3.15).

**Proposition 4.** *The value function  $\hat{V}$  from (3.15) can be expressed as*

$$(4.5) \quad \hat{V}(\varphi, x) = \inf_\tau \mathbf{E}_{\varphi, x}^0 \left[ \left( \frac{1}{\delta_0} + \frac{1}{\delta_1} \Phi_\tau \right) X_\tau^2 + (a\Phi_\tau \wedge b) \right] - \left( \frac{1}{\delta_0} + \frac{1}{\delta_1} \varphi \right) x^2$$

for  $(\varphi, x) \in [0, \infty) \times [0, \infty)$  where the infimum is taken over all stopping times  $\tau$  of  $(\Phi, X)$ .

**Proof.** From (3.16)+(3.17) we read that the infinitesimal generator of  $(\Phi, X)$  is given by

$$(4.6) \quad \mathbb{L}_{\Phi, X} = \frac{\delta_0 - 1}{2x} \partial_x + \gamma \frac{\varphi}{x} \partial_{\varphi x} + \frac{1}{2} \frac{\gamma^2 \varphi^2}{x^2} \partial_{\varphi\varphi} + \frac{1}{2} \partial_{xx}.$$

Defining  $K(\varphi, x) = (1/\delta_0 + \varphi/\delta_1)x^2$  it is easily verified that

$$(4.7) \quad \mathbb{L}_{\Phi, X}(K)(\varphi, x) = 1 + \varphi$$

for all  $(\varphi, x) \in [0, \infty) \times [0, \infty)$ . By Itô's formula we thus get

$$(4.8) \quad \begin{aligned} K(\Phi_t, X_t) &= K(\Phi_0, X_0) + \int_0^t \mathbb{L}_{\Phi, X}(K)(\Phi_s, X_s) ds \\ &\quad + \int_0^t \left[ K_\varphi(\Phi_s, X_s) \gamma \frac{\Phi_s}{X_s} + K_x(\Phi_s, X_s) \right] dB_s \\ &= K(\Phi_0, X_0) + \int_0^t (1 + \Phi_s) ds + \int_0^t \left[ \frac{2}{\delta_0} + \frac{1}{\delta_1} (2 + \gamma) \Phi_s \right] X_s dB_s \end{aligned}$$

for  $t \geq 0$  where the final term on the right-hand side defines a continuous local martingale. Making use of a localisation sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (4.8) that

$$(4.9) \quad \mathbf{E}_{\varphi, x}^0 [K(\Phi_\tau, X_\tau)] = K(\varphi, x) + \mathbf{E}_{\varphi, x}^0 \left[ \int_0^\tau (1 + \Phi_t) dt \right]$$

for  $(\varphi, x) \in [0, \infty) \times [0, \infty)$  and all (bounded) stopping times  $\tau$  of  $(\Phi, X)$ . Inserting (4.9) into (3.15) we obtain (4.5) as claimed and the proof is complete.  $\square$

Note that the Mayer reformulation (4.5) of the optimal stopping problem (3.15) is specific to the setting of Bessel processes. To find a Mayer reformulation of (3.15) in the sequential testing problems for diffusion processes  $X$  solving (2.2) when the drift equals either  $\mu_0$  or  $\mu_1$  depending on the outcome of the unobservable random variable  $\theta$ , one needs to find a particular solution  $K$  to the equation (4.7) where from (3.6)+(3.7) we read that the infinitesimal generator  $\mathbb{L}_{\Phi, X}$  of  $(\Phi, X)$  is given by

$$(4.10) \quad \mathbb{L}_{\Phi, X} = \mu_0(x) \partial_x + \varphi \rho(x) \sigma(x) \partial_{\varphi x} + \frac{1}{2} \varphi^2 \rho^2(x) \partial_{\varphi\varphi} + \frac{1}{2} \sigma^2(x) \partial_{xx}$$

and  $\rho$  is given by (2.7) above. Note that if  $\rho$  is constant then one needs to look for a solution  $K$  to (4.7) that is a function of  $\varphi$  only since in this case  $\Phi$  is a one-dimensional (strong) Markov process.

## 5. Reduction to a time-space problem

Recall that we have reduced the initial sequential testing problem (2.4) to the optimal stopping problem (3.15) for the strong Markov process  $(\Phi, X)$  solving (3.6)+(3.7) which in the setting of Bessel processes become (3.16)+(3.17). A key difficulty in this setting is that  $X$

enters the diffusion coefficient of the stochastic differential equation (3.16). This makes the applicability of available comparison theorems for  $(\Phi, X)$  more challenging. To tackle the problem in this section we make use of the known fact that a Bessel process of dimension  $\delta \geq 2$  can be time changed into a geometric Brownian motion. We show that this has a dramatic effect on the optimal stopping problem (3.15) since the stochastic differential equations for the time-changed process  $(\hat{\Phi}, \hat{X})$  get completely decoupled and moreover reduce to two geometric Brownian motions driven by the same standard Brownian motion. Solving the stochastic differential equations for  $\hat{\Phi}$  and  $\hat{X}$  explicitly, we then show that the problem (3.15) reduces to an optimal stopping problem for the time-space process  $(t+s, \Phi_s)_{s \geq 0}$  where the initial time  $t$  is expressed in terms of the initial points of  $\Phi$  and  $X$  and as such can also be negative.

To understand these steps within a more general context, note that since the stochastic differential equations (3.16)+(3.17) are driven by the same Brownian motion, we know that the resulting infinitesimal generator equation must be of parabolic type. It follows therefore that reducing this equation to its canonical form by means of a diffeomorphic transformation to be found replaces the process  $(\Phi, X)$  by the process  $(U, \Phi)$  where  $U$  is a process of bounded variation. We will see below that the process  $U$  happens to coincide with the additive functional  $A$  which is used to time change  $(\Phi, X)$  to  $(\hat{\Phi}, \hat{X})$  through its inverse. Moreover applying  $A$  to the closed form expressions for  $\hat{\Phi}$  and  $\hat{X}$  enables us to determine the diffeomorphic transformation itself explicitly. This provides probabilistic arguments for the reduction of the infinitesimal generator equation to its canonical form. Using standard analytic arguments for this reduction first and then applying the time change arguments would yield the same outcome. The resulting time-space problem will be studied in Section 7 below.

1. *Time change.* Consider the additive functional  $A = (A_t)_{t \geq 0}$  defined by

$$(5.1) \quad A_t = \int_0^t \frac{ds}{X_s^2}$$

and note that  $t \mapsto A_t$  is continuous and strictly increasing with  $A_0 = 0$  and  $A_t \uparrow \infty$  as  $t \uparrow \infty$  (the latter property is well known for Bessel processes  $X$  of dimension  $\delta \geq 2$  but will also be verified below). Hence the same properties hold for its inverse  $T = (T_t)_{t \geq 0}$  defined by

$$(5.2) \quad T_t = A_t^{-1}$$

for  $t \geq 0$ . Since  $A$  is adapted to  $(\mathcal{F}_t^X)_{t \geq 0}$  it follows that each  $T_t$  is a stopping time with respect to  $(\mathcal{F}_t^X)_{t \geq 0}$  so that  $T = (T_t)_{t \geq 0}$  defines a time change relative to  $(\mathcal{F}_t^X)_{t \geq 0}$ . The fact that  $t \mapsto T_t$  is continuous and strictly increasing with  $T_t < \infty$  for  $t \geq 0$  (or equivalently  $A_t \uparrow \infty$  as  $t \uparrow \infty$ ) implies that standard time change transformations are applicable to continuous semimartingales and their stochastic integrals without extra conditions on their sample paths (see e.g. [9, pp 7-9 & pp 179-181]) and they will be used below without explicit mention. Moreover, since  $(\Phi, X)$  is a strong Markov process by the well-known result dating back to [13] (see e.g. [10, p. 175] for a modern exposition) we know that the time-changed process  $(\hat{\Phi}, \hat{X}) = ((\hat{\Phi}_t, \hat{X}_t))_{t \geq 0}$  defined by

$$(5.3) \quad (\hat{\Phi}_t, \hat{X}_t) = (\Phi_{T_t}, X_{T_t})$$

for  $t \geq 0$  is a Markov process under  $\mathbf{P}_{\varphi, x}^0$  for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . It is possible to verify that  $(\Phi, X)$  is a Feller process and hence by the same well-known result we could also conclude

that  $(\hat{\Phi}, \hat{X})$  is a strong Markov process, however, we will make no use of the former fact while the latter fact will also follow from the existence and uniqueness of a weak solution to the system of stochastic differential equations for  $(\hat{\Phi}, \hat{X})$  derived below. Moreover from (5.1) one can read off that the infinitesimal generator of  $(\hat{\Phi}, \hat{X})$  is given by

$$(5.4) \quad \mathbb{L}_{\hat{\Phi}, \hat{X}} = x^2 \mathbb{L}_{\Phi, X}$$

where  $\mathbb{L}_{\Phi, X}$  is the infinitesimal generator of  $(\Phi, X)$ . Note also that  $\sigma = A_\tau$  is a stopping time of  $(\hat{\Phi}, \hat{X})$  if and only if  $\tau = T_\sigma$  is a stopping time of  $(\Phi, X)$  (where we recall that the natural filtration of  $(\hat{\Phi}, \hat{X})$  coincides with the time-changed natural filtration of  $(\Phi, X)$  given by  $\hat{\mathcal{F}}_t^{\Phi, X} = \mathcal{F}_{T_t}^{\Phi, X}$  for  $t \geq 0$ ). Finally, in addition to (5.1) it is easily seen using (5.2) that

$$(5.5) \quad T_t = \int_0^t \hat{X}_s^2 ds$$

for  $t \geq 0$ . Below we will make use of this relation too.

2. Recalling that the process  $(\Phi, X)$  solves the system of stochastic differential equations (3.16)+(3.17) and making use of these equations we find that

$$(5.6) \quad \hat{\Phi}_t = \Phi_{T_t} = \Phi_0 + \int_0^{T_t} \gamma \frac{\Phi_s}{X_s} dB_s = \Phi_0 + \int_0^t \gamma \frac{\Phi_{T_s}}{X_{T_s}} dB_{T_s} = \hat{\Phi}_0 + \int_0^t \gamma \hat{\Phi}_s d\tilde{B}_s$$

$$(5.7) \quad \begin{aligned} \hat{X}_t = X_{T_t} &= X_0 + \int_0^{T_t} \frac{\delta_0 - 1}{2X_s} ds + \int_0^{T_t} dB_s = X_0 + \int_0^t \frac{\delta_0 - 1}{2X_{T_s}} dT_s + \int_0^t dB_{T_s} \\ &= \hat{X}_0 + \int_0^t \frac{\delta_0 - 1}{2} \hat{X}_s ds + \int_0^t \hat{X}_s d\tilde{B}_s \end{aligned}$$

where the process  $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$  is defined by

$$(5.8) \quad \tilde{B}_t = \int_0^t \frac{dB_{T_s}}{X_{T_s}} = \int_0^t \frac{d\hat{B}_s}{\hat{X}_s} = \int_0^{T_t} \frac{dB_s}{X_s} = M_{T_t}$$

upon setting  $M_t = \int_0^t dB_s/X_s$  for  $t \geq 0$ . Since  $M = (M_t)_{t \geq 0}$  is a continuous local martingale with respect to  $(\mathcal{F}_t^X)_{t \geq 0}$  it follows that  $\tilde{B} = (\tilde{B}_t)_{t \geq 0}$  is a continuous local martingale with respect to  $(\hat{\mathcal{F}}_t^X)_{t \geq 0}$ . Note moreover that  $\langle \tilde{B}, \tilde{B} \rangle_t = \langle M_{T_t}, M_{T_t} \rangle_t = \langle M, M \rangle_{T_t} = \int_0^{T_t} ds/X_s^2 = A_{T_t} = t$  for  $t \geq 0$ . Hence by Lévy's characterisation theorem (see e.g. [9, p. 150]) we can conclude that  $\tilde{B}$  is a standard Brownian motion with respect to  $(\hat{\mathcal{F}}_t^X)_{t \geq 0}$ . It follows therefore that (5.6)+(5.7) can be written as the following stochastic differential equations

$$(5.9) \quad d\hat{\Phi}_t = \gamma \hat{\Phi}_t d\tilde{B}_t$$

$$(5.10) \quad d\hat{X}_t = \left( \frac{\delta_0 - 1}{2} \right) \hat{X}_t dt + \hat{X}_t d\tilde{B}_t$$

under  $\mathbb{P}_{\varphi, x}^0$  for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . This shows that  $\hat{\Phi}$  and  $\hat{X}$  are fully decoupled geometric Brownian motions (driven by the same standard Brownian motion) whose unique strong solutions under  $\mathbb{P}_0$  are given by

$$(5.11) \quad \hat{\Phi}_t^\varphi = \varphi \exp\left(\gamma \tilde{B}_t - \frac{\gamma^2}{2} t\right)$$

$$(5.12) \quad \hat{X}_t^x = x \exp\left(\tilde{B}_t + \left(\frac{\delta_0}{2} - 1\right)t\right)$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . Recalling known sufficient conditions (see e.g. [10, pp 166–173]) we formally see that the system of stochastic differential equations (5.9)+(5.10) has a unique weak solution and hence by the well-known result (see e.g. [10, pp 158–163]) we can conclude that  $(\hat{\Phi}, \hat{X})$  is a (time-homogeneous) strong Markov process under  $\mathbf{P}_{\varphi, x}^0$  for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$  (this can also be verified directly using (5.11)+(5.12) above). These facts will be useful in the subsequent analysis of the optimal stopping problem (3.15) since both (5.11) and (5.12) provide Markovian representations of the solutions where initial points are expressed explicitly.

3. We can now make use of the previous facts and derive a time-changed version of the optimal stopping problem (3.15) above.

**Proposition 5.** *The value function  $\hat{V}$  from (3.15) satisfies the identity*

$$(5.13) \quad \hat{V}(\varphi, x) = \inf_{\sigma} \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_t) \hat{X}_t^2 dt + \hat{M}(\hat{\Phi}_{\sigma}) \right]$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$  where the infimum is taken over all stopping times  $\sigma$  of  $(\hat{\Phi}, \hat{X})$ .

**Proof.** Recall that  $\tau = T_{\sigma}$  is a stopping time of  $(\Phi, X)$  if and only if  $\sigma = A_{\tau}$  is a stopping time of  $(\hat{\Phi}, \hat{X})$ . Thus if either  $\tau$  or  $\sigma$  is given we can form  $\sigma$  or  $\tau$  respectively and using (5.5) note that

$$(5.14) \quad \begin{aligned} \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\tau} (1 + \Phi_t) dt + \hat{M}(\Phi_{\tau}) \right] &= \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{T_{\sigma}} (1 + \Phi_t) dt + \hat{M}(\Phi_{T_{\sigma}}) \right] \\ &= \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_{T_t}) dT_t + \hat{M}(\hat{\Phi}_{\sigma}) \right] = \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_t) \hat{X}_t^2 dt + \hat{M}(\hat{\Phi}_{\sigma}) \right]. \end{aligned}$$

Taking the infimum over all  $\tau$  and/or  $\sigma$  on both sides of (5.14) as above we see that (5.13) holds as claimed and the proof is complete.  $\square$

It follows from Proposition 5 that the optimal stopping problem (3.15) is equivalent to the optimal stopping problem defined on the right-hand side of (5.13) for the strong Markov process  $(\hat{\Phi}, \hat{X})$  solving the system of stochastic differential equations (5.9)+(5.10) under  $\mathbf{P}_{\varphi, x}$  and given explicitly by (5.11)+(5.12) under  $\mathbf{P}_0$  for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . This equivalence will be exploited in Section 7 when deriving basic properties of the optimal stopping boundaries.

4. *Reduction to a time-space problem.* We now show that the problem (5.13) can be further reduced to a time-space problem of optimal stopping in dimension one. For this, note from (5.11) and (5.12) that the following identity holds

$$(5.15) \quad \hat{X}_t^x = \frac{x}{\varphi^{1/\gamma}} (\hat{\Phi}_t^{\varphi})^{1/\gamma} e^{\frac{\kappa}{2}t}$$

where we set  $\kappa = (\delta_1 + \delta_0 - 4)/2$  (the scaling of this constant in (5.15) is motivated by the presence of  $\hat{X}_t^2$  in (5.13) above). Note that  $\kappa > 0$  since  $\delta_1 > \delta_0 \geq 2$ . Replacing  $t$  in (5.15) by  $A_t$  from (5.1) above, and setting  $A_t^a = a + A_t$ , we see that (5.15) is equivalent to

$$(5.16) \quad X_t^x = (\Phi_t^{\varphi})^{1/\gamma} e^{\frac{\kappa}{2}A_t^a}$$

where  $a = (2/\kappa) \log(x/\varphi^{1/\gamma})$  for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . It is easily verified (using purely analytic arguments) that the mapping defined by (5.16) is a diffeomorphism which reduces the infinitesimal generator equation of  $(\Phi, X)$  to its canonical form. This replaces the process  $(\Phi, X)$  by the process  $(U, \Phi)$  where the bounded variation process  $U$  happens to coincide with the process  $A$  used to time change  $(\Phi, X)$  to  $(\hat{\Phi}, \hat{X})$  in (5.3) above. The time-changed version (5.15) of (5.16) does the same job for the process  $(\hat{\Phi}, \hat{X})$  which then gets replaced by the time-space process  $(t+s, \hat{\Phi}_s)_{s \geq 0}$  upon noting that (5.15) can be rewritten as

$$(5.17) \quad \hat{X}_s^x = (\hat{\Phi}_s^\varphi)^{1/\gamma} e^{\frac{\kappa}{2}(t+s)}$$

where we set the initial time to be

$$(5.18) \quad t = \frac{2}{\kappa} \log\left(\frac{x}{\varphi^{1/\gamma}}\right)$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . Note that  $t$  in (5.18) can also be negative. Recalling that  $\varphi = \pi/(1-\pi)$  we see that the initial time  $t$  is determined by the initial values of  $x$  and  $\pi$  in the sequential testing problem (2.4). Once set in motion at the state  $(t, \varphi)$ , the time-space process  $(t+s, \hat{\Phi}_s^\varphi)_{s \geq 0}$  travels only forward in time (thus exhibiting a pure parabolic nature), and the negativity of  $t$  plays no role afterwards. The identification (5.18) plays an important conceptual role in placing the optimal stopping problem (3.15) in the solvable setting.

5. We can now make use of the previous facts and derive a time-space version of the optimal stopping problem (3.15) above. In addition to  $\hat{M}(\varphi) = a\varphi \wedge b$  defined above we also set  $\hat{L}(\varphi) = (1+\varphi)\varphi^{2/\gamma}$  for  $\varphi \in [0, \infty)$  in what follows.

**Proposition 6.** *The value function  $\hat{V}$  from (3.15) satisfies the identity*

$$(5.19) \quad \hat{V}(\varphi, x) = \tilde{V}\left(\frac{2}{\kappa} \log\left(\frac{x}{\varphi^{1/\gamma}}\right), \varphi\right)$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$  where the value function  $\tilde{V}$  is defined by

$$(5.20) \quad \tilde{V}(t, \varphi) = \inf_{\sigma} \mathbf{E}_0 \left[ \int_0^{\sigma} e^{\kappa(t+s)} \hat{L}(\hat{\Phi}_s^\varphi) ds + \hat{M}(\hat{\Phi}_\sigma^\varphi) \right]$$

for  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$  and the infimum is taken over all stopping times  $\sigma$  of  $\hat{\Phi}$ .

**Proof.** We see from (5.13) using (5.17) that

$$(5.21) \quad \begin{aligned} \hat{V}(\varphi, x) &= \inf_{\sigma} \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_s) \hat{X}_s^2 dt + \hat{M}(\hat{\Phi}_\sigma) \right] \\ &= \inf_{\sigma} \mathbf{E}_0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_s^\varphi) (\hat{X}_s^x)^2 dt + \hat{M}(\hat{\Phi}_\sigma^\varphi) \right] \\ &= \inf_{\sigma} \mathbf{E}_0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_s^\varphi) (\hat{\Phi}_s^\varphi)^{2/\gamma} e^{\kappa(t+s)} dt + \hat{M}(\hat{\Phi}_\sigma^\varphi) \right] \\ &= \inf_{\sigma} \mathbf{E}_0 \left[ \int_0^{\sigma} e^{\kappa(t+s)} \hat{L}(\hat{\Phi}_s^\varphi) ds + \hat{M}(\hat{\Phi}_\sigma^\varphi) \right] = \tilde{V}(t, x) \end{aligned}$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$  with  $t$  given in (5.18) above. This completes the proof.  $\square$

We undertake the study of the optimal stopping problem (5.20) in Section 7 below by deriving basic properties of the optimal stopping boundaries.

## 6. Lagrange and Mayer time-space formulations

The optimal stopping problem (5.20) is Bolza formulated. In this section we derive its Lagrange and Mayer reformulations which are helpful in the subsequent analysis of the problem.

1. We first consider the Lagrange reformulation of the optimal stopping problem (5.20).

**Proposition 7.** *The value function  $\tilde{V}$  from (5.20) can be expressed as*

$$(6.1) \quad \tilde{V}(t, \varphi) = \inf_{\sigma} \mathbf{E}_0 \left[ \int_0^{\sigma} e^{\kappa(t+s)} \hat{L}(\hat{\Phi}_s^{\varphi}) ds - \frac{a}{2} \ell_{\sigma}^{b/a}(\hat{\Phi}^{\varphi}) \right] + \hat{M}(\varphi)$$

for  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$  where  $\ell_{\sigma}^{b/a}(\hat{\Phi}^{\varphi})$  is the local time of  $\hat{\Phi}^{\varphi}$  at  $b/a$  and  $\sigma$  given by

$$(6.2) \quad \ell_{\sigma}^{b/a}(\hat{\Phi}^{\varphi}) = \mathbf{P}\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\sigma} I\left(\frac{b}{a} - \varepsilon \leq \hat{\Phi}_t^{\varphi} \leq \frac{b}{a} + \varepsilon\right) d\langle \hat{\Phi}^{\varphi}, \hat{\Phi}^{\varphi} \rangle_t$$

and the infimum in (6.1) is taken over all stopping times  $\sigma$  of  $\hat{\Phi}$ .

**Proof.** This can be derived in exactly the same way as Proposition 3 above (replacing the process  $\Phi$  by its time-changed version  $\hat{\Phi}$ ).  $\square$

The Lagrange reformulation (6.1) of the optimal stopping problem (5.20), similarly to (4.1) in relation to (3.15), reveals the underlying rationale for continuing vs stopping in a clearer manner. This again can be seen by recalling that the local time process  $s \mapsto \ell_s^{b/a}(\hat{\Phi}^{\varphi})$  strictly increases only when  $\hat{\Phi}_s^{\varphi}$  is at  $b/a$ , and that  $\ell_s^{b/a}(\hat{\Phi}^{\varphi}) \sim \sqrt{s}$  is strictly larger than  $\int_0^s e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^{\varphi}) dr \sim s$  for small  $s$ , we see from (6.1) that it should never be optimal to stop at  $\varphi = b/a$  and the incentive for stopping should increase the further away  $\hat{\Phi}_s^{\varphi}$  gets from  $b/a$ . We will see in Section 7 below that these informal conjectures can be formalised as stated in the paragraph following the proof of Proposition 3 above.

2. We next consider the Mayer reformulation of the optimal stopping problem (5.20).

**Proposition 8.** *The value function  $\tilde{V}$  from (5.20) can be expressed as*

$$(6.3) \quad \tilde{V}(t, \varphi) = \inf_{\sigma} \mathbf{E}_0 \left[ e^{\kappa(t+\sigma)} \left( \frac{1}{\delta_0} + \frac{1}{\delta_1} \hat{\Phi}_{\sigma}^{\varphi} \right) (\hat{\Phi}_{\sigma}^{\varphi})^{2/\gamma} + (a \hat{\Phi}_{\sigma}^{\varphi} \wedge b) \right] - e^{\kappa t} \left( \frac{1}{\delta_0} + \frac{1}{\delta_1} \varphi \right) \varphi^{2/\gamma}$$

for  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$  where the infimum is taken over all stopping times  $\sigma$  of  $\hat{\Phi}$ .

**Proof.** This can be derived by time changing (4.5) via (5.2) and using (5.17) above. Alternatively we see from (5.9) that the infinitesimal generator of  $(t+s, \hat{\Phi}_s^{\varphi})_{s \geq 0}$  is given by

$$(6.4) \quad \partial_t + \mathcal{L}_{\hat{\Phi}} = \partial_t + \frac{\gamma^2}{2} \varphi^2 \partial_{\varphi\varphi}.$$



Defining  $\hat{K}(t, \varphi) = e^{\kappa t}((1/\delta_0) + (\varphi/\delta_1))\varphi^{2/\gamma}$  it is easily verified that

$$(6.5) \quad (\partial_t + \mathbb{L}_{\hat{\Phi}})(\hat{K})(t, \varphi) = e^{\kappa t} \hat{L}(\varphi)$$

for  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$  where  $\hat{L}$  is defined above Proposition 6. By Itô's formula we thus get

$$(6.6) \quad \begin{aligned} \hat{K}(t+s, \hat{\Phi}_s^\varphi) &= \hat{K}(t, \varphi) + \int_0^s (\partial_t + \mathbb{L}_{\hat{\Phi}})(\hat{K})(t+r, \hat{\Phi}_r^\varphi) dr + \int_0^s \hat{K}_\varphi(t+r, \hat{\Phi}_r^\varphi) \gamma \hat{\Phi}_r^\varphi d\tilde{B}_r \\ &= \hat{K}(t, \varphi) + \int_0^s e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^\varphi) dr + \int_0^s \gamma e^{\kappa(t+r)} \left( \frac{2}{\gamma \delta_0} + \frac{2+\gamma}{\gamma \delta_1} \hat{\Phi}_r^\varphi \right) (\hat{\Phi}_r^\varphi)^{2/\gamma} d\tilde{B}_r \end{aligned}$$

for  $t \geq 0$  where the final term on the right-hand side defines a continuous local martingale. Making use of a localising sequence of stopping times for this local martingale if needed, and applying the optional sampling theorem, we find from (6.6) that

$$(6.7) \quad \mathbb{E}_0[\hat{K}(t+\sigma, \hat{\Phi}_\sigma^\varphi)] = \hat{K}(t, \varphi) + \mathbb{E}_0 \left[ \int_0^\sigma e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^\varphi) dr \right]$$

for all (bounded) stopping times  $\sigma$  of  $\hat{\Phi}$ . Inserting (6.7) into (5.20) we obtain (6.3) as claimed and the proof is complete.  $\square$

Note that the Mayer reformulation (6.3) of the optimal stopping problem (5.20) is specific to the setting of Bessel processes.

## 7. Properties of the optimal stopping boundaries

In this section we establish the existence of an optimal stopping time in (3.15) and derive basic properties of the optimal stopping boundaries. Given that the optimal stopping problem (3.15) stands in one-to-one correspondence with the optimal stopping problem (5.20) as shown in Proposition 6 above, these facts then translate from (3.15) to (5.20) in a straightforward manner. In the first part of this section we focus on the former problem.

1. Looking at (3.15) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$(7.1) \quad C = \{ (\varphi, x) \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi, x) < \hat{M}(\varphi) \}$$

$$(7.2) \quad D = \{ (\varphi, x) \in [0, \infty) \times [0, \infty) \mid \hat{V}(\varphi, x) = \hat{M}(\varphi) \}$$

respectively. Time changing (4.5) by (5.2) and recalling that (5.11)+(5.12) define Markovian functionals of the initial points, we see that the expectation in (4.5) defines a continuous function of the initial point  $(\varphi, x)$  for every (bounded) stopping time  $\tau$  given and fixed. Taking the infimum over all (bounded) stopping times  $\tau$  we can thus conclude that the value function  $\hat{V}$  is upper semicontinuous. From (4.5) we see that the loss function is continuous and hence lower semicontinuous too. It follows therefore by [8, Corollary 2.9] that the first entry time of the process  $(\Phi, X)$  into the closed set  $D$  defined by

$$(7.3) \quad \tau_D = \inf \{ t \geq 0 \mid (\Phi_t, X_t) \in D \}$$

is optimal in (3.15) whenever  $\mathbb{P}_{\varphi,x}(\tau_D < \infty) = 1$  for all  $(\varphi, x) \in [0, \infty) \times [0, \infty)$ . In the sequel we will establish this and other properties of  $\tau_D$  by analysing the boundary of  $D$ .

2. We first show that the vertical line  $\varphi = b/a$  is contained in  $C$ . This fact is usually established using the scale function and speed measure techniques (see [8, pp 292-293] and [3, pp 523-524]). Motivated by the Lagrange reformulation (4.1) of (3.15) we now give a new proof of this fact based on the local time as discussed following the proof of Proposition 3 above.

**Lemma 9.** *The set  $\{(\varphi, x) \in [0, \infty) \times [0, \infty) \mid \varphi = b/a\}$  is contained in the continuation set  $C$  of the optimal stopping problem (3.15).*

**Proof.** Time changing (3.15) as in the proof of (5.13) and passing from  $\hat{M}(\hat{\Phi})$  to  $\ell^{b/a}(\hat{\Phi})$  as in the proof of (6.1) we see that

$$(7.4) \quad \hat{V}(\varphi, x) = \inf_{\sigma} \mathbb{E}_0 \left[ \int_0^{\sigma} (1 + \hat{\Phi}_s^{\varphi}) (\hat{X}_s^x)^2 ds - \frac{a}{2} \ell_{\sigma}^{b/a}(\hat{\Phi}^{\varphi}) \right] + \hat{M}(\varphi)$$

for  $(\varphi, x) \in (0, \infty) \times (0, \infty)$ . By Itô-Tanaka's formula we find using (5.9) that

$$(7.5) \quad |\hat{\Phi}_t^{b/a} - b/a| = M_t + \ell_t^{b/a}(\hat{\Phi}^{b/a})$$

for  $t \geq 0$  where  $M = (M_t)_{t \geq 0}$  is a continuous martingale. Recalling (5.11) we see that the left-hand side in (7.5) equals

$$(7.6) \quad \left| \frac{b}{a} \left( e^{\gamma \tilde{B}_t - \frac{\gamma^2}{2} t} - 1 \right) \right| = \frac{b}{a} \left| \sum_{n=1}^{\infty} \frac{(\gamma \tilde{B}_t - \frac{\gamma^2}{2} t)^n}{n!} \right|$$

for  $t \geq 0$ . Taking  $\mathbb{E}_0$  on both sides of (7.5)+(7.6) and using that  $\tilde{B}_t \sim \sqrt{t} \tilde{B}_1$  we get

$$(7.7) \quad \mathbb{E}_0[\ell_t^{b/a}(\hat{\Phi}^{b/a})] = \frac{b}{a} \mathbb{E}_0 \left| \sqrt{t} \left( \gamma \tilde{B}_1 - \frac{\gamma^2}{2} \sqrt{t} \right) \sum_{n=1}^{\infty} \frac{(\sqrt{t} \gamma \tilde{B}_1 - \frac{\gamma^2}{2} t)^{n-1}}{n!} \right|$$

for  $t \geq 0$ . Dividing both sides of (7.7) by  $\sqrt{t} > 0$  and letting  $t \downarrow 0$  this shows that

$$(7.8) \quad \lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \mathbb{E}_0[\ell_t^{b/a}(\hat{\Phi}^{b/a})] = \gamma \frac{b}{a} \mathbb{E}_0 |\tilde{B}_1| = \gamma \frac{b}{a} \sqrt{\frac{2}{\pi}}.$$

It means that  $\mathbb{E}_0[\ell_t^{b/a}(\hat{\Phi}^{b/a})] \sim \sqrt{t}$  as  $t \downarrow 0$ . On the other hand, it is clear from (5.11) and (5.12) that  $\mathbb{E}_0[\int_0^t (1 + \hat{\Phi}_s^{b/a}) (\hat{X}_s^x)^2 ds] \sim t$  as  $t \downarrow 0$ . Combining these two facts it is evident that the expectation in (7.4) is strictly negative when  $\varphi = b/a$  if  $\sigma = t$  is taken sufficiently small. This shows that each point  $(b/a, x)$  belongs to  $C$  for  $x > 0$  and the proof is complete.  $\square$

3. Moving from the vertical line  $\varphi = b/a$  outwards let us formally define the (least) boundaries between  $C$  and  $D$  by setting

$$(7.9) \quad b_0(x) = \sup \{ \varphi \in [0, \frac{b}{a}] \mid (\varphi, x) \in D \} \quad \& \quad b_1(x) = \inf \{ \varphi \in [\frac{b}{a}, \infty] \mid (\varphi, x) \in D \}$$

for every  $x > 0$  given and fixed. Clearly  $b_0(x) < \frac{b}{a} < b_1(x)$  for all  $x > 0$  and the supremum and infimum in (7.9) are attained since  $D$  is closed.

**Lemma 10.** *The mapping  $x \mapsto b_0(x)$  is increasing and the mapping  $x \mapsto b_1(x)$  is decreasing with  $0 < b_0(x) < \frac{b}{a} < b_1(x) < \infty$  for all  $x > 0$  and we have*

$$(7.10) \quad C = \{ (\varphi, x) \in [0, \infty) \times [0, \infty) \mid b_0(x) < \varphi < b_1(x) \}$$

$$(7.11) \quad D = \{ (\varphi, x) \in [0, \infty) \times [0, \infty) \mid 0 \leq \varphi \leq b_0(x) \text{ or } b_1(x) \leq \varphi < \infty \}.$$

**Proof.** 1. Note from (5.11)-(5.13) that

$$(7.12) \quad \varphi \mapsto \hat{V}(\varphi, x) \text{ is increasing and concave on } [0, \infty)$$

$$(7.13) \quad x \mapsto \hat{V}(\varphi, x) - \hat{M}(\varphi) \text{ is increasing on } [0, \infty)$$

for each  $x > 0$  and  $\varphi > 0$  given and fixed. Concavity of  $\varphi \mapsto \hat{V}(\varphi, x)$  in (7.12) combined with non-negativity and piecewise linearity of  $\varphi \mapsto \hat{M}(\varphi)$  in (3.15) implies that if  $(\varphi, x) \in D$  with  $\varphi < b/a$  and  $\varphi_1 < \varphi$  then  $(\varphi_1, x) \in D$  as well as that if  $(\varphi, x) \in D$  with  $\varphi > b/a$  and  $\varphi_2 > \varphi$  then  $(\varphi_2, x) \in D$ . This shows that  $b_0$  and  $b_1$  from (7.9) alone separate  $C$  and  $D$  fully and hence (7.10) and (7.11) are valid as claimed. Moreover, if  $(\varphi, x_1) \in D$  and  $x_2 \geq x_1$  then by (7.13) we see that  $0 = \hat{V}(\varphi, x_1) - \hat{M}(\varphi) \leq \hat{V}(\varphi, x_2) - \hat{M}(\varphi) \leq 0$  so that  $\hat{V}(\varphi, x_2) - \hat{M}(\varphi) = 0$  and hence  $(\varphi, x_2) \in D$  as well. This shows that  $x \mapsto b_0(x)$  is increasing and  $x \mapsto b_1(x)$  is decreasing on  $(0, \infty)$  as claimed.

2. We show that  $b_0(x) > 0$  for all  $x > 0$ . For this, suppose that there exists  $x_1 > 0$  such that  $b_0(x_1) = 0$ . By the increase of  $b_0$  we then know that  $[0, \varphi_1] \times [0, x_1]$  is contained in  $C$  where we let  $\varphi_1$  stand for  $b/a$ . Set  $x_0 = x_1/3$  and  $x = 2x_1/3$ , choose any  $\varphi$  in  $(0, \varphi_1)$ , and consider the stopping times

$$(7.14) \quad \sigma_{\varphi_1; x_0, x_1}^{\varphi, x} = \inf \{ t \geq 0 \mid \hat{\Phi}_t^\varphi \geq \varphi_1 \text{ or } \hat{X}_t^x \notin (x_0, x_1) \}$$

$$(7.15) \quad \sigma_{x_0, x_1}^x = \inf \{ t \geq 0 \mid \hat{X}_t^x \notin (x_0, x_1) \}.$$

Then  $\sigma_{\varphi_1; x_0, x_1}^{\varphi, x} \leq \sigma_D$  where  $\sigma_D = \inf \{ t \geq 0 \mid (\hat{\Phi}_t, \hat{X}_t) \in D \}$  is an optimal stopping time so that from (5.13) we find that

$$(7.16) \quad \begin{aligned} \hat{V}(\varphi, x) &= \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma_D} (1 + \hat{\Phi}_t) \hat{X}_t^2 dt + \hat{M}(\hat{\Phi}_{\sigma_D}) \right] \geq \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_1; x_0, x_1}^{\varphi, x}} (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^x)^2 dt \right] \\ &\geq x_0^2 \mathbf{E}_0 [\sigma_{\varphi_1; x_0, x_1}^{\varphi, x}] \rightarrow x_0^2 \mathbf{E}_0 [\sigma_{x_0, x_1}^x] > 0 \end{aligned}$$

as  $\varphi \downarrow 0$ . This shows that taking  $\varphi > 0$  sufficiently small we get  $\hat{V}(\varphi, x) > a\varphi = \hat{M}(\varphi)$  which is a contradiction since  $\hat{V} \leq \hat{M}$ . Hence  $b_0(x) > 0$  for all  $x > 0$  as claimed.

3. We show that  $b_1(x) < \infty$  for all  $x > 0$ . For this, suppose that there exists  $x_1 > 0$  such that  $b_1(x_1) = \infty$ . By the decrease of  $b_1$  we then know that  $[b/a, \infty) \times [0, x_1]$  is contained in  $C$ . Set  $x = x_1/2$ , choose any  $\varphi_0 < \varphi$  in  $[b/a, \infty)$ , and consider the stopping times

$$(7.17) \quad \sigma_{\varphi_0; x_1}^{\varphi, x} = \inf \{ t \geq 0 \mid \hat{\Phi}_t^\varphi \leq \varphi_0 \text{ or } \hat{X}_t^x \geq x_1 \}$$

$$(7.18) \quad \sigma_{x_1}^x = \inf \{ t \geq 0 \mid \hat{X}_t^x \geq x_1 \}.$$

Then  $\sigma_{\varphi_0; x_1}^{\varphi, x} \leq \sigma_D$  so that from (5.13) we find that

$$(7.19) \quad \begin{aligned} \hat{V}(\varphi, x) &= \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma_D} (1 + \hat{\Phi}_t) \hat{X}_t^2 dt + \hat{M}(\hat{\Phi}_{\sigma_D}) \right] \geq \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_0; x_1}^{\varphi, x}} (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^x)^2 dt \right] \\ &\geq (1 + \varphi_0) \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_0; x_1}^{\varphi, x}} (\hat{X}_t^x)^2 dt \right] \rightarrow (1 + \varphi_0) \mathbf{E}_0 \left[ \int_0^{\sigma_{x_1}^x} (\hat{X}_t^x)^2 dt \right] > b \end{aligned}$$

as  $\varphi \rightarrow \infty$  if  $\varphi_0$  is taken large enough for the final/strict inequality to hold (upon noting that the final expectation is strictly positive). This shows that taking  $\varphi > 0$  sufficiently large we get  $\hat{V}(\varphi, x) > b = \hat{M}(\varphi)$  which is a contradiction since  $\hat{V} \leq \hat{M}$ . Hence  $b_1(x) < \infty$  for all  $x > 0$  as claimed and the proof is complete.  $\square$

**Lemma 11.** *The following relations hold*

$$(7.20) \quad \lim_{x \downarrow 0} b_0(x) = 0 \quad \& \quad \lim_{x \downarrow 0} b_1(x) = \infty$$

$$(7.21) \quad \lim_{x \rightarrow \infty} b_0(x) = \lim_{x \rightarrow \infty} b_1(x) = \frac{b}{a}.$$

**Proof.** 1. For (7.20) in view of Lemma 10 it is enough to show that for every  $\varphi > 0$  (small and large) there exists  $x > 0$  small enough such that  $(\varphi, x)$  belongs to  $C$ . For this, fix any  $\varphi > 0$  and note from (5.13) that

$$(7.22) \quad \hat{V}(\varphi, x) \leq \mathbf{E}_0 \left[ \int_0^s (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^x)^2 dt + \hat{M}(\hat{\Phi}_s^\varphi) \right]$$

for all  $x > 0$  with any  $s > 0$  given and fixed. Since the random variable  $\hat{\Phi}_s^\varphi$  has for its support the entire  $(0, \infty)$ , it is easily verified that the Jensen inequality is strict

$$(7.23) \quad \mathbf{E}_0[\hat{M}(\hat{\Phi}_s^\varphi)] < \hat{M}(\varphi)$$

where we note that  $\varphi \mapsto \hat{M}(\varphi) = a\varphi \wedge b$  is concave on  $(0, \infty)$ . Recalling (5.12) we see that

$$(7.24) \quad \mathbf{E}_0 \left[ \int_0^s (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^x)^2 dt \right] = x^2 \mathbf{E}_0 \left[ \int_0^s (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^1)^2 dt \right]$$

can be made arbitrarily small in  $(0, \infty)$  by choosing  $x > 0$  sufficiently small. Combining (7.22)-(7.24) we see that  $\hat{V}(\varphi, x) < \hat{M}(\varphi)$  for  $x > 0$  sufficiently small so that  $(\varphi, x)$  belongs to  $C$  as needed and the proof of (7.20) is complete.

2. For (7.21) suppose that  $b_0(\infty) := \lim_{x \rightarrow \infty} b_0(x) < b/a$  and fix any  $\varphi \in (b_0(\infty), b/a)$  (e.g. the mid point). Consider the stopping times

$$(7.25) \quad \sigma_{\varphi_0, \varphi_1; x_0}^{\varphi, x} = \inf \{ t \geq 0 \mid \hat{\Phi}_t^\varphi \notin (\varphi_0, \varphi_1) \text{ or } \hat{X}_t^x \leq x_0 \}$$

$$(7.26) \quad \sigma_{\varphi_0, \varphi_1}^\varphi = \inf \{ t \geq 0 \mid \hat{\Phi}_t^\varphi \notin (\varphi_0, \varphi_1) \}$$

with  $x > x_0$  in  $(0, \infty)$  given and fixed where we set  $\varphi_0 := b(\infty)$  and  $\varphi_1 = b/a$ . Then  $\sigma_{\varphi_0, \varphi_1; x_0}^{\varphi, x} \leq \sigma_D$  so that from (5.13) we find that

$$(7.27) \quad \begin{aligned} \hat{V}(\varphi, x) &= \mathbf{E}_{\varphi, x}^0 \left[ \int_0^{\sigma_D} (1 + \hat{\Phi}_t) \hat{X}_t^2 dt + \hat{M}(\hat{\Phi}_{\sigma_D}) \right] \geq \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_0, \varphi_1; x_0}^{\varphi, x}} (1 + \hat{\Phi}_t^\varphi) (\hat{X}_t^x)^2 dt \right] \\ &\geq x^2 \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_0, \varphi_1; x_0}^{\varphi, x}} (\hat{X}_t^1)^2 dt \right] \rightarrow \infty^2 \mathbf{E}_0 \left[ \int_0^{\sigma_{\varphi_0, \varphi_1}^{\varphi}} (\hat{X}_t^1)^2 dt \right] = \infty \end{aligned}$$

as  $x \rightarrow \infty$  (upon noting that the final expectation is strictly positive) which is a contradiction since  $\hat{V} \leq \hat{M} \leq b < \infty$ . This shows that  $b_0(\infty) = b/a$  as claimed. The case  $(b/a, b_1(\infty)) \neq \emptyset$  can be disproved similarly and this completes the proof.  $\square$

4. The results of Lemmas 9-11 translate from the optimal stopping problem (3.15) to its time-space version (5.20) in a straightforward manner using the diffeomorphic transformation described in (5.15)-(5.18). This can be summarised as follows (see Figure 2 below).

**Corollary 12.** *The continuation and stopping sets in the optimal stopping problem (5.20) are given by the following expressions*

$$(7.28) \quad \begin{aligned} \tilde{C} &= \{ (t, \varphi) \in \mathbb{R} \times [0, \infty) \mid \tilde{V}(t, \varphi) < \hat{M}(\varphi) \} \\ &= \{ (t, \varphi) \in \mathbb{R} \times [0, \infty) \mid \tilde{b}_0(t) < \varphi < \tilde{b}_1(t) \} \end{aligned}$$

$$(7.29) \quad \begin{aligned} \tilde{D} &= \{ (t, \varphi) \in \mathbb{R} \times [0, \infty) \mid \tilde{V}(\varphi, x) = \hat{M}(\varphi) \} \\ &= \{ (t, \varphi) \in \mathbb{R} \times [0, \infty) \mid 0 \leq \varphi \leq \tilde{b}_0(t) \text{ or } \tilde{b}_1(t) \leq \varphi < \infty \} \end{aligned}$$

respectively, where the mapping  $t \mapsto \tilde{b}_0(t)$  is increasing and the mapping  $t \mapsto \tilde{b}_1(t)$  is decreasing with  $0 \leq \tilde{b}_0(t) < \frac{b}{a} < \tilde{b}_1(t) < \infty$  for all  $t \in \mathbb{R}$  and the following relations hold

$$(7.30) \quad \lim_{t \rightarrow -\infty} \tilde{b}_0(t) = 0 \quad \& \quad \lim_{t \rightarrow -\infty} \tilde{b}_1(t) = \infty$$

$$(7.31) \quad \lim_{t \rightarrow \infty} \tilde{b}_0(t) = \lim_{t \rightarrow \infty} \tilde{b}_1(t) = \frac{b}{a}.$$

**Proof.** All claims follow directly from the facts proved in Lemmas 9-11 using the equivalence between the problem (3.15) and its time-changed version (5.13) combined with diffeomorphic transformation (5.18) which realises the equivalence between the problem (5.13) and its time-space version (5.20) as established in Proposition 6 above.  $\square$

**Remark 1.** The conclusion that  $\tilde{b}_0(t) > 0$  for all  $t \in \mathbb{R}$  in Corollary 12 cannot be derived directly from the fact that  $b_0(x) > 0$  for all  $x > 0$  with  $\lim_{x \downarrow 0} b_0(x) = 0$  using (5.18) because this implication would require some information on the rate of convergence in the latter limit. We now show that  $\tilde{b}_0(t) > 0$  for all  $t \in \mathbb{R}$  when  $\gamma > 2$ . This can be done through a direct analysis of the optimal stopping problem (5.20) as follows. Suppose that  $\tilde{b}_0(t_1) = 0$  for some  $t_1 \in \mathbb{R}$  and fix any  $t_0 < t_1$ . Set  $\delta = t_1 - t_0$  and consider the stopping time

$$(7.32) \quad \sigma_{b/a}^\varepsilon = \inf \{ s \in [0, \delta] \mid \hat{\Phi}_s^\varepsilon \geq b/a \}$$

for  $\varepsilon > 0$  given and fixed. Since  $[t_0, t_1] \times [0, b/a] \subseteq \tilde{C}$  we see that  $\sigma_D^{t_0, \varepsilon} \geq \sigma_{b/a}^\varepsilon$  where  $\sigma_D^{t_0, \varepsilon} =$

$\inf \{ s \geq 0 \mid (t_0 + s, \hat{\Phi}_s^\varepsilon) \in \tilde{D} \}$  is the optimal stopping time in (5.20). It follows therefore that

$$(7.33) \quad \tilde{V}(t_0, \varepsilon) \geq \mathbf{E}_0 \left[ \int_0^{\sigma_{\tilde{b}/a}^\varepsilon} e^{\kappa(t_0+s)} (1 + \hat{\Phi}_s^\varepsilon) (\hat{\Phi}_s^\varepsilon)^{2/\gamma} ds \right] \geq \varepsilon^{2/\gamma} \mathbf{E}_0 \left[ \int_0^{\sigma_{\tilde{b}/a}^\varepsilon} e^{\kappa(t_0+s)} e^{2\tilde{B}_s - \gamma s} ds \right]$$

where we use (5.11) and note that

$$(7.34) \quad \mathbf{E}_0 \left[ \int_0^{\sigma_{\tilde{b}/a}^\varepsilon} e^{\kappa(t_0+s)} e^{2\tilde{B}_s - \gamma s} ds \right] \rightarrow \mathbf{E}_0 \left[ \int_0^\delta e^{\kappa(t_0+s)} e^{2\tilde{B}_s - \gamma s} ds \right] =: I > 0$$

as  $\varepsilon \downarrow 0$  since  $\sigma_{\tilde{b}/a}^\varepsilon \uparrow \delta$  due to  $\Phi^\varepsilon \downarrow 0$ . From (7.33) and (7.34) we see that taking  $\varepsilon > 0$  sufficiently small we get  $\tilde{V}(t_0, \varepsilon) \geq (I/2) \varepsilon^{2/\gamma} > a\varepsilon = \hat{M}(\varepsilon)$  which is a contradiction since  $\tilde{V} \leq \hat{M}$ . Hence  $\tilde{b}_0(t) > 0$  for all  $t \in \mathbb{R}$  when  $\gamma > 2$  as claimed.

**Remark 2.** We will see in Section 9 below that the optimal stopping boundaries  $\tilde{b}_0$  &  $\tilde{b}_1$  can be characterised as the unique solution to a coupled system of nonlinear Volterra integral equations. These equations can be used to find the optimal stopping boundaries  $\tilde{b}_0$  &  $\tilde{b}_1$  numerically (as shown in Figure 2 below). Using (5.18) it is easily seen that

$$(7.35) \quad b_0^{-1}(\varphi) = \varphi^{1/\gamma} \exp\left(\frac{\kappa}{2} \tilde{b}_0^{-1}(\varphi)\right) \quad \& \quad b_1^{-1}(\varphi) = \varphi^{1/\gamma} \exp\left(\frac{\kappa}{2} \tilde{b}_1^{-1}(\varphi)\right)$$

for  $\varphi \geq 0$  and these identities can then be used to find  $b_0$  &  $b_1$  numerically (as shown in Figure 3 below). We will return to this point at the end of Section 9 below.

## 8. Free-boundary problems

In this section we derive the free-boundary problems that stand in one-to-one correspondence with the optimal stopping problems (3.15) and (5.20) respectively. The two free-boundary problems are equivalent and the latter problem can be seen as a canonical time-changed reformulation of the former problem. Using results derived in the previous sections we show that the value functions and their optimal stopping boundaries  $(\hat{V}; b_0, b_1)$  and  $(\tilde{V}; \tilde{b}_0, \tilde{b}_1)$  from (3.15) and (5.20) solve the free-boundary problems respectively. This establishes the existence of a solution. Its uniqueness in natural classes of functions will follow from a more general uniqueness result established in Section 9 below. This will also yield a double-integral representation for the value function  $\tilde{V}$  expressed in terms of the optimal stopping boundaries  $\tilde{b}_0$  &  $\tilde{b}_1$ . A similar integral representation also holds for the value function  $\hat{V}$  expressed in terms of the optimal stopping boundaries  $b_0$  &  $b_1$  but we will not state it explicitly.

1. We first consider the optimal stopping problem (3.15) where the strong Markov process  $(\Phi, X)$  solves the system of stochastic differential equations (3.16)+(3.17) under the measure  $\mathbf{P}_{\varphi, x}$  with  $(\varphi, x) \in [0, \infty) \times [0, \infty)$ . Recalling that the infinitesimal generator  $\mathcal{L}_{\Phi, X}$  of  $(\Phi, X)$  is given by (4.10) with (2.3)+(2.8) above and relying on other properties of  $\hat{V}$  and  $b_0$  &  $b_1$  derived in Section 7 above, we are naturally led to formulate the following free-boundary problem for finding  $\hat{V}$  and  $b_0$  &  $b_1$ :

$$(8.1) \quad \frac{\delta_0 - 1}{2x} \hat{V}_x + \gamma \frac{\varphi}{x} \hat{V}_{\varphi x} + \frac{\gamma^2}{2} \frac{\varphi^2}{x^2} \hat{V}_{\varphi\varphi} + \frac{1}{2} \hat{V}_{xx} = -L \quad \text{in } C$$

$$(8.2) \quad \hat{V}(\varphi, x) = \hat{M}(\varphi) \quad \text{for } (\varphi, x) \in D \quad (\text{instantaneous stopping})$$

$$(8.3) \quad \hat{V}_\varphi(\varphi, x) = \hat{M}'(\varphi) \quad \text{for } \varphi = b_0(x) \ \& \ \varphi = b_1(x) \quad \text{with } x > 0 \quad (\text{smooth fit})$$

$$(8.4) \quad \hat{V}_x(\varphi, x) = 0 \quad \text{for } \varphi = b_0(x) \ \& \ \varphi = b_1(x) \quad \text{with } x > 0 \quad (\text{smooth fit})$$

where we set  $L(\varphi) = 1 + \varphi$  for  $\varphi \in [0, \infty)$  and the (continuation) set  $C$  and the (stopping) set  $D$  are given by (7.10) and (7.11) respectively. Clearly the global condition (8.2) can be replaced by the local condition  $\hat{V}(\varphi, x) = \hat{M}(\varphi)$  for  $\varphi = b_0(x) \ \& \ \varphi = b_1(x)$  with  $x > 0$  so that the free-boundary problem (8.1)-(8.4) needs to be considered on the closure of  $C$  only (extending  $\hat{V}$  to the rest of  $D$  as  $\hat{M}$  being then evident).

2. We next consider the optimal stopping problem (5.20) where the strong Markov process  $(t+s, \hat{\Phi}_s^\varphi)_{s \geq 0}$  in its second/spatial component solves the stochastic differential equation (5.9) under the measure  $\mathbf{P}_0$  yielding the explicit representation (5.11) for  $\varphi > 0$ . Recalling that the infinitesimal generator  $\partial_t + \mathbf{L}_{\hat{\Phi}}$  of  $(t+s, \hat{\Phi}_s^\varphi)_{s \geq 0}$  is given by (6.4) and relying on the connection between  $(t+s, \hat{\Phi}_s^\varphi)_{s \geq 0}$  and  $(\Phi, X)$  realised through (5.13) and (5.15)-(5.18) combined with other properties of  $\tilde{V}$  and  $\tilde{b}_0 \ \& \ \tilde{b}_1$  derived in Section 7 above, we are naturally led to formulate the following free-boundary problem for finding  $\tilde{V}$  and  $\tilde{b}_0 \ \& \ \tilde{b}_1$ :

$$(8.5) \quad \left( \tilde{V}_t + \frac{\gamma^2}{2} \varphi^2 \tilde{V}_{\varphi\varphi} \right)(t, \varphi) = -e^{\kappa t} \hat{L}(\varphi) \quad \text{for } (t, \varphi) \in \tilde{C}$$

$$(8.6) \quad \tilde{V}(t, \varphi) = \hat{M}(\varphi) \quad \text{for } (t, \varphi) \in \tilde{D} \quad (\text{instantaneous stopping})$$

$$(8.7) \quad \tilde{V}_t(t, \varphi) = 0 \quad \text{for } \varphi = \tilde{b}_0(t) \ \& \ \varphi = \tilde{b}_1(t) \quad \text{with } t \in \mathbb{R} \quad (\text{smooth fit})$$

$$(8.8) \quad \tilde{V}_\varphi(t, \varphi) = \hat{M}'(\varphi) \quad \text{for } \varphi = \tilde{b}_0(t) \ \& \ \varphi = \tilde{b}_1(t) \quad \text{with } t \in \mathbb{R} \quad (\text{smooth fit})$$

where we recall that  $\hat{L}(\varphi) = (1 + \varphi)\varphi^{2/\gamma}$  for  $\varphi \in [0, \infty)$  and the (continuation) set  $\tilde{C}$  and the (stopping) set  $\tilde{D}$  are given by (7.28) and (7.29) respectively. Clearly the global condition (8.6) can be replaced by the local condition  $\tilde{V}(t, \varphi) = \hat{M}(\varphi)$  for  $\varphi = \tilde{b}_0(t) \ \& \ \varphi = \tilde{b}_1(t)$  with  $t \in \mathbb{R}$  so that the free-boundary problem (8.5)-(8.8) needs to be considered on the closure of  $\tilde{C}$  only (extending  $\tilde{V}$  to the rest of  $D$  as  $\hat{M}$  being then evident).

3. To formulate the existence and uniqueness result for the free-boundary problem (8.1)-(8.4) we let  $\mathcal{C}$  denote the class of functions  $(F; a_0, a_1)$  such that

$$(8.9) \quad F \text{ belongs to } C^1(\bar{C}_{a_0, a_1}) \cap C^2(C_{a_0, a_1}) \text{ and is bounded on } [0, \infty) \times [0, \infty)$$

$$(8.10) \quad a_0 \text{ is continuous and increasing on } (0, \infty) \text{ with } a_0(0+) = 0 \text{ and } a_0(\infty) = b/a$$

$$(8.11) \quad a_1 \text{ is continuous and decreasing on } (0, \infty) \text{ with } a_1(0+) = \infty \text{ and } a_1(\infty) = b/a$$

where we set  $C_{a_0, a_1} = \{ (\varphi, x) \in [0, \infty) \times [0, \infty) \mid a_0(x) < \varphi < a_1(x) \}$ .

**Theorem 13.** *The free-boundary problem (8.1)-(8.4) has a unique solution  $(\hat{V}; b_0, b_1)$  in the class  $\mathcal{C}$  where  $\hat{V}$  is given by (3.15) and  $b_0 \ \& \ b_1$  are defined in (7.9).*

**Proof.** The first fact to note is that the boundary points between  $C$  and  $D$  are regular for  $D$  relative to  $(\Phi, X)$  and  $(\hat{\Phi}, \hat{X})$  in the sense that

$$(8.12) \quad \tau_D^{\varphi_n, x_n} \rightarrow 0 \quad \& \quad \sigma_D^{\varphi_n, x_n} \rightarrow 0$$

with  $\mathbb{P}_0$ -probability one whenever  $(\varphi_n, x_n)$  from  $C$  tends to  $(\varphi, x)$  at its boundary  $\partial C$  specified by  $\varphi = b_0(x)$  or  $\varphi = b_1(x)$  for  $x > 0$  as  $n \rightarrow \infty$ . Recall in (8.12) that  $\tau_D^{\varphi_n, x_n}$  is the first entry time of  $(\Phi^{\varphi_n, x_n}, X^{x_n})$  into  $D$  and  $\sigma_D^{\varphi_n, x_n}$  is the first entry time of  $(\hat{\Phi}^{\varphi_n, x_n}, \hat{X}^{x_n})$  into  $D$  for  $n \geq 1$ . It is well known (because both  $(\Phi, X)$  and  $(\hat{\Phi}, \hat{X})$  are strong Feller processes) that (8.12) is equivalent to the fact that the first hitting times of  $(\Phi, X)$  and  $(\hat{\Phi}, \hat{X})$  to  $D$  defined by  $\tilde{\tau}_D = \inf \{ t > 0 \mid (\Phi_t, X_t) \in D \}$  and  $\tilde{\sigma}_D = \inf \{ t > 0 \mid (\hat{\Phi}_t, \hat{X}_t) \in D \}$  are equal to zero with  $\mathbb{P}^\infty$ -probability one whenever  $(\varphi, x)$  belongs to  $\partial C$ . Given that the time change  $t \mapsto T_t$  in (5.2), which builds  $(\hat{\Phi}, \hat{X})$  from  $(\Phi, X)$ , is strictly increasing on  $[0, \infty)$ , we thus see that the boundary points in  $\partial C$  are regular for  $D$  relative to  $(\Phi, X)$  if and only if they are regular relative to  $(\hat{\Phi}, \hat{X})$ . The latter process however is just a pair of geometric Brownian motions (5.11) and (5.12) for which (upon recalling that  $b_0$  &  $b_1$  are monotone) the regularity at each point  $\varphi$  or  $x$  for  $[\varphi, \infty)$  or  $[x, \infty)$  respectively is evident from the regularity of 0 for  $[0, \infty)$  relative to standard Brownian motion (with drift). These arguments establish (8.12) and equipped with this fact we can then adapt the proof of Proposition 13 from [4] and infer the global  $C^1$  regularity of the value function  $\hat{V}$  in the sense that

$$(8.13) \quad (\varphi, x) \mapsto \hat{V}_\varphi(\varphi, x) \text{ is continuous on } [0, \infty) \times [0, \infty)$$

$$(8.14) \quad (\varphi, x) \mapsto \hat{V}_x(\varphi, x) \text{ is continuous on } [0, \infty) \times [0, \infty).$$

Moreover, since the problem (3.15) stands in one-to-one correspondence with the problem (5.20) for the time-space process whose infinitesimal generator has only one partial derivative with respect to time, and this correspondence is established by the diffeomorphic transformation (5.18) (in addition to the time change (5.2)), we see that the same arguments as in the proof of Corollary 14 from [4] imply that

$$(8.15) \quad \tilde{V}_{\varphi\varphi} \text{ admits a continuous extension from } \tilde{C} \text{ to } \text{cl}(\tilde{C})$$

where for notational reasons we let  $\text{cl}(\tilde{C})$  denote the closure of  $\tilde{C}$ . For the same reason we see that the analogous arguments as in the proof of Proposition 15 from [4] imply the basic regularity of  $b_0$  &  $b_1$  in the sense that

$$(8.16) \quad x \mapsto b_0(x) \text{ \& } x \mapsto b_1(x) \text{ are continuous on } (0, \infty).$$

Combined with other properties derived in Section 7 above, this shows that the triple  $(\hat{V}; b_0, b_1)$  belongs to the class  $\mathcal{C}$ . Moreover, from the Bolza formulation (3.15) we know that  $\hat{V}$  solves (8.1) and from (8.13)+(8.14) we know that  $\hat{V}$  satisfies (8.3) and (8.4). Since  $\hat{V}$  evidently satisfies (8.2) this show that the triple  $(\hat{V}; b_0, b_1)$  is a solution to the free-boundary problem (8.1)-(8.4) in the class  $\mathcal{C}$ . To derive uniqueness of the solution we will first see in the next section that any solution  $(\tilde{F}; \tilde{a}_0, \tilde{a}_1)$  to the free-boundary problem (8.5)-(8.8) in the class  $\tilde{\mathcal{C}}$  defined analogously to  $\mathcal{C}$  (to be specified below) admits a closed double-integral representation for  $\tilde{F}$  in terms of  $\tilde{a}_0$  &  $\tilde{a}_1$ , which in turn solve a coupled system of nonlinear Volterra integral equations, and we will see that this system cannot have other solutions satisfying the required properties. Recalling that the problem (3.15) stands in one-to-one correspondence with the problem (5.20) and putting these facts together we can conclude that there cannot be more than one solution to (8.1)-(8.4) in the class  $\mathcal{C}$  as claimed.  $\square$



4. To formulate the existence and uniqueness result for the free-boundary problem (8.5)-(8.8) we let  $\tilde{\mathcal{C}}$  be defined in exactly the same ways as  $\mathcal{C}$  above with the domains  $[0, \infty) \times [0, \infty)$  and  $(0, \infty)$  being replaced by the domains  $\mathbb{R} \times [0, \infty)$  and  $\mathbb{R}$  respectively (the right-hand limits at 0 in (8.10) and (8.11) becoming the limits at  $-\infty$  with the same values).

**Corollary 14.** *The free-boundary problem (8.5)-(8.8) has a unique solution  $(\tilde{V}; \tilde{b}_0, \tilde{b}_1)$  in the class  $\tilde{\mathcal{C}}$  where  $\tilde{V}$  is given by (5.20) and  $\tilde{b}_0$  &  $\tilde{b}_1$  are defined in (7.32).*

**Proof.** This follows from Theorem 13 using the fact that the value function  $\tilde{V}$  from (5.20) and its optimal stopping boundaries  $\tilde{b}_0$  &  $\tilde{b}_1$  are a canonical time-changed reformulation of the value function  $\hat{V}$  from (3.15) and its optimal stopping boundaries  $b_0$  and  $b_1$  obtained by means of the diffeomorphic transformation (5.18) as explained in Section 5 above.  $\square$

## 9. Nonlinear integral equations

In this section we show that the optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$  from (7.29) can be characterised as the unique solution to a coupled system of nonlinear Volterra integral equations. This also yields a closed double-integral representation of the value function  $\tilde{V}$  from (5.20) expressed in terms of the optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$ . Analogous results also hold for the optimal stopping boundaries  $b_0$  and  $b_1$  from (7.9) and the value function  $\hat{V}$  from (3.15) but we will not state them explicitly. As a consequence of the existence and uniqueness result for the coupled system of nonlinear Volterra integral equations we also obtain uniqueness of the solution to the free-boundary problems (8.1)-(8.4) and (8.5)-(8.8) as explained in the proofs of Theorem 13 and Corollary 14 above. Finally, collecting the results derived throughout the paper we conclude our exposition at the end of this section by disclosing the solution to the initial problem.

From (5.11) we easily find that the probability density function of  $\hat{\Phi}_t^\varphi$  is given by

$$(9.1) \quad f(\varphi; t, \psi) = \frac{1}{\gamma \sqrt{2\pi t} \psi} \exp \left[ -\frac{1}{2\gamma^2 t} \left( \log \left( \frac{\psi}{\varphi} \right) + \frac{\gamma^2}{2} t \right)^2 \right]$$

for  $t > 0$  and  $\varphi$  &  $\psi$  in  $(0, \infty)$ . Having  $f$  we can evaluate the expression of interest appearing in the theorem below as follows

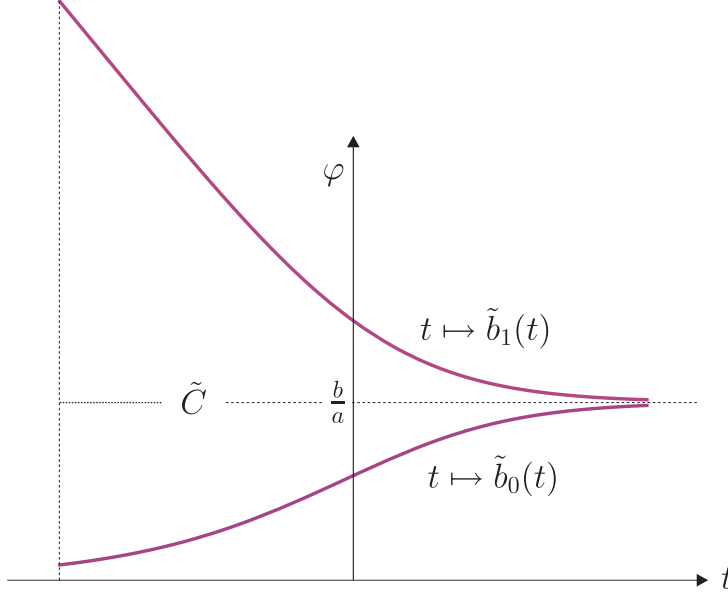
$$(9.2) \quad K(\varphi; t, \varphi_1, \varphi_2) := \mathbb{E}_0 [\hat{L}(\Phi_t^\varphi) I(\varphi_1 < \Phi_t^\varphi < \varphi_2)] = \int_{\varphi_1}^{\varphi_2} \hat{L}(\psi) f(\varphi; t, \psi) d\psi$$

for  $t > 0$  and  $\varphi$ ,  $\varphi_1$  &  $\varphi_2$  in  $(0, \infty)$  where  $\hat{L}(\psi) = (1+\psi)\psi^{2/\gamma}$  for  $\psi > 0$ .

**Theorem 15 (Existence and uniqueness).** *The optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$  in the problem (5.20) can be characterised as the unique solution to the coupled system of nonlinear Volterra integral equations*

$$(9.3) \quad a\tilde{b}_0(t) = \int_0^\infty e^{\kappa(t+s)} K(\tilde{b}_0(t); s, \tilde{b}_0(t+s), \tilde{b}_1(t+s)) ds$$

$$(9.4) \quad b = \int_0^\infty e^{\kappa(t+s)} K(\tilde{b}_1(t); s, \tilde{b}_0(t+s), \tilde{b}_1(t+s)) ds$$



**Figure 2.** Location of the optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$  for the Bessel motion from Figure 1.

in the class of continuous functions  $\tilde{b}_0$  and  $\tilde{b}_1$  on  $\mathbb{R}$  where  $t \mapsto \tilde{b}_0(t)$  is increasing and  $t \mapsto \tilde{b}_1(t)$  is decreasing with  $0 \leq \tilde{b}_0(t) < \frac{b}{a} < \tilde{b}_1(t) < \infty$  for  $t \in \mathbb{R}$ . The value function  $\tilde{V}$  in the problem (5.20) admits the following representation

$$(9.5) \quad \tilde{V}(t, \varphi) = \int_0^\infty e^{\kappa(t+s)} K(\varphi; s, \tilde{b}_0(t+s), \tilde{b}_1(t+s)) ds$$

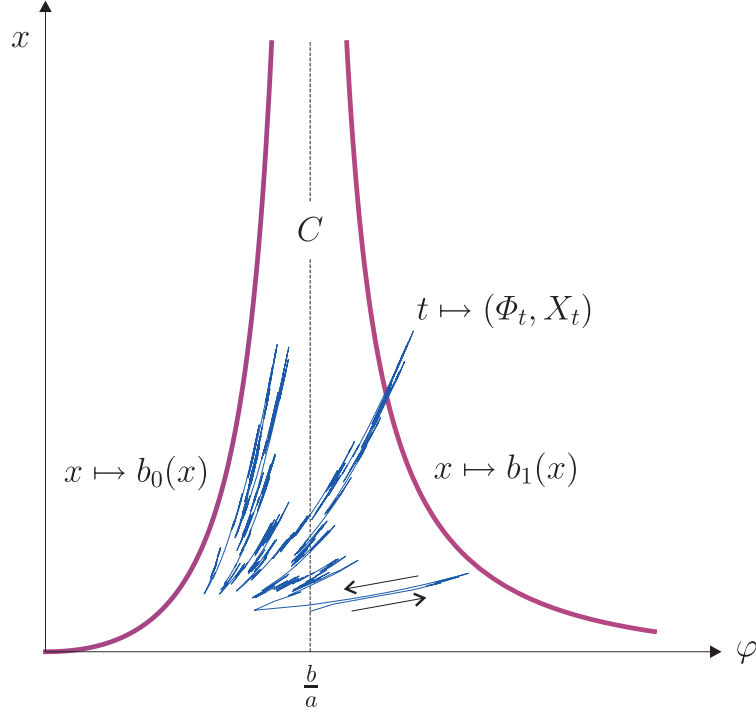
for  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$ . The optimal stopping time in the problem (5.20) is given by

$$(9.6) \quad \sigma_{\tilde{b}_0, \tilde{b}_1} = \inf \{ s \geq 0 \mid \hat{\Phi}_s^\varphi \notin (\tilde{b}_0(t+s), \tilde{b}_1(t+s)) \}$$

under  $\mathbb{P}_0$  with  $(t, \varphi) \in \mathbb{R} \times (0, \infty)$  given and fixed (see Figure 2 above).

**Proof.** 1. *Existence.* We first show that the optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$  in the problem (5.20) solve the system (9.3)+(9.4). Recalling that  $\tilde{b}_0$  and  $\tilde{b}_1$  satisfy the properties stated following (9.3)+(9.4) as established above, this will prove the existence of the solution to (9.3)+(9.4). For this, we will first note that Itô's formula is applicable to  $\tilde{V}$  composed with  $(t+s, \hat{\Phi}_s^\varphi)$  for  $s \geq 0$  with  $t \in \mathbb{R}$  and  $\varphi \in (0, \infty)$  given and fixed. Indeed, recalling that  $\tilde{V}$  is  $C^{1,2}$  on the closure of  $\tilde{C}$  and equals  $\hat{M}$  on  $\tilde{D}$  (which also is  $C^{1,2}$  since the line  $\varphi = b/a$  at which  $\hat{M}$  as a function of two arguments is non-smooth belongs to  $\tilde{C}$  as established above) we see that the local time-space formula from [7] is applicable to  $\tilde{V}$  composed with  $(t+s, \hat{\Phi}_s^\varphi)$  for  $s \geq 0$  and moreover this formula reduces to Itô's formula due to the smooth fit condition (8.8). Using (8.5)+(8.6) this yields

$$(9.7) \quad \begin{aligned} \tilde{V}(t+s, \hat{\Phi}_s^\varphi) &= \tilde{V}(t, \varphi) + \int_0^s (\tilde{V}_t + \mathbb{L}_{\hat{\Phi}} \tilde{V})(t+r, \hat{\Phi}_r^\varphi) dr + \int_0^s \tilde{V}_\varphi(t+r, \hat{\Phi}_r^\varphi) \gamma \hat{\Phi}_r^\varphi d\tilde{B}_r \\ &= \tilde{V}(t, \varphi) - \int_0^s e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^\varphi) I(\tilde{b}_0(t+r) < \hat{\Phi}_r^\varphi < \tilde{b}_1(t+r)) dr + M_s \end{aligned}$$



**Figure 3.** Kinematics of the process  $(\Phi, X)$  associated with the Bessel motion from Figure 1 and location of the optimal stopping boundaries  $b_0$  and  $b_1$ .

where  $M_s = \int_0^s \tilde{V}_\varphi(t+r, \hat{\Phi}_r^\varphi) \gamma \hat{\Phi}_r^\varphi d\tilde{B}_r$  is a continuous local martingale for  $s \geq 0$ . Taking a localisation sequence of stopping times  $(\tau_n)_{n \geq 1}$  for  $M$ , replacing  $s$  on both sides of (9.7) by  $s \wedge \tau_n$ , applying the optional sampling theorem and letting  $n \rightarrow \infty$ , we obtain

$$(9.8) \quad \mathbf{E}_0[\tilde{V}(t+s, \hat{\Phi}_s^\varphi)] = \tilde{V}(t, \varphi) - \mathbf{E}_0\left[\int_0^s e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^\varphi) I(\tilde{b}_0(t+r) < \hat{\Phi}_r^\varphi < \tilde{b}_1(t+r)) dr\right]$$

for  $s \geq 0$ . Letting  $s \rightarrow \infty$  and noting that  $0 \leq \tilde{V}(t+s, \hat{\Phi}_s^\varphi) \leq \hat{M}(\hat{\Phi}_s^\varphi) = a\hat{\Phi}_s^\varphi \wedge b \rightarrow 0$  we see that the dominated and monotone convergence theorems yield

$$(9.9) \quad \tilde{V}(t, \varphi) = \mathbf{E}_0\left[\int_0^\infty e^{\kappa(t+r)} \hat{L}(\hat{\Phi}_r^\varphi) I(\tilde{b}_0(t+r) < \hat{\Phi}_r^\varphi < \tilde{b}_1(t+r)) dr\right]$$

which establishes the representation (9.5) upon recalling (9.2) above. Recalling that  $\tilde{V}(t, \tilde{b}_0(t)) = \hat{M}(\tilde{b}_0(t)) = a\tilde{b}_0(t)$  and  $\tilde{V}(t, \tilde{b}_1(t)) = \hat{M}(\tilde{b}_1(t)) = b$  we see that (9.5) implies (9.3) and (9.4). This shows that  $\tilde{b}_0$  and  $\tilde{b}_1$  solve the system (9.3)+(9.4) as claimed.

*2. Uniqueness.* To show that  $\tilde{b}_0$  and  $\tilde{b}_1$  are a unique solution to the system (3.3)+(3.4) one can adopt the four-step procedure from the proof of uniqueness given on [1, Theorem 4.1] extending and further refining the original arguments from [6, Theorem 3.1] in the case of a single boundary. Given that the present setting creates no additional difficulties we will omit further details of this verification and this completes the proof.  $\square$

The coupled system of nonlinear Volterra integral equations (9.3)+(9.4) can be used to find the optimal stopping boundaries  $\tilde{b}_0$  and  $\tilde{b}_1$  numerically. Note that the identity (9.8) can be

used to produce a finite horizon approximation to the system obtained by replacing  $s$  with  $T-t$  in (9.8) which yields (9.9) and hence (9.3)+(9.4) as well with  $T-t$  in place of  $\infty$  as the upper limit of integration (making (9.3)+(9.4) solvable numerically by backward recursion). Having found  $\tilde{b}_0$  and  $\tilde{b}_1$  the identities (7.35) can be used to calculate  $b_0$  and  $b_1$  numerically. Collecting the results derived throughout we now disclose the solution to the initial problem.

**Corollary 16.** *With the initial point  $x > 0$  of the process  $X$  solving (2.2)+(2.3) given and fixed, the value function of the initial problem (2.4) is given by*

$$(9.10) \quad V(\pi) = (1-\pi) \tilde{V} \left( \frac{2}{\kappa} \log \left[ \left( \frac{1-\pi}{\pi} \right)^{1/\gamma} x \right], \frac{\pi}{1-\pi} \right)$$

for  $\pi \in (0,1)$  with  $\gamma = (\delta_1 - \delta_0)/2$  and  $\kappa = (\delta_1 + \delta_0 - 4)/2$  where the function  $\tilde{V}$  is given by (9.5) above. The optimal stopping time in the initial problem (2.4) is given by

$$(9.11) \quad \tau_* = \inf \left\{ t \geq 0 \mid \frac{\pi}{1-\pi} \left( \frac{X_t}{x} \right)^\gamma \exp \left( -\frac{\kappa\gamma}{2} \int_0^t \frac{ds}{X_s^2} \right) \notin (b_0(X_t), b_1(X_t)) \right\}$$

(see Figure 3 above upon noting that the random variable on the left-hand side from the non-element sign equals  $\Phi_t^\varphi$  with  $\varphi = \pi/(1-\pi)$  for  $\pi \in (0,1)$  fixed) where  $b_0$  &  $b_1$  are expressed in terms of  $\tilde{b}_0$  &  $\tilde{b}_1$  by (7.35) respectively and  $\tilde{b}_0$  &  $\tilde{b}_1$  are a unique solution to the coupled system of nonlinear Volterra integral equations (9.3)+(9.4). The optimal decision function  $d_{\tau_*}$  equals  $i$  and we conclude that the dimension of the observed process  $X$  is  $\delta_i$  if the stopping in (9.11) happens at  $b_i$  for  $i = 0,1$ .

**Proof.** The identity (9.10) follows by combining (3.9)+(3.10) in Proposition 2 with (5.19) in Proposition 6 and the result of Theorem 15. The explicit form (9.11) follows from (9.6) in Theorem 15 combined with (5.15)-(5.18). The final claim on the optimal decision function follows from the general argument invoked following (2.6) above completing the proof.  $\square$

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